Planning and Reasoning

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1 Modal Logic

In propositional logic we can formalize things as propositional facts that can be either true or false (example: x = *Today is raining*, y = *Tomorrow will rain*, etc.). This is fine if we have a short number of conditions, but the problem is that we can't express more general facts without introducing an infinite number of variables (example: if we want to say that it will not rain forever, we would have to explicitily define that it won't rain on day x, for an infinite number of days). Additionally, facts in propositional logic can be completely uncorrelated and we would have no way to tell. In modal logic, instead, we can have this distinction. Temporal logic is one of the main applications of modal logic. We can also have a situation where we have a set of possible scenarios that are not related to time which may be possible, so we wouldn't know in which situation we are in, but we know that it's one of the situations in the set. This is useful to express that something is true in one or multiple situations.

The motivation for modal logic is that we want to model a case in which we have a set of facts that could be true or false not in just one situation but in a set of situations (which could be alternative situations, situations over time, the possible outcome of a program). In particular for programs, we can use modal logic to model the outcome of the program from the set of inputs.

1.1 Syntax

We have:

- Propositional variables (elementary conditions); usually indipendent from each other unless we have additional information about them
- Connectives: $\neg, \wedge, \vee$
- Unary modal operators: $\Diamond, \Box$:
  - $\Box x$ means that, in all possible situations, x is always true
  - $\Diamond x$ means that, in all possible situations, x is or will be true

Example: $x = Itrains$. If we want to say that today it rains, we simply use x. If we want to say that today it rains but it won't rain forever we use $x \wedge \neg\Box x$. This could also be expressed as $x \wedge \Diamond\neg x$ (there is a one or more scenarios where x is false).

We can apply modal operators to both literals and more complex formulas (also nested operators). For example, $\Box\Diamond x$ can be interpreted as "Whichever day I consider, it rains in the future of that day".

We mainly use temporal modal logic because it's easier to understand since the set of scenario lies on the line of time, whereas in other modal logics the set may simply be a group.

Rules for precedence:

- Unary operators $\neg, \Box$ and $\Diamond$ have the same precedence
- Binary connectives $\wedge$ and $\vee$ have less precedence than unary operators

Note that $\Diamond$ is complementary to $\Box$: $\Diamond F = \neg\Box\neg F$.

1.2 Semantics

A formula F cannot be evaluated with a single interpretation but always a set of interpretations. The scenario of the interpretation depends on the application: for example, $\Box x$ can be interpreted as "x is true in all situations considered possible", "x is true now and in all future time points" (temporal modal logic), "x ought to be true", "x is true after the execution of a program".

1.3 Types of modal logic

We mainly consider two types:

1. S5: truth in a set of possible scenarios (they all refer to the same time)
2. TL: truth in the future (only one scenario at the time)

Both have a set of interpretations.

1.4 Kripke Model

We can use modal logic for multiple purposes:

- Evaluation: knowing the sitations, is $\Box x$ true? This is not very common, since it requires perfect knowledge of the future.
- Given a formula in modal logic, check if it is satisfiable (check if it could be true)
- If $\Box(x \wedge y)$ is true, is $\Box x$ true?
- Check if a certain complex condition is true

For the last 3 cases, we don't want to consider them separately, we want to evaluate them regardless of the type of modal logic required; we want to use some unified semantics.

The problem is not defining a semantic, but having a unified semantic (so we could have an unified solver).

The situations we consider may be undifferentiated alternatives to each other (S5) or in a line (TS); the Kripke model is a generalization to all of them, it encompasses all these different cases. The Kripke model is a triplet, consisting a set of possible worlds (elements, for example: integers), for each world there is a propositional interpretation and the information about how the worlds are linked (this changes according to the type of modal logic: S5: all worlds are linked to another, TL: the links roughly follow a line).

