Basis A *basis* of a topology on set X is a collection $\mathcal{B}$ of subsets of X s.t.

- For each $x \in X$, there is $B \in \mathcal{B}$ s.t $x \in B$ (i.e. $\bigcup_{B \in \mathcal{B}} B = X$)
- For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is $B \in \mathcal{B}$ w/ $x \in B \subset B_1 \cap B_2$.

The topology generated by a basis $\mathcal{B}$ is the set $\mathcal{T}$ s.t.
$U \in \mathcal{T}$ if, for every $x \in U$ there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.

Claim: $\mathcal{T}$ is a topology.

Pf:
- $\emptyset \in \mathcal{T}$ Trivially.
- For all $x \in X$ there is $B \in \mathcal{B}$ s.t. $x \in B \subset X$. Thus, $X \in \mathcal{T}$.
- Take $U_i \in \mathcal{T}$ for $i \in J$.
  Let $x \in \bigcup_{i \in J} U_i$, then $x \in U_j$, for some $j \in J$.
  Since $U_j \in \mathcal{T}$, there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U_j \subset \bigcup_{i \in J} U_j$
  Hence, $\bigcup_{i \in J} U_i \in \mathcal{T}$.
- Let $U_1, U_2 \in \mathcal{T}$.
  Take $x \in U_1 \cap U_2$:
  We know $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$ for $B_1, B_2 \in \mathcal{B}$
  Thus $x \in B_1 \cap B_2$, so there is $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$
  So $B_1 \cap B_2 \in \mathcal{T}$. ∎

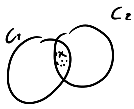
Lemma: Let $\mathcal{B}$ be a basis for a top. $\mathcal{T}$ on X then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Pf: $\boxed{\supseteq}$ Take $\{B_i\}_{i \in J} \subset \mathcal{B}$. Take $x \in \bigcup_{i \in J} B_i$, then $x \in B_j \subset \bigcup_{i \in J} B_i$ for some $B_j \in \mathcal{B}$, thus $\bigcup_{i \in J} B_i \in \mathcal{T}$.

$\boxed{\subseteq}$ Take $U \in \mathcal{T}$. We know that for all $x \in U$, there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.
Thus, $U = \bigcup_{x \in U} B_x$ ∎

Lemma: Let $(X, \mathcal{T})$ top. space, let $\mathcal{C}$ be a collection of open sets s.t. for each open set $U \in \mathcal{T}$ and each $x \in U$, there is $C \in \mathcal{C}$ s.t. $x \in C \subset U$.
Then, $\mathcal{C}$ is a basis and generates $\mathcal{T}$.

Pf: i) Clearly, for each $x \in X$ there is $C \in \mathcal{C}$ w/ $x \in C$.
And for $C_1, C_2 \in \mathcal{C}$, $x \in C_1 \cap C_2$
$C_1 \cap C_2$ open (since C_1, C_2 open), so we have $C \in \mathcal{C}$ w/ $x \in C \subset C_1 \cap C_2$. Proving $\mathcal{C}$ is a basis.



ii) Let $\mathcal{T}'$ be the top. generated by $\mathcal{C}$.
Claim: $\mathcal{T}' = \mathcal{T}$
Pf: $\boxed{\subset}$ Take $U \in \mathcal{T}'$. That is $U = \bigcup_{C \in \mathcal{C}'} C$, for some $\mathcal{C}' \subset \mathcal{C}$
Since all $C \in \mathcal{C}$ open, we have U open, i.e. $U \in \mathcal{T}$.
$\boxed{\supset}$ Take $U \in \mathcal{T}$. Take $x \in U$, then for some $C \in \mathcal{C}$ we have $x \in C \subset U$
Thus, $U \in \mathcal{T}'$. ∎

Subbasis: If X is a set, a subbasis for X is a collection $\mathcal{S} \subset X$ s.t. for each $x \in X$, there is $S \in \mathcal{S}$ w/ $x \in S$.

$$(\mathcal{S} \subset \mathcal{B} \subset \mathcal{T})$$

$$\underset{\mathcal{S}}{\text{subbasis}} \xrightarrow[\text{intersection}]{\text{finite}} \underset{\mathcal{B}}{\text{basis}} \xrightarrow[\text{unions}]{\text{arbitrary}} \underset{\mathcal{T}}{\text{topology}}$$