Basis A *basis* of a topology on set X is a collection $\mathcal{B}$ of subsets of X s.t.

- For each $x \in X$, there is $B \in \mathcal{B}$ s.t. $x \in B$ (i.e. $\bigcup_{B \in \mathcal{B}} B = X$)
- For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is $B \in \mathcal{B}$ w/ $x \in B \subset B_1 \cap B_2$.

The topology generated by a basis $\mathcal{B}$ is the set τ s.t.

$U \in \tau$ if, for every $x \in U$ there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.

Claim: τ is a topology.

Pf:
- $\emptyset \in \tau$ Trivially.
- For all $x \in X$ there is $B \in \mathcal{B}$ s.t. $x \in B \subset X$. Thus, $X \in \tau$.
- Take $U_i \in \tau$ for $i \in J$.
  Let $x \in \bigcup_{i \in J} U_i$, then $x \in U_j$, for some $j \in J$.
  Since $U_j \in \tau$, there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U_j \subset \bigcup_{i \in J} U_j$
  Hence, $\bigcup_{i \in J} U_i \in \tau$.
- Let $U_1, U_2 \in \tau$.
  Take $x \in U_1 \cap U_2$:
  We know $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$ for $B_1, B_2 \in \mathcal{B}$
  Thus $x \in B_1 \cap B_2$, so there is $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$
  So $B_1 \cap B_2 \in \tau$. ∎

Lemma: Let $\mathcal{B}$ be a basis for a top. τ on X then τ is the collection of all unions of elements of $\mathcal{B}$.

Pf: $\boxed{\supseteq}$ Take $\{B_i\}_{i \in J} \subset \mathcal{B}$. Take $x \in \bigcup_{i \in J} B_i$, then $x \in B_j \subset \bigcup_{i \in J} B_i$ for some $B_j \in \mathcal{B}$, thus $\bigcup_{i \in J} B_i \in \tau$.

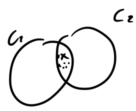
$\boxed{\subseteq}$ Take $U \in \tau$. We know that for all $x \in U$, there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.

Thus, $U = \bigcup_{x \in U} B_x$ ∎

Lemma: Let (X, τ) top. space, let $\mathcal{C}$ be a collection of open sets s.t. for each open set $U \in \tau$ and each $x \in U$, there is $C \in \mathcal{C}$ s.t. $x \in C \subset U$.

Then, $\mathcal{C}$ is a basis and generates τ.

Pf: i) Clearly, for each $x \in X$ there is $C \in \mathcal{C}$ w/ $x \in C$.
And for $C_1, C_2 \in \mathcal{C}$, $x \in C_1 \cap C_2$
$C_1 \cap C_2$ open (since C_1, C_2 open), so we have $C \in \mathcal{C}$ w/ $x \in C \subset C_1 \cap C_2$. Proving $\mathcal{C}$ is a basis.



ii) Let τ' be the top. generated by $\mathcal{C}$.

Claim: $\tau' = \tau$

Pf: $\boxed{\subset}$ Take $U \in \tau'$. That is $U = \bigcup_{C \in \mathcal{C}'} C$, for some $\mathcal{C}' \subset \mathcal{C}$
Since all $C \in \mathcal{C}$ open, we have U open, i.e. $U \in \tau$.
$\boxed{\supset}$ Take $U \in \tau$. Take $x \in U$, then for some $C \in \mathcal{C}$ we have $x \in C \subset U$
Thus, $U \in \tau'$. ∎

Subbasis: If X is a set, a subbasis for X is a collection $\mathcal{S} \subset X$ s.t. for each $x \in X$, there is $S \in \mathcal{S}$ w/ $x \in S$.

$$(\mathcal{S} \subset \mathcal{B} \subset \tau)$$

$$\underset{\text{subbasis}}{\mathcal{S}} \xrightarrow[\text{intersection}]{\text{finite}} \underset{\text{basis}}{\mathcal{B}} \xrightarrow[\text{unions}]{\text{arbitrary}} \underset{\text{topology}}{\tau}$$

Order Topology

Given a total order $<$ on a set X:

Then, $\mathcal{B} = \{(x,y) : x<y,\ x,y\in X\} \cup \{[x,y) : x<y,\ x,y \in X\} \cup \{(x,y] : x<y,\ x,y\in X\}$

is a basis which generates the order topology on X (assoc. w/ $<$).

E.g.

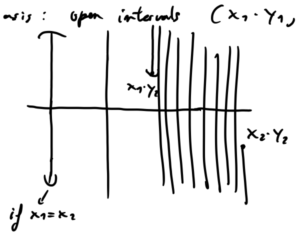
$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ w/ lexicographic order:

Let $x_1\cdot y_1, x_2\cdot y_2 \in \mathbb{R}^2$, then

$x_1\cdot y_1 < x_2 \cdot y_2$ iff

$x_1 < x_2$ or $x_1 = x_2 \wedge y_1 < y_2$.

Basis: open intervals $(x_1\cdot y_1, x_2\cdot y_2)$



Product Topology

Let X, Y be top. spaces.

Then $\mathcal{B} = \{U\times V : U \text{ open in } X,\ V \text{ open in } Y\}$

is a basis which generates the product topology on $X\times Y$.

Lemma: If B_X is a basis for X and B_Y is a basis for Y then $B' = \{U'\times V' : U\in B_X, V\in B_Y\}$ is a basis for the product topology.

Pf: apply a lemma:

Let W be open in $X\times Y$ and $x\cdot y\in W$. By def. of the product top., there is $U\times V\in B$ s.t. $x\cdot y \in U\times V\subset W$

So $x\in U$, $y \in V$. Since U open in X we have $U'\in B_X$ s.t. $x\in U' \subset U$,

similarly we have $V'\in B_Y$ s.t. $y\in V'\subset V$

So $x\cdot y\in U'\times V' \subset U\times V$

Thus, B' is a basis generating the product top. ∎

E.g.

$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$

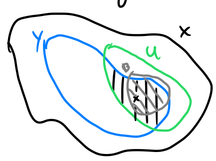
the std top. on $\mathbb{R}^2$ is the product top. of the std. top on $\mathbb{R}$.

Subspace Topology

For X top. space, $Y\subset X$, $\tau_Y = \{Y\cap U : U \text{ open}\}$ is the subspace topology of Y as a subspace of X.

Lemma: If $\mathcal{B}$ is a basis for X, then $\mathcal{B}_Y = \{B\cap Y : B\in\mathcal{B}\}$ is a basis for Y.

Pf picture:

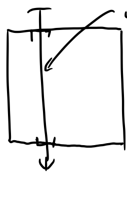


E.g. $I = [0,1]\subset\mathbb{R}$

$\underbrace{I\times I}_{\text{lexicographic order}} \subset \mathbb{R}\times\mathbb{R}$

the ordered squared I_o^2 ($[0,1]^2$ w/ ordered top.)

Observation: the subspace topology of $I\times I$ (as subspace of $\mathbb{R}\times\mathbb{R}$ w/ ordered top) is strictly finer than the ordered square top.



open in the subspace top.
but not open in the ordered top.

Closed sets and limit pnts.

X top. space, $A \subset X$ is closed if $X - A$ is open

thm: - $X, \emptyset$ are closed
- finite unions of closed sets are closed
- arbitrary intersection of closed sets are closed.

Lemma: $Y \subset X$ subspace, then $A \subset Y$ is closed iff $A = Y \cap C$ for some C closed in X.

Pf: $\boxed{\Leftarrow}$ Sps. $A = C \cap Y$ w/ C closed in X.
So $X - C$ is open in X. Hence $(X - C) \cap Y$ is open in Y. And $(X - C) \cap Y = Y - (C \cap Y) = Y - A$. Thus A closed in Y.
$\boxed{\Rightarrow}$ Sps A is closed in Y. Then $Y - A$ open, thus $Y - A = U \cap Y$ for U open in X. So $X - U$ closed in X. And $(X - U) \cap Y = Y - (U \cap Y) = Y - (Y - A) = A$ ■

Def: $A \subset X$ subset
$\mathrm{Int}(A) = \mathring{A} = \bigcup \{ U \text{ open in } X : U \subset A \}$, Clearly $\mathring{A}$ open in X. $\mathring{A}$ is the largest open subset of X contained in A.
$\mathrm{cl}(A) = \bar{A} = \bigcap \{ C \text{ closed in } X : A \subset C \}$, $\bar{A}$ closed in X, and is smallest closed set containing A.

Lemma: Y subspace X, $A \subset Y$ subset. Then the closure of A in Y is $\bar{A} \cap Y$ ($\bar{A}$ closure in X).

Pf: Let B the closure of A in Y.
$\boxed{\subseteq}$ $\bar{A}$ closed in X, hence $\bar{A} \cap Y$ is closed in Y. $A \subset (\bar{A} \cap Y)$. Thus $B \subset (\bar{A} \cap Y)$.
$\boxed{\supseteq}$ B is closed in Y, hence $B = C \cap Y$ for C closed in X. Thus, $A \subset (C \cap Y)$ and $A \subset C$, so $\bar{A} \subset C$. Which means $\bar{A} \cap Y \subset C \cap Y = B$ ■

Prop. $A \subset X$, $x \in X$
i) $x \in \bar{A}$ iff every neigh. U of x intersects A ($U \cap A \neq \emptyset$)
ii) $\mathcal{B}$ basis of X, $x \in \bar{X}$ iff every $B \in \mathcal{B}$ containing x intersects A.

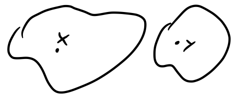
Pf: i) if $x \notin \bar{A}$, then for $U = X - \bar{A}$ we have $x \in U$ and U open, thus U neigh of x s.t. $U \cap \bar{A} = \emptyset$.
if U is neigh. of x s.t. $U \cap \bar{A} = \emptyset$ then $C = X - U$ is a closed set s.t. $A \subset C$ and $x \notin C$. Thus $x \notin \bar{A}$.
ii) Follows from i) and the fact that every open set is the union of base sets. ■

Def: For $A \subset X$, $x \in X$. x is <u>limit point</u> of A if every neigh. U of x intersects A in a point different than x.

Lemma: $\bar{A} = A \cup A'$, w/ A' is the set of limit pnts. of A.

Pf: $\boxed{\supseteq}$ $A \subset \bar{A}$. And for all $x \in A'$ and all neigh U of x $U \cap A \neq \emptyset$. So $A' \subset \bar{A}$.
$\boxed{\subseteq}$ Take $x \in \bar{A}$, $x \notin A$. Let U neigh of x, we know $U \cap A \neq \emptyset$, but $x \notin A$. So U intersects A in a pnt. different from x. Thus $x \in A'$.
So $\bar{A} \subset A \cup A'$. ■

Hausdorf Space A top. space X is said to be Hausdorff if every two distinct pnts. x, y has disjoint neighbourhoods.



Prop. Every finite set of a Hausdorff space is closed.

Pf: Every finite set is a finite union of singletones.

Take $x \in X$. Let $y \in X$, $y \neq x$. Since X Hausdorff y has neigh. not containing x, thus $y \notin \overline{\{x\}}$. So only $\{x\} = \overline{\{x\}}$. $\{x\}$ closed. ■

Separation axioms:

T_2: Hausdorff space

T_1: for all $x, y \in X$, y has a neigh. not containing x and viceversa.

Prop: X is T_1 $\Leftrightarrow$ pnts. are closed.

Pf: $\boxed{\Rightarrow}$ Former proof.

$\boxed{\Leftarrow}$ Sps. all pnts. are closed in X. Take $x, y \in I$. Since $\{x\}$ is closed, $y \notin \overline{\{x\}}$ which means that y has a neigh not cointaining x. ■

The finite complement topology is always T_1 (since for $x, y \in X$, $X \setminus \{y\}$ open) but not T_2. if X is infinite.

Sps. $x, y \in X$, $x \in U_1$ open, $y \in U_2$ open and $U_1 \cap U_2 = \emptyset$. We know $X = X \setminus (U_1 \cap U_2) = X \setminus U_1 \cup X \setminus U_2$ but this implies X finite.

Def: X top. space, a sequence $x_1, x_2, x_3, \dots$ in X converges to $x \in X$: if for every neigh. U of x there exist $N \in \mathbb{N}$ s.t. $x_n \in U$ for all $n > N$

Observation: If X is Hausdorff then the limit of all convergent sequence are unique.

Continuous function

Def: X, Y top. spaces a function $f: X \to Y$ is continuous if the pre-image of open sets are open.
i.e. for all V open in Y, $f^{-1}(V) = \{x \in X : f(x) \in V\}$ open in X.
Locally, f is continuous at $x \in X$ if for every neig. V of $f(x)$ there is a neigh. U of x s.t. $f(U) \subset V$.

Prop. f is continuous iff f is continuous in every pnt. $x \in X$.
Pf $\boxed{\Rightarrow}$ Sps. f is continuous, Take $x \in X$ and let V neigh. of $f(x)$. Since V open $f^{-1}(V)$ open, and $f(x) \in V$ implies $x \in f^{-1}(V)$. Thus $f^{-1}(V)$ is neig. of x s.t. $f(f^{-1}(V)) = V \subset V$.
$\boxed{\Leftarrow}$ Sps. f cont. at x, for all $x \in X$. Take V non-empty open set of Y and take $x \in f^{-1}(V)$. Since $f(x) \in V$ and V open, V is neig. of $f(x)$. Thus there is open $U_x \subset X$ s.t. $x \in U_x$, $f(U_x) \subset V$. Take the the union of such U_x: $U = \bigcup_{x \in f^{-1}(V)} U_x$. Clearly U open.

Notice for all $x \in f^{-1}(V)$, $x \in U_x \subset U$, and thus $f^{-1}(V) \subset U$
Also, since $f(U_x) \subset V$, $f(U) \subset V$. Hence $U \subset f^{-1}(V)$.
And we have $f^{-1}(V) = U$ open. ∎

Remark: X set, τ_1, τ_2 two topologys on X. The ident map $id: (X, \tau_1) \to (X, \tau_2)$, $x \mapsto x$ is continuous iff τ_1 is finer than τ_2

Def: $f: X \to Y$ is an homeomorphism if f is continuous, bijective and $f^{-1}: Y \to X$ is also continuous.

