Mathematical background on variational calculus

1 Variation of a functional

We will first introduce the *variational operator*. Let $\mathbf{H}\{\mathbf{u}\}$ be a quantity that depends on $\mathbf{u}$. Let $\mathbf{w}(\mathbf{x})$ be an arbitrary function and let ζ a real variable in the range $-\zeta_0 < \zeta < \zeta_0$, with ζ_0 a small parameter such that $\|\zeta\mathbf{w}(\mathbf{x})\| < h$. We define the variation of $\mathbf{H}\{\mathbf{u}\}$ with respect to $\mathbf{w}$

$$\delta\mathbf{H}\{\mathbf{u};\mathbf{w}\} := \frac{d}{d\zeta}\mathbf{H}\{\mathbf{u}+\zeta\mathbf{w}\}\bigg|_{\zeta=0} \tag{1}$$

The function $\zeta\mathbf{w}$ is often called *test function*.

The function $\mathbf{w}$ corresponds to a variation of $\mathbf{u}$: suppose that $\mathbf{H}$ is the identity operator $\mathbf{H}\{\mathbf{u}\} = \mathbf{u}$, then

$$\delta\mathbf{H} = \delta\mathbf{u} := \frac{d}{d\zeta}\mathbf{H}\{\mathbf{u}+\zeta\mathbf{w}\}\bigg|_{\zeta=0} = \frac{d}{d\zeta}(\mathbf{u}+\zeta\mathbf{w})\bigg|_{\zeta=0} = \mathbf{w} \tag{2}$$

From now on we can write:

$$\delta\mathbf{H}\{\mathbf{u};\mathbf{w}\} = \delta\mathbf{H}\{\mathbf{u};\delta\mathbf{u}\} = \delta\mathbf{H}\{\mathbf{u}\} \tag{3}$$

Let us take a known function $\mathbf{F}(\mathbf{x}, t, \mathbf{u}(\mathbf{x}, t), \partial_x\mathbf{u}(\mathbf{x}, t), \partial_t\mathbf{u}(\mathbf{x}, t))$. For the sake of simplicity we remove the time dependency. We define the *functional*

$$I\{\mathbf{u}(\mathbf{x})\} = \int_{\mathbf{x_0}}^{\mathbf{x_1}} \mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}))d\mathbf{x} \tag{4}$$

where $\mathbf{u}'(\mathbf{x}) = \partial_x \mathbf{u}(\mathbf{x})$.

We know that

$$I : \mathbf{C} \to \mathbb{R}$$

where $\mathbf{C}$ is the space of distribution of the possible functions $\mathbf{u}(\mathbf{x})$.

The variation of the functional I is written as it follows:

$$\delta I\{\mathbf{u}; \delta\mathbf{u}\} = \int_{x_0}^{x_1} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}} \delta\mathbf{u} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}'} \delta\mathbf{u}' \right) d\mathbf{x} \tag{5}$$

2 Functional minimization problem

At this point, we need to find a function $\mathbf{u}(\mathbf{x})$, over a set of functions $\mathbf{C}$, which minimizes the functional $I\{\mathbf{u}(\mathbf{x})\}$ defined in equation 4.

In order to minimize our functional, we need to clarify our definition of *functional minimization.*

1. Let $\mathbf{w}(\mathbf{x})$ be an arbitrary function and let ζ a real variable in the range $-\zeta_0 < \zeta < \zeta_0$, with ζ_0 a small parameter such that $\|\zeta \mathbf{w}(\mathbf{x})\| < h$.
2. A function $\mathbf{u}(\mathbf{x})$ is said to uniquely minimize the functional $I\{\mathbf{u}(\mathbf{x})\}$ over the set $\mathbf{C}$, in a neighborhood of size h around $\mathbf{u}(\mathbf{x})$, if

$$I\{\mathbf{u}(\mathbf{x}) + \zeta \mathbf{w}(\mathbf{x})\} \geq I\{\mathbf{u}(\mathbf{x})\}$$

It can be demonstrated that the necessary condition for $\mathbf{u}(\mathbf{x})$ to be a minimizer of I is

$$I\{\mathbf{u}; \mathbf{w}\} = 0 \quad \forall \text{ admissible } \mathbf{w} \tag{6}$$

which, according to equation 5, can be rewritten as it follows:

$$\delta I\{\mathbf{u}; \mathbf{w}\} = \int_{x_0}^{x_1} \left(\frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}')}{\partial \mathbf{u}} \mathbf{w} + \frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}')}{\partial \mathbf{u}'} \mathbf{w}' \right) d\mathbf{x} \quad \forall \text{ admissible } \mathbf{w} \tag{7}$$