Linear elasticity and weak formulation of the elastostatic problem

1 Linear elasticity

1.1 Total energy of a system

The total energy of a mechanical system consists of:

- *internal potential energy*;
- *kinetic energy*
- *work done by external forces*;

Let us take an elastic body B, with its boundary being ∂B. We can associate a vector $\mathbf{x}$ to every point of the body B. We will divide the boundary ∂B in two parts S_u and S_t, such that $S_u \cup S_t = \partial B$ and $S_u \cap S_t = 0$. On S_t we impose an external mechanical traction $\mathbf{t}(\mathbf{x})$ (it corresponds to a contact force per unit area). On S_u we impose a constraint on the displacement field, such that $\mathbf{u} = \hat{\mathbf{u}}$. We can also apply mechanical forces per unit volume inside the body B: we define them with the field $\mathbf{b}(\mathbf{x})$.

The total energy of the body B will be

$$\Pi = E_{int} + W_{ext} \tag{1}$$

where E_{int} is the internal potential energy of the body, whereas W_{ext} is the work done by external forces. We have neglected the kinetic energy, as we assume that the behavior of the body B is *quasi-static*.

Let $\mathbf{u}$ be one of the possible displacement field of the body, and $\boldsymbol{\sigma}$ the stress associated with the displacement field, the internal potential energy can be defined as

$$E_{int}\{\nabla\mathbf{u}, \boldsymbol{\sigma}\} = \frac{1}{2}\int_B \boldsymbol{\sigma} \cdot \nabla\mathbf{u} \ \ dv \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

Since $\boldsymbol{\sigma}$ is symmetric, if we define $\boldsymbol{\epsilon} = \mathbf{sym}\{\nabla\mathbf{u}\} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$, we can rewrite equation 2:

$$E_{int}\{\nabla\mathbf{u}, \boldsymbol{\sigma}\} = \frac{1}{2}\int_B \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \ \ dv \tag{3}$$

Furthermore, with the constitutive relation of linear elasticity

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon} \tag{4}$$

with $\mathbb{C}$ being a symmetric fourth-order tensor, we notice that the internal energy E_{int} does only depend on the field $\nabla\mathbf{u}$:

$$E_{int}\{\nabla\mathbf{u}\} = \frac{1}{2}\int_B \mathbb{C}\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \ \ dv \tag{5}$$

The work done by external forces W_{ext} can be easily written as

$$W_{ext}\{\mathbf{u}\} = -\int_B \mathbf{b} \cdot \mathbf{u} \ \ dv - \int_{S_t} \mathbf{t} \cdot \mathbf{u} \ \ da \tag{6}$$

where we have assumed that forces and displacement are related.

Now we have an expression for the total energy of an elastostatic system:

$$\Pi\{\mathbf{u}\} = E_{int}\{\nabla\mathbf{u}\} + W_{ext}\{\mathbf{u}\} = \frac{1}{2}\int_B \mathbb{C}\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \ \ dv - \int_B \mathbf{b} \cdot \mathbf{u} \ \ dv - \int_{S_t} \mathbf{t} \cdot \mathbf{u} \ \ da \tag{7}$$

1.2 Principle of minimum potential energy

The *principle of minimum potential energy* states that every system tends to achieve a state which minimizes its total energy.

Let $\Pi\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be the total energy of a system, which depends on the generic fields $\mathbf{x}, \mathbf{y}, \mathbf{z}$. If $\overline{\mathbf{x}}, \overline{\mathbf{y}}, \overline{\mathbf{z}}$ are the actual fields which allow the system to be in equilibrium, then $\delta\Pi\{\overline{\mathbf{x}}, \overline{\mathbf{y}}, \overline{\mathbf{z}}; \delta\mathbf{x}, \delta\mathbf{y}, \delta\mathbf{z}\} = 0 \ \forall$ admissible $\delta\mathbf{x}, \delta\mathbf{y}, \delta\mathbf{z}$.

That said, if the displacement field $\mathbf{u}$ is actually the equilibrium field of our system, then, recalling equation 7:

$$\delta\Pi\{\mathbf{u}; \delta\mathbf{u}\} = \delta E_{int}\{\nabla\mathbf{u}; \nabla\delta\mathbf{u}\} + \delta W_{ext}\{\mathbf{u}; \delta\mathbf{u}\} = 0 \quad \forall \text{ admissible } \delta\mathbf{u} \tag{8}$$

noted that $\delta(\nabla\mathbf{u}) = \nabla\delta\mathbf{u}$. In this case with the expression $\forall$ admissible $\delta\mathbf{u}$ we mean all the possible fields for which $\mathbf{u} = \hat{\mathbf{u}}$ on S_u. If $\tilde{\mathbf{u}} = \mathbf{u} + \zeta\delta\mathbf{u}$, also for the variation $\tilde{\mathbf{u}} = \hat{\mathbf{u}}$ on S_u. It is therefore obvious that $\delta\mathbf{u} = 0$ on S_u.

Remembering how to compute the variation of a functional (**??**), then

$$\delta\Pi\{\mathbf{u}; \delta\mathbf{u}\} = \frac{1}{2}\int_B \frac{\partial(\mathbb{C}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon})}{\partial(\nabla\mathbf{u})}\cdot\nabla\delta\mathbf{u} \ dv - \int_B \frac{\partial}{\partial\mathbf{u}}(\mathbf{b}\cdot\mathbf{u})\cdot\delta\mathbf{u} \ dv - \int_{S_t}\frac{\partial}{\partial\mathbf{u}}(\mathbf{t}\cdot\mathbf{u})\cdot\delta\mathbf{u} \ da \tag{9}$$

We will avoid the passages that lead to the final expression to not weigh the paragraph down. That said, we can finally exploit the *weak formulation of the linear elastostatic problem.*

$$\int_B \boldsymbol{\sigma}\cdot\nabla\delta\mathbf{u} \ dv - \int_B \mathbf{b}\cdot\delta\mathbf{u} \ dv - \int_{S_t}\mathbf{t}\cdot\delta\mathbf{u} \ da = 0 \quad \forall \text{ admissible } \delta\mathbf{u} \tag{10}$$

It can be easily demonstrated that the *weak formulation* is equivalent to the *local balance equations* and to the *principle of virtual power.*