Dynamic Models with Latent Variables: An analysis of 'Stationarity and Ergodicity of Markov Switching Positive Conditional Mean Models'

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1 Introduction

The purpose of this report is to explore and review the theoretical and practical aspects of the paper *Stationarity and Ergodicity of Markov Switching Positive Conditional Mean Models* by Aknouche A. and Francq C. (2022) [2]. This work focuses on a class of Markov Switching Autoregressive Conditional Mean models, which are particularly relevant for non-negative time series, such as count data, durations, and proportions. These models allow flexibility of Markov switching mechanisms to capture regime changes, thereby enhancing their possibility of application to real-world time series displaying regime-specific dynamics. Furthermore, the intent of this work is also to replicate some of the examples proposed in the paper.

Objectives and Structure of the Report

This project aims to achieve various objectives; firstly, we provide a concise yet comprehensive synthesis and analysis of the contributions of the theoretical results and most methodologies presented in the paper. We go through the structure of the paper, listing all the results which we considered fundamental. We proceed by conducting a critical analysis on the paper's assumptions and theoretical results, delving deeper into some specifics of the proofs, seeking to evaluate the clarity of the proposed methods, exploring the assumptions for the proofs, and finally, assessing their applicability to practical scenarios and their relevance to the context of the field. Finally, we focus on performing numerical experiments, inspired directly by examples presented in the paper. These experiments include the implementation of simulations and the validation the Markov Switching Positive Conditional Mean Models. The Poisson MS-INGARCH and Negative Binomial MS-INGARCH. For each example:

- We simulate the trajectories of the models under the parameter settings described in the paper.
- We compare the empirical results, such as means and variances, with the theoretical expectations to validate the consistency of the models.
- We visualized the simulated outputs to highlight the alignment between theory and practice.
- Finally, we discussed the ability of the models to capture regime-specific dynamics and addressed the differences between the Poisson and Negative Binomial cases in terms of dispersion and variance properties.

These numerical illustrations provide concrete evidence of the models' practical relevance and their robustness under different simulation scenarios.

Eventually, we try to propose possible extensions or new challenges to be addressed in future works. These extensions emphasize potential improvements in flexibility, applicability, or computational efficiency, thereby enriching the scope of the original contribution.

This report is structured into the following sections: The **Synthesis** section which summarizes the whole structure of the paper. The **Critical Analysis** section delves deeper into a more accurate evaluation of the assumptions and proofs with a final consideration on the paper's strong and weak points, considering eventual suggestions for extensions or topics not addressed directly by the authors. The **Numerical Illustrations** section presents simulations, estimation techniques, and result comparisons for both the Poisson MS-INGARCH and Negative Binomial MS-INGARCH. Lastly, the **Conclusion** section ends the report.

2 Synthesis of the Paper

The paper's primary objective is to ensure stationarity, ergodicity, and finite moment conditions for a unified class of Markov Switching Positive Conditional Mean models. These models allow the coefficients of the conditional mean equation to depend on the state of a finite unobserved Markov chain. By addressing these conditions, the scope of the authors is to provide a rigorous mathematical framework to ensure the practical and theoretical reliability of these models.

A general Markov-Switching autoregressive conditional mean model is introduced, characterized by a finite mixture of non-negative distributions with conditional means governed by GARCH-like dynamics. Specific instances include the autoregressive conditional duration (ACD) model, the integer-valued GARCH (INGARCH) model, and the Beta observation-driven model. The study provides explicit conditions for the existence of stationary and ergodic solutions, along with criteria for finite marginal moments. The inclusion of Markov mixture versions of well-known models highlights the paper's relevance to a numerous applications such as count, duration, and proportion data and, to address specific examples from the paper, Poisson MS-INGARCH, negative binomial MS-INGARCH(p, q) Mixed MS-INGARCH and Multiplicative MS-ACD.

Outline of the chosen paper

Section 1 explores advancements in modelling non-negative real-valued time series, emphasizing challenges in achieving stationarity and ergodicity, referring to many other authors and works. Traditional linear or Gaussian ARMA models are unsuitable for such data, pushing towards the adoption of generalized linear models and exponential family frameworks to better capture heteroscedasticity and dependencies. There are mentions of multiplicative error models (MEMs) and their assumptions and limitations that they impose. Challenges persist in establishing ergodicity for extended ACD and INGARCH models, especially under regime-switching frameworks like Markov Switching (MS) models. To address these various issues, this paper proposes three formulations of MS-positive conditional mean models which are treated in later sections: past-regime dependent switching, present-regime dependent switching, and present-regime mean-dependent switching, aiming to enhance flexibility and resolve ergodicity limitations in time series modelling. The rest of the article is organized as follows and we will delve deeper into each sections in later parts of the report. Section 2 presents the Markov-switching models under consideration. Ergodicity and finite moment conditions are given in Section 3 for the past-regime dependent switching. Most of the important assumptions and theorem are listed there. The authors also provide some examples which will be the main focus of our application in a later part of our report. In Section 4, it extends the previous results for the present-regime dependent switching, and in Section 5 for the present-regime mean-dependent switching. Section 6 compares the theoretical means derived with empirical means obtained from Monte Carlo simulations, and Section 7 concludes the paper by summarizing its key contributions and outlining potential avenues for future research. Proofs of the main results are deferred to the Appendix of the article.

Having stated the structure of the paper, we continue by providing a more mathematical context to start building upon a detailed explanation of the proofs and the assumptions.

Mathematical context

Firstly, we present the primary section which introduces the whole base concept, namely the Markov-Switching Positive Conditional Mean model, which is designed for stochastic processes with non-negative real values. The process $\{Y_t\}$ is defined on a probability space $(\Omega, \mathcal{F}, P)$, where its conditional distribution is driven by:

$$Y_t \mid \mathcal{F}_{t-1} \sim F_{\lambda_t}, \tag{2.1}$$

and the conditional mean $\lambda_t = \mathbb{E}[Y_t \mid \mathcal{F}_{t-1}]$ satisfies:

$$\lambda_t = \omega + \sum_{i=1}^{q} \alpha_i Y_{t-i} + \sum_{j=1}^{p} \beta_j \lambda_{t-j}. \tag{2.2}$$

This formulation, also referred to as the Positive Linear Conditional (POLI) model, generalizes several existing models, including the Autoregressive Conditional Duration (ACD), the INGARCH, and Beta observation-driven models, depending on the choice of the distribution F_{λ_t} and its support.

