EQUAÇÕES DIFERENCIAIS PARCIAIS

EX: $\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t}$ $\qquad E = E(x,t)$

EDO: $\frac{dy}{dx} = y \Rightarrow \begin{matrix} y_1(x) = e^x \\ y_2(x) = 4e^x \end{matrix}$

$\Rightarrow \{ce^x\}$

$E = E(x - vt)$

Ex$_1$: $E(x,t) = E_0 \cos(x - vt)$

$$\frac{\partial E}{\partial x} = -E_0 \operatorname{sen}(x - vt) \cdot 1 \qquad \frac{\partial E}{\partial t} = -E_0 \operatorname{sen}(x - vt) \cdot (-v)$$

$$\frac{\partial^2 E}{\partial x^2} = -E_0 \cos(x - vt) \qquad \frac{\partial^2 E}{\partial t^2} = E_0 v \cos(x - vt) \cdot (-v)$$

$$\Rightarrow \frac{\partial^2 E}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2}.$$

Ex$_2$: $E(x,t) = \ln(x - vt)$

Ex$_3$: $E(x,t) = \sqrt{x - vt}$

* EDP's LINEARES DE 2ª ORDEM

$$\boxed{a\frac{\partial^2 f}{\partial x^2} + 2b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = g\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

$f = f(x,y)$

$$a f_{xx} + 2b f_{xy} + c f_{yy} = g(x, y, f, f_x, f_y)$$

CLASSIFICAÇÃO:

i) EDP ELÍPTICA : $b^2 - ac < 0$

EX: EQUAÇÃO DO POISSON : $\nabla^2 V = -\rho/\epsilon_0$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho}{\epsilon_0}$$

ii) EDP PARABÓLICA : $b^2 - ac = 0$

Ex: EQUAÇÃO DA PROPAGAÇÃO DO CALOR:

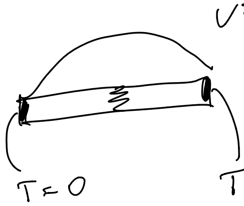
$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x} \qquad \text{OU} \qquad \frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$$

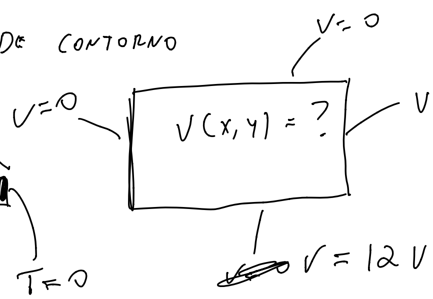
$$T = T(x,t)$$

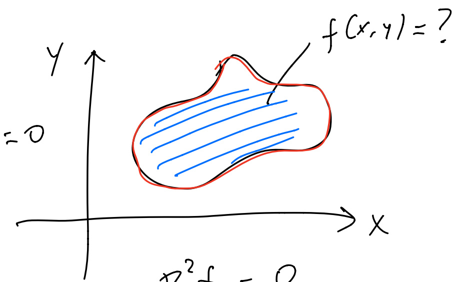
iii) EDP HIPERBÓLICA: $b^2 - ac > 0$

$$\frac{\partial^2 f}{\partial t^2} = \alpha^2 \frac{\partial^2 f}{\partial x^2} \quad ; \quad f = f(x,t).$$

* CONDIÇÕES DE CONTORNO

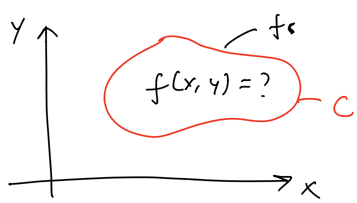




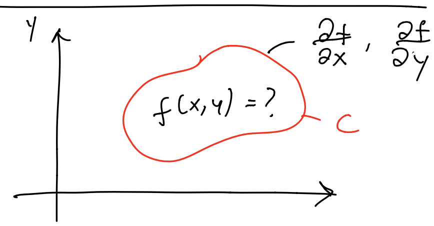


$$\nabla^2 f = 0$$

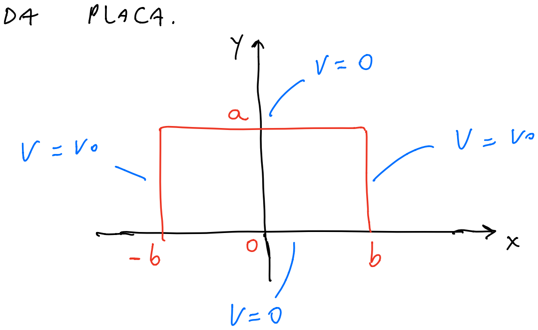
PROBLEMA DE DIRICHLET



PROBLEMA DE NEUMANN



Ex: UMA PLACA DE METAL ATERRADA NOS LADOS SUPERIOR E INFERIOR É MANTIDA SOB UM POTENCIAL CONSTANTE V_0 NOS LADOS ESQUERDO E DIREITO. ENCONTRE O POTENCIAL NO INTERIOR DA PLACA.



