

Tableaux for first-order logic

we extend the rules for propositional logic

principle is the same:

break the formula into its components until contradiction can be detected by simple inspection

Rules

propositional:	① $\frac{A \wedge B}{A}$ B	② $\frac{A \vee B}{A \mid B}$
	③ $\frac{\forall x F}{F[t/x]}$ <p>where t is a ground term</p>	④ $\frac{\exists x F}{F[c/x]}$ <p>where c is a new constant</p>

③ Existential quantification rule

$$\frac{\exists x F}{F[c/x]}$$

c is a new constant

this is Skolemization

$\exists x F$ is satisfiable if and only if $F[c/x]$ is so

the second formula is "almost" **equivalent** to the first
(almost=apart from c , not in the original formula)

③ Universal quantification rule

$$\frac{\forall x F}{F[t/x]}$$

$\forall x F$ means: for every object of the domain, F is true

a term t is evaluated to an object of the domain

therefore, $F[t/x]$ is true

this is a **consequence** of the first formula
(note: not equivalent, just a consequence)

New constants

when expanding $\forall x F$ to $F[t/x]$ term t may contain all function symbols and constants, including the ones created by Skolemization

Example

$$\{\forall x P(x), \exists x \neg P(f(x))\}$$

Example, first step

$$\begin{array}{c} \forall x P(x) \\ | \\ \exists x \neg P(f(x)) \end{array}$$

place formulae in column

Example, expansion of existential quantifier

$$\begin{array}{c} \forall x P(x) \\ | \\ \exists x \neg P(f(x)) \\ | \\ \neg P(f(c)) \end{array} \quad \text{expand } \exists$$

A dotted arrow points from the existential quantifier $\exists x$ in the second formula to the constant c in the third formula.

create a new constant c

next: expand \forall

Example, expansion of universal quantifier

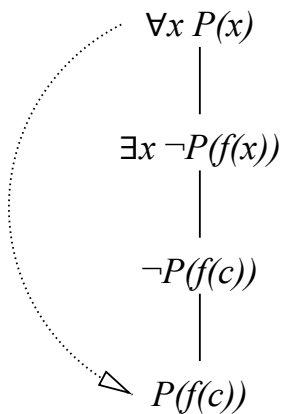
to expand $\forall x P(x)$, we can use any ground term t

possible choices:

- c
- $f(c)$
- $f(f(c))$
- ...

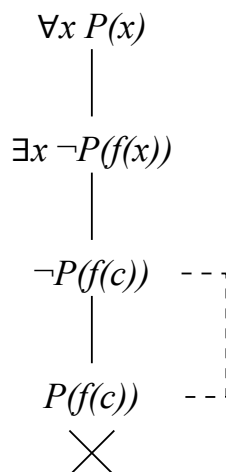
we choose $f(c)$ for the obvious reasons

Example, expansion of universal quantifier



expand $\forall x P(x)$ using $f(c)$ as the term

Closure



the branch contains $\neg P(f(c))$ and $P(f(c))$

opposite literals \Rightarrow the branch can be closed

all branches closed (only one exists!) \Rightarrow set is unsatisfiable

Second example

$$\{\forall x \neg P(x), \exists x (P(x) \vee P(f(x)))\}$$

First step

$$\begin{array}{c} \forall x \neg P(x) \\ | \\ \exists x (P(x) \vee P(f(x))) \end{array}$$

which formula to expand first?

no constant in the tableau so far, so no ground term

in order to apply the rule for \forall , we would have to create one

Second step

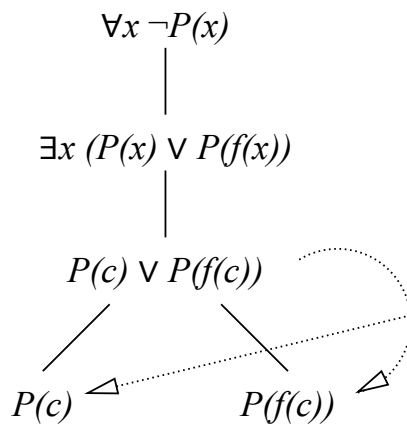
$$\begin{array}{c} \forall x \neg P(x) \\ | \\ \exists x (P(x) \vee P(f(x))) \\ | \\ \triangleleft P(c) \vee P(f(c)) \end{array}$$