Lemma: discreteness is a topological property.
Pf: Let X discrete top. space and Y top. space.
If $f: X \to Y$ is homeomorphism.
Take $y \in Y$. By bijectivity of f there is $x \in X$ s.t. $f(x) = 1$.
Notice $\{x\}$ open in X, by continuity of f^{-1} and the fact that $(f^{-1})^{-1} = f$.
we have $\{y\}$ open in Y. ∎

Prop. $\mathbb{R}$ has a countable basis, i.e. $\mathcal{B} = \{(a,b) : a < b,\ a, b \in \mathbb{Q}\}$
Pf: Take U open set in $\mathbb{R}$, i.e. $U = (c,d)$ for some $c, d \in \mathbb{R}$, $c < d$.
Let $x \in U$ and take $a, b \in \mathbb{Q}$ s.t. $c < a < x < b < d$
Thus $x \in (a,b) \subset (c,d)$. ∎

Def: A top. space is second countable if has a countable basis. This is a topological property.

Prop. X, Y top. spaces
$f: X \to Y$ map, then the following are equivalent:
i) f continuous
ii) for every $A \subset X$, one has $f(\bar{A}) \subset \overline{f(A)}$
iii) preimages of closed sets are closed.

Pf: $\boxed{i) \Rightarrow ii)}$ Sps. f cont.
Let $x \in \bar{A}$ and let V be neigh. of $f(x)$, then $f^{-1}(V)$ is neigh. of x.
$x \in \bar{A}$ implies $f^{-1}(V) \cap A \neq \emptyset$. Hence $f(f^{-1}(V) \cap A) \neq \emptyset$, and $f(f^{-1}(V) \cap A) \subset V \cap f(A)$. Thus $V \cap f(A) \neq 0$.
So all neigh. of $f(x)$ intersect $f(A)$, meaning $f(x) \in \overline{f(A)}$.
$\boxed{ii \Rightarrow iii}$ Sps. $A \subset X$ implies $f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$.
Let $B \subset Y$ closed and $A = f^{-1}(B)$.
Take $x \in \bar{A}$, so $f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$.
Thus $x \in f^{-1}(B) = A$. Hence $\bar{A} = A$, A closed.
$\boxed{iii \Rightarrow i}$ Sps f^{-1} sends closed sets to closed sets.
Take $V \subset Y$ open. $Y - V$ closed in Y.
Hence $C = f^{-1}(Y - V)$ closed in X. Let $U = X - C$, clearly U open in X.
Take $x \in f^{-1}(V)$, then $f(x) \in V$, $f(x) \notin Y - V$. Hence $x \notin f^{-1}(Y-V) = C$, so $x \in U$. And we have $f^{-1}(V) \subset U$.
Now take $x \in U$, then $x \notin C$, i.e. $x \notin f^{-1}(Y - V)$. So $f(x) \notin Y - V$. So $f(x) \in V$. Meaning $x \in f^{-1}(V)$ and $U \subset f^{-1}(V)$,
So $f^{-1}(V) = U$ open. ∎

Remarks:

1. constant funct. are continuous
2. if $A \subset X$ subspace, the inclusion
   $$i : A \hookrightarrow X$$
   is continuous

Pf: $f : X \to Y$, for $a \in Y$ $f(x) = a$ $(x \in X)$

Take U open in Y. If $a \in U$, $f^{-1}(U) = X$ which is open.
if $a \notin U$ $f^{-1}(U) = \emptyset$ which is open.

For U open in X, $i^{-1}(U) = U \cap A$ which by def. is open in subspace ■

The pasting lemma: Let $X = A \cup B$ w/ A, B closed in X.
Let $f : A \to Y$ and $g : B \to Y$ be continuous
s.t. $f(x) = g(x)$ for all $x \in A \cap B$.
Then $h : X \to Y$
$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$
is well-defined and continuous

Pf: Let $C \subset Y$ be closed
$$h^{-1}(C) = \underbrace{f^{-1}(C)}_{\text{closed in } A} \cup \underbrace{g^{-1}(C)}_{\text{closed in } B}$$
Thus, $f^{-1}(C), g^{-1}(C)$ closed in X.
$f^{-1}(C) \cup g^{-1}(C)$ closed in X
$\Rightarrow h$ continuous

Generalized Product Topology

cartesian product
of top. spaces $\{X_\alpha\}_{\alpha \in J}$:
$$\prod_{\alpha \in J} X_\alpha = \{ (x_\alpha)_{\alpha \in J} \mid x_\alpha \in X_\alpha \}$$

Let $\beta \in J$, $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the projection onto the β^{th} component
$$(x_\alpha)_{\alpha \in J} \mapsto x_\beta$$

we want projection to be continuous. So
For each $\beta \in J$
$$S_\beta = \{ \pi_\beta^{-1}(U) : U \text{ open in } X_\beta \}$$
consists of open set let
$$S = \bigcup_{\beta \in J} S_\beta$$

Def: Product topology on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbase S. That is, the product topology is the coarsest topology where all projection are continuous

$S \xrightarrow{\text{finite intersection}} \mathcal{B}$

Spr. $\beta_1, \beta_2, \dots, \beta_n \in J$ are all different.
$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) = \prod_{\alpha \in J} U_\alpha \quad \text{w/ } U_\alpha = \begin{cases} U_{\beta_i} & \text{if } \alpha = \beta_i, \ i = 1, \dots, n \\ X_\alpha & \text{if } \alpha \neq \beta_1, \beta_2, \dots, \beta_n \end{cases}$$

So $\mathcal{B} = \{ \prod_{\alpha \in J} U_\alpha : U_\alpha = X_\alpha \text{ except for finitely many indices } \beta_1, \dots, \beta_n \in J$
U_α open in $X_\alpha \}$

Consider now:
$$\mathcal{B}' = \{ \prod_{\alpha \in J} : U_\alpha \text{ open in } X_\alpha \}$$
is also a basis on $\prod_{\alpha \in J} X_\alpha$ that generates the box topology

The box topology is finer than the product topology.

Prop. Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$, f_α are the component or coordinate functions, i.e. $f_\alpha = \pi_\alpha \circ f$.
Then f is continuous iff f_α is cont. for all $\alpha \in J$.

Pf: $\boxed{\Leftarrow}$
Let $\pi^{-1}(U_\beta) \in \mathcal{S}$ subbasis for the prod. top.
$f^{-1}(\pi_\beta^{-1}(U_\beta)) = (f^{-1} \circ \pi_\beta^{-1})(U_\beta) = f_\beta^{-1}(U_\beta)$ open in A.
Hence f is continuous
$\boxed{\Rightarrow}$ Sps. f cont. π_β is cont.
Then $\pi_\alpha \circ f = f_\alpha$ cont.

$\boxed{E.G.}$ $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots = \{(x_n)_{n \in \mathbb{N}} : (x_1, x_2, \ldots),\ x \in \mathbb{R}\}$
i.e. $\mathbb{R}^\omega$ is the set of real sequences.

Let $f: \mathbb{R} \to \mathbb{R}^\omega$
$t \mapsto (t, t, t, \cdots)$
if $\mathbb{R}^\omega$ has product top. f is cont.
if $\mathbb{R}^\omega$ has the box top. then
$\prod_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) \subset \mathbb{R}^\omega$
$f^{-1}\left(\prod_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

So on $\mathbb{R}^\omega$ the box top. is strictly finer than the product top.

Obs. $\prod_{n \in \mathbb{N}} (-1, 1) = (-1, 1)^\omega$ is open in the box top. (product of open sets in $\mathbb{R}$)
but does not contain any basis element of the product top.
Thus, $\operatorname{Int}((-1,1)^\omega) = \emptyset$ in the product top.

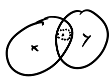
Metric Spaces

A metric space is a set X together with a metric $d: X \times X \to \mathbb{R}$ s.t.
1. $d(x, y) \geq 0$, for all $x, y \in X$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Def: An ε-ball $(\varepsilon > 0)$ around a pnt $x \in X$ is the set $\{y \in X \mid d(x,y) < \varepsilon\}$ and denoted $B_d(x, \varepsilon)$

Lemma: Let $x, y \in X$, $\varepsilon > 0$. Then, if $y \in B(x, \varepsilon)$ exists an $\delta > 0$ s.t. $B(y, \delta) \subset B(x, \varepsilon)$
Pf:
Take $\delta = \varepsilon - d(x, y)$. Consider $z \in B(y, \delta)$
$d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + \delta = \varepsilon$
So, $z \in B(x, \varepsilon)$. ∎

Corollary: $\mathcal{B} = \{B(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$ are a basis.



Pf: 1. Clearly for all $x \in X$, $x \in B(x, \varepsilon) \in \mathcal{B}$
2. Take $B(x, \varepsilon_1), B(y, \varepsilon_2) \in \mathcal{B}$ w/ non-empty intersection,
Take $z \in B(x, \varepsilon_1) \cap B(x, \varepsilon_2)$
By the lemma exist $\delta_1, \delta_2 > 0$ s.t.
$B(z, \delta_1) \subset B(x, \varepsilon_1)$
$B(z, \delta_2) \subset B(x, \varepsilon_2)$
Let $\delta = \min\{\delta_1, \delta_2\}$ notice
$B(z, \delta) \subset B(x, \varepsilon_1) \cap B(y, \varepsilon_2)$. ∎

Def: A top. space (X, τ) is said to metrizable if there is a metric d on X s.t. the metric top. assoc. to d is equal to τ.

Problem: Find necessary and sufficient conditions s.t. a top. space is metrizable.

E.g. discrete space : is metrizable by $d(x,y) = \begin{cases} 1 & , x \neq y \\ 0 & , x = y \end{cases}$

indiscrete space not metrizable (by not being Hausdorff and following lemma)

Lemma: X metrizable implies X Hausdorff

Pf: Sps. $d: X \times X \to \mathbb{R}$ metric. Take $x, y \in X$ and let $\varepsilon = d(x,y)/2$.

Sps. bwoc. $z \in B_d(x,\varepsilon) \cap B_d(y,\varepsilon)$

then, $d(x,y) \leq d(x,z) + d(z,y) < 2\varepsilon = d(x,y)$

So, $d(x,y) < d(x,y)$ ※

Hence $B_d(x,\varepsilon) \cap B_d(y,\varepsilon) = \emptyset$

Since $B_d(x,\varepsilon), B_d(x,\varepsilon)$ open in X, x,y arbitrary X Hausdorff ∎

The uniform metric and the Euclidean metric on $\mathbb{R}^n$ has the same topology, i.e. the standard or the product.

Question: is $\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} \mathbb{R}$ metrizable?

Lemma: $d: X \times X \to \mathbb{R}$ is a metric on X, then $\bar{d}: X \times X \to \mathbb{R}$, $\bar{d}(x,y) = \min\{d(x,y), 1\}$ ("the metric d cut at 1") is a metric on X, the bound metric assoc. with d. Moreover d and $\bar{d}$ induce the same topology in X.

If $\varepsilon < 1$, $B_{\bar{d}}(x,\varepsilon) = B_d(x,\varepsilon)$

if $\varepsilon > 1$, $B_{\bar{d}}(x,\varepsilon) = X$.

Def: Let J be any index set and consider $\mathbb{R}^J = \prod_J \mathbb{R} = \{(r_\alpha)_{\alpha \in J} \mid r_\alpha \in \mathbb{R}\}$.

$\bar{\rho}: \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$

$((x_\alpha)_{\alpha \in J}, (y_\alpha)_{\alpha \in J}) \mapsto \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$

w/ $\bar{d}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto \max\{|x-y|, 1\}$

$\bar{\rho}$ is the uniform metric on $\mathbb{R}^J$, $\bar{\rho}$ induces the uniform top. on $\mathbb{R}^\omega$.

Prop: the uniform topology on $\mathbb{R}^J$ is finer than the product topology. and stricly finer if J is infinite.

Pf: Let $\prod_{\alpha \in J} U_\alpha$ be a basis element for the product topology.

So, $U_\alpha \neq \mathbb{R}$ for finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n \in J$, U_α open in $\mathbb{R}$ for all $\alpha \in J$.

let $x = (x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha$

for $\alpha_1, \alpha_2, \dots, \alpha_n \in J$, choose $\varepsilon_i > 0$ s.t.

$$B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$$

take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$

then,

$$x \in B_{\bar{\rho}}(x,\varepsilon) \subset \prod_{\alpha \in J} U_\alpha \qquad (1)$$

Hence, the uniform top. is finer than the product top.

Proof of (1): take $y = (y_\alpha)_{\alpha \in J} \in B_{\bar{\rho}}(x,\varepsilon)$. Then $\bar{\rho}(x,y) < \varepsilon$, $\sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\} < \varepsilon$

Thus, $d(x_\alpha, y_\alpha) < \varepsilon \leq \varepsilon_i$ for all $\alpha \in J$

and $y_{\alpha_i} \in U_{\alpha_i}$, $i = 1, \dots, n$

for $\alpha \neq \alpha_1, \alpha_2, \dots, \alpha_n$ $U_\alpha = \mathbb{R}$ and $y_\alpha \in U_\alpha$.

stricly finer : $B_{\bar{\rho}}(0,1)$ is open in the uniform topology but not in the product topology.

In fact not a single basis element of the product topology is contain in $B_{\bar{\rho}}(0,1)$.

Which means $\mathrm{Int}(B_{\bar{\rho}}(0,1)) = \emptyset$ in the product topology.

For index set J:

Consider $\mathbb{R}^J$

$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$

$x = (x_\alpha)_{\alpha\in J}$ w/ $\bar{d}(x_\alpha, y_\alpha) = \min\{|x_\alpha - y_\alpha|, 1\}$

$y = (y_\alpha)_{\alpha \in J}$

J infinite $\Rightarrow$ product top. $\subsetneq$ uniform top $\subsetneq$ box topology

<u>Theorem</u>: Let $x = (x_i)_{i\in\mathbb{N}}, y = (y_i)_{i\in\mathbb{N}} \in \mathbb{R}^\omega$, let $D(x,y) = \sup\{\frac{\bar{d}(x_i,y_i)}{i}, i \in \mathbb{N}\}$

Then D is a metric on $\mathbb{R}^\omega$ which induces the <u>product top.</u>

<u>Sequence lemma</u>: X top. space, $A \subset X$. If there is a sequence in A which converges to $x \in X$, then $x \in \bar{A}$.

