# Complex Manifolds

## Sheaves, Vector Bundles and other topics

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# Some kind of introduction

A *complex manifold* is a topological manifold equipped with an atlas of charts onto open disks[1] in $\mathbb{C}^n$, such that the transition maps are biholomorphic. Consequently, each complex manifold is in particular a real differentiable manifold. Moreover, since biholomorphic maps are orientation-preserving, a complex manifold is *canonically oriented* (not just orientable).

We denote by $\mathcal{O}_X$ the *sheaf of holomorphic functions* on a given complex manifold $X$. In other words, for any open subset $U$ of $X$, we get a $\mathbb{C}$-algebra

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}.$$

For simplicity we often just write $\mathcal{O}$ in place of $\mathcal{O}_X$, when it is clear from the context to which underlying complex manifold $X$ we are referring to.

## Complex manifolds vs. Real manifolds

The theories of real (differentiable) manifolds and complex manifolds are substantially different, the main reason being that holomorphic functions are much more rigid than *smooth* (i.e. $\mathcal{C}^\infty$) functions. For example there exists no holomorphic function on a *compact* complex manifold, apart from the trivial case of constant functions. Indeed, compact complex manifolds are much closer to *algebraic varieties* than to differentiable manifolds (see the end of section 3.3).

The Whitney embedding theorem tells us that any real manifold can be (diffeomorphically) embedded in $\mathbb{R}^N$, while "most" complex manifolds do not admit any holomorphic embedding into $\mathbb{C}^N$ (nor in $\mathbb{P}^N$, in the compact case).

The classification of complex manifolds is more complicated than that of real manifolds. For example, a given topological manifold $X$ admits only finitely many differentiable structures if $\dim X \neq 4$, while a given complex manifold often admits uncountably many complex structures. As a matter of fact, the set of all complex structures (up to equivalence) on a given complex manifold, forms itself a continuous space, and in fact can be given the structure of a complex algebraic variety, called *moduli space*.

[1] these are subsets of the form $U = \{z \in \mathbb{C}^n : |z| < 1\}$ (also called open unit "balls"). One has to use open disks instead of the whole $\mathbb{C}^n$, because these are not isomorphic as complex manifolds, in contrast with the case of real (differentiable) manifolds. Equivalently, one can use any open subset $U \subset \mathbb{C}^n$ as local models (since open disks form a topological basis of $\mathbb{C}^n$).

## The holomorphic tangent space

The definition of *holomorphic tangent space*, is a bit trickier than one would expect from the differentiable analogue.

Let $X$ be a $n$-dimensional complex manifold. We start by considering the underlying differentiable manifold of (real) dimension $2n$, which we still denote by $X$. Consider the *sheaf of smooth functions* $\mathcal{C}^\infty$ on $X$, i.e. the sets $\mathcal{C}^\infty(U)$ of differentiable (or smooth) functions $f : U \to \mathbb{R}$, for each open subset $U$ of $X$.

At each point $a \in X$ we have the "stalk" of $\mathcal{C}^\infty$ at $a$, which we denote by $\mathcal{C}_a^\infty$. It consists of (equivalence classes of) smooth functions $f$ defined locally around $a$ (two such functions are considered equivalent when they coincide on some small neighbourhood of $a$). An element $f_a \in \mathcal{C}_a^\infty$ is called *germ at $a$* of any smooth function $f$ for which $f_a$ is a representative. For simplicity we often drop the lower $a$ and just write $f \in \mathcal{C}_a^\infty$, with some abuse of notation.

The sets $\mathcal{C}^\infty(U)$, and consequently $\mathcal{C}_a^\infty$, naturally carry the structure of real vector spaces and in fact of $\mathbb{R}$-algebras, with the natural multiplication of functions $(fg)(x) := f(x)g(x)$.

The dual vector space $(\mathcal{C}_a^\infty)^*$, contains a particularly important subspace: the space of derivations. A *derivation (at $a$)* is a linear map $v : \mathcal{C}_a^\infty \to \mathbb{R}$ which satisfies the following "Leibniz rule" for multiplication of functions:

$$v(fg) = f(a)v(g) + g(a)v(f) \qquad (f, g \in \mathcal{C}_a^\infty)$$

The *tangent space* $T_aX$ is by definition the space of derivations at $a \in X$. Its complexification, called the *complexified tangent space*, will be denoted by

$$T_aX_{\mathbb{C}} := T_aX \otimes_{\mathbb{R}} \mathbb{C} = \{v + iw \mid v, w \in T_aX\}.$$

The complexified tangent space extends the above construction of $T_aX$, in the following sense. We could go the other way around and start by taking the complexification of the sheaf of *smooth real-valued* functions $\mathcal{C}^\infty$, so to get the sheaf of *smooth complex-valued* functions, which we denote by $\mathcal{A}^0$. Concretely,

$$\mathcal{A}^0(U) := \{f : U \to \mathbb{C} \mid f = g + ih, \text{ where } g, h \in \mathcal{C}^\infty(U)\}.$$

A derivation $v : \mathcal{C}_a^\infty \to \mathbb{R}$ (i.e. $v \in T_aX$) extends $\mathbb{C}$-linearly to a derivation $\mathcal{A}_a^0 \to \mathbb{C}$, in the natural way $g + ih \mapsto v(g) + iv(h)$. In fact one can see that *all* complex derivations $\mathcal{A}_a^0 \to \mathbb{C}$ are uniquely obtained this way from $T_aX$.