existentially quantified formula expanded

the universal quantification expansion rule could be now applied

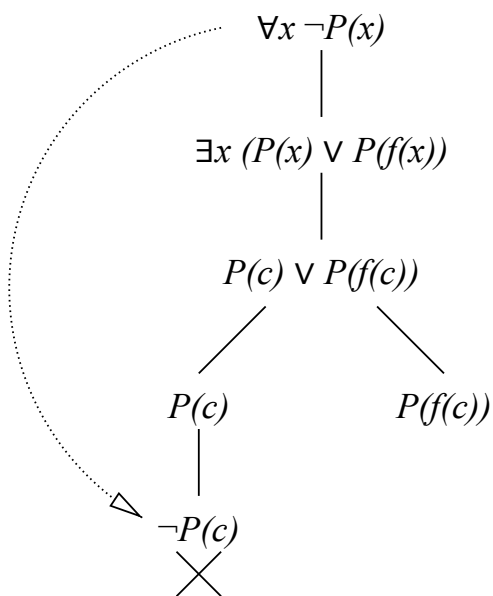
also the disjunction

Disjunction



disjunction expanded

Closure



first branch closed applying the rule for universal quantification

still an unclosed branch!

is set satisfiable?

Unsatisfiability of the set

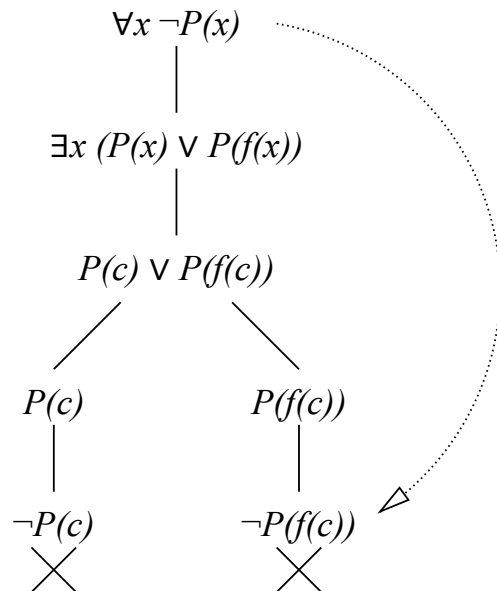
the set contains $\forall x \neg P(x)$

we have $P(f(c))$ in the unclosed branch

the two contradict each other: P cannot be always false but at the same time true for the element of the domain $f(c)$

we need to apply the rule for universal quantifiers again, generating $\neg P(f(c))$

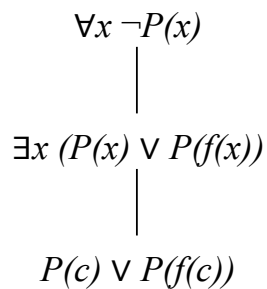
Multiple applications



$\forall x \neg P(x)$ expanded again,
now on the second branch and with a different ground term

all branches closed \Rightarrow set is unsatisfiable

Alternative development

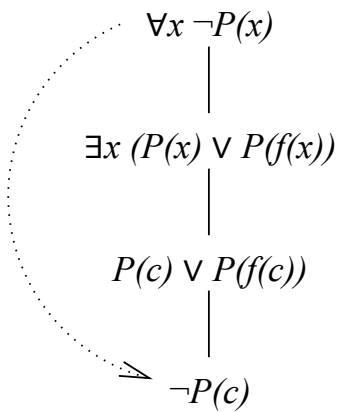


there was a choice, at this point:

