Equações Diferenciais Parciais

* Equações elípticas

$$a\frac{\partial^2 f}{\partial x^2} + 2b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = g\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

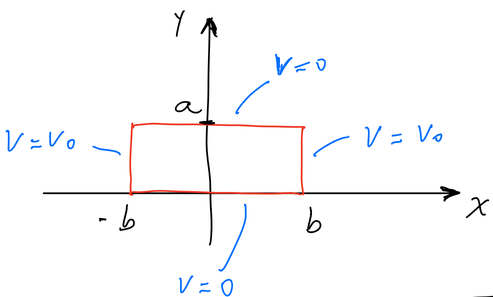
$$b^2 - ac < 0$$

Ex: $\nabla^2 f = f_0$

$\nabla^2 V = -\rho/\epsilon_0$ (eletrostática) $V = V(x,y)$

$\nabla^2 T = 0$ (distribuição de temperatura) $T = T(x,y)$

Ex: Duas placas de metal aterradas e de comprimento infinito são conectadas em $x = \pm b$ por tiras de metal mantidas em potencial constante V_0, como mostra a figura. Encontre o potencial dentro do tubo resultante. (Resolvido analiticamente)



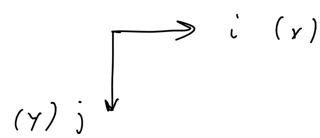
$$\nabla^2 V = 0$$

$$\nabla^2 \to \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m\cosh(m\pi b/a)}\cosh\left(\frac{m\pi x}{a}\right)\operatorname{sen}\left(\frac{m\pi y}{a}\right)}$$

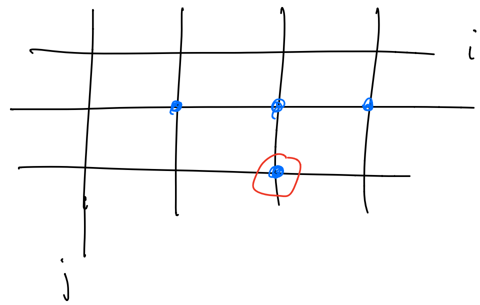
$$\begin{cases} a = 1 \\ b = 1 \\ V_0 = 10 \end{cases}$$



$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2} \longrightarrow \dfrac{1}{h_x^2}\left(V_{i+1,j} - 2V_{ij} + V_{i-1,j}\right) \\[2ex] \dfrac{\partial^2 V}{\partial y^2} \longrightarrow \dfrac{1}{h_y^2}\left(V_{i,j+1} - 2V_{ij} + V_{i,j-1}\right) \end{cases}$$

$$\frac{1}{h_x^2}\left(V_{i+1,j} - 2V_{ij} + V_{i-1,j}\right) + \frac{1}{h_y^2}\left(V_{i,j+1} - 2V_{ij} + V_{i,j-1}\right) = 0$$

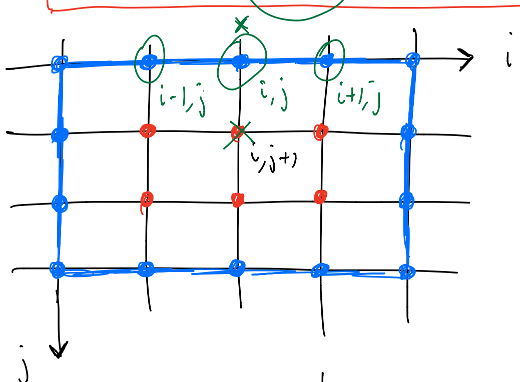
$$h_y^2\left(V_{i+1,j} - 2V_{ij} + V_{i-1,j}\right) + h_x^2\left(V_{i,j+1} - 2V_{ij} + V_{i,j-1}\right) = 0$$

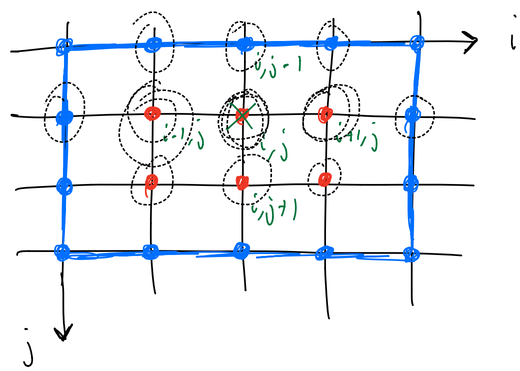


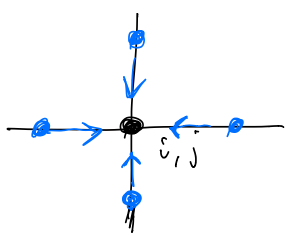
$$\boxed{h_x^2 V_{i,j+1} - 2\left(h_x^2 + h_y^2\right) V_{ij} + h_y^2\left(V_{i-1,j} + V_{i+1,j}\right) + h_x^2 V_{i,j-1} = 0}$$

