Natural transformation

meu

July 2, 2025

Contents

1 Definition

If F and G are functors between the categories C and D, then a **natural transformation** η from F to G is a family of morphisms satisfying two requirements: 1. The natural transformation must associate, to every object X in C, a morphism $\eta_X : F(X) \to G(X)$ between objects of D. The morphism η_X is called the **component** of η at X. 2. Components must be such that for every morphism $f : X \to Y$ in C the following diagram commutes

$$\begin{array}{ccc} X & \quad & F(X) \xrightarrow{\eta_X} G(X) \\ \Big\downarrow f & & \Big\downarrow F(f) \qquad \Big\downarrow G(f) \\ Y & & F(Y) \xrightarrow{\eta_Y} G(Y) \end{array}$$

2. is called the **naturality condition**.

2 Example

In Vect we define the following morphism between a vector space V and its double dual V^{**}:

$$\mathcal{E}_V : V \to V^{**} : v \mapsto (f \in V^* \mapsto f(v))$$

Then, the family $(\mathcal{E}_V)_{V\in\mathrm{Vect}}$ defines a natural transformation $\mathcal{E} : \mathbb{1}_{\mathrm{Vect}} \to (\cdot)^{**}$. In this case, the naturality condition comes down to

$$\begin{array}{ccc} V & & V \xrightarrow{\mathcal{E}_V} V^{**} \\ \Big\downarrow l & & \Big\downarrow l \qquad\qquad \Big\downarrow F\in V^{**}\mapsto(g\in W^{*}\mapsto F(g\circ l)) \\ W & & W \xrightarrow{\mathcal{E}_W} W^{**} \end{array}$$