expand the disjunction or the universal quantifier

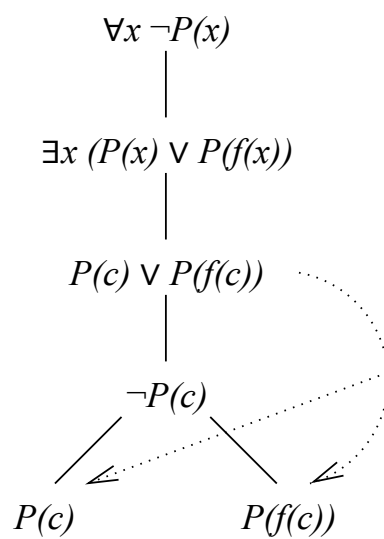
make the second choice now

Other choice



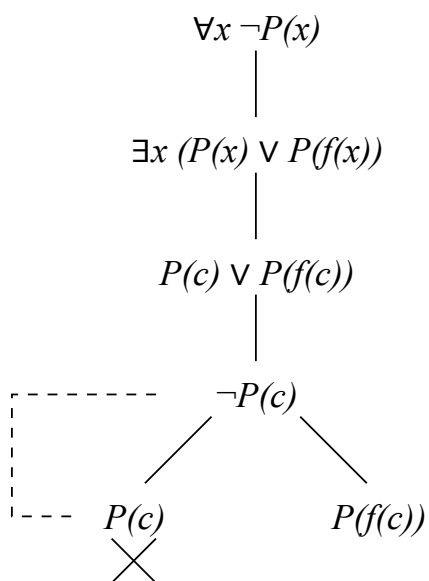
$P(c)$ could be useful to closing some branch
use c as the ground term

Alternative, one step more



disjunction expanded

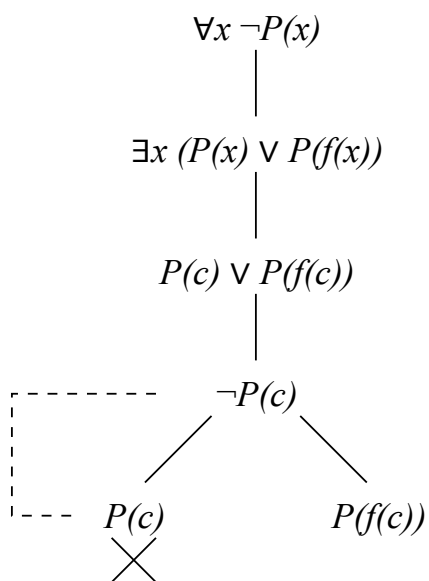
Closure



the first branch can be closed

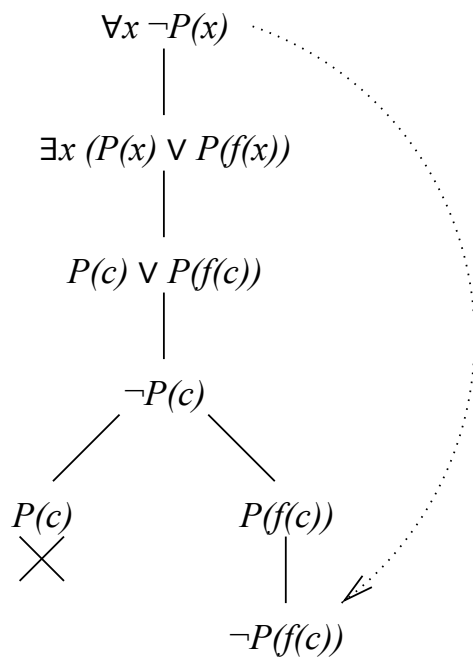
still an unclosed branch

Multiple application



$\forall x \neg P(x)$ already expanded *on the second branch*
(expansion done in the path from root to $P(f(c))$)

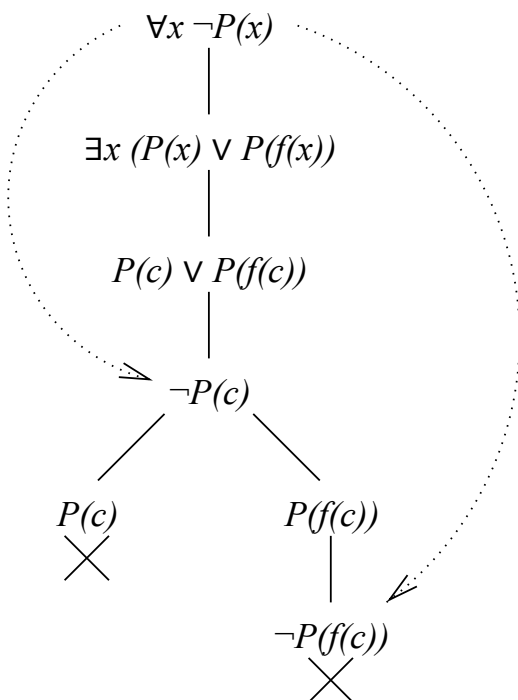
Alternative: unsatisfiability



expand universally quantified formula again

the second branch can now be closed

The complete tableau



formula $\forall x \neg P(x)$ was expanded **twice on the same branch**

necessary for closing the tableau

still a valid tableau for the set (proves unsatisfiability)