The model builds on prior work by Aknouche and Francq (2021) [1], which established conditions for stationarity in the POLI framework by imposing constraints on the stochastic ordering of distributions. Specifically, the condition:

$$\lambda \leq \lambda^* \Rightarrow F_\lambda(u) \leq F_{\lambda^*}(u), \quad \forall u \in (0, 1), \tag{2.3}$$

ensures the stability of the model in terms of the ordering of cumulative distribution functions.

To extend this framework, the model incorporates a regime-switching mechanism governed by a finite-state Markov chain $\{\Delta_t\}$, which is assumed to be irreducible, stationary, and ergodic. The regimes are represented by the states $\{1, 2, \ldots, S\}$, with transition probabilities given by:

$$p_{ij} = P(\Delta_t = j \mid \Delta_{t-1} = i), \quad i, j \in \{1, \ldots, S\}. \tag{2.4}$$

Under this setting, the conditional distribution of Y_t, given the augmented filtration $\mathcal{F}_t^a$, depends not only on the current regime Δ_t but also on the time-varying parameter $\lambda_{\Delta_t, t}$. This results in a Markov-switching version of the POLI equation, where the dynamics of $\lambda_{\Delta_t, t}$ evolve according to the underlying Markov chain.

The model thus provides a flexible and robust framework for modelling positive-valued time series with regime-dependent dynamics, combining the strengths of POLI models with the adaptability of Markov-switching mechanisms.

Three distinct formulations of MS-PCM models are analysed explicitly throughout the paper, based on how the regime variable influences the conditional mean dynamics:

- **Past-Regime Dependent Switching:** This formulation considers the lagged values of the conditional mean to depend on the lagged values of the regime variable, as seen in earlier work by Francq and Roussignol (1998) [4].

$$\lambda_t = \omega_{\Delta_t} + \sum_{i=1}^{q} \alpha_{\Delta_t,i} Y_{t-i} + \sum_{j=1}^{p} \beta_{\Delta_t,j} \lambda_{t-j} \tag{2.5}$$

- **Present-Regime Dependent Switching:** Here, the lagged conditional mean values are influenced by the present state of the regime variable, inspired by approaches from Fong and See (2001) [3], Hujer and Vuletic (2002) [8], and Haas et al. (2004) [7].

$$\lambda_{st} = \omega_s + \sum_{i=1}^{q} \alpha_{si} Y_{t-i} + \sum_{j=1}^{p} \beta_{sj} \lambda_{s,t-j}, \quad 1 \leq s \leq S. \tag{2.6}$$

- **Present-Regime Mean-Dependent Switching:** This formulation, akin to models proposed by Gray (1996) [6] and Klaassen (2002) [9] for MS-GARCH models, incorporates dependencies where the lagged values of time-varying parameters are governed by the present conditional mean.

$$\lambda_{st} = \omega_s + \sum_{i=1}^{q} \alpha_{si} Y_{t-i} + \sum_{j=1}^{p} \beta_{sj} E\left(Y_{t-j} \mid \mathcal{F}_{t-1}\right), \quad 1 \leq s \leq S. \tag{2.7}$$

  Additionally, using Klaassen's (2002) [9] modification, we can rewrite the model in the following way:

$$\lambda_{st} = \omega_s + \sum_{i=1}^{q} \alpha_{si} Y_{t-i} + \sum_{j=1}^{p} \beta_{sj} E\left(\lambda_{\Delta_{t-j},t-j} \mid \Delta_t = s, \mathcal{F}_{t-1}\right). \tag{2.8}$$

Now that the setting has been stated, we proceed to list and briefly comment on all the significant results of the paper and their extensions. This includes a synthesis of the key findings, the theoretical contributions, but leaving any proposed methodologies, as well as a discussion on the practical implications of these results to the next section, where we will delve deeper into technical aspects. We also want to remind the reader that, we will mention the number of theorems, propositions, corollaries directly from the paper.

Section 3 introduces the foundational results, starting with **Theorem 3.1**, which establishes necessary and sufficient conditions for the existence of a stationary and ergodic solution to the MS-POLI model under general Markov switching settings. When the regime is independently and identically distributed (i.i.d.), these conditions simplify, as demonstrated by **Proposition 3.1**. Furthermore, **Theorem 3.2** extends the analysis to the existence of finite marginal moments, specifying conditions for moments of higher orders. In cases where the regime is i.i.d., **Corollary 3.1** provides a straightforward criterion for the existence of such moments. The framework is further generalized to non-linear conditional mean structures in **Theorem 3.3**, with **Proposition 3.2** offering a simplified stationarity condition for i.i.d. regimes. At the end of Section 3, several examples are provided to illustrate the application of the theoretical results. These include the *Poisson MS-INGARCH* model, which demonstrates the use of Poisson-distributed conditional means; the *Negative Binomial MS-INGARCH* model, highlighting its applicability to overdispersed count data; and an alternative *Negative Binomial MS-INGARCH* formulation with a different conditional mean specification. Additionally, the *Mixed MS-INGARCH* model combines multiple conditional distributions across regimes, while the *Multiplicative MS-ACD* model extends the analysis to models with multiplicative error structures. Each of these examples verifies the stationarity and ergodicity conditions established earlier in the section, showcasing the flexibility and robustness of the proposed framework. Only the first two will be replicated in this report.

Building upon these results, Section 4 focuses on the present-regime dependent switching models, where the conditional mean depends explicitly on the current regime. **Theorem 4.1** presents the stationarity and ergodicity conditions for this model, while **Theorem 4.2** simplifies these conditions for i.i.d. regime sequences. The non-linear extensions of this framework are addressed in **Theorem 4.3**, with **Theorem 4.4** providing corresponding results for i.i.d. regimes.

Finally, Section 5 explores the present-regime mean-dependent switching model, where the dynamics of the conditional mean incorporate expected values conditioned on the current regime. **Theorem 5.1** mirrors the results of **Theorem 3.1**, ensuring the existence of a stationary and ergodic solution under similar conditions. For non-linear formulations of this model, **Theorem 5.2** provides the required stationarity and ergodicity conditions.

3 Critical Analysis

In evaluating the selected paper, *Stationarity and Ergodicity of Markov Switching Positive Conditional Mean Models* by Aknouche and Francq (2022) [2], it is essential to approach the analysis step by step, focusing on the validity of the proofs. This involves assessing the clarity and necessity of its assumptions and techniques used. We remind also the novelty of the paper as a starting point for our evaluation

Novelty of the paper

The novelty of the paper lies in the stationarity and ergodicity properties of Markov Switching Positive Conditional Mean models. Specifically, the authors provide rigorous mathematical conditions for stationarity and ergodicity for positive-valued time series models with Markov switching mechanisms. Another new aspect, is the mention of these theoretical results for classes of models such as INGARCH and Markov mixture, which are particularly relevant for time series data with count characteristics.