Conversely, sps. A is metrizable, if $x \in \bar{A}$ then there is a sequence in A which converges to $x \in X$.

<u>Pf</u>: $\boxed{\Rightarrow}$ Take U neigh. of x. Then for some $N \in \mathbb{N}$ we have $a_n \in U$ for $n > N$. Since all $a_i \in A$ we get $U \cap A \neq \emptyset$

Thus $x \in \bar{A}$.

$\boxed{\Leftarrow}$ Sps. $d: X \times X \to \mathbb{R}$ metric inducing X.

We know $B_d(x, \frac{1}{n}) \cap A \neq \emptyset$, since the ball is a neigh. of x and $x \in \bar{A}$.

Take $a_n \in B_d(x, \frac{1}{n})$

Let U be a neigh. of x. There is $\varepsilon > 0$ s.t.

$B_d(x, \varepsilon) \subset U$. Let $N \in \mathbb{N}$, $\frac{1}{N} < \varepsilon$

So, $B_d(x, \frac{1}{N}) \subset U$ and for all $n > N$

$B_d(x, \frac{1}{n}) \subset B_d(x, \frac{1}{N}) \subset U$

Thus, $a_n \in U$. ∎

Prop. If J is uncountable then $\mathbb{R}^\omega$ w/ the <u>product</u> top. is not metrizable.

<u>Pf</u>: Let $A = \{(x_\alpha)_{\alpha\in J} \in \mathbb{R}^J$ s.t.

$x_\alpha = 1$ for almost all α and

$x_\alpha = 0$ for any other $\alpha\}$

Denote $0 = (0)_{\alpha \in J} \in \mathbb{R}^J$.

We have $0 \in \bar{A}$.

Let $\prod_{\alpha \in J} U_\alpha$ be a basis element of $\mathbb{R}^J$ containing 0.

Notice $U_\alpha = \mathbb{R}$ for almost all $\alpha \in J$.

Let $\alpha_1, \dots, \alpha_n$ the finitely many indices s.t. $U_{\alpha_i} \neq \mathbb{R}$.

Let $(x_\alpha)_{\alpha \in J} \in \mathbb{R}^J$ s.t.

$$x_\alpha = \begin{cases} 1 & \text{if } \alpha \neq \alpha_1, \dots, \alpha_n \\ 0 & \text{if } \alpha \in \{\alpha_1, \dots, \alpha_n\} \end{cases}$$

then $(x_\alpha)_{\alpha\in J} \in A \cap \prod_{\alpha \in J} U_\alpha \neq \emptyset$

Hence $0 \in \bar{A}$.

Now let a_n be any sequence in $A \subset \mathbb{R}^J$

$a_n = (x_{\alpha n})_{\alpha \in J} \in \mathbb{R}^J$

Let $J_n = \{\alpha \in J : x_{\alpha n} = 0\}$, notice J_n finite by def of A.

$\bigcup_{n\in\mathbb{N}} J_n$ countable (countable union of finite sets)

so, $\bigcup_{n \in \mathbb{N}} J_n \subsetneq J$

choose $\beta \in J$, $\beta \notin \bigcup_{n\in\mathbb{N}} J_n$. Then $x_{\beta n} = 1$, $\forall n \in \mathbb{N}$

Notice

$U = \pi_\beta^{-1}((-1,1))$

is a neigh. of 0 s.t. $a_n \notin U$ for all $n \in \mathbb{N}$

this is b/c $x_{\beta n} = 1 \notin (-1,1)$.

So we've proven for $0 \in \bar{A}$ that no sequence in A converges to 0. thus by the sequence lemma $\mathbb{R}^J$ is not metrizable. ∎

Def. A top. sapce X is first countable at a pnt. $x \in X$ if there is a countable system of neigh.s $U_1, U_2, \dots$ s.t. given an arbitrary neigh. U of x there is $n \in \mathbb{N}$ s.t. $U_n \subset U$.

We can also assume that $U_1 \supset U_2 \supset U_3 \supset \dots$

$$(U_1 \supset U_1 \cap U_2 \supset U_1 \cap U_2 \cap U_3 \supset \dots)$$

Proposition: Second-countable implies first-countable

Pf: Let $\mathcal{B}$ be a comtable basis for X.

Let $x \in X$.

Take $\mathcal{B}' = \{ B \in \mathcal{B} : x \in B \}$

$\mathcal{B}'$ countable, since $\mathcal{B}$ countable and $\mathcal{B}' \subset \mathcal{B}$

Write $\mathcal{B}' = \{ B_1, B_2, \dots \}$

Let $U_1 = B_1$, $U_2 = B_1 \cap B_2$, $U_3 = B_1 \cap B_2 \cap B_3$, ...

If U is neigh. of x there there is $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

obv. $B \in \mathcal{B}'$. ∎

Def. X, Y top spaces. A function $f: X \to Y$ is a quotient map if f continuous, surjective and the following holds:

$V \subset Y$ is open $\Leftrightarrow$ $f^{-1}(V)$ open in X

Connectivity

Def: A separation of a top. space X is a pair $U, V \subset X$ s.t.

1. $X = U \cup V$.
2. $U \cap V = \emptyset$
3. $U, V \notin \{X, \emptyset\}$

Def. X is connected if the only clopen sets are $X, \emptyset$.

Prop. X connected iff a separation does not existe

Pf "$\Rightarrow$" Sps. P clopen in X. Let $Q = X \setminus Q$. Since P closed, Q open.
$(P \neq X, \emptyset)$

Notice $P \cup Q = X$.

"$\Leftarrow$" Sps. P, Q sep. for X.

Then, $P = X \setminus Q$, and Q open, thus P closed.

So we have P clopen and not X or $\emptyset$. ∎

Lemma: If X top.space C, D sep. of X and $Y \subset X$ subspace is connected.

Then $Y \subset C$ or $Y \subset D$.

Pf: $Y = (Y \cap C) \cup (Y \cap D)$

both open and disjoint. But Y connected, hence $Y \cap D = \emptyset$ or $Y \cap C = \emptyset$. Thus $Y \subset D$ or $Y \subset C$. ∎

Lemma: Sps. $A_\alpha \subset X$, $\alpha \in J$ subspaces all connected, and $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$

then $\bigcup_{\alpha \in J} A_\alpha$ is connected.

Pf: Let $p \in \bigcap_{\alpha \in J} A_\alpha$, $Y = \bigcup_{\alpha \in J} A_\alpha$

Sps. $Y = C \cup D$ open and disjoint

W.l.o.g. sps. $p \in C$.

Then for for all $\alpha \in J$, $p \in C$ and A_α connected implies $A_\alpha \subset C$.

Thus, $\forall \alpha \in J : A_\alpha \subset C$. Thus $\bigcup_{\alpha \in J} A_\alpha \subset C$, implies $D = \emptyset$.

And Y connected. ∎

Prop. Let $A \subset X$ be connected and $A \subset B \subset \bar{A}$, then B is connected. In particular, the closure of a connected subset is connected.

Pf: Sps. $B = C \,\dot\cup\, D$

by lemma, $A \subset C$ or $A \subset D$. W.l.o.g. sps. $A \subset C$.

Then, $\bar{A} \subset \bar{C}$, thus $B \subset \bar{C}$.

We have, $B \subset (\bar{C} \cap B)$

but $\bar{C}$ is closed in B so $\bar{C} = C$ and $B \subset C$.

Thus, $D = \emptyset$, B connected. ∎

Lemma: If $f: X \to Y$ continuous and X connected, then $f(X)$ connected.

Pf: Sps. $f(X)=Y$. If $Y = C \dot\cup D$ sep.

then, $X = f^{-1}(C) \cup f^{-1}(D)$ separation. ∎

Theorem: A product $\prod_{\alpha \in J} X_\alpha$ of connected spaces is connected.

Pf:

i) if J is finite can be proven inductively from:

$X \times Y$ product of connected spaces.

Fix $b \in Y$, then $X \times \{b\} \cong X$, so it's connected.

Now for $x \in X$ then $\{x\} \times Y \cong X$ so it's connected.

Then $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

is connected (not empty intersection)

Notice $X \times Y = \bigcup_{x \in X} T_x$

and $\bigcap_{x \in X} T_x = X \times \{b\} \neq \emptyset$

so $X \times Y$ is connected.

ii) if J is infinite: $X = \prod_{\alpha \in J} X_\alpha$

Fix $b = (b_\alpha)_{\alpha \in J} \in X$

for any finite subset $\alpha_1, \alpha_2, \dots, \alpha_n \in J$, let $X(\alpha_1, \dots, \alpha_n) := \{ (x_\alpha)_{\alpha \in J} \in J : x_\alpha = b_\alpha \text{ if } \alpha \neq \alpha_1, \dots, \alpha_n \}$

So $X(\alpha_1, \dots, \alpha_n) \cong X_{\alpha_1} \times \cdots \times X_{\alpha_n}$, which is connected by i).

Let $Y = \bigcup_{\substack{\alpha_1, \dots, \alpha_n \in J \\ n \in \mathbb{N}}} X(\alpha_1, \dots, \alpha_n)$

Notice $\bigcap_{\substack{\alpha_1, \dots, \alpha_n \in J \\ n \in \mathbb{N}}} X(\alpha_1, \dots, \alpha_n) \ni b$, hence the intersection is not empty.

Thus, Y is connected.

claim: Y is dense in X, i.e. $X = \overline{Y}$.

Pf: Let $x = (x_\alpha)_{\alpha \in J} \in X$ and take $U = \prod_{\alpha \in J} U_\alpha$ be a neigh. (from the basis) of x.

We know $U_\alpha = X_\alpha$ for almost all $\alpha \in J$, let $\alpha_1, \dots, \alpha_n \in J$ the finite many indices s.t. $U_{\alpha_i} \neq X_{\alpha_i}$

Consider $y = (y_\alpha)_{\alpha \in J}$ w/ $y_\alpha = \begin{cases} b_\alpha, & \alpha \neq \alpha_1, \dots, \alpha_n \\ x_\alpha, & \alpha = \alpha_1, \dots, \alpha_n \end{cases}$

Thus, $y \in \left(\prod_{\alpha \in J} U_\alpha \right) \cap Y$

So, $X = \overline{Y}$. So, X connected. ∎

Example: $\mathbb{R}^\omega$ w/ box top. is not connected.

$\mathbb{R}^\omega = \{\text{bounded seq.}\} \dot\cup \{\text{unbounded seq.}\}$

Pf: Let $(r_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ bounded seq.

then $\prod_{n \in \mathbb{N}} (r_n - 1, r_n + 1)$ is a neigh of $(r_n)_{n \in \mathbb{N}}$ in the box top. which consists of bounded seq.

Making the union for all such seq. we got that the set of bounded seq. are open.

Similarly, the seq. of unbounded seq. is open. ∎

Def: An ordered set X is a <u>linear continuum</u> if the following conditions hold:

1. Any non empty bounded subset of X has a least upper bound.
2. For all $x, y \in X$ s.t. $x < y$ there exist $z \in X$ s.t. $x < z < y$.

e.g.
- $\mathbb{R}$ (yes)
- $\mathbb{Q}$ (no, Dedekind's cut proves 1. does not hold)
- $\mathbb{Z}$ (no)

Theorem: If X is a linear continuum, then X is connected and, also, every interval and ray Y in X is connected.

Pf: Let A, B be disjoint, non-empty, open subsets of $Y \subseteq X$.

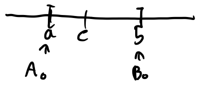
Choose, $a \in A$, $b \in B$ and sps. w.l.o.g. $a < b$.

Then $[a, b] \subset Y$

Let $A_0 = A \cap [a, b]$, $B_0 = B \cap [a, b]$

Let $c = \sup A_0$

Case 1: if $c \in B_0$

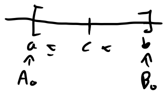


B_0 is open in $[a, b]$

So, there is $d < c$ s.t. $(d, c] \subset B_0$

Hence d upper bound of A_0 # (c lup.)

Case 2: if $c \in A_0$



A_0 open, so exists $d > c$ s.t. $[c, d) \in A_0$

And we have z s.t. $c < z < d$, then $z \in A_0$

But $z > c = \sup A_0$ #

Thus, Y has no separation. ∎

Corollary: $\mathbb{R}$ is connected.

Intermediate Value Theorem: Let $f: X \to Y$ be cont. w/ X is connected and Y is ordered with the order top.
Then, if $f(a) < r < f(b)$ for $a, b \in X$, then there is $c \in X$ s.t. $f(c) = r$.

Pf: Sps. $r \notin f(X)$. Then

$$X = f^{-1}((-\infty, r)) \cup f^{-1}((r, \infty))$$

Notice:

1. $a \in f^{-1}((-\infty, r))$, $b \in f^{-1}((r, \infty))$
   Thus both not empty.
2. Open, f continuous and rays are open.
3. Disjoint

So this is a separation #. ∎

Def A path in a top space X is a cont. map $\gamma: [0, 1] \to X$ from $\gamma(0)$ (initial pt) to $\gamma(1)$ (final pt).

A top. space is path-connected if for every two pts $a, b \in X$, there is a path from a to b.

Prop. X path-connected implies X connected

Pf: Let $X = A \cup B$, for A, B open in X

Take $a \in A$, $b \in B$. Let $f: [0, 1] \to X$ be a path in X from a to b.

Then, $[0, 1] = f^{-1}(A) \cup f^{-1}(B)$

By the connectedness of the interval and the fact $0 \in f^{-1}(A)$, $1 \in f^{-1}(B)$ both open

we must have $f^{-1}(A) \cap f^{-1}(B) \neq \emptyset$ and thus $A \cap B \neq \emptyset$ for all possible separations of X

Hence, X connected. ∎

The converse is not true,

I_o^2 is connected, since it is a linear cont.
but not path-connected.

