COMSOL® implementation

Let us suppose we don't know from the start that the radial component of the displacement field u_r has the specific form given in the previous paper ($u_r = Ar + \frac{B}{r}$), but we still wish to solve the cylindrical pressure vessel problem, in case of plain strain ($\epsilon_0 = 0$). The *Weak form PDE* environment in COMSOL® allows us to set the problem. We found the *2D axisymmetric* workspace more helpful in order to analyze the final results (the 2D coordinates will be called r, z instead of x, y and the software provides a revolution of the geometry around the straight line $r = 0$ when the solution is generated).

We chose a steel thick-walled cylinder, subjected to an external pressure p_0 and an internal pressure p_i. Here are the values of the parameters:

1. Geometry:
   - internal radius $a = 25\,\mathrm{mm}$;
   - external radius $b = 50\,\mathrm{mm}$;
   - axial length $l = 200\,\mathrm{mm}$;
2. Material (*Lamé moduli*):
   - $\lambda = 118.715\,\mathrm{GPa}$;
   - $\mu = 79.143\,\mathrm{GPa}$;
3. Forces:
   - $p_0 = 1.013\,\mathrm{bar}$ (atmospheric pressure);
   - $p_i = 50\,\mathrm{bar}$:

The strain-displacement relation and the stress-strain constitutive relation (remembering that $\boldsymbol{\epsilon_z} \equiv \boldsymbol{\epsilon_0} = 0$) are:

$$\epsilon_{rr} = \frac{du_r}{dr}, \qquad \epsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \epsilon_{zz} = \frac{du_z}{dz}. \tag{1}$$

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{1} \tag{2}$$

Let us call the body B, the internal lateral surface S_i and the external lateral surface S_0. We can now state the weak formulation:

$$\int_B (\sigma_{rr}\delta\epsilon_{rr} + \sigma_{\theta\theta}\delta\epsilon_{\theta\theta})\, 2\pi r dr dz - \int_{S_i} p_i \delta u_r 2\pi r dz + \int_{S_0} p_0 \delta u_r 2\pi r dz = 0 \tag{3}$$

The software operates an integration over a 2D plane (coordinates r, z). The 2D axisymmetric environment, in fact does only provide a revolution of the geometry and it is unable to operate integration in cylindrical coordinates. Therefore, we must make explicit that

$$drdydz \rightarrow rdrd\theta dz = 2\pi r dr dz \tag{4}$$

Where we have defined y the coordinate that refers to the third (non-existent) dimension; we have also assumed the integrating variables (equation 3) being independent from θ.

In order to obtain more accurate results, we chose an *extremely dense mesh* (figure 1).

We will now show the solutions of the problem.

Figure 2a shows the three-dimensional view of the cylinder. As the solution is identical in every plane that is perpendicular to the rotation axis, we can display a cut plane to visualize the cross-section of the cylinder (figure 2b).

At this point the we want to compare the trends of $u_r, \sigma_{rr}, \sigma_{\theta\theta}$ and σ_{zz} along the cut line shown in figure 4 with the ones given by equations 8 and 9 on the analitical solution paper.

The exact trends (figure 6) have been plotted using MATLAB®.