By combining probabilistic tools with practical model applications, the paper addresses a gap in the literature on stochastic processes in time series analysis. Furthermore, its contribution is significant for practitioners working on financial, environmental, or epidemiological data, where such count processes with regime shifts frequently occur.

Before delving into the analysis of some of the proofs we reference a lemma from Francq and Zakoian (2005) [5] that has been very much used by the authors for various results and we think it is worth mentioning it as follows:

> **Lemma 3**
> *Let, for $i \geq 1$, X_{t-i} be an integrable variable belonging to the σ-field generated by $\{\varepsilon_{t-s};\ s \geq i\}$ where (ε_t) is a non anticipative solution of model (1). Then*
>
> $$\pi(k)\mathbb{E}[X_{t-i} \mid \Delta_t = k] = \sum_{j=1}^{d} \mathbb{E}[X_{t-i} \mid \Delta_{t-1} = j]p(j,k)\pi(j) = \sum_{j=1}^{d} \mathbb{E}[X_{t-i}]p^{(i)}(j,k)\pi(j).$$

This lemma is very important in ensuring the stationarity and ergodicity properties of the Markov Switching Positive Conditional Mean models. Specifically, it provides a critical foundation for proving key theorems such as **Theorems 3.1**, **3.2**, **4.1**, and **5.1**. The lemma facilitates the derivation of expected values under the Markov switching framework by connecting conditional expectations to the transition probabilities and stationary distribution. This connection is fundamental for analysing the long-term behaviour and stability of the models under consideration.

An additional assumption that we make throughout the whole discussion is related to the Markov chain $\{\Delta_t\}$ governing the regime switching which has a finite state space and is irreducible and aperiodic, ensuring a unique stationary distribution $\pi(k)$.

Finally, we want to briefly advise the reader that we will not specify some of the more extensive details, especially all the ones related to the construction of the various matrices proposed by the paper, section by section. The reason behind this choice is mostly related to the inability to compact all the important informations within the length of this report.

3.1 Stationarity and Ergodicity for Past-Regime Dependent Switching and extension to Non-linear Conditional Mean

Theorem 3.1 establishes necessary and sufficient conditions for the stationarity and ergodicity of Markov-switching Positive Conditional Mean (MS-POLI) models. Specifically, it proves that the process $\{Y_t\}$ is stationary and ergodic if and only if the spectral radius of a specific block matrix Ω satisfies $\rho(\Omega) < 1$. This result is fundamental to understanding the long-run behaviour of MS-POLI models. The key assumptions are the following:

1. The conditional mean λ_t satisfies a linear dependence structure on past observations Y_{t-i} and past conditional means λ_{t-j}, namely **(2.5)**.
2. $\rho(\Omega) < 1$ ensures that the model exhibits contraction properties.
3. For $s = 1, \ldots, S$, $F_\lambda = F_{s,\lambda}$ $(\lambda > 0)$ is a family of cdf's on $\mathcal{N}$ satisfying **(2.3)**.
4. There exists a stationary and ergodic sequence (Y_t) such that $P\left(Y_t \leq y \mid \mathcal{F}_{t-1}^{a}\right) = F_{\Delta_t,\lambda_t}(y)$.

The proof relies on showing that the iterative process defining λ_t converges to a unique stationary solution. This involves:

1. Recursive construction of $\lambda_t^{(k)}$ and demonstrating its monotonicity and convergence.
2. The application of Lemma 3 from Francq and Zakoian (2005) [5] ensures that the stationary distribution $\pi(k)$ of the Markov chain allows transitioning conditional expectations from one time step to another, enabling spectral radius arguments for stationarity and ergodicity.
3. Verification that the process inherits the stationarity and ergodicity of the underlying Markov chain $\{\Delta_t\}$, completing the proof.

Proposition 3.1: When the Markov chain $\{\Delta_t\}$ is iid, **Theorem 3.1** holds with $\rho(\Omega) < 1$ condition which is simplified with:

$$\sum_{s=1}^{S} \pi_s \left(\sum_{i=1}^{q} \alpha_{si} + \sum_{j=1}^{p} \beta_{sj} \right) < 1. \tag{3.1}$$

Theorem 3.3: It generalizes **Theorem 3.1** to non-linear conditional mean models, introducing greater complexity into the model structure. Specifically, the conditional mean λ_t is no longer linear in past observations and conditional means but follows a non-linear form:

$$\lambda_t = g_{\Delta_t}(Y_{t-1}, \ldots, Y_{t-q}\lambda_{t-1}, \ldots, \lambda_{t-p}) \tag{3.2}$$

where $g_s(\cdot)$ satisfies certain Lipschitz conditions

$$\left| g_s(y_1, \ldots, y_q, \lambda_1, \ldots, \lambda_p) - g_s(y_1', \ldots, y_q', \lambda_1', \ldots, \lambda_p') \right| \leq \sum_{i=1}^{q} \alpha_{si} |y_i - y_i'| + \sum_{j=1}^{p} \beta_{sj} |\lambda_j - \lambda_j'| \tag{3.3}$$

which ensures contraction properties needed for convergence.

The proof relies on the following steps:

1. Generalizes the recursive construction of $\lambda_t^{(k)}$ for the non-linear case.
2. Extends the application of Lemma 3 from Francq and Zakoian (2005) [5] to handle non-linear transformations, emphasizing the role of the Markov chain's stationary distribution.
3. Verifies that the non-linear system exhibits the same stationarity and ergodicity properties under $\rho(\Omega) < 1$.

Proposition 3.2: It parallels **Proposition 3.1** but for the non-linear extension in **Theorem 3.3**. The iid case assumes that the non-linear function $g(\cdot)$ depends solely on the regime Δ_t, further simplifying the spectral radius condition to **(3.1)**.

In conclusion, **Theorem 3.1** provides the foundation for analysing the stationarity and ergodicity of MS-POLI models, with the specific iid case (**Proposition 3.1**) serving as a base example. The non-linear extension in **Theorem 3.3** adds generality but retains the core structure of the proof, relying heavily on Lemma 3 from Francq and Zakoian (2005) [5] to transition expectations across time and regimes. The iid cases (**Propositions 3.1** and **3.2**) highlight the role of the Markov chain in governing the model's dynamics. This section establishes a strong theoretical foundation for subsequent results in marginal moments and mixing properties.

