Fixed Points

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1 Definition

Definition 1.1. A fixed point of a function f is a value x of its domain satisfying:

$$f(x) = x$$

One seemingly obvious way to find fixed point if we allow infinitely many application of the function is:

$$x = f(f(f(f(...))))$$

Proposition Let $f : \mathbb{R} \to \mathbb{R}$ be *continuos* and $x_1 \in \mathbb{R}$. If the sequence

$$x_1, x_2 = f(x1), x_3 = f(x_2), ...$$

converges to $x \in \mathbb{R}$, then $f(x) = x$.

2 Applications on logic

For a formula ϕ in the language of Peano Arithmetic (PA), let $\ulcorner \phi \urcorner$ denote its Göldel number.

Lemma (Carnap) For any formula $\phi(x)$, there is a sentence ψ such that

$$PA \vdash \psi \leftrightarrow \phi(\ulcorner \psi \urcorner)$$

i.e. ψ says "I have property ϕ".
In the book *An Introduction to Godel's Theorems* from Peter Smith (Logic Matters) one can find a constructive proof os this Lemma (called **Diagonal Lemma**).

2.1 Gödel's Incompleteness Theorem

Theorem 2.1 (Gödel)**.** *If PA is consistent , then it admits true but unprovable statements.*

Proof. By previous lemma, there is ψ for which

$$PA \vdash \psi \leftrightarrow \neg Prov(\psi)$$

Where $Prov(x)$ is the formula constructed such that $Prov(\phi)$ if ϕ is provable in PA. □

3 Applications on computer science

The fixed point of a function can be used to achieve infinite loops via recursion in something like λ-calculus.
First consider

$$\omega = \lambda x.xx$$

and

$$\Omega = \omega\omega$$

Ω reduces to itself:

$$\begin{aligned} \Omega &= \omega\omega && \text{By definition of } \Omega \\ &= (\lambda x.xx)\omega && \text{by definition of } \omega \\ &= \omega\omega && \text{by } \beta\text{-reduction} \\ &= \Omega \end{aligned} \tag{1}$$

If we introduce F into ω, we can generate an F at each reduction step of Ω. Let $\omega_F = \lambda x.F(xx)$ and $\Omega_F = \omega_F\omega_F$. Then

$$\Omega_F = \omega_F\omega_F = (\lambda x.F(xx))\omega_F = F(\omega_F\omega_F) = F(\Omega_F)$$

In other words, Ω_F is a fixed point of F.
We can generalize this to obtain a fixed-point combinator:
Theorem (Curry): There is a combinator Y such that

$$YF = F(YF)$$

is a fixed point of F for any term F.
Pf :

$$Y = \lambda x.\Omega_f = \lambda x.f(xx)(\lambda x.(fxx)) \qquad \square$$