OBSERVE QUE SE $h_x = h_y = h \Rightarrow$

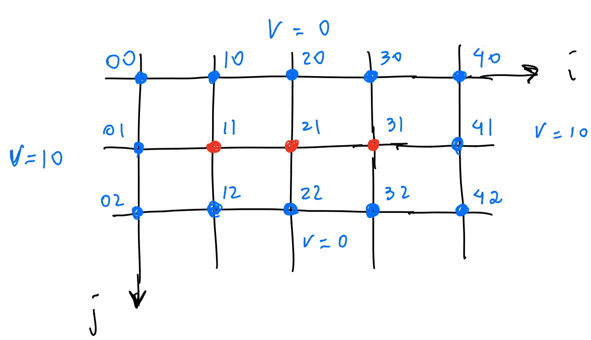
$$\boxed{V_{i,j+1} - 4V_{ij} + V_{i-1,j} + V_{i+1,j} + V_{i,j-1} = 0 \,, \quad h_x = h_y}$$

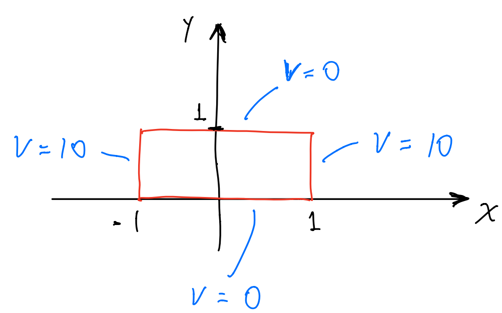






EX: PROBLEMA DO POTENCIAL ELÉTRICO. $h_x = h_x = 0{,}5$





11: $-4V_{11} + V_{10} + V_{01} + V_{12} + V_{21} = 0$

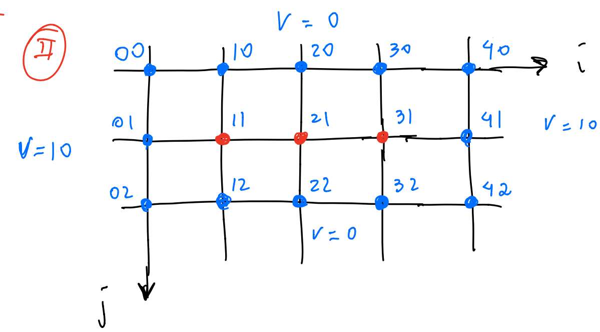
$-4V_{11} + 0 + 10 + 0 + V_{21} = 0$

$\boxed{-4V_{11} + V_{21} = -10}$ (I)

21: $-4V_{21} + V_{20} + V_{11} + V_{22} + V_{31} = 0$

$-4V_{21} + 0 + V_{11} + 0 + V_{31} = 0$

$\boxed{V_{11} - 4V_{21} + V_{31} = 0}$ (II)



31: $-4V_{31} + V_{30} + V_{21} + V_{32} + V_{41} = 0$

$-4V_{31} + 0 + V_{21} + 0 + 10 = 0$

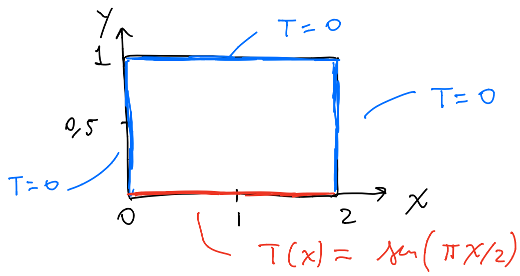
$\boxed{V_{21} - 4V_{31} = -10}$ (III)

SISTEMA LINEAR:
$$\begin{cases} -4V_{11} + V_{21} = -10 \\ V_{11} - 4V_{21} = 0 \\ V_{21} - 4V_{31} = -10 \end{cases}$$

$$\Rightarrow \underbrace{\begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 1 & -4 \end{pmatrix}}_{\bar{\bar{A}}} \underbrace{\begin{pmatrix} V_{11} \\ V_{21} \\ V_{31} \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} -10 \\ 0 \\ -10 \end{pmatrix}}_{\vec{b}} \qquad \bar{\bar{A}}\,\vec{x} = \vec{b}$$

SOLUÇÃO:
$$\begin{cases} V_{11} = 2,667 \\ V_{21} = 0,667 \\ V_{31} = 2,667 \end{cases}$$