$$\nabla^2 V = -\frac{\rho}{\epsilon_0} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V = V(x,y) = X(x)\,Y(y)$$ (SEPARAÇÃO DE VARIÁVEIS)

$$\frac{\partial^2}{\partial x^2}\left[X(x)Y(y)\right] + \frac{\partial^2}{\partial y^2}\left[X(x)Y(y)\right] = 0$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \quad : XY$$

$$\frac{Y}{XY}\frac{d^2X}{dx^2} + \frac{X}{XY}\frac{d^2Y}{dy^2} = 0$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)} = 0$$

$$f(x) + g(y) = 0 \qquad \forall\ x, y.$$

$$f(x) = k^2$$
$$\Rightarrow g(y) = -k^2 \quad \Rightarrow \quad \begin{cases} \frac{1}{X}\frac{d^2X}{dx^2} = k^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 \end{cases}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = k^2$$
$$\frac{d^2X}{dx^2} = k^2X$$

$$\boxed{X(x) = Ae^{kx} + Be^{-kx}}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2$$
$$\frac{d^2Y}{dy^2} = -k^2Y$$

$$\boxed{Y(y) = C\cos(ky) + D\,\text{sen}(ky)}$$

Condições de contorno:

i) $y = 0 \Rightarrow V = 0$

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\cos(ky) + D\,\text{sen}(ky)\right)$$

$$V(x,0) = \left(Ae^{kx} + Be^{-kx}\right).\left(C.1 + D.0\right) = 0$$

$$\Rightarrow \boxed{C = 0}$$

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right).D\,\text{sen}(ky) = 0$$

ii) $y = a \Rightarrow V = 0.$

$$V(x,a) = (Ae^{kx} + Be^{-kx}) \cdot D\,\text{sen}(ka) = 0$$

$$\text{sen}(ka) = 0$$

$$\Rightarrow ka = 0, \pm\pi, \pm 2\pi, \dots$$

$$\boxed{k = \frac{n\pi}{a}}, \quad n = 0, 1, 2, 3, \dots$$

iii) $x = b \Rightarrow V = V_0$:

$$V(b,y) = (Ae^{kb} + Be^{-kb}) \cdot D\,\text{sen}(ky) = V_0$$

iv) $x = -b \Rightarrow V = V_0$:

$$V(-b,y) = (Ae^{-kb} + Be^{kb}) \cdot D\,\text{sen}(ky) = V_0$$

$$\begin{cases} (Ae^{kb} + Be^{-kb}) \cdot D\,\text{sen}(ky) = V_0 \\ (Ae^{-kb} + Be^{kb}) \cdot D\,\text{sen}(ky) = V_0 \end{cases}$$

$$\Rightarrow Ae^{kb} + Be^{-kb} = Ae^{-kb} + Be^{kb}$$

$$\Rightarrow (A-B)e^{kb} = (A-B)e^{-kb}$$

$$A - B = (A-B)e^{-2kb}$$

mas $k \neq 0$, e agora? $\boxed{A = B.}$ $\Rightarrow$

$$V(x,y) = (Ae^{kx} + Ae^{-kx})\,D\,\text{sen}(ky)$$

$$= \underset{\substack{\parallel \\ F}}{AD}\,(\underbrace{e^{kx} + e^{-kx}})\,\text{sen}(ky)$$

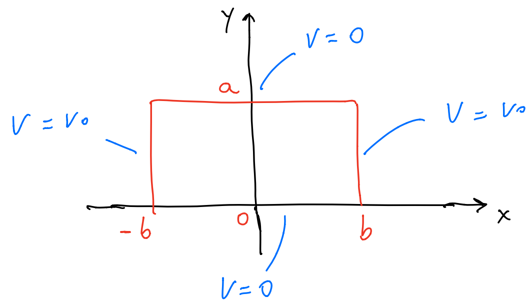
$$V(x,y) = F\cosh(kx)\,\text{sen}(ky)$$

$$V(x,y) = F\cosh\left(\frac{n\pi x}{a}\right)\text{sen}\left(\frac{n\pi y}{a}\right), \quad \text{mas } F = ?$$