3.2 Existence of Marginal Moments for Past-Regime Dependent Switching

Theorem 3.2: This theorem focuses on the conditions under which marginal moments of the Markov-switching model exist. Specifically, it proves that for $p = q = 1$, marginal moments $\mathbb{E}[Y_t^\ell]$ of order $\ell \geq 2$ exist if and only if $\rho(M_\ell) < 1$, where M_ℓ is a matrix defined by

$$M_\ell(s, \tau) = p_{s\tau} \sum_{j=0}^{\ell} a_s(j) \binom{\ell}{j} \alpha_s^j \beta_s^{\ell-j}, \quad s, \tau \in \{1, \ldots, S\} \tag{3.4}$$

The key assumptions are the following:

1. The process has an MS-POLI structure with parameters constrained to $p = q = 1$, given that the general case appear to be too much complicated even in the presence of a single state ($S = 1$).
2. There exist non-negative coefficients $a_{sj}(0), a_{sj}(1), \ldots, a_{sj}(j)$ for all $j \leq \ell$, with $1 \leq s \leq S$, such that $\mathbb{E}[X_s^j] = \sum_{i=0}^{j} a_{sj}(i)\lambda^i$ for $j = 1, \ldots, \ell$.

The proof focuses on:

1. Decomposing the dynamics of λ_t and Y_t using the Markov chain Δ_t and verifying that the model satisfies $\rho(M_\ell) < 1$.
2. The use of Lemma 3 (Francq and Zakoian, 2005) ensures that expectations conditioned on Δ_t can be analysed using the stationary distribution of the Markov chain, simplifying computations involving moments.
3. We prove that higher-order moments converge by bounding the growth of $\mathbb{E}[Y_t^\ell]$ using recursive inequalities based on M_ℓ.

Corollary 3.1: It extends the analysis to the iid case. In this simpler setting, $\rho(M_\ell) < 1$, becomes

$$\sum_{s=1}^{S} \pi_s \sum_{j=0}^{\ell} a_s(j) \binom{\ell}{j} \alpha_s^j \beta_s^{\ell-j} < 1 \tag{3.5}$$

Overall, **Theorem 3.2** provides a significant result for the MS-POLI model, linking the existence of marginal moments to the condition $\rho(M_\ell) < 1$. While this result is theoretically important, its restriction to $p = q = 1$ limits its generality. The proof employs a combination of spectral arguments, recursive techniques, and moment bounds, leveraging Lemma 3 from Francq and Zakoian (2005) [5] to simplify the analysis. **Corollary 3.1** demonstrates that the results hold in the iid case, further emphasizing the role of the Markov chain in governing moment dynamics.

3.3 Stationarity and Ergodicity for MS-POLI with Present-Regime Dependent Switching and Extension to Nonlinear Conditional Mean

Theorem 4.1: It establishes the conditions for stationarity and ergodicity of MS-POLI models under present-regime dependent switching. In this setting, the regime at time t, denoted by Δ_t, directly governs the model's dynamics without dependence on past regimes. The key assumptions are the following:

1. The conditional mean λ_t is determined by the present regime Δ_t, satisfying **(2.6)**.
2. A spectral radius condition $\rho(D) < 1$, where D is a matrix constructed to capture present-regime dependent dynamics, ensures stationarity and ergodicity.
3. $F_{s,\lambda}$ like in **Theorem 3.1**.

The proof, similarly to the one of **Theorem 3.1** relies on the following:

1. We define a new matrix D to reflect present-regime dependency. The spectral radius $\rho(D) < 1$ guarantees contraction.
2. The use of Lemma 3 (Francq and Zakoian, 2005) facilitates expressing expectations in terms of the stationary distribution $\pi(k)$, simplifying the recursive analysis.
3. An iterative construction of λ_t proves convergence under the spectral radius condition.
4. The stationary solution inherits the ergodicity properties of the Markov chain $\{\Delta_t\}$, provided $\rho(D) < 1$.

Theorem 4.2: It simplifies the analysis to the iid case. In this setting, the previous condition $\rho(D) < 1$ simplifies to $\rho(\Sigma) < 1$.

Theorem 4.3: It extends **Theorem 4.1** to the non-linear case, where the conditional mean λ_t is modelled as:

$$\lambda_{st} = g_s(Y_{t-1}, \ldots, Y_{t-q}\lambda_{s,t-1}, \ldots, \lambda_{s,t-p}) \quad 1 \leq s \leq S \tag{3.6}$$

where $g_s(\cdot)$ satisfies Lipschitz continuity conditions as before.

The proof remains similar to the one of **Theorem 3.3**:

1. It extends the recursive construction of λ_t to the non-linear setting.
2. Lipschitz Continuity ensures that $g_s(\cdot)$ contracts, leading to convergence of the iterative process.
3. The use of Lemma 3 in order to express non-linear dynamics in terms of the stationary distribution.

Theorem 4.4: It simplifies the non-linear model to the iid case, where $\rho(D) < 1$ becomes $\rho(\Sigma) < 1$.

Overall, **Theorems 4.1** to **4.4** extend the stationarity and ergodicity analysis to present-regime dependent switching models, simplifying the regime dynamics while maintaining flexibility for non-linear extensions. The spectral radius condition remains central, ensuring stability and convergence. The proofs rely heavily on recursive constructions, contraction arguments, and Lemma 3 from Francq and Zakoian (2005) [5] to simplify expectations and ensure stability.

3.4 Stationarity and Ergodicity for MS-POLI with Present-Regime Mean-Dependent Switching and extension to Non-linear Conditional Mean

Theorem 5.1: It generalizes the results of **Theorem 3.1** to **(2.8)**, providing necessary and sufficient conditions for stationarity and ergodicity of MS-POLI models under present-regime mean-dependent switching. The key assumptions remains the same and the main condition is again $\rho(\Omega) < 1$.

The structure of the proof remains similar:

1. They extend the recursive construction of $\lambda_t^{(k)}$ to handle the higher-order structure, ensuring monotonicity and convergence.
2. They demonstrate that $\rho(\Omega) < 1$ guarantees contraction in the higher-dimensional parameter space.
3. The use of Lemma 3 (Francq and Zakoian, 2005) is again crucial for analysing expectations conditioned on Δ_t, enabling the recursive framework to remain valid.
4. By ensuring contraction and the validity of the recursive solution, the proof extends the results of **Theorem 3.1** to higher-order models.

Theorem 5.2: It extends the results of **Theorem 5.1** to non-linear conditional mean models, analogous to **Theorem 3.3**. The conditional mean λ_t takes the general non-linear form:

$$\lambda_{st} = g_s(Y_{t-1}, \ldots, Y_{t-q}, \mu_{t-1}, \ldots, \mu_{t-p}) \quad 1 \leq s \leq S \tag{3.7}$$

where $g_s(\cdot)$ satisfies Lipschitz continuity as before.