in general:

the same universally quantified formula may need to be expanded on the same branch more than once

Multiple application

in the previous case, the set

- had a tableau where a formula is expanded more than once on the same branch
- but also a tableau where this is not the case

example that cannot be closed without multiple applications:

$\{P(a), \forall x (P(x) \wedge \neg P(f(x)))\}$

At some point...

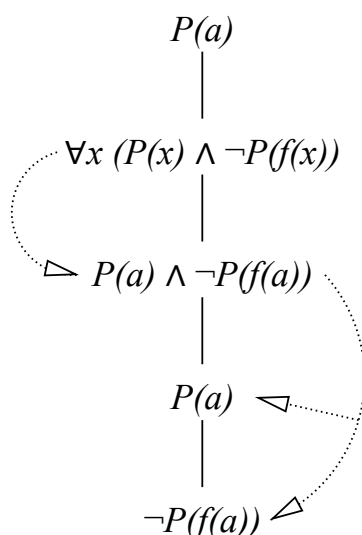
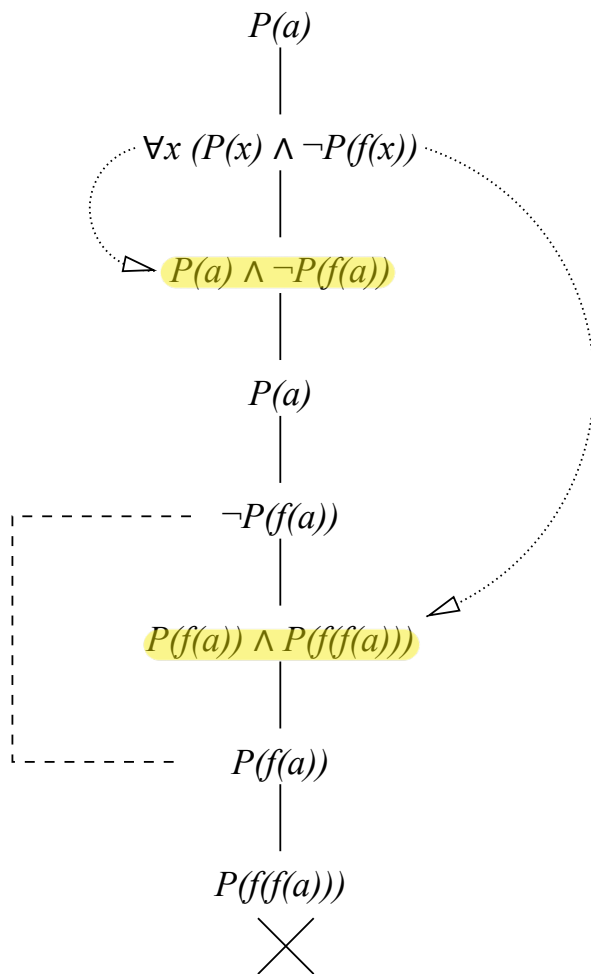


tableau after expanding every formula in every branch once

not closed

Multiple application, closure



closure requires expansion of the universally quantified formula a second time

Multiple application: conclusion

- some tableaux of some sets cannot be closed until the same existentially quantified formula is expanded on the same branch more than once (e.g., the previous example)
 - for some sets, this is the case for all tableaux (e.g., this example)
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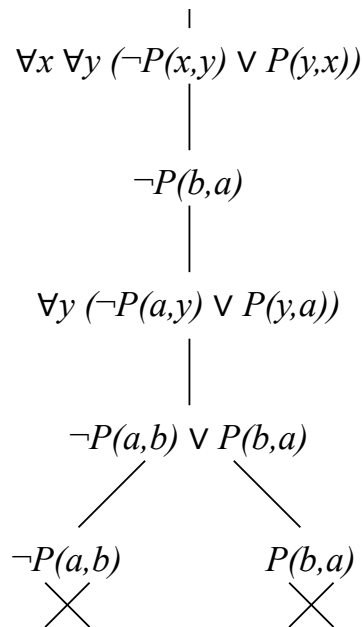
Example

