Productive and creative sets

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1 Productive sets

We define the notions of **productive** and **creative** sets, both are varieties of non-computable subsets of the natural numbers.
We first notice that the following are equivalent for a subset $A \subset \mathbb{N}$.

- A is not computably enumerable.
- For all comutable enumerable subsets $W_e \subset A$ there exists an $n \in A \backslash W_e$

Here the subindex $(W_e)_{e \in \mathbb{N}}$ stands for an enumeration of the c.e. subsets of A.

Definition 1.1. A set $A \subset \mathbb{N}$ is **productive** if there exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $e \in \mathbb{N}$, $W_e \subset A$ we have $g(e) \in A \backslash W_e$.

The function g is call **outsider finder** for A.
Crearly, every productive set is not c.e. Moreover, In the effective topos a set A

- is not c.e. iff $\forall f : A^{\mathbb{N}}.\neg(f \text{ is surjective})$.
- is productive iff $\forall f : A^{\mathbb{N}}.\exists a : A(a \notin \text{image} f)$

Being the second one stronger in the intuitionistic logic that is model by this topos.

2 Creative sets

Definition 2.1. A set $A \subset \mathbb{N}$ is **creative** if it is c.e. and its complements if productive.

Proposition 1. *The halting set K is creative.*

3 Example from the language of arithmetic

Let T be a theory with a effective axiomatization, then the set w$\{S \mid T \vdash S\}$ of provable sentences is c.e.. it can be proven that the set $\{s \mid t \vDash S\}$ of true senteces is productive. This can be used to prove Gödel first incompleteness theorem.