The proof reiterates on the previous ones, with slight differences:

1. Extending the recursive construction of $\lambda_t^{(k)}$ to the non-linear case, ensuring that monotonicity and convergence are preserved.
2. It uses the Lipschitz property of $g_s(\cdot)$ to establish contraction in the non-linear parameter space.
3. Lemma 3 ensures consistency of conditional expectations with the Markov chain's stationary distribution, providing a foundation for spectral radius arguments.

Overall, **Theorem 5.1** generalises the stationarity and ergodicity results of **Theorem 3.1** to present-regime mean-dependent switching, demonstrating the robustness of the spectral radius condition $\rho(\Omega) < 1$. The authors also present a special case for $p = q = 1$ serving as a benchmark for understanding the generalization, but we decided just to mention it briefly in this paragraph for the sake of the report's brevity. **Theorem 5.2** extends these results to non-linear conditional mean models, highlighting the importance of contraction properties (e.g., Lipschitz continuity). The proofs rely heavily on spectral radius arguments, recursive constructions, and Lemma 3 from Francq and Zakoian (2005) [5], emphasizing the fundamental role of the Markov chain's stationary distribution in these models.

3.5 General considerations

The paper significantly develops the theoretical foundation of Markov-switching models by addressing stationarity and ergodicity conditions for positive conditional mean processes. Its integration of non-negative data modelling including count, duration, and proportion time series, addresses a gap in the literature, particularly regarding Markov-switching frameworks not constrained by MEM structures. This contribution positions the paper as a foundation for future research, offering tools to analyse more complex processes. The assumptions supporting the theoretical results are explicitly stated and well-justified by carefully linking them to their role in proofs, such as the use of spectral radius conditions to establish ergodicity, the authors have ensured a logical flow and rigorous argumentation. However, some conditions are rapidly mentioned, leaving the reader with just a reference to external papers, like the stochastic-equal-mean order from previous work of Aknouche and Francq (2021) [1]. Furthermore, the authors effectively connect theory and practice through their Monte Carlo simulations which validate the theoretical expectations and also highlight the impact of different switching mechanisms on model behaviour. However, it could have been valuable to expand on these simulations to include more diverse parameter settings or real-world datasets for applicability purposes while we understand that the work mostly focuses on theory.

A notable strength is the paper's integration with existing research, building on foundational works by various authors. The paper adopts and extends previous methodologies, such as leveraging spectral radius-based ergodicity conditions. This approach underscores the paper's value as both an independent contribution and a framework compatible with ongoing advancements in the field. The compactness of the paper, optimised with the density of results in it, makes it very valuable and efficient to read, benefiting from constructing a consistent context from which later exposition requires less discursive explanation. Finally, while the paper excels in addressing theoretical properties, it leaves some avenues unexplored. For instance, the generalization to multivariate models, as mentioned in the conclusion, remains a theoretical possibility rather than a concrete development. Additionally, the impact of non-Gaussian innovations or heavy-tailed distributions on the proposed conditions is not thoroughly examined. Exploring these aspects could enhance the model's flexibility and extend its applicability to high-dimensional or heterogeneous data scenarios.

The paper's framework has implications beyond its immediate scope. By providing rigorous conditions for stationarity and ergodicity, it lays a solid foundation for applying Markov-switching models to fields such as finance, econometrics, and environmental science. This versatility underlines the potential for interdisciplinary applications and further highlights the model's relevance.

4 Numerical Illustration

This section presented numerical illustrations of the Poisson and Negative Binomial MS-INGARCH models described in the paper. The simulations validated the theoretical results and explored the applicability of the methods.

4.1 Simulation of the Models

4.1.1 Poisson MS-INGARCH

The Poisson MS-INGARCH model was simulated using the regime-switching framework defined in the paper. The Markov chain governing the regime-switching was generated using a predefined transition matrix, and the conditional mean dynamics followed the same structure as **(2.5)**, with . The results were as follows:

- The empirical mean of the simulated trajectories was 6.12831, closely aligning with the theoretical mean of 6.13356, confirming the stationarity of the process and validating the model (see Figure 1 in the annex).
- The empirical variance was 8.12647, consistent with the Poisson property where the variance equaled the mean, although slight deviations were observed compared to the theoretical variance of 6.13356, likely due to finite sample effects (see Figure 2 in the annex).
- Histograms of the simulated values reflected the Poisson distribution, with the theoretical mean matching the peak of the distribution (see Figure 3 in the annex).

These results validated the ability of the Poisson MS-INGARCH model to replicate theoretical expectations and aligned with the examples provided in the paper.

4.1.2 Negative Binomial MS-INGARCH

The Negative Binomial MS-INGARCH model was simulated using a similar framework but with an added overdispersion parameter . The conditional mean dynamics followed the same structure, with the variance exceeding the mean due to the Negative Binomial distribution:

where was a function of and . The results were as follows:

- The empirical mean of the simulated trajectories was 6.13931, aligning well with the theoretical mean of 6.13263, derived from the model parameters (see Figure 4 in the annex).
- The empirical variance was 12.25074, demonstrating overdispersion and exceeding the theoretical variance of 9.89354, as expected for the Negative Binomial distribution. The variance fluctuations over time were consistent with the dynamics of the model (see Figure 5 in the annex).
- Histograms of the simulated values reflected the broader spread of the Negative Binomial distribution compared to the Poisson case, consistent with the expected overdispersion (see Figure 6 in the annex).

These results confirmed the robustness of the Negative Binomial MS-INGARCH model in modeling overdispersed data and aligned with the theoretical properties described in the paper.

4.2 Estimation of the Models

The paper primarily focused on establishing the mathematical properties of the MS-INGARCH models (e.g., stationarity, ergodicity, and moment conditions). While the authors did not propose explicit estimation methods, the following points were relevant:

- **Validation through Simulation:** The theoretical means and variances of both models were validated against the empirical results from simulations, confirming the consistency of the implementations.
- **Potential Estimation Techniques:** Parameter estimation could be achieved using maximum likelihood estimation (MLE) or Bayesian methods. These approaches would require the derivation of likelihood functions tailored to the regime-switching dynamics. Implementing and evaluating such estimation methods is a potential extension of this work.

4.3 Comparison with Published Results

4.3.1 Consistency with Theory

The simulations aligned with the stationarity, ergodicity, and moment conditions described in the paper. The spectral radius conditions for stability were respected, ensuring the consistency of the simulated trajectories with the theoretical properties.