TRUQUE DE FOURIER:

$$\int_0^a \operatorname{sen}\left(\frac{n\pi x}{a}\right) \operatorname{sen}\left(\frac{n'\pi x}{a}\right) dx = \begin{cases} 0, & n \neq n' \\ \frac{a}{2}, & n = n' \end{cases}$$

$\hookrightarrow$ $x = b$, $V = V_0$.



$$V(b, y) = F \cosh\left(\frac{n\pi b}{a}\right) \operatorname{sen}\left(\frac{n\pi y}{a}\right) = V_0$$

$$\int_0^a F \cosh\left(\frac{n\pi b}{a}\right) \underbrace{\operatorname{sen}\left(\frac{n\pi y}{a}\right) \operatorname{sen}\left(\frac{n'\pi y}{a}\right)} dy = \int_0^a V_0 \operatorname{sen}\left(\frac{n'\pi y}{a}\right) dy$$

$$V(x, y) = F \cosh\left(\frac{n\pi x}{a}\right) \operatorname{sen}\left(\frac{n\pi y}{a}\right) \quad , \quad n = 0, 1, 2, 3, \dots$$

SOLUÇÃO GERAL:

$$V(x, y) = \sum_{n=1}^{\infty} F_n \cosh\left(\frac{n\pi x}{a}\right) \operatorname{sen}\left(\frac{n\pi y}{a}\right) \quad , \quad F_n = ?$$

$x = b$, $V = V_0$:

$$V(b, y) = \sum_{n=1}^{\infty} F_n \cosh\left(\frac{n\pi b}{a}\right) \operatorname{sen}\left(\frac{n\pi y}{a}\right) = V_0$$

$$\int_0^a \sum_{n=1}^{\infty} F_n \cosh\left(\frac{n\pi b}{a}\right) \operatorname{sen}\left(\frac{n\pi y}{a}\right) \operatorname{sen}\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0 \operatorname{sen}\left(\frac{n'\pi y}{a}\right) dy$$

$$\sum_{n=1}^{\infty} F_n \cosh\left(\frac{n\pi b}{a}\right) \cdot \frac{\cancel{a}}{2} \delta_{nn'} = V_0 \frac{\cancel{a}}{n'\pi} \left[1 - \cos(n'\pi)\right]$$

$$F_{n'} \cosh\left(\frac{n'\pi b}{a}\right) \cdot \frac{1}{2} = \frac{V_0}{n'\pi} \left[1 - \cos(n'\pi)\right]$$

$$F_n = \frac{2V_0}{n\pi} \frac{1 - \cos(n\pi)}{\cosh\left(\frac{n\pi b}{a}\right)} \Rightarrow$$

$$\boxed{V(x,y) = \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n} \, \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \, \text{sen}\left(\frac{n\pi y}{a}\right)}$$

* EDP'S PARABÓLICAS

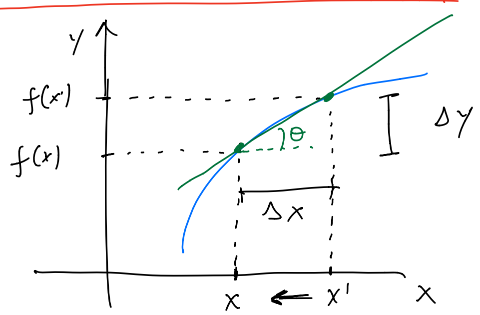
ESTAMOS INTERESSADOS EM RESOLVER A EQUAÇÃO DO TIPO

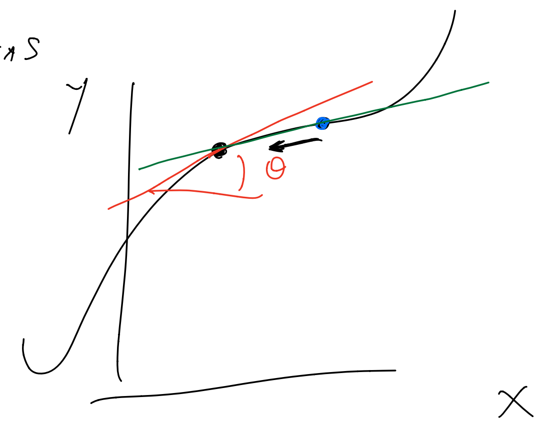
$$\frac{\partial f}{\partial t} = \alpha^2 \frac{\partial^2 f}{\partial x^2} \qquad \text{(EQ. DO CALOR)},$$

ONDE $f = f(x,t)$.