Pf: Sps there is a path $f: I \to I_o^2$ from 0×0 to 1×1.

Since I connected, by the IVT, f is surjective

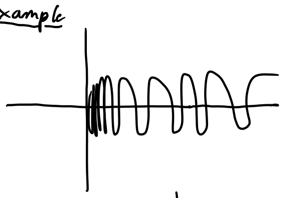
Take $x \in I$, $x \times (0, 1) = (x \times 0, x \times 1)$ open interval in I_o^2

$A_x = f^{-1}(x \times (0, 1))$ is open and non-empty (f surjective)

Clearly for $x \neq y$, $A_x \cap A_y = \emptyset$

Each A_x contains a rational, so varying x in I we get uncountably many rationals #. ∎

example



$G = \{ x \times \sin(\frac{1}{x}),\ 0 < x \leq c\}$

G is path-connected

$\overline{G} = G \cup \{0 \times [-1, 1]\}$

connected

but not path connected

Sps $f : I \to \overline{G}$ is a path w/ $f(0) \in (0 \times [-1, 1])$
$f(1) \in G$

We can assume that $f(t) \in G$ for $t > 0$

so only $p = f(0) \in (0, [-1, 1])$

f is cont in 0.

$U = B(p, \varepsilon) \cap \overline{G}$, there is a neigh. $[0, \delta)$ of 0 in I

s.t. $f([0, \delta)) \subset U$

And $f([0, \delta))$ is connected

But clearly $f([0, \delta))$ is not connected in U $\#$.

<u>Def.</u> A space X is <u>locally connected</u> if, for every point $x \in X$ there is a neigh. U of x which is connected.

<u>Def:</u> A <u>component</u> of a top. space X is a maximal connected subset of X

<u>fact:</u> 2 components are either equal or disjoint.

Compact spaces

<u>Def:</u> A top. space X is <u>compact</u> if every open covering of X contains a finite subcollection which covers X.

<u>Lemma:</u> Let $Y \subset X$ be a subspace. Then Y is compact iff every <u>open cover</u> of Y in X contains a finite subcover.

<u>Pf:</u> $\boxed{\Rightarrow}$ Sps. Y is compact

Let $\mathcal{U} = \{U_\alpha : \alpha \in J\}$

be an open cover of Y in X

$(Y \subset \bigcup_{\alpha \in J} U_\alpha)$

$\mathcal{U}' = \{U_\alpha \cap Y : \alpha \in J\}$

open cover of Y

Since Y compact we have

$U_{\alpha_1} \cap Y \cup \cdots \cup U_{\alpha_n} \cap Y$

covers Y for $\alpha_1, \dots, \alpha_n \in J$

And this is also cover in X.

$\boxed{\Leftarrow}$ Similarly. ∎

Example

Let X be a set with the finite complement topology. Then, any subset Y of X is compact.

Pf: Let $\mathcal{U} = \{U_\alpha : \alpha \in J\}$ be an open cover for Y in X. We can assume $Y \neq \emptyset$ (otherwis Y trivially compact).

Take $y_0 \in Y$ fixed.

Let $y_0 \in U_{\alpha_0}$

but $X - U_{\alpha_0}$ finite

so $Y \cap (X - U_{\alpha_0}) = \{y_1, \dots, y_N\}$ for finitely many $y_1, \dots, y_N \in Y$.

Let $y_n \in U_{\alpha_n}$, for $1 \leq n \leq N$.

So, $U_{\alpha_1}, \dots, U_{\alpha_N}$ is a finite subcover. ∎

Notice: in this topology the closed sets are only the finite sets.

Obs: compact subsets need not to be closed.

Prop: A closed subset of a compact space is compact.

Pf: Let $Y \subset X$, X compact, Y closed.

Let $\mathcal{U}$ open cover of Y in X.

then, $\mathcal{U} \cup \{X - Y\}$ is an open cover for X.

By compactness of X we have a finite subcover $U_1, U_2, \dots, U_n \in \mathcal{U}$ (maybe plus $X - Y$).

Notice $U_1, \dots, U_n$ is a finite subcover for Y in X. ∎

Prop: Every compact subset of a Hausdorff space is closed.

Pf: Let $Y \subset X$, X Hausdorff, Y compact.

Let $x_0 \in X \setminus Y$.

For each $y \in Y$ chose disjoints neigh.

U_y of x_0

V_y of y.

Then, $\{V_y : y \in Y\}$ is an open cover for Y in X.

Since Y compact, exists finite subcover $V_{\alpha_1}, \dots, V_{\alpha_m}$, $Y \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_m}$.

Let $U = U_{\alpha_1} \cap \dots \cap U_{\alpha_m}$, notice U neighbor. of x_0 s.t. $U \subset X \setminus Y$.

U can be obtained in this way for each $x \in X \setminus Y$, hence $X \setminus Y$ open and Y closed. ∎

Lemma: If $f : X \to Y$ is a cont. map and X is compact, then $f(X)$ compact in Y.

Pf: Let $\mathcal{A}$ be an open cover for $f(X)$ in Y, then

$$f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$$

is an open cover for X.

Since X compact, $f^{-1}(\mathcal{A})$ has a finite subcover which gives us an open subcover for $f(X)$ by applying f. ∎

Theorem: Let $f : X \to Y$ be a cont. bijection. If X is compact and Y is Hausdorff then f is a homeomorphism.

Pf: w.t.s. f^{-1} cont. iff f open $\underset{\text{since } f \text{ bijective}}{\text{iff}}$ f is closed

Let A be closed in X, then A is compact. We have $f(A)$ compact, since Y Hausdorff $f(A)$ closed. ∎

Theorem (Tube Lemma):

Consider $X \times Y$, with Y compact. If N is a neigh. of a slice $\{x_0\} \times Y$, $x_0 \in X$ in $X \times Y$, then there is a neigh. W of x_0 in X s.t. the tube $W \times Y \subset N$.

Pf: We cover $\{x_0\} \times Y$ by basis elements $U \times V \subset N$, so we have an open cover of $\{x_0\} \times Y \cong Y$, so finitely many suffices.

$U_1 \times V_1, \dots, U_m \times V_m$ and we can assume $x_0 \in U_i$ for $1 \leq i \leq m$.

Then, $W = U_1 \cap \dots \cap U_m$ is a neigh. of x_0 s.t. $W \times Y \subset N$. ∎

Theorem: A product of finitely many compact spaces is compact.

Pf: It suffices to prove $X \times Y$ compact for X, Y compact.

Let $\mathcal{A}$ be an open cover of $X \times Y$, for each $x \in X$, the slice $\{x\} \times Y \cong Y$ is compact and hence cover by finitely many elements $A_1, \dots, A_n \in \mathcal{A}$. Hence $N_x = A_1 \cup \dots \cup A_n$ is a neigh. of $\{x\} \times Y$. By the tube lemma there is a neigh. W_x of x st. $W_x \times Y \subset N$.

Now $\{W_x : x \in X\}$ is an open cover for X, by compactness $W_{x_1}, \dots, W_{x_m}$ suffices.

$$\begin{aligned} X \times Y &= (W_{x_1} \cup \dots \cup W_{x_m}) \times Y \\ &= (W_{x_1} \times Y) \cup \dots \cup (W_{x_m} \times Y) \\ &= N_{x_1} \cup \dots \cup N_{x_m} \end{aligned}$$

which by construction is a finite union of elements of $\mathcal{A}$. ∎

Theorem (Tychonoff):

An arbitrary product of compact spaces is compact.

Theorem: Let X be an ordered set with the least upper bound property. In the ordered top. each closed interval is compact.

Max and min value theorem: Let $f : X \to Y$ be a continuous map w/ Y is ordered with the ordered topology. If X is compact then there are $c, d \in X$ s.t.

$$f(c) \leq f(x) \leq f(d) \quad \forall x \in X$$

Pf: X compact, f cont. implies $f(X)$ compact.

$\{(-\infty, a) : a \in f(X)\}$ is an open cover for $f(X)$ in Y.

Sps. twoc f has no maximum. But notice $f(X) \subset (-\infty, a_0)$, $a_0 \in f(X)$. # ∎

Def: i) A space X is limit pnt compact if every infinite subset has a limit pnt.
ii) X is sequentially compact if every sequence in X has a convergent subsequence.

Prop. X compact $\Rightarrow$ X limit pnt. compact.

Pf: Let A be a subset of X which has no limit pnt

$\bar{A} = A \cup A' = A$

Hence A is closed, and thus also compact.

For each $a \in A$ a is not a limit pnt.

So, there is a neighbh. U_a of a s.t. $U_a \cap A = \{a\}$.

$\{U_a : a \in A\}$ is an open cover of A in X

Since A compact.

$A \subset U_{a_1} \cup \cdots \cup U_{a_n}$

for finitely many $a_1, \ldots, a_n$

And then

$A \subset \{a_1, \ldots, a_n\}$

Hence A finite. ∎

Lemma: X metric and limit pnt cpt. implies X sequentially compact.

Pf: Let x_n be a seq. in X if $\{x_n : n \in \mathbb{N}\}$ is finite then there is a constant subsequence.

So assume $S = \{x_n : n \in \mathbb{N}\}$ is infinite

We know S has a limit pnt $x_0 \in X$.

Take

$x_{n_1} \in B_d(x_0, 1) \cap S$ (non-empty and infinite by Hausdorff)

$x_{n_2} \in B_d(x_0, \frac{1}{2}) \cap S$ "

s.t. $n_2 > n_1$

$x_{n_3} \in B_d(x_0, \frac{1}{3}) \cap S$

s.t. $n_3 > n_2 > n_1$

$\vdots$

we get a subsequence convergent to x_0.

Lebesgue Number Lemma: Let $\mathcal{A}$ be an open covering of the compact, metric space X. Then there exists a $\delta > 0$ (Lebesgue number assoc. to the cover $\mathcal{A}$) s.t. every subset of X with a diameter less than δ is contained in at least one $A \in \mathcal{A}$.

(sequentially compact suffices)

Pf:

Sps. no such $\delta > 0$ exists, there is a subset A_δ of X whose diameter is smaller than δ but is is not contained in any $A \in \mathcal{A}$.

Consider $\delta = \frac{1}{n}$, $n \in \mathbb{N}$:

$A_{1/n}$

Let $x_n \in A_{1/n}$

Since X seg. compact x_n has a convergent subsequence $x_{n_i} \xrightarrow{i \to \infty} x_0 \in X$.

Let $x_0 \in A \in \mathcal{A}$

Hence, there is $B_d(x_0, \varepsilon) \subset A$ for $\varepsilon > 0$.

Choose n_i s.t.

$$d(x_{n_i}, x_0) < \frac{\varepsilon}{2}$$

and

$$\frac{1}{n_i} < \frac{\varepsilon}{2}$$

Let $y \in A_{1/n_i}$.

$$\text{So, } d(y, x_0) \leq d(x_{n_i}, y) + d(x_{n_i}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $y \in B_d(x_0, \varepsilon) \subset A$

Thus, $A_{1/n_i} \subset A$. #

∎

Theorem: Let X be metriazable, th the following are equivalent:

i) X is compact.

ii) X is limit point compact

iii) X is sequentially compact.

Pf: i) $\Rightarrow$ ii)

general fact

ii) $\Rightarrow$ iii)

preceding lemma

(no need metric)

Lemma: Sps. X is metric and sequentially compact. then for each $\varepsilon > 0$, there is a finite covering of ε-balls of X.

Pf: Sps that there is $\varepsilon > 0$ s.t. there is no finite covering of X by ε-balls.

Let $x_1 \in X$, $B_d(x_1, \varepsilon) \subsetneq X$

Take $x_2 \in X - B_d(x_1, \varepsilon)$

Take $x_3 \in X - B_d(x_1, \varepsilon) - B_d(x_2, \varepsilon)$

Recursively construct sequence x_n s.t. $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$.

Hence x_n cannot have any convergent subsequence

Implying X not sequentially compact. ∎

Pf: iii) $\Rightarrow$ i)

Let $\mathcal{A}$ be an open cover of X. X metric, seq. compact

By the Lebesgue number lemma there is Lebesgue number $\delta > 0$ assoc. w/ covering $\mathcal{A}$.

And by preceding lemma, there is a finite open cover of balls of radius $\varepsilon = \frac{\delta}{3}$.

All this balls has diameter $\frac{2\delta}{3} < \delta$ so are contained in an element of $\mathcal{A}$.

Hence, finitely many elements of $\mathcal{A}$, the ones assoc w/ each ε-ball, cover X. ∎

Def: An ordered set is well-ordered if every non-empty subset has a minimum

e.g.

$\mathbb{N}$ yes

$\mathbb{Z}$ no

Let X be an ordered set and $\alpha \in X$. $S_\alpha = (-\infty, \alpha) = \{x \in X : x < \alpha\}$ is called a section of α.

axiom of choice $\Leftrightarrow$ Zorn's lemma $\Leftrightarrow$ well-ordered theorem (Gödel - Cohen proving independence from ZF)

well-ordered theorem: Every set X can be well-ordered

Theorem: There is an uncountable, well-ordered set such that every section of it is countable.

Pf: Using the well-ordered theorem let X be any uncountable well-ordered set.

If every section S_x for $x \in X$ is countable then we are done

Otherwise, let

$Y = \{x \in X : S_x \text{ is uncountable}\}$

Since X is well-ordered Y has a minimum $\Omega \in Y \subset X$.