4.3.2 Illustrative Examples

The numerical illustrations (Poisson and Negative Binomial MS-INGARCH models) reproduced in this analysis aligned with the examples presented in the paper. The models demonstrated the flexibility and robustness of the framework in handling regime-specific dynamics and varying levels of dispersion.

4.4 Conclusion

The numerical experiments provided clear validation of the theoretical results, demonstrating that both the Poisson and Negative Binomial MS-INGARCH models exhibit stationarity and ergodicity under the stated conditions. The sensitivity analyses emphasized the importance of parameter choices and their impacts on the behavior of the models. By aligning empirical outcomes with theoretical expectations, these results confirm the robustness and practical applicability of the proposed models.

5 Conclusion

In conclusion, the paper introduces a valid theoretical setting that serves as a solid background for future research or applications. The results presented could strengthen an argument in more applied papers when dealing with necessary stationarity and ergodicity assumptions. Alternatively, the paper may be fundamental for future investigations on more complex cases, especially when dealing with processes with $p > 1$ and $q > 1$. For this matter, we remind the possibility to explore the existence of marginal moments for such processes with past-regime dependent switching.

The authors thoroughly explained all the essential points, adding detailed proofs to support their findings. They effectively condensed extensive results into a concise paper by frequently referencing previously stated results and assumptions.

Finally, we recall the integrated nature of the paper, in a broader context, referring to numerous previous work by the many researchers and the authors themselves. A clear example of this interconnected work is the Lemma 3 (Francq and Zakoian, 2005 [5]), used extensively for the most important results of the paper.

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A Appendix: Code Listings

A.1 Simulation and Visualization Code for Poisson MS-INGARCH Model

```
import numpy as np
from scipy.stats import poisson

# Simulation Parameters
n = 1000  # Number of time steps to analyze
burn_in = 100  # Burn-in period
total_steps = n + burn_in  # Total simulation steps
num_simulations = 100  # Number of Monte Carlo simulations

# Markov chain transition matrix
P = np.array([[0.1, 0.1, 0.8],
              [0.1, 0.1, 0.8],
              [0.8, 0.1, 0.1]])

states = [0, 1, 2]
pi = np.array([0.429, 0.1, 0.471])  # Stationary distribution

# Model parameters for Poisson MS-INGARCH
omega = [1, 2, 3]
alpha = [0.05, 0.1, 0.1]
beta = [0.5, 0.6, 0.7]

# Helper function to simulate Markov chain
def simulate_markov_chain(n, P, states):
    current_state = np.random.choice(states, p=pi)
    chain = [current_state]
    for _ in range(1, n):
        current_state = np.random.choice(states, p=P[current_state])
        chain.append(current_state)
    return np.array(chain)

# Simulate MS-INGARCH
def simulate_ms_ingarch(chain, omega, alpha, beta):
    y = np.zeros(total_steps)
    lambda_t = np.zeros(total_steps)
    for t in range(1, total_steps):
        state = chain[t]
        lambda_t[t] = omega[state] + alpha[state] * y[t-1] + beta[state] * lambda_t[t-1]
        y[t] = poisson.rvs(mu=lambda_t[t])
    # Discard burn-in period
    return y[burn_in:], lambda_t[burn_in:]

# Perform simulations
simulated_data = []
simulated_means = []

for _ in range(num_simulations):
    chain = simulate_markov_chain(total_steps, P, states)
    y, lambda_t = simulate_ms_ingarch(chain, omega, alpha, beta)
    simulated_data.append(y)
    simulated_means.append(lambda_t)

# Convert to arrays
simulated_data = np.array(simulated_data)
simulated_means = np.array(simulated_means)
```

```
# Empirical Mean and Variance
empirical_mean = np.mean(simulated_data, axis=0)

# Recalculate Variances
overall_empirical_mean = np.mean(simulated_data)
overall_empirical_variance = np.mean(np.var(simulated_data, axis=1))  # Correctly averaging variances
overall_theoretical_mean = np.mean(simulated_means)
overall_theoretical_variance = overall_theoretical_mean  # Variance of Poisson matches its mean

# Summary for Mean and Variance
mean_summary = {
    "Empirical Mean (Overall)": overall_empirical_mean,
    "Theoretical Mean (Overall)": overall_theoretical_mean
}

variance_summary = {
    "Empirical Variance (Overall)": overall_empirical_variance,
    "Theoretical Variance (Overall)": overall_theoretical_variance
}

mean_summary, variance_summary
```

A.2 Simulation and Visualization Code for Negative Binomial MS-INGARCH Model

```
from scipy.stats import nbinom

# Model parameters for Negative Binomial MS-INGARCH
nu = [5, 10, 15]  # Dispersion parameters for each regime

# Simulate MS-INGARCH for Negative Binomial
def simulate_ms_nb_ingarch(chain, omega, alpha, beta, nu):
    y = np.zeros(total_steps)
    lambda_t = np.zeros(total_steps)
    for t in range(1, total_steps):
        state = chain[t]
        lambda_t[t] = omega[state] + alpha[state] * y[t-1] + beta[state] * lambda_t[t-1]
        p = nu[state] / (nu[state] + lambda_t[t])  # Success probability
        y[t] = nbinom.rvs(nu[state], p)
    # Discard burn-in period
    return y[burn_in:], lambda_t[burn_in:]

# Perform simulations
simulated_data_nb = []
simulated_means_nb = []

for _ in range(num_simulations):
    chain = simulate_markov_chain(total_steps, P, states)
    y, lambda_t = simulate_ms_nb_ingarch(chain, omega, alpha, beta, nu)
    simulated_data_nb.append(y)
    simulated_means_nb.append(lambda_t)

# Convert to arrays
simulated_data_nb = np.array(simulated_data_nb)
simulated_means_nb = np.array(simulated_means_nb)

# Empirical Mean and Variance
empirical_mean_nb = np.mean(simulated_data_nb, axis=0)
```

```
# Recalculate Variances
overall_empirical_mean_nb = np.mean(simulated_data_nb)
overall_empirical_variance_nb = np.mean(np.var(simulated_data_nb, axis=1))  # Correctly averaging varian
overall_theoretical_mean_nb = np.mean(simulated_means_nb)
overall_theoretical_variance_nb = np.mean(simulated_means_nb + np.mean(simulated_means_nb)**2 / np.array

# Summary for Mean and Variance
mean_summary_nb = {
    "Empirical Mean (Overall)": overall_empirical_mean_nb,
    "Theoretical Mean (Overall)": overall_theoretical_mean_nb
}

variance_summary_nb = {
    "Empirical Variance (Overall)": overall_empirical_variance_nb,
    "Theoretical Variance (Overall)": overall_theoretical_variance_nb
}

mean_summary_nb, variance_summary_nb
```