- MÉTODO DAS DIFERENÇAS FINITAS

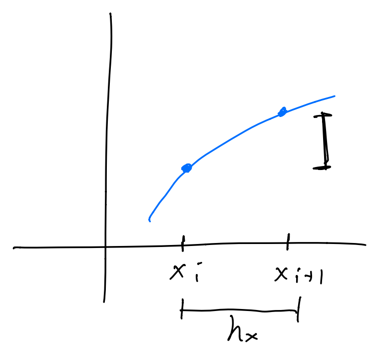
$$\boxed{\frac{df}{dx} = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x}}$$





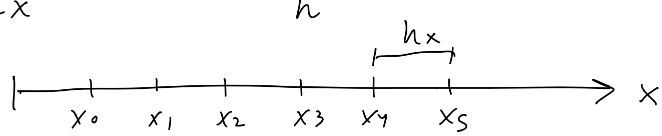
$$\text{tg}\,\theta = \text{INCLINAÇÃO} = \frac{\Delta y}{\Delta x}$$

$$\frac{df}{dx} \simeq \frac{\Delta y}{\Delta x} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$



$$x_{i+1} - x_i \equiv h$$

$$\frac{df}{dx} = \frac{f(x_{i+1}) - f(x_i)}{h}$$



$$x_i \equiv i\,h_x \Rightarrow f(x_i) \equiv f_i$$

$$\boxed{\frac{\partial f}{\partial x} \longrightarrow \frac{f_{i+1} - f_i}{h_x}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{f(x_{i+1}) - f(x_i)}{h_x}\right]$$

$$= \frac{1}{h_x}\frac{\partial}{\partial x}\left[f(x_{i+1}) - f(x_i)\right]$$

$$= \frac{1}{h_x}\left[\frac{f(x_{i+1}) - f(x_i)}{h_x} - \frac{f(x_i) - f(x_{i-1})}{h_x}\right]$$

$$= \frac{1}{h_x^2}\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right]$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} \longrightarrow \frac{1}{h_x^2}\left(f_{i+1} - 2f_i + f_{i-1}\right)}$$

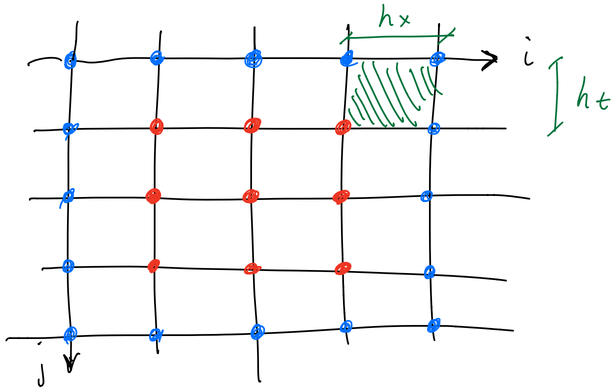
$$\frac{\partial f}{\partial t} = \alpha^2 \frac{\partial^2 f}{\partial x^2} \qquad f(x,t)$$

$$\boxed{\frac{\partial f}{\partial t} \longrightarrow \frac{1}{h_t}\left(f_{j+1} - f_j\right)}$$

$i \rightarrow x$ (PARTE ESPACIAL)
$j \rightarrow t$ (PARTE TEMPORAL)

$$\Rightarrow \qquad f(x,t) \longrightarrow f_{i,j}$$

$$\frac{1}{h_t}\left(f_{i,j+1} - \underline{f_{i,j}}\right) = \frac{\alpha^2}{h_x^2}\left(f_{i+1,j} - \underline{2f_{ij}} + f_{i-1,j}\right)$$



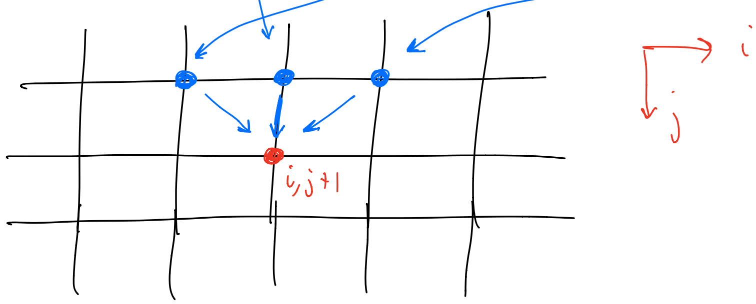
MESH

$$f_{i,j+1} - f_{ij} = \frac{\alpha^2 h_t}{h_x^2}\left(f_{i+1,j} - 2f_{ij} + f_{i-1,j}\right)$$

$$f_{i,j+1} = f_{ij} + \frac{\alpha^2 h_t}{h_x^2}\left(f_{i+1,j} - 2f_{ij} + f_{i-1,j}\right)$$

$$\lambda \equiv \frac{\alpha^2 h_t}{h_x^2} \Rightarrow$$

$$\boxed{f_{i,j+1} = (1-2\lambda) f_{ij} + \lambda\left(f_{i-1,j} + f_{i+1,j}\right)}$$