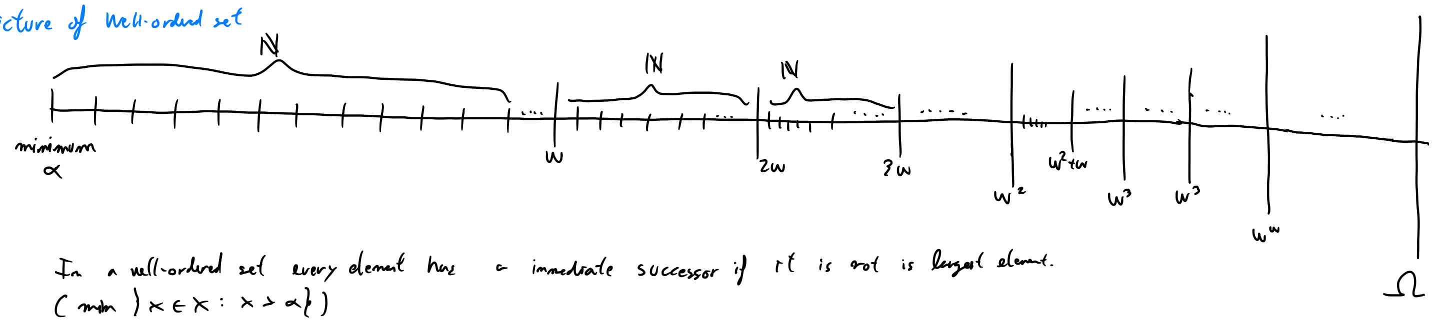
then S_Ω is uncountable but every section of S_Ω is countable.

So S_Ω is our desidered set. ∎

$$\overline{S_\Omega} := S_\Omega \cup \{\underbrace{\Omega}_{\text{largest element of } \overline{S_\Omega}}\}$$

then $\overline{S_\Omega}$ is well-ordered.

Picture of well-ordered set



In a well-ordered set every element has a immediate successor if it is not is largest element.
(min { x ∈ X : x > α })

S_Ω : minimal uncountable well-ordered set

Lemma: If $A \subset S_\Omega$ is countable, then A has an upper bound in S_Ω.

Pf: $\underbrace{\bigcup_{a \in A} S_a}_{\text{countable union of countable sets. Hence, countable}} \subsetneq S_\Omega$

Every element $S_\Omega - \bigcup_{a\in A} S_a$ is an upper bound of A. ∎

Proposition: S_Ω is a Hausdorff space which is not compact and limit pnt compact.

Pf: "not compact"

Consider

$\{ S_x : x \in S_\Omega \}$, is an open cover, since S_Ω has no largest element.

There is no finite subcover, since otherwise it would be an open section $\underset{a}{S_a} = \underset{a}{S_\Omega}$.

"limit pnt. compact"

Let A be an infinite subset of S_Ω.

If A is uncountable we choose a countable subset.

So assume A countable.

We know A has a upper bound $b \in S_\Omega$ by lemma.

$A \subset [\alpha, b]$

S_Ω has the least upper bound property, since it is well-ordered.

So $[\alpha, b]$ is compact (it is closed) and thus limit pnt compact.

So A has a limit pnt in $[\alpha, b]$ hence it has it in S_Ω. ∎

Corollary: S_Ω is not metrizable.

Prop. S_Ω is first countable but no second-countable.

Pf: "first countable"

Let $a \in S_\Omega$, and a' the immediate successor of a.

$\{ (x, a') : x < a \}$

is a fundamental system of neigh. of a.

"not second countable"

Let $\mathcal{B}$ be any basis of S_Ω

for each $x \in S_\Omega$, $x \in [\alpha, x')$ w/ x' is the immediate successor of x.

This is a neighb. of x. Hence exists $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset [\alpha, x')$

$$\varphi : S_\Omega \longrightarrow \mathcal{B}$$
$$x \longmapsto B_x$$

Spz $x, y \in S_\Omega$, sps. $x < y$.

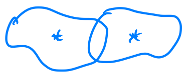
$B_x \subset [\alpha, x')$

$B_y \subset [\alpha, y')$

$y \notin B_x$ b/c $y \notin [\alpha, x')$

Then $B_x \neq B_y$. Hence φ is injective. Thus $\mathcal{B}$ is uncountable ∎

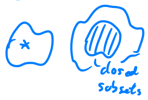
The separation axioms



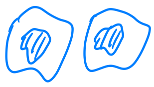
T_1

$\Leftrightarrow$ pts are closed

T_2 Hausdorff



T_3



T_4

regular: T_3 and T_1

normal: T_4 and T_1

normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff $\Rightarrow T_1$

Theorem: X metric $\Rightarrow$ X normal

Pf: Sps. X metric, then X Hausdorff. Hence X is T_1.

Let A, B be disjoint closed subsets. For each $a \in A$, there is open ball $B_d(a, \varepsilon_a)$ disjoint from B.

For each $b \in B$ there is an open ball $B_d(b, \varepsilon_b)$ disjoint from A.

$U = \bigcup_{a \in A} B_d(a, \frac{\varepsilon_a}{2})$ neigh of A

$V = \bigcup_{b \in B} B_d(b, \frac{\varepsilon_b}{2})$ neigh. of B

If $x \in U \cap V$, then $x \in B_d(a_0, \frac{\varepsilon_{a_0}}{2})$ and $x \in B_d(b_0, \frac{\varepsilon_{b_0}}{2})$ for $a_0 \in A, b_0 \in B$ and $\varepsilon_{a_0} \geq \varepsilon_{b_0}$

Notice $d(a_0, b_0) \leq d(a, x) + d(x, b)$

$$< \frac{\varepsilon_{a_0}}{2} + \frac{\varepsilon_{b_0}}{2} < \varepsilon_{a_0}$$

$\Rightarrow b_0 \in B_d(a_0, \varepsilon_{a_0})$

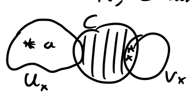
$\Rightarrow B \cap B_d(a_0, \varepsilon_{a_0}) \neq \emptyset$ $\lightning$

Hence, $U \cap V = \emptyset$. ∎

Theorem: X compact Hausdorff implies X normal.

Pf: First we show X is T_3.

Take $a \in X$, C closed subset of X.



For each $x \in C$ we have disjoint open neigh. U_x of a and V_x of x.

Since X compact and C closed we know C compact.

And $\{V_x : x \in C\}$ is an open cover for C.

Hence $C \subset V_{x_1} \cup \dots \cup V_{x_n} =: V$ for finitely many V_{x_i}

So V is a neigh. of A. Also $U = U_{x_1} \cap \dots \cap U_{x_n}$ is neigh. for a.

And clearly, $V \cap U = \emptyset$. So X is T_3.

Now for T_4. Let A, B closed subsets of X. Since X compact A, B also compact.

Take $x \in A$, since X is T_3 we have disjoint open neigh. U_x of B, V_x of x.

Again, $\{V_x : x \in A\}$ is open covering for A, so finitely many $V_{x_1}, \dots, V_{x_n}$ suffices.

Let $V = V_{x_1} \cup \dots \cup V_{x_n}$, V neigh. of A. Let $U = U_{x_1} \cap \dots U_{x_n}$ neigh. of B.

And $V \cap U = \emptyset$. Hence X is T_4. ∎

Lemma: i) X is T_3 iff for every $x \in X$ and neigh. U of x, there is a neigh. V of x s.t. $\overline{V} \subset U$.

ii) X is T_4 iff for each closed subset A of X and neigh. U of A, there is a neigh. V of A s.t. $\overline{V} \subset U$.

Pf i) "$\Rightarrow$" Take $x \in X$ and neigh. U of x. Then $X \setminus U$ is closed, by T_3 there is a neigh. V of x s.t. $V \subset U$ and there is another neigh. of $X \setminus U$ disjoint to V. Hence $\overline{V} \subset U$.

"$\Leftarrow$" Take $x \in X$, A closed in X. Since $x \notin A$ and $X \setminus A$ open, $X \setminus A$ is a open neigh. of x. Hence, there is a neigh. V of x s.t. $\overline{V} \subset X \setminus A$. Also $X \setminus \overline{V}$ is neigh of A. And clearly $V \cap X \setminus \overline{V} = \emptyset$.

ii) Similar ∎

Theorem (Urysohn's Lemma)

Let X be T_4 (or normal) and A, B disjoint closed subsets of X. Then there exists a continuous map

$$f : X \to [0,1]$$

s.t. $f(A) = \{0\}$ & $f(B) = \{1\}$.

We say that A, B can be separated by a cont. real-valued function.

Obs. The converse is also true. Since $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$ are disjoint neigh. of A and B.

Pf:

Let $P = \mathbb{Q} \cap [0,1]$ all rationals in $[0,1]$.

For each $p \in P$, we want to define open set U_p in X s.t. whenever $p < q$, $q \in P$ then $\overline{U_p} \subset U_q$.

Arrange P in a sequence starting with $1, 0, \underbrace{\ldots}_{\text{arbitrary}}$

We define U_p recursively, following this sequence:

Let $U_1 = X - B$ neigh. of A.

X is T_4, so lemma implies there is a neigh. U_0 of A s.t. $\overline{U_0} \subset U_1$.

Now, inductively spse. U_p is already defined for the first n rat. numbers P_n of the sequence.

Let r be the next rat. number of the seq.

Let p be the immediate predecessor of r in P_n, in the natural order of $[0,1]$

Let q be the immediate successor of r in P_n, in the natural order of $[0,1]$.

Notice, U_p and U_q are already defined s.t. $\overline{U_p} \subset U_q$.

So U_q is a neigh of $\overline{U_p}$.

Hence, by X being T_4 and the lemma there exist a neigh. U_r of $\overline{U_p}$ s.t. $\overline{U_r} \subset U_q$. Notice $\overline{U_p} \subset U_r$.

Thus, by inductive U_p is defined for all $p \in P$.

Now, define U_p for all $p \in \mathbb{Q}$ setting $U_p = \emptyset$ if $p < 0$ and $U_p = X$ if $p > 1$.

Now we wish to define f.

For $x \in X$ let

$$Q(x) = \{ p \in \mathbb{Q} : x \in U_p \}$$

$$f(x) := \inf Q(x)$$

Clearly, $0 \le f(x) \le 1$, since $p \notin Q(x)$ for all $p < 0$. And $p \in Q(x)$ for all $p \ge 1$, hence the inf is always at most 1.

Consider $f(A)$, notice U_0 is a neigh. of A, hence $0 \in Q(x)$ for all $x \in A$. Thus, $f(A) = \{0\}$.

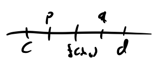
Now for $f(B)$, we know every U_p, $p \in [0,1]$ is disjoint to B. Hence the first U_p w/ $x \in B$, $x \in U_p$ are the $p > 1$. Hence $f(B) = \{1\}$.

Only continuity remains.

Observation: $x \in \overline{U_r} \Rightarrow f(x) = \inf Q(x) \le r$
(since $\overline{U_r} \subset U_q$, $r < q$)

$x \notin U_r \Rightarrow f(x) = \inf Q(x) \ge r$

Let $x_0 \in X$, let $f(x_0) \in \underbrace{(c,d)}_{\substack{\text{neigh of} \\ f(x_0)}} \subset \mathbb{R}$



Let $p, q \in \mathbb{Q}$ s.t. $c < p < f(x_0) < q < d$.

Let $U = U_q - \overline{U_p}$ open. We have $x_0 \in U_q$, $x_0 \in \overline{U_p}$ (look for a while at previous obs.)

So, U neigh. of x. Also $f(U) \subset [p,q] \subset (c,d)$

Thus f continuous ∎

Prop: Every regular space with a countable basis is normal.

Urysohn metrization theorem: X regular with a countable basis implies X metrizable.

necessary cond.

$$\begin{array}{ccl} X \text{ metrizable} & \Rightarrow & X \text{ Hausdorff, first countable, normal} \\ & \not\Leftarrow & \\ & \uparrow & \\ & \mathbb{R}_\ell, \mathbb{R}_\ell^2, S_\Omega & \end{array}$$

sufficient cond.

$$\begin{array}{ccl} X \text{ second countable, normal} & \Rightarrow & X \text{ metrizable} \\ & \not\Leftarrow & \\ & \uparrow & \\ & \text{take any discrete uncountable space} & \end{array}$$

Uryshonn metrizable theorem: X normal, countable basis $\Rightarrow$ X metrizable

Pf:

Lemma: There is a countable collection of cont. functions $f_n : X \to [0,1]$, $n \in \mathbb{N}$ s.t. for all $x_0 \in X$, neigh. U of x_0 there is $n \in \mathbb{N}$ s.t. $f_n(x_0) > 0$ and $f_n(X \setminus U) = \{0\}$.

Pf:

Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis for X. For each pair of indices $n, m \in \mathbb{N}$ s.t. $\overline{B_n} \subset B_m$, by the Uryschon lemma chose

$$g_{n,m} : X \to [0,1]$$

s.t. $g_{nm}(\overline{B_n}) = \{1\}$ & $g_{nm}(X \setminus B_m) = \{0\}$

Now let $x_0 \in X$, U neigh of x_0.

Choose a basis element B_m s.t. $x_0 \in B_m \subset U$.

X regular implies (by prev. lemma) there is $B_n \in \mathcal{B}$ s.t. $x_0 \in \overline{B_n} \subset B_m$.

So, $g_{nm}(x_0) = 1$ and $g_{nm}(X \setminus U) = \{0\}$. ∎

We define $F : X \to \mathbb{R}^\omega$, w/ $\mathbb{R}^\omega$ is equipped with the prod. top.

$$x \mapsto (f_1(x), \dots) = (f_n(x))_{n \in \mathbb{N}}$$

We claim F is an embedding (i.e. an homeomorphism onto its image).

Then, $X \cong f(X) \subset \mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ metrizable and a subspace of a metrizable space is itself metrizable we are done.

Now we prove the claim.

First, F is injective: let $x, y \in X$, $x \neq y$.

X is T_1, hence we have neigh U of x s.t. $y \notin U$.

As before, there are $B_n, B_m \in \mathcal{B}$, $x \in \overline{B_n} \subset B_m$.

So $g_{nm}(x) = 1$, $g_{nm}(y) = 0$ [$y \in X \setminus B_m$].

Hence, $F(x) \neq F(y)$

Second, F is cont. since f_n cont. for all $n \in \mathbb{N}$.

Finally, it remain to prove F is open, namely for $Z = F(X)$ we have for each open set $U \subset X$, $f(U)$ open in Z.