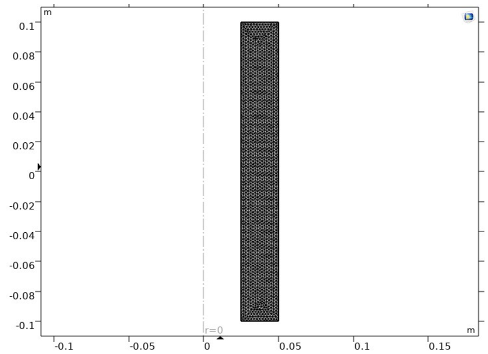

Figure 1: Mesh

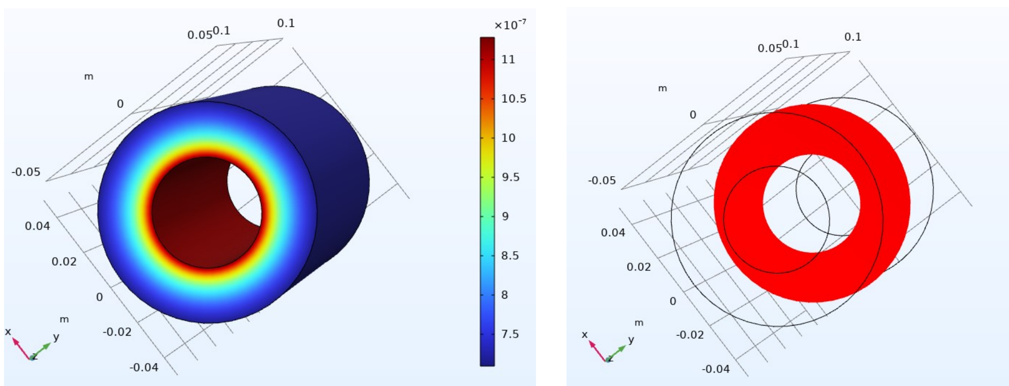

Figure 2: from the left: a. Radial displacement u_r b. Cut plane

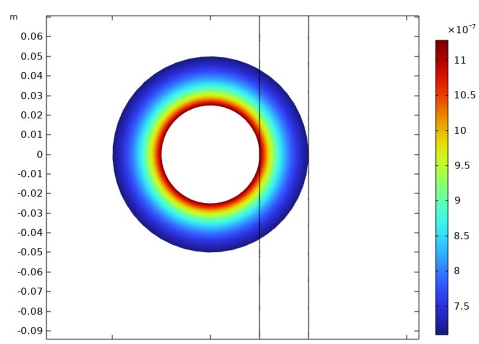

Figure 3: Cross-section

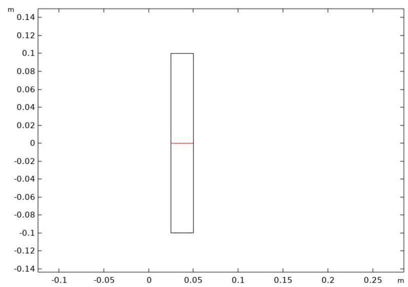

Figure 4: Cut line

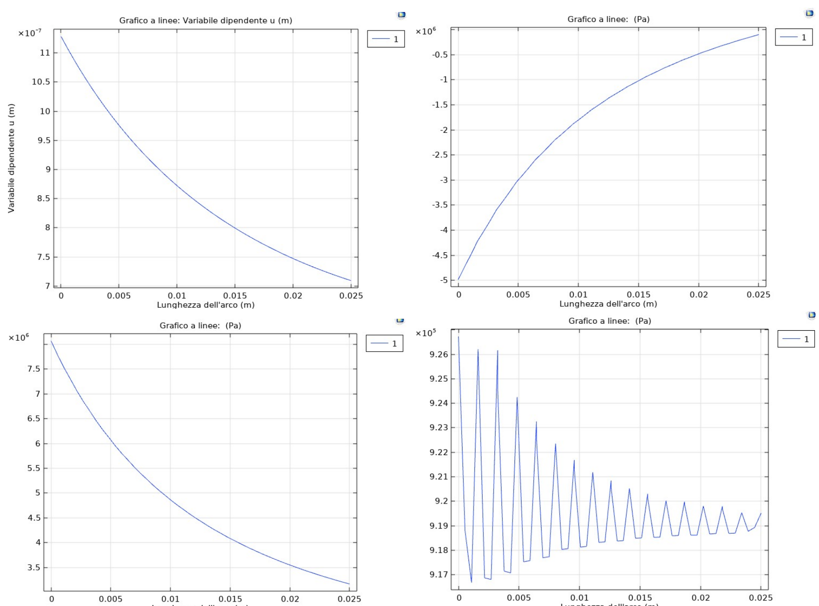

Figure 5: Numerical solutions. From the upper left: a. Radial displacement u_r b. Radial stress σ_{rr} c. Circumferential stress $\sigma_{\theta\theta}$ d. Axial stress σ_{zz}