Why having worlds and intepretations? We could have a set of propositional interpretations and their corresponding links, but this wouldn't work in TL because the same interpretation may occur multiple times. For example, in the sequence of states $ab, \neg ab, a\neg b, ab, ab, ab, ...$ (ab repeats forever), we have that ab occurs in different time points (so they have the same interpretation), but the modal formula $\Box a$ is false in the first state and true in the fourth.This is representated by having a set of time points (worlds) and the interpretation attached to each of them. In S5 this is not strictly necessary, though since we want an unified semantic, we use worlds in S5 as well.

Note: in S5 the transition is bidirectional from one state to every other and itself (loop), in TL the transition is directional:

- from one state to the next
- from one state to itself (loop)
- from one state to another (transition)

A Kripke model is a set of points connected with arrows, labeled with their interpretation; it's a directed graph with labels in each node. Different modal logic differ only on the arrows. The formal definition follows that of a graph with the interpretation attached to each node: $KM = \langle R, S, V\rangle$, where R is the set of worlds, V is the assignment function from S to the set of propositional interpretations, R is the set of edges (a subset of $S \times S$).

In TL we only allow models in which the accessibility relation is aligned with the propositional interpretation. In S5 we only allow models in which the accessibility relation is universal: R must contain every pair of worlds.

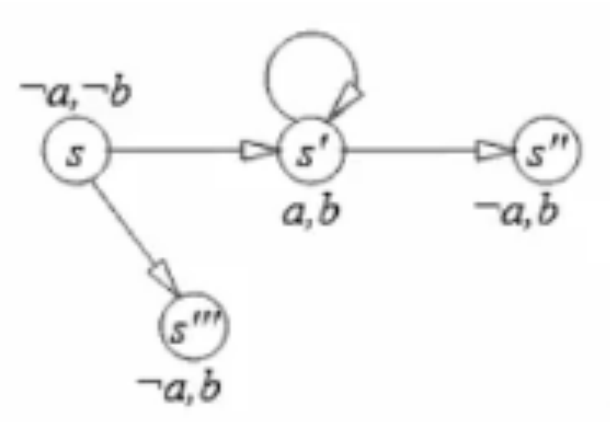
To evaluate a formula, we need both a KM and a specific world (usually, the current one); this is necessary because every formula that does not contain the modal operator is evaluated in the world.

- A variable x is true in $\langle R, S, V\rangle, s$ if x is true in the current interpretation of the world $V(s)$
- A formula $F \wedge G$ is true in $\langle R, S, V\rangle, s$ if both F and G are true in $\langle R, S, V\rangle, s$ (same for $\neg$ and $\vee$)
- The modal formula $\Box F$ is true in $\langle R, S, V\rangle, s$ if F is true in $\langle R, S, V\rangle, s', \forall s' \in S | \langle s, s'\rangle \in R$. We evaluate every subformula in all linked states; if F is true in all of them, then it's true in the initial state.
- The modal formula $\Diamond F$ is true in s if F is true in a world s' such that $\langle s, s'\rangle \in R$.

Notation: $\langle R, S, V\rangle, s \models F$

Remember: we only check successors at depth 1 from the current world!

Example:



This is not an S5 or TL model, but still a valid KM. We evaluate the following formulae in this KM and the current state s:

- $\neg a$: it's true in s because $\neg a$ is true in $V(s)$
- $\neg a \vee b$: also true in s because $\neg a$ is true in s
- $\Box b$: b must be true at the end of each arrow starting from s. The arrows go to s' and s''', in both cases b is true, so $\Box b$ is true in s (though b isn't true in s).
- $\Box a$: it's not true in s since, following the arrows, a is true in s' but false in s'''.
- $\Box \neg a$: still false in s because, following the arrows, a is true in s' but false in s'''.
- $\Box \neg a$: is false in s' because s' is linked to itself, where $\neg a$ is false
- $\Box b$: is true in s' because b is true in both s' and s''.

Note that both $\Box a$ and $\Box \neg a$ are false in s.

If a formula is true in all the states of a model, then the formula is true in a model: $\langle R, S, V \rangle \models F$ iff $\langle R, S, V \rangle, s \models F, \forall s \in S$.

Depending on the modal logic, the same formula may be true in all models. For example $\Diamond F \Rightarrow \Box \Diamond F$ is always true in S5 but may be false in TL.