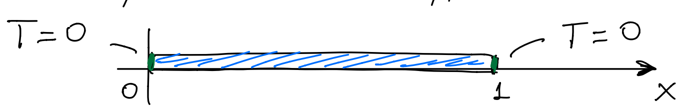
CONDIÇÃO DE ESTABILIDADE:

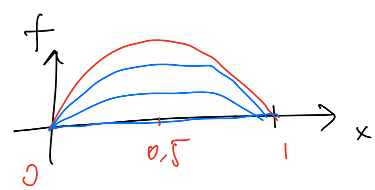
O MÉTODO É ESTÁVEL SE $\boxed{\lambda = \frac{\alpha^2 h_t}{h_x^2} \leq \frac{1}{2}}$

EX: CONSIDERE UMA BARRA DE COMPRIMENTO $L = 1$ E $\alpha = 0{,}5$ COM AS EXTREMIDADES FIXAS COM TEMPERATURA $T = 0$. A TEMPERATURA INICIAL DA BARRA É DADA PELA FUNÇÃO

$$T(x) = 10 \operatorname{sen}(\pi x).$$

CALCULE A EVOLUÇÃO DA TEMPERATURA DA BARRA ENTRE $t = 0$ E $t = 0{,}5$. USE $h_t = 0{,}1$ E $h_x = 0{,}25$.





$$\lambda = \frac{\alpha^2 h_t}{h_x^2} = (0{,}5)^2 \cdot 0{,}1 \cdot \frac{1}{(0{,}25)^2}$$

$$= \frac{1}{4} \cdot \frac{1}{10} \cdot 16 = \frac{16}{40} =$$

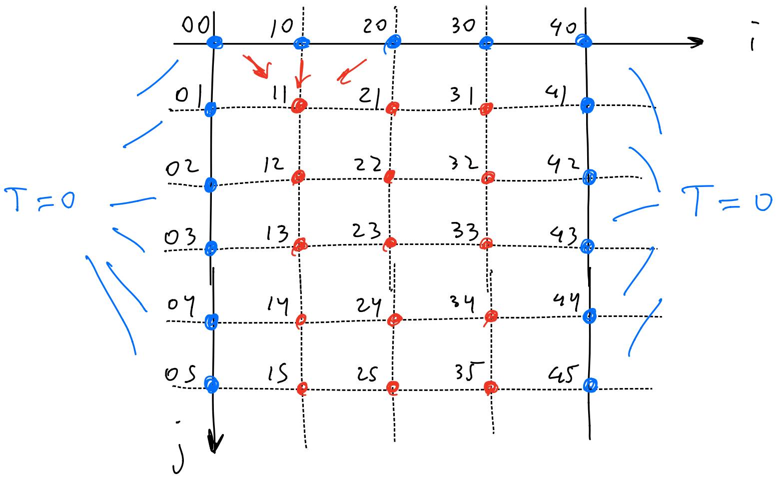
$$\boxed{\lambda = 0{,}4 < 0{,}5 \Rightarrow \text{ESTÁVEL.}}$$

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2}$$

$$1 - 2 \cdot 0{,}4$$

$$f_{i,j+1} = (1-2r)\, f_{ij} + r\,(f_{i-1,j} + f_{i+1,j}) \Rightarrow$$

$$\boxed{T_{i,j+1} = 0{,}2\, T_{ij} + 0{,}4\,(T_{i-1,j} + T_{i+1,j})}$$



$T(x) = 10 \sin(\pi x)$

$\boxed{T(0) = 0}$

$\boxed{T(0{,}25) = 10 \sin\left(\frac{\pi}{4}\right)}$

$\boxed{T(0{,}25) \simeq 7}$

$\boxed{T(0{,}5) = 10 \sin\left(\frac{\pi}{2}\right) = 10}$

$\boxed{T(0{,}75) \simeq 7}$

$\boxed{T(1) = 0}$