The finite element method (FEM)

We will now give a few information about the FEM method, analyzing a particular case for it to be applied (linear elastostatic problem).

First we introduce the *local balance equations*:

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \forall\ \mathbf{x} \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{t} \quad \text{on } S_t \\ \mathbf{u} = \hat{\mathbf{u}} \quad \text{on } S_u \end{aligned} \tag{1}$$

Let us define a *shape function* $\boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}})$. We will now multiply the first of the equations 1 by the shape function, and integrate over the domain B.

$$\int_B \left[(\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}}) + \mathbf{b} \cdot \boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}}) \right] dv = 0 \quad \forall\ \boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}}) \tag{2}$$

but knowing that

$$\nabla \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\varphi} = \nabla \cdot (\boldsymbol{\sigma}\boldsymbol{\varphi}) - \boldsymbol{\sigma} \cdot \nabla\boldsymbol{\varphi} \tag{3}$$

we can rewrite equation 2

$$\int_B \left[\nabla \cdot (\boldsymbol{\sigma}\boldsymbol{\varphi}) - \boldsymbol{\sigma} \cdot \nabla\boldsymbol{\varphi} + \mathbf{b} \cdot \boldsymbol{\varphi} \right] dv = 0 \quad \forall\ \boldsymbol{\varphi} \tag{4}$$

Then, by applying the *divergence theorem* and the *Neumann condition* ($\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}$ on S_t),

$$\int_B (\boldsymbol{\sigma} \cdot \nabla \boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}}) - \mathbf{b} \cdot \boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}}))\, dv - \int_{S_t} \mathbf{t} \cdot \boldsymbol{\varphi}(\mathbf{x}, \overline{\mathbf{x}}) \;\; da = 0 \quad \forall\, \overline{\mathbf{x}} \tag{5}$$

We will now operate a first *discretization*: we will only choose a finite number of shape functions (n), such that

$$\int_B (\boldsymbol{\sigma} \cdot \nabla \boldsymbol{\varphi_i}(\mathbf{x}) - \mathbf{b} \cdot \boldsymbol{\varphi_i}(\mathbf{x}))\, dv - \int_{S_t} \mathbf{t} \cdot \boldsymbol{\varphi_i}(\mathbf{x}) \;\; da = 0 \quad \forall\, i \tag{6}$$

The second *discretization* is done as it follows: we define a *test function* (often polinomial) $\boldsymbol{\psi}(\mathbf{x})$ such that

$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{n} a_i \boldsymbol{\psi_i}(\mathbf{x}) \tag{7}$$

By choosing $\boldsymbol{\varphi_i} \equiv \boldsymbol{\psi_i}$, equation 6 will eventually become

$$a_j \int_B \mathbb{C}(\nabla \boldsymbol{\psi_j}) \cdot \nabla \boldsymbol{\psi_i} \;\; dv - \int_B \mathbf{b} \cdot \boldsymbol{\psi_i} \;\; dv - \int_{S_t} \mathbf{t} \cdot \boldsymbol{\psi_i} \;\; da = 0 \quad \forall\, i \tag{8}$$

which we can turn in a matrix form equation

$$\mathbf{Ka} - \mathbf{b} - \mathbf{t} = 0 \tag{9}$$

where

$$\begin{aligned} K_{ij} &= \int_B \mathbb{C}(\nabla \boldsymbol{\psi_i}) \cdot \nabla \boldsymbol{\psi_j} \;\; dv \\ b_i &= \int_B \mathbf{b} \cdot \boldsymbol{\psi_i} \;\; dv \\ t_i &= \int_{S_t} \mathbf{t} \cdot \boldsymbol{\psi_i} \;\; da \end{aligned}$$

We will obtain the unknown vector $\mathbf{a}$ by solving the algebraic system 9. We can finally calculate the discretized displacement field (see equation 7).