Analitical solution of the pressure vessel problem

We consider an isotropic thick-walled cylinder with inner and outer radii a and b, respectively. The cylinder is subjected to an internal pressure p_i, and external pressure p_0. We wish to determine the displacement, strain and stress fields in the cylinder. To that end we will employ a cylindrical coordinate system centered upon the centerline of the cylinder. Let us recall that the gradient operator in cylindrical coordinates (r, θ, z) has this form:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{1}$$

It is clear that because the material is isotropic, and the geometry and loading are perfectly symmetric, the displacement field has the special form:

$$u_\theta = 0, \qquad u_r = \hat{u}_r(r), \qquad u_z = \hat{u}_z(z). \tag{2}$$

The only non-zero strain components corresponding to this displacement field are

$$\epsilon_{rr} = \frac{du_r}{dr}, \qquad \epsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \epsilon_{zz} = \frac{du_z}{dz}. \tag{3}$$

We know that a fourth-order tensor $\mathbb{G}$ which maps symmetric tensors to symmetric tensors is isotropic if and only if there are scalars μ and λ such that

$$\mathbb{G}\mathbf{A} = 2\mu\mathbf{A} + \lambda(\operatorname{tr}\mathbf{A})\mathbf{1} \tag{4}$$

Thus, if the body is isotropic then $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon}$ has the specific form

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda(\mathrm{tr}\,\boldsymbol{\epsilon})\mathbf{1} \tag{5}$$

The stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ follow equation 5 and $\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{z\theta} = 0$.

It can be demonstrated that for the thick-walled cylinder we find that the equilibrium equations are satisfied provided

$$\frac{du_z}{dz} = \epsilon_{zz} \equiv \epsilon_0 \qquad \text{(a constant)} \tag{6}$$

and that the displacement field $u_r = \hat{u}_r(r)$ is of the form

$$u_r = Ar + \frac{B}{r} \tag{7}$$

The problem admits an algebraic solution:

$$u_r = \frac{(1+\nu)}{E} \frac{r}{\left(\frac{b}{a}\right)^2 - 1} \left\{ (1-2\nu) \left[p_i - p_0 \left(\frac{b}{a}\right)^2 \right] + (p_i - p_0) \left(\frac{b}{a}\right)^2 \right\} \tag{8}$$

and

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{\left(\frac{b}{a}\right)^2 - 1} \left[p_i - p_0 \left(\frac{b}{a}\right)^2 - (p_i - p_0) \left(\frac{b}{r}\right)^2 \right] \\
\sigma_{\theta\theta} &= \frac{1}{\left(\frac{b}{a}\right)^2 - 1} \left[p_i - p_0 \left(\frac{b}{a}\right)^2 + (p_i - p_0) \left(\frac{b}{r}\right)^2 \right] \\
\sigma_{zz} &= \frac{2\nu}{\left(\frac{b}{a}\right)^2 - 1} \left[p_i - p_0 \left(\frac{b}{a}\right)^2 \right] + E\epsilon_0
\end{aligned} \tag{9}$$