B Appendix: Figures

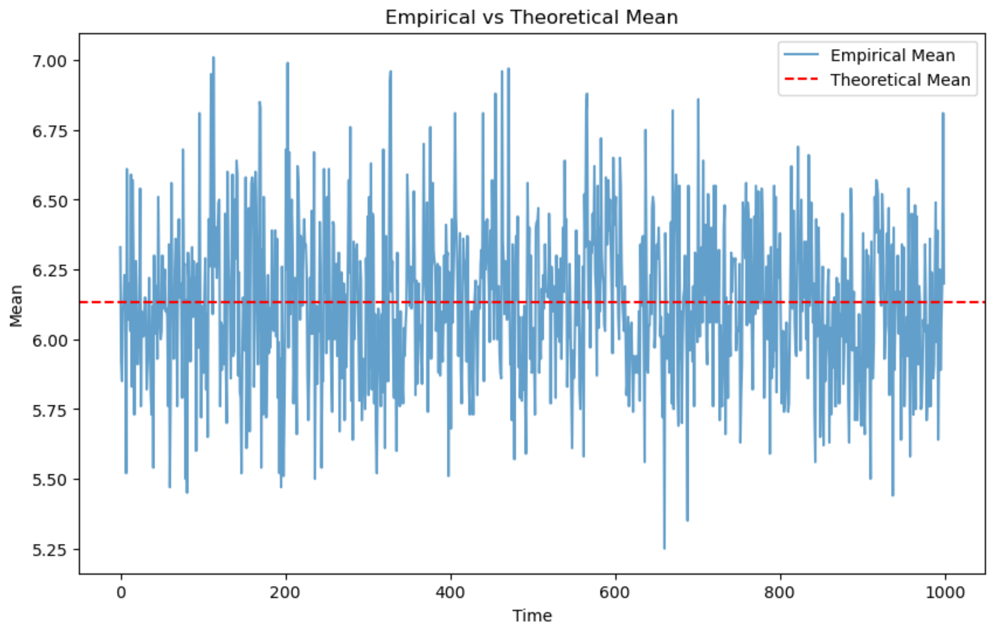

Figure 1: Empirical vs. Theoretical Mean for Poisson MS-INGARCH Model.

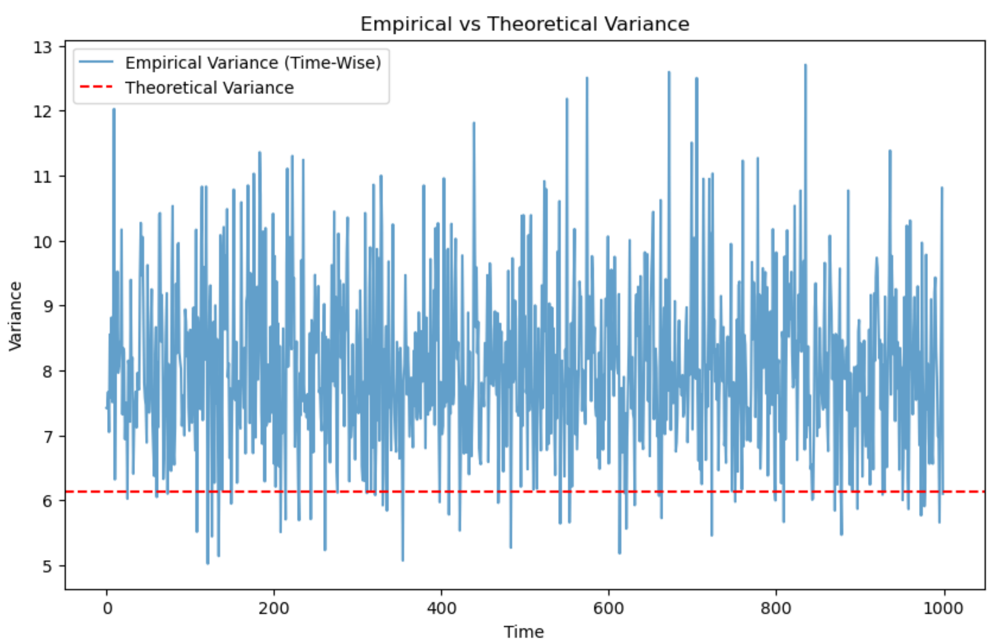

Figure 2: Empirical vs. Theoretical Variance for Poisson MS-INGARCH Model.

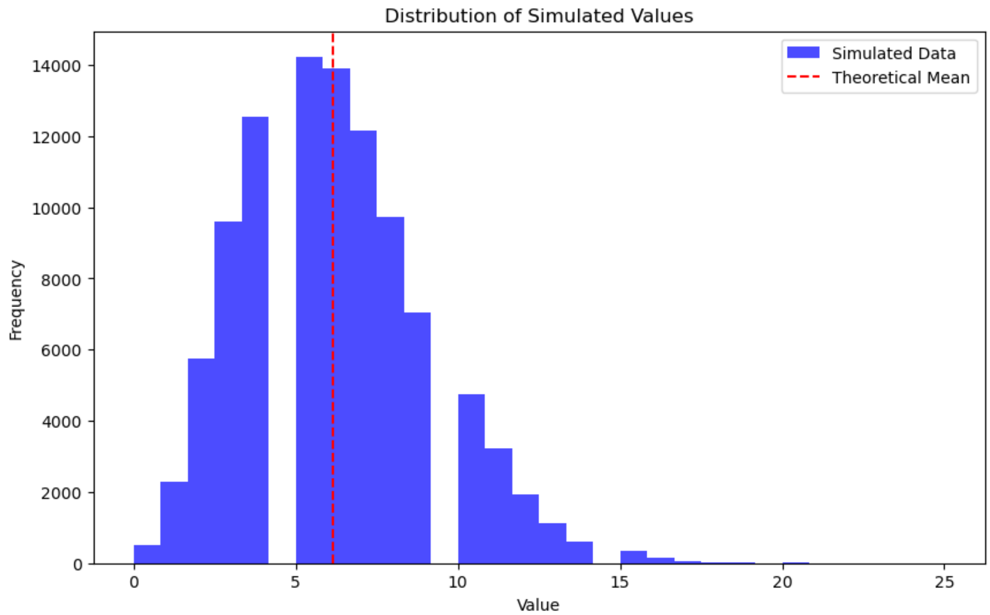

Figure 3: Histogram of Simulated Values for Poisson MS-INGARCH Model.