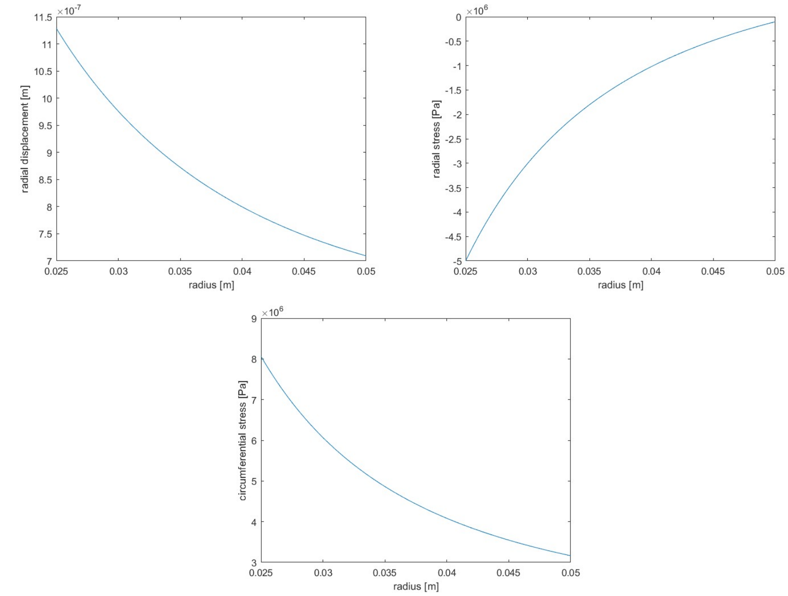

Figure 6: Algebraic solutions. From the upper left: a. Radial displacement u_r b. Radial stress σ_{rr} c. Circumferential stress $\sigma_{\theta\theta}$

As we can see, the solutions match almost perfectly. Let us note that, according to the third of equations 9 mentioned before, the value of σ_{zz} is independent from the radius r: its exact value was calculated to be $\sigma_{zz} = 0.9184\,\text{MPa}$; the numerical solution (figure 5d) oscillates around this value within a range of few kPa.

Now that we have confirmed the match between the solutions, we can see how the body behaves when we increase the internal pressure p_i. To do this, we employed a *sweep* parameter (`range(1,1,7)`, which means an array of 7 ordered numbers from 1 to 7) which will be multiplied by p_i within its weak expression. The range of pressures p_i will then go from 50 bar to 350 bar. The results are shown in figures 7a-d.

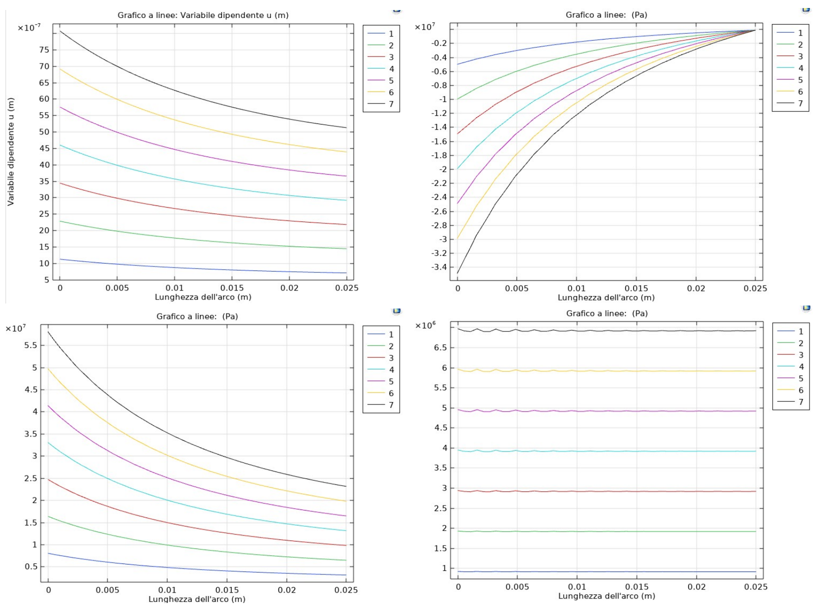

Figure 7: From the upper left: a. u_r b. σ_{rr} c. $\sigma_{\theta\theta}$ d. σ_{zz}