EX: Considere a placa ao lado, onde 3 lados são mantidos em $T=0$, enquanto que o lado inferior está mantido a uma temperatura



$$T(x) = \operatorname{sen}(\pi x/2), \quad \text{com} \quad 0 \le x \le 2.$$

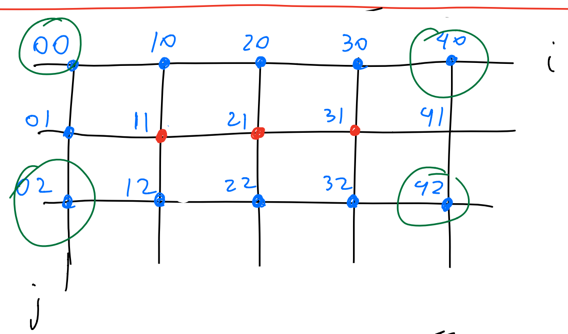
Calcule numericamente a temperatura dentro da placa dividindo a malha com $h_x = 0,5$ e $h_y = 0,5$, e sabendo que $T(x,y)$ obedece a equação

$$\nabla^2 T(x,y) = 0.$$

Solução analítica:

$$T(x,y) = \frac{1}{\operatorname{senh}(\pi/2)} \operatorname{sen}\left(\frac{\pi x}{2}\right) \operatorname{senh}\left[\frac{\pi(1-y)}{2}\right]$$

$$\boxed{V_{i,j+1} - 4V_{i,j} + V_{i-1,j} + V_{i+1,j} + V_{i,j-1} = 0, \quad h_x = h_y}$$



$$\begin{cases} x_i = i\,h_x \\ y_j = j\,h_y \end{cases}$$

$T_{10} = ?$ $\quad i = 1, \; j = 0$

$i = 1 \Rightarrow x = 0,5$

$$T(x) = \operatorname{sen}\left(\frac{\pi x}{2}\right)$$

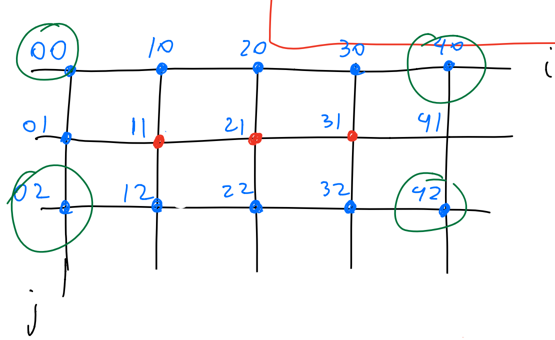
$T(0,5) = T_{10} = \operatorname{sen}\left(\frac{\pi \cdot 0,5}{2}\right) = \operatorname{sen}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$T(1) = T_{20} = \operatorname{sen}\left(\frac{\pi \cdot 1}{2}\right) = 1$

$T(3/2) = T_{30} = \frac{\sqrt{2}}{2}.$ $\qquad T_{12} = 0$

$T_{01} = 0$ $\qquad T_{22} = 0$

$T_{41} = 0$ $\qquad T_{32} = 0$



$$-4T_{11} + T_{10} + \cancelto{0}{T_{01}} + \cancelto{0}{T_{12}} + T_{21} = 0$$

$$-4T_{11} + T_{21} = -\sqrt{2}/2 \quad \text{I}$$

$$-4T_{21} + T_{11} + T_{31} + T_{20} + \cancelto{0}{T_{22}} = 0$$

$$T_{11} - 4T_{21} + T_{31} = -1 \quad \text{II}$$

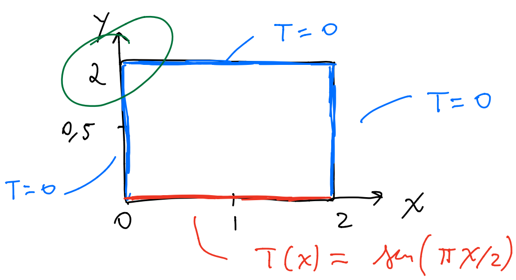
$$-4T_{31} + T_{21} + \cancelto{0}{T_{41}} + T_{30} + \cancelto{0}{T_{32}} = 0$$

$$T_{21} - 4T_{31} = -\sqrt{2}/2 \quad \text{III}$$

$$\Rightarrow \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ -1 \\ -\sqrt{2}/2 \end{pmatrix}$$

EX: CONSIDERE A PLACA AO LADO, ONDE 3 LADOS SÃO MANTIDOS EM $T=0$, ENQUANTO QUE O LADO INFERIOR ESTÁ MANTIDO A UMA TEMPERATURA

$$T(x) = \operatorname{sen}(\pi x/2), \quad \text{com} \quad 0 \le x \le 2.$$



CALCULE NUMERICAMENTE A TEMPERATURA DENTRO DA PLACA DIVIDINDO A MALHA COM $h_x = 0,5$ E $h_y = 0,5$, E SABENDO QUE $T(x,y)$ OBEDECE A EQUAÇÃO

$$\nabla^2 T(x,y) = 0.$$