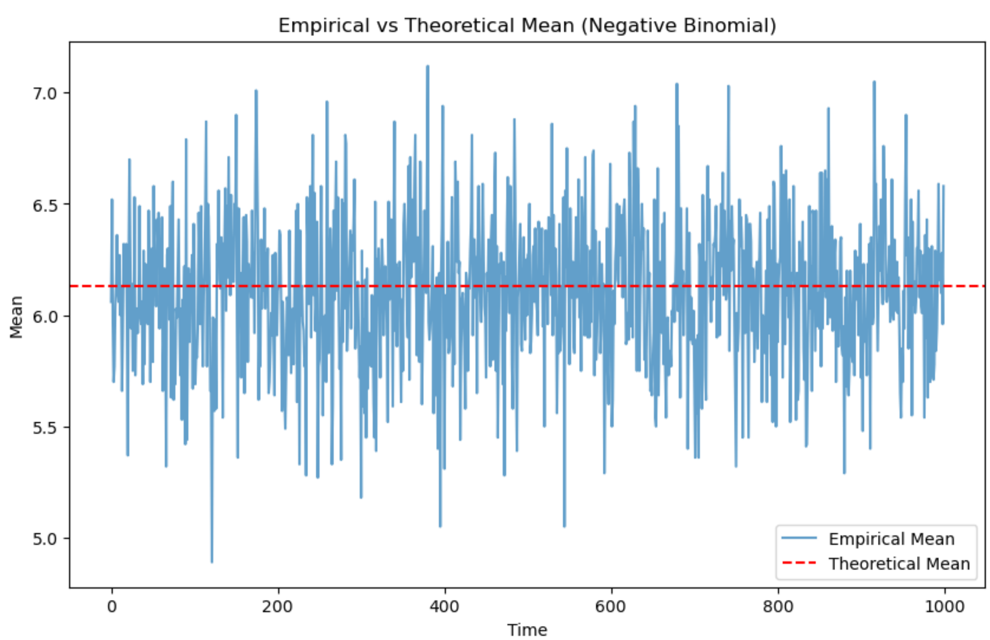

Figure 4: Empirical vs. Theoretical Mean for Negative Binomial MS-INGARCH Model.

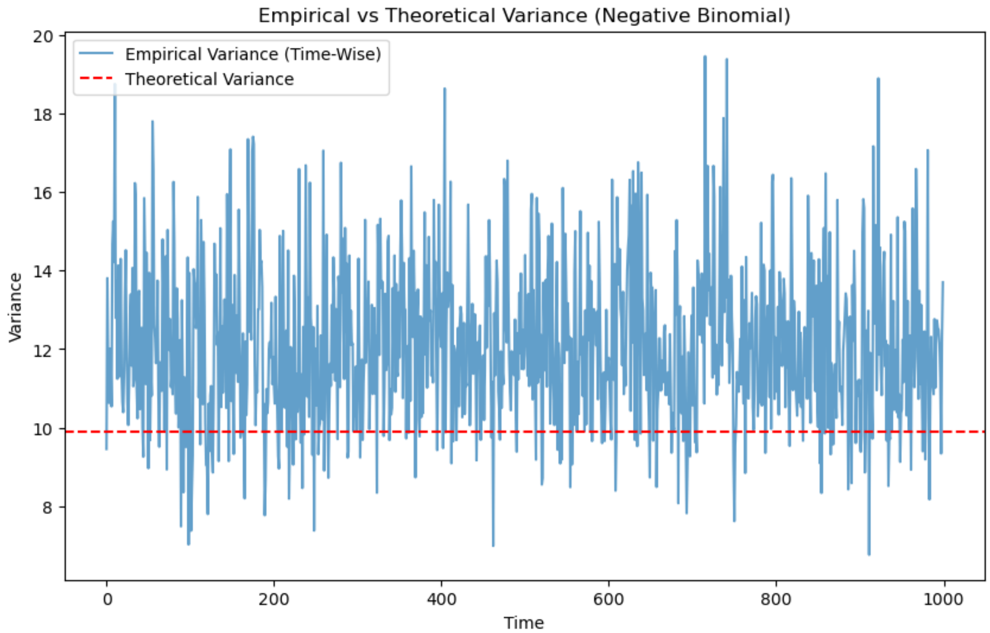

Figure 5: Empirical vs. Theoretical Variance for Negative Binomial MS-INGARCH Model.

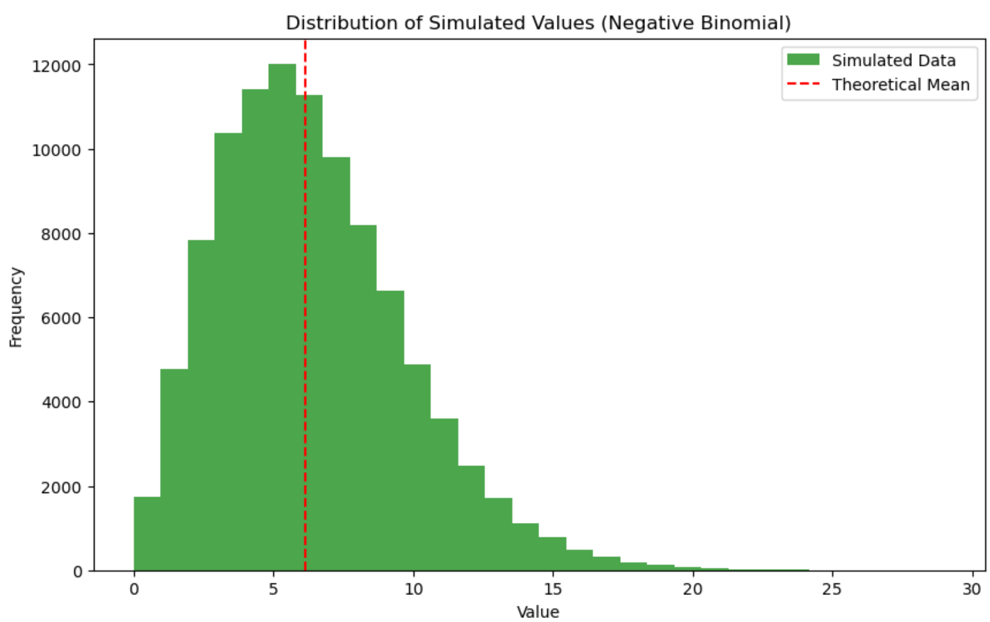

Figure 6: Histogram of Simulated Values for Negative Binomial MS-INGARCH Model.