Let $z_0 \in F(U)$. Let $x_0 \in U$ s.t. $F(x_0) = z_0$.

Choose $n \in \mathbb{N}$ s.t. $f_n(x_0) > 0$ and $f_n(X \setminus U) = \{0\}$ (by lemma exists).

Let $V := \pi_n^{-1}((0, \infty))$, w/ π_n is the projection onto the nth-component of $\mathbb{R}^\omega$.

Clearly, V open in $\mathbb{R}^\omega$. Then $W = V \cap Z$ open in Z.

Since $\pi_n(z_0) = \pi_n(F(x_0)) = f_n(x_0) > 0$, $z_0 \in V$, so $z_0 \in W$.

Let $z \in W$, $z = F(x)$ for $x \in X$ and $\pi_n(z) \in (0, \infty)$

since $0 \neq \pi_n(z) = \pi_n(F(x)) = f_n(x)$ and f_n vanishes outside of U, meaning $x \in U$.

And $z \in F(U)$.

Proving $z_0 \in W \subset F(U)$, hence $F(U)$ open.

So we've prove F embedding and the lemma. ∎

Theorem (Urysohn)

For a compact and Hausdorff space X.

$$X \text{ metrizable} \iff X \text{ second-countable}$$

Pf: "$\Leftarrow$"

X compact implies X normal. Then X normal and second-countable, by Urysohn metrizable lemma X is metrizable

"$\Rightarrow$"

Now sps X is metrizable.

Let $\mathcal{B}_n$ be a finite cover of X by open balls of radius $\frac{1}{n}$.

Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$, clearly $\mathcal{B}$ is countable.

Let U open in X, $x \in U$. Since X is metric, there is $B_d(x, \frac{1}{n}) \subset U$ for some $n \in \mathbb{N}$.

We know $x \in B_d(y, \frac{1}{2n}) \in \mathcal{B}_{2n} \in \mathcal{B}$ for some $y \in X$.

$B_d(y, \frac{1}{2n}) \subset B_d(x, \frac{1}{n})$, since for every $z \in B_d(y, \frac{1}{2n})$ we know $d(z,x) \leq d(z,y) + d(y,x) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$.

Proving $\mathcal{B}$ basis for X. ■

Historical note: Urysohn (1898 – 1924)

The general metrizable theorem from Nagata and Smirnov is from circa 1950.

Homotopy and fundamental group

Def: The continuous map

$$f, f' : X \to Y$$

are homotopic if there exists a cont. map

$$F : X \times [0,1] \to Y$$

s.t.

$F(x, 0) = f(x)$
$F(x, 1) = f'(x)$

for all $x \in X$. We denote it by $f \underset{F}{\simeq} f'$.

Obs. Homotopy induces a equivalence relation on the set of cont. maps from X to Y.

Pf:
- $f \underset{F}{\simeq} f$ via $F(x,t) = f(x)$
- $f \underset{F}{\simeq} g \Rightarrow g \underset{G}{\simeq} f$ w/ $G(x,t) = F(x, 1-t)$
- $f \underset{F}{\simeq} g \wedge g \underset{G}{\simeq} h \Rightarrow f \underset{H}{\simeq} h$ w/ $H(x,t) = \begin{cases} F(x, 2t), & 0 \le t \le \frac{1}{2} \\ G(x, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$

H cont. by pasting lemma.

Examples:

i) Every map $f : X \to \mathbb{R}^n$ is homotopic to the constant map

Let C_{z_0} be the cte. map mapping to z_0.
Consider $H(x,t) = (1-t) f(x_0) + t z_0$.
Clearly, $f \underset{H}{\simeq} C_{z_0}$.

H is the straight line homotopy

ii) Every $f : [0,1] \to X$ is homotopic to a constant path.

$H(s,t) = f((1-t)s)$

$f \underset{H}{\simeq} C_{f(0)}$

Modification of homotopy for pathes:

Def: (path-homotopy)

Let $f, g : [0,1] \to X$ be pathes s.t. $f(0) = g(0)$ and $f(1) = g(1)$.
Then, f and g are (path-)homotopic if there exists a cont. map

$$F : I \times I \to X$$

s.t.

$\left.\begin{array}{l} F(s,0) = f(s) \\ F(s,1) = g(s) \end{array}\right\}$ general homotopy

$\left.\begin{array}{l} F(0,t) = f(0) = g(0) \\ F(1,t) = f(1) = g(1) \end{array}\right\}$ path homotopic

We write $f \simeq g$.

Remark: Path-homotopy induces a equivalence relation in the set of pathes from a fixed init pnt. to a fix end pnt in X.

Def: Let f be a path in X from x_0 to x_1 and g be a path from x_1 to x_2.

Then, $f * g : I \to X$

$$t \mapsto \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

is the product of f times g. Is also a path by the pasting lemma.

Def: For a path $f: I \to X$, let $[f]$ be the equivalence class of f w.r.t. the path-homotopy.

Let f, g be paths with $f(1) = g(0)$.

Define $[f] * [g] = [f * g]$.

Well-defined:

Spr. $[f] = [f']$, $[g] = [g']$ for paths f, f', g, g'.

So, $f \underset{F}{\simeq} f'$, $g \underset{G}{\simeq} g'$ for some F, G.

Consider
$$H(s,t) = \begin{cases} F(2s, t) & , 0 \le s \le \frac{1}{2} \\ G(2s-1, t) & , \frac{1}{2} \le s \le 1. \end{cases}$$

notice $H(s, 0) = (f * g)(s)$

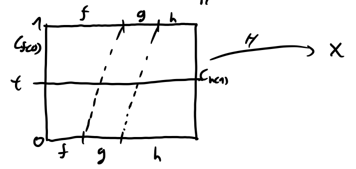
$H(s, 1) = (f' * g')(s)$

by def. of H and $*$.

Hence $[f * g] \underset{H}{\simeq} [f' * g']$. ∎

<u>Lemma 1</u>: $([f] * [g]) * [h] = [f] * ([f] * [g])$, for suitable f, g, h.

Pf: w.t.s. $(f * g) * h \underset{H}{\simeq} f * (g * h)$



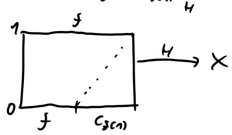
This defines a reparametrization from $(f * g) * h$ to $f * (g * h)$

<u>Lemma 2</u>: Let $f: I \to X$ path

$[f] * [c_{f(1)}] = [f]$

$[c_{f(0)}] * [f] = [f]$

Pf: w.t.s. $f * c_{f(1)} \underset{H}{\simeq} f$



, $H(s,t) = \begin{cases} f(2s-ts) , & 0 \le s \le \frac{t+1}{s} \\ f(1) , & \frac{t+1}{2} \le s \le 1 \end{cases}$

Similarly we get $c_{f(0)} * f \underset{H'}{\simeq} f$. ∎

<u>Lemma 3</u>: Let $f: I \to X$ be a path in X

define $f^{-1}: I \to X$

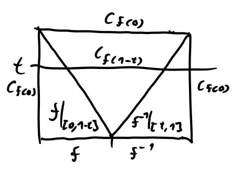
by $f^{-1}(t) = f(1-t)$

then, $[f] * [f^{-1}] = [c_{f(0)}]$

$[f^{-1}] * [f] = [c_{f(1)}]$

Pf:

w.t.s. $f * f^{-1} \underset{H}{\simeq} c_{f(0)}$



Def: Let X be a top. space, $x_0 \in X$.

$\pi_1(X, x_0) = \{ [f] \mid f: I \to X, f(0) = f(1) = x_0 \}$
(closed path)

This is called the <u>fundamental group</u> of X with <u>based pnt.</u> x_0.

$[f] * [g] = [f * g]$ is the group operation

By lemma 1 $*$ is associative, by lemma 2 $[c_{x_0}] \in \pi_1(X, x_0)$ is the identity element, and by lemma 3 inverse exists

<u>Example</u>:

$\pi_1(\mathbb{R}, 0) = \{ [c_0] \}$ is the trivial group.

Def: Let α be a path in X from x_0 to x_1
define
$$\hat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$$
$$\hat{\alpha}([f]) = [\alpha^{-1} * f * \alpha]$$

Prop. $\hat{\alpha}$ is an isomorphism of groups.

Pf:
$$\hat{\alpha}([f] * [g]) = [\alpha^{-1}] * [f * g] * [\alpha]$$
$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] * [g] * [\alpha]$$
$$= [\alpha^{-1}] * [f] * [\alpha * \alpha^{-1}] * [g] * [\alpha]$$
$$= [\alpha^{-1}] * [f * g] * [\alpha],$$
for all paths f, g. Hence $\hat{\alpha}$ is a group homomorphism.

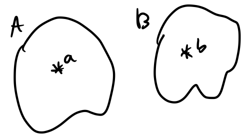
Now consider $\widehat{\alpha^{-1}}$, similarly $\widehat{\alpha^{-1}}$ is a group homomorphism.
Notice $(\widehat{\alpha^{-1}} \circ \hat{\alpha})[f] = [\alpha] * [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] = [f]$.

Hence $\hat{\alpha}$ has an inverse. ■

Obs: If X is path-connected, then for all $x_0, x_1 \in X$
$$\pi_1(X, x_0) \cong \pi_1(X, x_0).$$
However, the isomorphism is not canonical, in general depends on the path α.

Def: X is said to be simply-connected and the fundamental group of X (which is equal at every pnt.) is trivial.

Remark: If X has two path-components A and B, then $\pi_1(X, a_0) = \pi_1(A, a_0)$ and $\pi_1(X, b_0) = \pi_1(B, b_0)$ (for $a_0 \in A, b_0 \in B$) are "independent".



Def: Let $h: X \to Y$ be cont. with $h(x_0) = y_0$
$$h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[f] \longmapsto [h \circ f]$$
is a group homomorphism, called the homomorphism induced by h.

Pf: h_* is well-defined:
If $f \underset{H}{\simeq} f'$, then $h \circ f \underset{h \circ H}{\simeq} h \circ f'$
$$h_*([f]) * h_*([g]) = [h \circ f] * [h \circ g] = [(h \circ f) * (h \circ g)] = [h \circ (f * g)] = h_*([f] * [g])$$
Hence, h_* is a homomorphism.

Functorial properties of the fundamental group.

Given
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$
$f(x_0) = y_0$, $g(y_0) = z_0$

We have,

(1) $(g \circ f)_* = g_* \circ f_*$

(2) $id_{X*} = id_{\pi(X, x_0)}$

Pf: (1) $(g \circ f)_*([w]) = [(g \circ f) \circ w] = [g \circ (f \circ w)] = g_*(f_*([w])) = (g_* \circ f_*)([w])$

(2) $id_*([w]) = [id_X \circ w] = [w]$ ■

Theorem: If $f: X \to Y$ is a homeomorphism with $f(x_0) = y_0$
then $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$
is a group isomorphism.

Pf: Let $g: X \to Y$ be the inverse of f.
then,
$$g \circ f = id_X$$
$$f \circ g = id_Y$$
Hence
$$f_* \circ g_* \; (f \circ g)_* = (id_X)_* = id_{\pi_1(X, x_0)}$$
$$g_* \circ f_* \; (g \circ f)_* = (id_Y)_* = id_{\pi_1(X, x_1)}$$
Hence g_* inverse of f_* proving f_* isomorphism. □

Remark: If X, Y two path-connected spaces, and $\pi_1(X, x_0)$ not isomorphic to $\pi_1(Y, y_0)$
then X not homeomorphic to Y.
So, the fundamental group is an example of a topological invariant for path-connected spaces.

In this sense, the fundamental group $(\pi_1(X, x_0), h_*)$ is a functor from the category of topological spaces with base points and topological base preserving maps to the category of groups and group homomorphism.

Coverings

Let $p: E \to B$ be a cont. surjective map. An open set U in B is said to be evenly cover by p if $p^{-1}(U)$ is a disjoint union of open sets V_α, $\alpha \in J$
$$p^{-1}(U) = \dot{\bigcup_{\alpha \in J}} V_\alpha$$
and its restriction
$$p|_{V_\alpha}: V_\alpha \to U$$
is a homeomorphism $\forall \alpha \in J$.

Def: A cont. surjective map $p: A \to B$ is a covering if each $b \in B$ has a neigh. U which is evenly cover.

Example:
$$p: \mathbb{R} \to S^1 \subset \mathbb{C}$$
$$(S^1 = \{x \in \mathbb{C} : |x| = 1\})$$
$$p(t) = e^{2\pi i t} = \cos(2\pi t) + i \sin(2\pi t) \in \mathbb{C}$$
This is clearly a cover.

Example:
Let $n \in \mathbb{N}$ be fixed.
$$p_n: S^1 \to S^1$$
$$z \mapsto z^n$$
So $p_n(e^{i\theta}) = e^{in\theta}$ for each $z = e^{i\theta} \in S^1$
And it's also a cover.

Remark: $p: E \to B$ covering.
then each fiber $p^{-1}(b)$, $b \in B$ has the discrete topology (as a subspace of E).

Pf: Let $b \in B$. Let U be an evenly cover neighborhood of b.
Then, $p^{-1}(U) = \dot{\bigcup_{\alpha \in J}} V_\alpha$, for open set $\{V_\alpha\}_{\alpha \in J}$.
and
$$p|_{V_\alpha}: V_\alpha \to U$$
is a homeomorphism.
This implies each V_α contains exactly one element of $p^{-1}(b)$.
In other words,
$$|V_\alpha \cap p^{-1}(b)| = 1$$
Since V_α open in E, pnts. are open in the fiber.

Exercise: $p: E \to B$ covering with B connected.
Then each fiber $p^{-1}(b)$, $b \in B$ has the same cardinality.

Pf: Fix $b_0 \in B$

Consider $X = \{ b \in X : |p^{-1}(b)| = |p^{-1}(b_0)| \}$

$X \neq \emptyset$, since $b_0 \in X$.

We prove that X is open in B.

Let $b \in X$, U neigh. of b

Then $p^{-1}(U) = \dot{\bigcup}_{\alpha \in J} V_\alpha$ and $p|_{V_\alpha}: V_\alpha \to B$ is a homeomorphism.