$\{ \forall x \neg P(x,a), P(a,b), \forall x \forall y (\neg P(x,y) \vee P(y,x)) \}$

Solution

$\forall x \neg P(x,a)$

$P(a,b)$



is this the only possible tableau for this set?

how did we reach it?

Meaning of formulae

$\forall x \forall y (\neg P(x,y) \vee P(y,x))$ means that P is "symmetric"

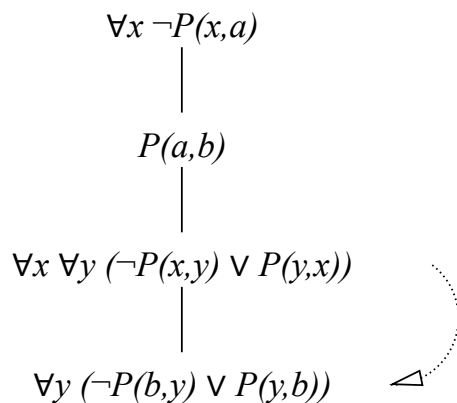
therefore, $P(a,b)$ and $\forall x \neg P(x,a)$ contradict each other

this information has been used when developing the tableau

- such kind of facts are easy to overlook
- difficult to encode in a general algorithm

what about making the wrong choice?

Another development for the same set

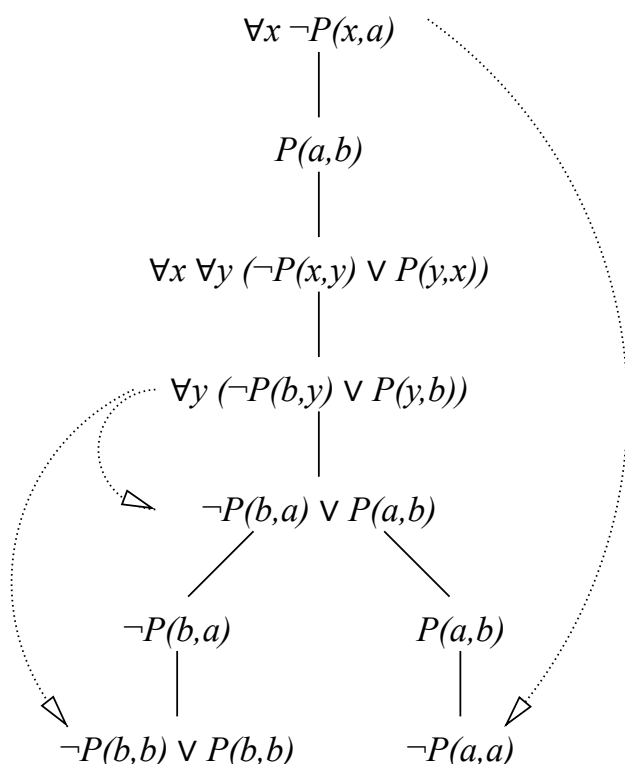


clearly useless

clearly = only knowing the way to prove unsatisfiability in this case

for a computer, b is a perfectly valid choice of a term: simple and mentioned in the tableau already = potentially useful

Further (wrong) development



not only lacks progress toward proving unsatisfiability

there are two branches now, instead of one

unsatisfiability is yet to be proved in both

making the wrong choices may make the problem even harder to solve

Choice of terms

rules for conjunction, disjunction and existential quantification are deterministic

only indeterminacy is their order of application

different for universal quantification:

$$\frac{\forall x F}{F[t/x]}$$

the problem is:

which term t should be used?