Each fiber $p^{-1}(b)$ of $b \in U$ satisfies $|p^{-1}(b)| = |J|$.

Hence $U \subset X$. Proving X open.

Now to prove X closed, i.e. $B \setminus X$ open.

Take $b \in B \setminus X$ and U neigh of b. By the same argument as before $\forall x \in U: |p^{-1}(x)| = |J|$.

So $U \subset B \setminus X$, and X closed.

Thus, X clopen and not $\emptyset$. By connectivity of B, $X = B$. ∎

Def: Given a cover $p: E \to B$. A *lifting* of the cont. function $f: X \to B$, is a continuous map $\tilde{f}: X \to E$ s.t. the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Path lifting lemma: Let $p: E \to B$ be a covering, $p(e_0) = b_0$. Then, any path

$$f: [0,1] \to B$$

with $f(0) = b_0$. Has a unique lifting

$$\tilde{f}: [0,1] \to E$$

s.t. $\tilde{f}(0) = e_0$

Pf: Choose a covering $\mathcal{U}$ of B by open set which are evenly cover.
Then the preimage

$$f^{-1}(\mathcal{U}) = \{ f^{-1}(U) : U \in \mathcal{U} \}$$

is an open cover of $[0,1]$. Since $[0,1]$ is a compact metric space, we apply the Lebesgue Number Lemma.

Then, we have a subdivision $s_0 = 0 < s_1 < \cdots < s_n = 1$ of $[0,1]$ s.t.

$f([s_i, s_{i+1}])$ is contained in at least one $U \in \mathcal{U}$.

Let $\tilde{f}(0) = e_0$. Sps by induction that $\tilde{f}$ is defined in the interval $[0, s_i]$.

Let $f([s_i, s_{i+1}]) \subset U \in \mathcal{U}$.

$p^{-1}(U) = \dot{\bigcup}_{\alpha \in J} V_\alpha$, $p|_{V_\alpha}: V_\alpha \to U$ homeomorphism.

Notice $\tilde{f}(s_i)$ is defined and is in $p^{-1}(U)$. Also, $\tilde{f}(s_i)$ lies in one V_α, $\alpha \in J$. Let $\tilde{f}(s_i) \in V_0$.

So, for $t \in [s_i, s_{i+1}]$ define

$$\tilde{f}(t) = ((p|_{V_0})^{-1} \circ f)(t)$$

By pasting lemma we've extended continuously $\tilde{f}$ to $[0, s_{i+1}]$.

Inductively, $\tilde{f}$ is define on the whole $[0,1]$.

Now we prove uniqueness.

$\tilde{f}(0) = e_0$ unique, no choice to be made,

Now sps. for induction that $\tilde{f}$ is unique in $[0, s_i]$.

As before, let $\tilde{f}(s_i) \in V_{\alpha_0}$

$$\left(p^{-1}(U) = \dot{\bigcup}_{\alpha \in J} V_\alpha \right)$$

Since $f([s_i, s_{i+1}]) \subset U \in \mathcal{U}$

$\tilde{f}([s_i, s_{i+1}]) \subset p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$

Hence $\tilde{f}([s_i, s_{i+1}]) \subset V_{\alpha_0}$ (since this union is disjoint and $\tilde{f}([s_i, s_{i+1}])$ connected)

So $\tilde{f}|_{[s_i, s_{i+1}]} = (p|_{V_{\alpha_0}})^{-1} \circ f|_{[s_i, s_{i+1}]}$ ∎

Homotopy lifting lemma:

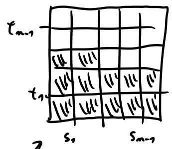
Let $p: E \to B$ be a covering, $p(e_0) = b_0$.
Let $F: I \times I \to B$ be cont. ($I = [0,1]$), w/ $F(0,0) = b_0$
then there is a lifting $\tilde{F}: I \times I \to E$ s.t. $\tilde{F}(0,0) = e_0$.
If F is a path-homotopy then $\tilde{F}$ is a path-homotopy.

Pf: Using the Lebesgue Numbering Lemma there are subdivisions

$$s_0 = 0 < s_1 < \dots < s_m = 1 \quad \text{and}$$
$$t_0 = 0 < t_1 < \dots < t_m = 1$$

of I s.t.

$F([s_i, s_{i+1}] \times [t_i, t_{i+1}])$ is contained in at least one U.



Now we construct $\tilde{F}$.

$\tilde{F}(0,0) = e_0$.

Sps. $\tilde{F}$ is defined already on $(I \times [0, t_j]) \cup ([0, s_i] \times [t_j, t_{j+1}])$

We wish to define $\tilde{F}$ on $[s_i, s_{i+1}] \times [t_j, t_{j+1}] \subset U$

$\tilde{F}$ is already defined on $\underbrace{(\{s_i\} \times [t_j, t_{j+1}]) \cup ([s_i, s_{i+1}] \times \{t_j\})}_{\text{connected}} = Y$

$$p^{-1}(U) = \dot{\bigcup_{\alpha \in J}} V_\alpha$$

$\Rightarrow \tilde{F}(Y) \subset V_{\alpha_0}$ for some $\alpha_0 \in J$

Define $\tilde{F}$ on $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ by $\tilde{F} = (p|_{V_{\alpha_0}})^{-1} \circ F$.

After finitely many steps we define $\tilde{F}$.

Now sps. F is a path-homotopy.

$\tilde{F}|_{\{0\} \times I}$ is a lifting of the constant path c_{b_0}, $b_0 \in B$.

The lifting of c_{b_0} remains on the fiber $p^{-1}(b_0)$, which is discrete.
Any path in a discrete space is constant (since I connected).
So $\tilde{F}|_{\{0\} \times I}$ is constant.
Similarly $\tilde{F}|_{\{1\} \times I}$ is constant.

Hence $\tilde{F}$ is a homotopy. ∎

Corollary: Let $p: E \to B$ be a covering, $p(e_0) = b_0$. Let f, g be paths in B from b_0 to b_1 which are path-homotopic.
And let $\tilde{f}, \tilde{g}$ be liftings to E with initial pnt. e_0. Then, also $\tilde{f}$ and $\tilde{g}$ are path-homotopic with same endpnt.

Pf: $f \underset{H}{\simeq} g$

By the homotopy lifting lemma, H lifts to $\tilde{H}$ which is also a path-homotopy.
by uniqueness of path-lifting with a given start pnt e_0

$$\tilde{f} \underset{\tilde{H}}{\simeq} \tilde{g}. \ ∎$$

Theorem: $\pi_1(S_1, 1) \cong \mathbb{Z}$

Pf: $\varphi: \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$

For $[w] \in \pi_1(S^1, 1)$, $w: I \to S^1$ w/ $w(0) = w(1) = 1 \in S^1$.

Take covering $p: \mathbb{R} \to S^1$
$t \mapsto e^{2\pi i t}$

Now let $\tilde{w}: I \to \mathbb{R}$ be the lifting of w to $\mathbb{R}$, w/ initial pnt $\tilde{w}(0) = 0 \in \mathbb{R}$.

Then $\tilde{w}(1) \in p^{-1}(1) = \mathbb{Z}$

This is the rotation number of w.

Then set

$$\varphi([w]) = \tilde{w}(1) \in \mathbb{Z}$$

If $[w] = [w']$, i.e. $w \underset{F}{\simeq} \hat{w}$

Let $\tilde{w}, \tilde{w}'$ be the lifting of $w, \hat{w}$. By prev. lemma $\tilde{w} \underset{H}{\simeq} \tilde{w}'$, and $[\tilde{w}] = [\tilde{w}']$.

So φ is well-defined.

i) φ is a group homomorphism

$$\varphi([w]) + \varphi([v]) = \tilde{w}(1) + \tilde{v}(1)$$

$$\varphi([w] * [v]) = \widetilde{w * v}(1)$$

$$\widetilde{w*v} = \tilde{w} * (\tilde{v} + \tilde{w}(1))$$

So $\widetilde{w*v}(1) = (\tilde{v} + \tilde{w}(1))(1)$

$= \tilde{v}(1) + \tilde{w}(1)$ ✓

ii) Let $[w] \in \operatorname{Ker} \varphi$. Notice $\tilde{w}(1) = 0$. Then $\tilde{w}$ is a closed path.

So $[\tilde{w}] \in \pi_1(\mathbb{R}, 0)$ i.e. $[\tilde{w}] = [c_0]$, so $\tilde{w} \underset{H}{\simeq} c_0$.

And $w = p \circ c_0 \underset{p \circ H}{\simeq} p \circ c_0 = c_1$

Hence, $[w] = [c_1]$. Proving $\operatorname{Ker} \varphi = \{[c_1]\}$ and φ injective.

iii) Let $n \in \mathbb{Z}$.

Let $\mu : I \to \mathbb{R}$ be any path in the reals s.t. $\mu(0) = 0, \mu(1) = n$.

Let $w = p \circ \mu$, notice is a closed path $(p(n) = p(0) = 1)$.

So $\varphi([w]) = \tilde{w}(1) = \mu(1) = n$

Hence φ surjective. ∎

Def: Let $A \subset X$ be a subspace. A cont. map $r : X \to A$ is a <u>retraction</u> if

$$r|_A = id_A$$

This is equivalent to

$$r \circ i = id_A$$

w/ $i : A \hookrightarrow X$ is the inclusion map.

e.g.: $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$

$$x \mapsto \frac{x}{\|x\|}$$

is a retraction.

Lemma: If $r : X \to A$ is a retraction then

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective

and

$$i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$$

is injective. $(a_0 \in A \subset X)$.

Pf: We know $r \circ i = id_A$ So

$$(r \circ i)_* = (id_A)_*$$

$$r_* \circ i_* = id_{\pi_1(A, a_0)}$$

By the two functorial properties.

Hence r_* is surjective and i_* injective

Prop: There is no retraction

$$r : B^2 \to S^1$$

w/ $B^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ ($S^1 = \partial B$).

Pf: Bwoc. if r is a retraction, then

$$r_* : \pi_1(B^2, 1) \to \pi_1(S^1, 1) \cong \mathbb{Z}$$

is surjective.

But $\pi_1(B^2, 1) \cong \{0\}$ trivial group. #

∎

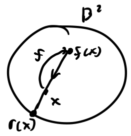
Remark: There is a retraction

$$r: B^2 \setminus \{0\} \to S^1$$
$$x \mapsto \frac{x}{\|x\|}$$

Theorem (Brouwer fixed point theorem in dim. 2):

Every cont. map
$$f: B^2 \to B^2$$
has a fixed point.

Pf: Sps. f has no fixed point.



Then, $\forall x \in B^2 : f(x) \neq x$. Let
$$r: B^2 \to S^1$$
s.t.
$$r(x) = \{tx + (1-t)f(x) : t \in \mathbb{R}\} \cap S^2$$

It is known that $r(x) = \dfrac{f(x)+x}{1 + x f(x)}$ w/ multiplication is done in $\mathbb{C}$.

Hence r is cont.

Clearly $r|_{S^1} = id_{S^1}$, and r is a retraction. ↯

Hence f has at least one fixed pnt. ∎

Lemma: If a cont. map $f: S^1 \to X$ extends a cont. map $F: B^2 \to X$ $(F|_{S^1} = f)$ then
$$f_*: \pi_1(S^1, 1) \to \pi_1(X, f(1))$$
is the trivial homomorphism.

Pf: Let $i: S^1 \hookrightarrow B^2$ be the inclusion.

Then $f = F \circ i$

So $f_* = F_* \circ i_*$, by the functorial properties.

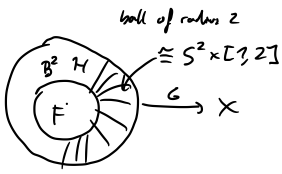
$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(S^1, 1) \arrow[r, "i_*"] \arrow[rr, bend right, "f_*"'] & \underbrace{\pi_1(B^2, 1)}_{\text{trivial}} \arrow[r, "F_*"] & \pi_1(X, F(1)) \end{tikzcd}$$
, commutes.

Hence f_* is trivial ∎

Lemma: Let $f \underset{H}{\cong} g$, $f, g: S^1 \to X$ cont.

If f extends for B^2 then als g extends to B^2. And f_* trivial implies g_* trivial

Pf: $H: S^1 \times [1,2] \to X$ is a homotopy between f and g.



By the pasting lemma G is cont. and extends g.
□

Theorem (Fundamental theorem of algebra):

Every non-constant complex polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has a root in $\mathbb{C}$.

Pf: Sps. $|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1$.

Sps. $p(x)$ has no root bwoc.

Let $g: B^2 \to \mathbb{C} \setminus \{0\}$,
$x \mapsto p(x)$

$f: S^1 \to \mathbb{C} \setminus \{0\}$
$x \mapsto p(x)$

So, $g = p|_{B^2}$, $f = g|_{S^1}$. So f extends to B^2. By prev. lemma f_* is trivial

$f_*: \pi_1(S^1, 1) \to \pi_1(\mathbb{C}\setminus\{0\}, f(1))$. So $\pi_1(S^1, 1) \cong \pi_1(\mathbb{C}\setminus\{0\}, f(1)) \cong \mathbb{Z}$.

Let $k: S^1 \to \mathbb{C}\setminus\{0\}$ (notice x^n is the dominant term of p).
$x \mapsto x^n$

We have

$$f \underset{H}{\simeq} k$$

w/ $H(x,t) = x^n + t(a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0)$

To check H correctly avoids 0 (since the codomain of k is $\mathbb{C}\setminus\{0\}$).

$$\begin{aligned} |H(x,t)| &\geq |x^n| - |t(a_{n-1}x^{n-1} + \cdots + a_0)| \\ &\geq 1 - t(|a_{n-1}x^{n-1}| + |a_{n-2}x^{n-2}| + \cdots + |a_0|) \\ &= 1 - t(\underbrace{|a_{n-1}| + \cdots + |a_0|}_{<1}) > 0 \end{aligned}$$

$\Rightarrow H(x,t) \neq 0$

f extends to $g: B^2 \to \mathbb{C}\setminus\{0\}$ and $f \underset{H}{\simeq} k$

Hence, k extends to a map $B^2 \to \mathbb{C}\setminus\{0\}$ by prev. lemma.