choose randomly

as shown in the previous example, this could lead to increase difficulty to close the tableau

wait until all other rules have been applied

the rule for universal quantification could be the only one left to apply

use only terms already in the tableau

done in the previous example, does not solve the problem

Infinite expansions

since the rule for universal quantification can be applied an unbounded number of times, it may lead to a forever-expanding tableau:

$$\{\forall x P(x), \neg P(f(a))\}$$

$$\begin{array}{c} \forall x P(x) \\ | \\ \neg P(f(a)) \\ | \\ P(a) \\ | \\ P(f(f(a))) \\ | \\ P(f(f(f(a)))) \\ | \\ \dots \end{array}$$

this is a valid (but useless) tableau for the set

Completeness

it can be proved that:

$S \text{ unsat} \Rightarrow \text{a closed tableau for } S \text{ exist}$

in the propositional case, a closed tableau is always keeping applying rules (even in a "wrong" order, such as disjunctions before conjunctions)

in the first order case, this is not always the case: as the previous example shows, it is possible to keep expanding the tableau forever even if the set is unsatisfiable

Complete proof procedure

an algorithm of application of the expansion rules that produces a closed tableau if the set is unsatisfiable

as seen before, applying rules at random yields to a complete proof procedure in the propositional case

reason: keeping applying rules leads to closure eventually, if the set is unsatisfiable

in the first-order case, applying rules at random does not yield to a complete proof procedure

reason: the tableau may not close even if the set is unsatisfiable

Fairness

general concept in computer science

more or less:

unfairness = avoiding doing something useful forever

in the case of tableau:

a policy of application of the rules is *unfair* if some tableau contains a branch where one of its formulae $\forall x F$ is never expanded with a specific ground term t

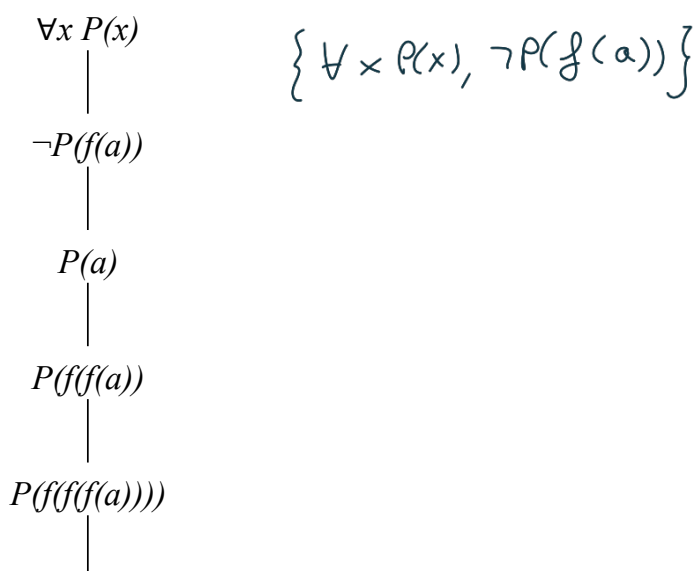
in other words: given a term t and branch of a tableau containing $\forall x F$, if the policy never expand the formula with t is unfair

fairness is the opposite of unfairness

given a term t and branch of a tableau containing $\forall x F$, the policy at some point will expand the formula using t

Effects of unfairness

unfairness means that the policy may forever "avoid" a term t while expanding $\forall x F$



...

in this case, the policy "skipped" $f(a)$

because of this, the tableau does not close even if the set is unsatisfiable

A complete proof procedure

a proof procedure that is fair is also complete

note that fairness concerns both $\forall x F$ and t , not just t

counterexample: $\{\forall x P(x), \forall x \neg P(x), Q(f(a))\}$

$$\begin{array}{c} \forall x P(x) \\ | \\ \forall x \neg P(x) \\ | \\ Q(f(a)) \\ | \\ P(a) \\ | \\ P(f(a)) \\ | \\ P(f(f(a))) \\ | \\ \dots \end{array}$$

if $\forall x \neg P(x)$ is never expanded, the tableau does not close

Choice of terms

a fair algorithm of choice of the application of the rules at some point will close all tableaux for unsatisfiable sets

still difficult to choose terms that lead to closure *fast*

terms that are useful are difficult to determine

solution: delay choice until usefulness of a term is clear

this is done employing unification (next page)