So, $k_*: \pi_1(S^1, 1) \to \pi_1(\mathbb{C}\setminus\{0\}, 1)$ is trivial also by lemma.

Consider the commutative diagram:

$$\begin{tikzcd} \pi_1(S^1,1) \arrow[r, "k_*"] \arrow[dr, "\cdot n"'] & \pi_1(\mathbb{C}\setminus\{0\},1) \arrow[d, "r_*"] \\ & \pi_1(S^1,1) \end{tikzcd} \quad \left(r(x) = \frac{x}{\|x\|}\right)$$

Since $n > 0$, the map is not trivial, but the diagram commute. $\lightning$

Now we reduce every other case to our assumption about the size of the coeffs.

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

$\iff c^n y^n + a_{n-1}c^{n-1}y^{n-1} + \cdots + a_0 = 0$ w/ $x = cy$, $c > 0$, $c \in \mathbb{R}$.

$$\iff y^n + \frac{a_{n-1}}{c}y^{n-1} + \frac{a_{n-2}}{c^2}y^{n-2} + \cdots + \frac{a_0}{c^n} = 0$$

Choosing c large we can make the coeffs. arbitrary small. ■

Fundamental Group of 2-manifolds

Theorem (special case of Van Kampen's theorem)

Let $X = A \cup B$, A, B open. Let $x_0 \in A \cap B$.
If $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ are trivial, then also $\pi_1(X, x_0)$ is trivial.

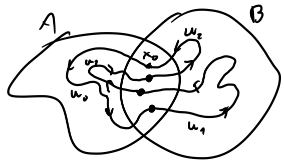
Pf: Let $[w] \in \pi_1(X, x_0)$

$w : [0,1] \to X$ path w/ $w(0) = w(1) = x_0$

By the Lebesgue number lemma applied to the open covering $\{w^{-1}(A), w^{-1}(B)\}$ of $I = [0,1]$ we have a subdivision $t_0 = 0 < t_1 < \cdots < t_{l-1} < t_l = 1$ of I s.t.

$$w([t_i, t_{i+1}]) \subset A \text{ or } B \quad \forall i = 1, \dots, l-1$$

alternatingly in A and B. So, $w(t_i), w(t_{i+1}) \in A \cap B$.



Let $w_i = w\big|_{[t_i, t_{i+1}]}$ reparametrized to $[0,1]$. So

$$w_i : [0,1] \to X$$

Hence,

$$w \simeq w_0 * w_1 * \cdots * w_{l-1}$$

Since the RHS is a reparametrization of w

Now let $k_i : I \to A \cap B$ be a path from x_0 to $w_i(1)$ for $i = 0, \dots, l-2$

So $\quad w \simeq w_0 * k_0^{-1} * k_0 * w_1 * k_1^{-1} * k_1 * \cdots * k_{l-2}^{-1} * k_{l-2} * w_{l-1}$

Notice $k_i^{-1} * k_i \simeq c_{x_0}$ for all i.

Using associativity

$$w \simeq (w_0 * k_0^{-1}) * (k_0 * w_1 * k_1^{-1}) * \cdots * (k_{l-2} * w_{l-1})$$

But each $(k_i * w_{i+1} * k_{i+1}^{-1}) \simeq c_{x_0}$ (since are in A or B).

Proving $w \simeq c_{x_0}$. ∎

Theorem For $n > 1$, $\pi_1(S^n, x_0)$ is trivial

Pf: Let $S^n_+ = S^n \setminus \{(0, 0, \dots, -1)\}$ and $S^n_- = S^n \setminus \{(0, 0, \dots, 1)\}$

Then $S^n_+, S^n_- \cong \mathbb{R}^n$ using the stereographic projection.

And, thus, have trivial fund. groups.

$S^n_+ \cap S^n_- \cong \mathbb{R}^n \setminus \{0\}$, which is path-connected.

By prev. theorem

$$\pi_1(S^n, x_0) \text{ trivial}$$

∎

Remark: $\pi_1(S^2, x_0)$ is trivial. Take $[w] \in \pi_1(S^2, x_0)$

If w is not surjective, then some pnt is not in the image.

Then, using the stereographic projection with that antipodal pnt, contract w, using great circles, to the cte path.

But there are surjective paths.

Examples: $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times \mathbb{R}_{>0}$

$$x \overset{\text{iso}}{\longmapsto} \frac{x}{\|x\|} \times \|x\|$$

Prop: $\pi_1(X\times Y, x_0\times y_0)\cong \pi_1(X,x_0)\times\pi_1(Y,y_0)$

Pf:

$$\pi_1(X\times Y, x_0\times y_0) \xrightarrow{\varphi} \pi_1(X,x_0)\times\pi_1(Y,y_0)$$
$$[w] \longmapsto [p_x\circ w]\times[p_y\circ w]$$

w/ $p_x: X\times Y\to X$

$p_y: X\times Y\to Y$

are the projections.

Thus φ is a well-defined, homomorphism of groups.

Also $\varphi^{-1}: \pi_1(X,x_0)\times\pi_1(Y,y_0)\to\pi_1(X\times Y, x_0\times y_0)$

$[w_1]\times[w_2]\longmapsto[w_1\times w_2]$

$(w_1\times w_2)(t) = w_1(t)\times w_2(t)$

is also a well-defined, group homomorphism and the inverse of φ. ∎

Corollary: $\pi_1(\mathbb{R}^2\setminus\{0\})\cong\pi_1(S^1,x_0)\times\underbrace{\pi_1(\mathbb{R}_{>0},1)}_{\text{trivial}}\cong\pi_1(S^1,x_0)\cong\mathbb{Z}$

Corollary: $S^1\times S^1\subset\mathbb{C}\times\mathbb{C}=\mathbb{R}^4$ torus

$$\pi_1(S^1\times S^1, 1\times 1)\cong\pi_1(S^1,1)\times\pi_1(S^1,1)\cong\mathbb{Z}\times\mathbb{Z}.$$

Corollary: S^2 and $S^1\times S^1$ are not homeomorphic

Real projective spaces $\mathbb{P}\mathbb{P}^n$

$\mathbb{R}\mathbb{P}^n=\{[x]_\sim \mid x\in\mathbb{R}^n\}$ w/ $x,y\in\mathbb{R}^n$, $x\sim y :\Leftrightarrow x=\lambda y$, for some $\lambda\in\mathbb{R}$.

Also:

$S^n\subset\mathbb{R}^{n+1}$, with $[x]=\{x,-x\}$

Then $\mathbb{P}\mathbb{P}^n=\{[x]\mid x\in S^n\}$

$p: S^n\longrightarrow\mathbb{P}\mathbb{P}^n$

$x\mapsto[x]$

So $V\subset\mathbb{P}\mathbb{P}^n$ open if $p^{-1}(V)$ open in S^n.

This is the quotient topology ($\mathbb{RP}^n=S^n/\sim$)

$p: S^n\to\mathbb{P}\mathbb{P}^n$ is a covering

In part, 2-fold covering, i.e. each fiber has 2 elements.

Theorem: $\pi_1(\mathbb{P}\mathbb{P}^n,[x_0])\cong\mathbb{Z}_2$, for $n>1$.

Pf: $p: S^n\longrightarrow\mathbb{P}\mathbb{P}^n$

$x\longmapsto[x]=\{x,-x\}$

2-fold covering.

$\phi: \pi_1(\mathbb{RP}^n,[x_0])\longrightarrow\mathbb{Z}_2\cong(\{1,-1\},\cdot)$

Take $[w]\in\pi_1(\mathbb{RP}^n,[x_0])$

$w: I\to\mathbb{RP}^n$

$w(0)=w(1)=[x_0]$.

Let $\tilde w: I\to S^2$ be a lifting of w s.t. $\tilde w(0)=x_0\in S^1$. Then $\tilde w(1)=\varepsilon x_0$, $\varepsilon\in\{1,-1\}$.

Let $\phi([w]):=\varepsilon\in\{1,-1\}$

i) ϕ is well-defined by the Homotopy lifting lemma

ii) ϕ is a homomorphism

iii) ϕ is injective.

Let $[w]\in\ker\phi$, then $\tilde w(1)=1\cdot x_0=x_0=\tilde w(0)$

Then $\tilde w$ is a closed path in S^2.

Thus, $[\tilde w]\in\pi_1(S^2,x_0)$, trivial. So $\tilde w\underset{h}{\cong}c_{x_0}$.

Since $w=p\circ\tilde w\underset{path}{\cong}p\circ c_{x_0}=c_{[x_0]}$

Thus $[w]=[c_{[x_0]}]$, hence $\ker\phi=\{[c_{[x_0]}]\}$

Proving ϕ injective.

(v) ϕ is surjective.

Let α be a path in S^n from x_0 to $-x_0$ and $w = p \circ \alpha$. So w is closed path

Then $[w] \in \pi_1(\mathbb{R}P^n, [x_0])$.

$\tilde{w}(1) = \alpha(1) = -x_0$.

$\varphi([w]) = -1$.

Notice $\varphi([c_{[x_0]}]) = 1$. ■

Homotopy equivalence

Def: Let $f: X \to Y$ be cont. f is <u>homotopy equivalence</u> if there is a cont. map $g: Y \to X$ s.t.

$$g \circ f \simeq id_X$$
$$f \circ g \simeq id_Y$$

Two spaces X and Y are homotopy equivalence (or homotopic) if there is a homotopy equivalence.

Def: A top. space X is contractible if id_X is homotopic to a constant map:

$$id_X \simeq c_{x_0}, \quad x_0 \in X$$

Prop. X is contractible iff $X \simeq \{x_0\}$

Pf: "$\Rightarrow$" Sps. X is contractible. then $id_X \simeq c_{x_0}$, $x_0 \in X$. Let

$f: X \longrightarrow \{x_0\}$

$g: \{x_0\} \hookrightarrow X$ inclusion

$f \circ g = id_x$

$g \circ f = c_{x_0} \simeq id_X$

Hence $X \simeq \{x_0\}$

"$\Leftarrow$" Sps. $f: X \to \{y_0\}$, $g: \{y_0\} \to X$ s.t.

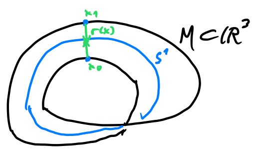
$$g \circ f = c_{g(y_0)} \simeq id_X$$ ■

Def: A retraction $r: X \longrightarrow A$, $A \subset X$ is a <u>deformation retraction</u> if

$$i \circ r \simeq id_X$$

Since for all retractions $r \circ i = id_A$, deformation retraction are homotopy equivalence.

e.g.: <u>Möbius band M</u>



$r: M \longrightarrow S^1$

r is a deformation retraction.

$$M \simeq S^1$$

Prop: Let $H: X \times I \to Y$ be a homotopy

$f_t(x) = H(x, t)$ for a fixed $t \in I$.

$$f_0 \underset{H}{\simeq} f_1$$

Fix $x_0 \in X$, let $\alpha: I \to Y$ be a path

$\alpha(t) = f_t(x_0) = H(x_0, t)$

Then, $\hat{\alpha} \circ (f_0)_* = (f_1)_*$, ie.

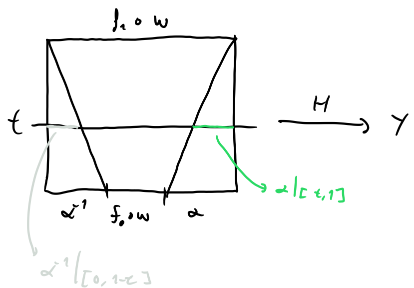
$$\begin{tikzcd}
& \pi_1(Y, f_0(x_0)) \arrow[d, "\cong", "\hat{\alpha}"'] \\
\pi_1(X, x_0) \arrow[ur, "(f_0)_*"] \arrow[r, "(f_1)_*"'] & \pi_1(Y, f_1(x_0))
\end{tikzcd}$$

commutes

Pf: Let $[w] \in \pi_1(X, x_0)$

W.t.s. $[\alpha^{-1}] * [f_0 \circ w] \circ [\alpha] = [f_1 \circ w]$

$\Leftrightarrow \alpha^{-1} * f_0 \circ w * \alpha \underset{H}{\simeq} f_1 \circ w$



Theorem: If $f: X \to Y$ is a homotopy equivalence, with $f(x_0) = y_0$, then $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Pf:

$f: X \to Y$, $g: Y \to X$

$g \circ f \simeq id_X$

$f \circ g \simeq id_Y$

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\cong"', "\hat{\alpha}"] & \pi_1(Y, y_0) \arrow[dl, "g_*"'] \\
\pi_1(X, g(x_0)) \arrow[r, "f'_*"'] & \pi_1(Y, f(g(y_0))) \arrow[u, "\hat{\beta}", "\cong"']
\end{tikzcd}$$

Notice, by the functorial properties:

$(g \circ f)_* = g_* \circ f_*$
$(id_X)_* = id_{\pi_1(X, x_0)}$ $\implies g_* \circ f_* = \hat{\alpha} \circ id_{\pi_1(X, x_0)} = \hat{\alpha}$ //

Similarly //

So, since $\pi_1(X, x_0) \cong \pi_1(X, g(y_0))$, g_* surjective

And, since $\pi_1(Y, y_0) \cong \pi_1(Y, f(g(y_0)))$, g_* injective.

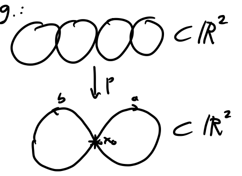
By simmetry ■

Corollary: If X is contractable then $\pi_1(X, x_0)$ is the trivial group.

Pf: $id_X \simeq c_{x_0} \iff X \simeq \{x_0\} \implies \pi_1(X, x_0) \cong \pi_1(\{x_0\}, x_0)$ trivial ■

Corollary: If X, Y are path-connected and homotopy equivalent then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for all $x_0 \in X$, $y_0 \in Y$.

e.g.:



$a * b \not\simeq b * a$