The complexification process doubles the dimension, so that $T_aX_{\mathbb{C}}$ has twice the dimension we desire for a "good" tangent space, which should have the same complex dimension as that of $X$, while $\dim_{\mathbb{C}}(T_aX_{\mathbb{C}}) = 2\dim_{\mathbb{C}}(X)$.

An important observation: so far, the complex structure on $X$ is irrelevant! Here it comes into play: the complex structure on $X$ gives rise to a *canonical* direct sum decomposition $T_aX_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$, where $T^{0,1}$ consists of the derivations which kill the holomorphic functions. Let us describe this. Let

$$\begin{aligned} x &= (x_1, \dots, x_n) \in \mathbb{R}^n \\ y &= (y_1, \dots, y_n) \in \mathbb{R}^n \\ z &= (z_1, \dots, z_n) \in \mathbb{C}^n \\ z_j &= x_j + iy_j, \ j = 1, \dots, n \end{aligned}$$

As real vector spaces, we identify $\mathbb{C}^n = \mathbb{R}^{2n}$ by $z \leftrightarrow (x, y)$. However, on $\mathbb{C}^n$ we have some additional structure, induced by multiplication and conjugation of complex numbers. For example, the automorphism $z \mapsto iz = (iz_1, \ldots, iz_n)$, can be viewed by the above identification, as the linear operator $J$ on $\mathbb{R}^{2n}$ given by $J\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)$. In other words

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I$ is the identity matrix. In particular $J^2 = -I$, and $J$ has two eigenvalues $\pm i$. When viewed as an operator on the real tangent space $T_aX \simeq \mathbb{R}^{2n}$, the linear operator $J$ is called *complex structure* of $X$. Often, it is simply denoted by $i$. We can naturally extend $J$ to a linear operator on the complexified tangent space $T_aX_{\mathbb{C}}$. Consider the two eigenspaces

$$T_a^{1,0} := \{w \in T_aX_{\mathbb{C}} : \; Jw = iw\}$$
$$T_a^{0,1} := \{w \in T_aX_{\mathbb{C}} : \; Jw = -iw\}$$

When we work in local coordinates $z = (x, y)$ around $a \in X$, a basis of $T_aX$ is given by the partial derivatives[2], $T_aX = \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \rangle_{\mathbb{R}}$. Then, complexification simply reads $T_aX_{\mathbb{C}} = \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \rangle_{\mathbb{C}}$. We define for $z = (x, y)$ the derivations

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \in T_a^{1,0}$$
$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \in T_a^{0,1}$$

which form a basis with respect to the orthogonal decomposition:

$$T_aX_{\mathbb{C}} = T^{1,0} \oplus T^{0,1} = \langle \frac{\partial}{\partial z_j} \rangle_{\mathbb{C}} \oplus \langle \frac{\partial}{\partial \bar{z}_j} \rangle_{\mathbb{C}}.$$

Note that $\frac{\partial}{\partial x_j} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}$ and $\frac{\partial}{\partial y_j} = i(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j})$. In particular, now the inclusion of the tangent space into its complexification takes the form

$$\begin{aligned} T_aX = \mathbb{C}^n &\longrightarrow T^{1,0} \oplus T^{0,1} \\ v = (v_1, \ldots, v_n) &\longmapsto v_1 \tfrac{\partial}{\partial z_1} + \cdots + v_n \tfrac{\partial}{\partial z_n} + \bar{v}_1 \tfrac{\partial}{\partial \bar{z}_1} + \cdots + \bar{v}_n \tfrac{\partial}{\partial \bar{z}_n} \end{aligned}$$

As mentioned above, $T_aX_{\mathbb{C}}$ is the space of derivations of complex-valued smooth functions on $X$. Now let $f \in \mathcal{A}^0(U)$, $f = g + ih$. Then

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}_j} = 0 \;\; (\forall j) &\iff 2\left(\frac{\partial g}{\partial x_j} - \frac{\partial h}{\partial y_j}\right) - 2i\left(\frac{\partial h}{\partial x_j} + \frac{\partial g}{\partial y_j}\right) = 0 \;\; (\forall j) \\ &\iff \frac{\partial g}{\partial x_j} - \frac{\partial h}{\partial y_j} = \frac{\partial h}{\partial x_j} + \frac{\partial g}{\partial y_j} = 0 \;\; (\forall j) \\ &\iff f \in \mathcal{O}_X(U). \end{aligned}$$

Thus, $T^{0,1}$ kills holomorphic functions! In other words, for any holomorphic function $f$ and for each derivation $w \in T^{0,1}$ we have $w(f) = 0$.

Since one is generally interested in the study of holomorphic functions on a complex manifold, we are naturally lead to the following definition.

[2] keep in mind that "evaluation at $a$", i.e. $\frac{\partial}{\partial x_j}(a)$, is understood in the notation here.

**Definition.** The *holomorphic tangent space at* $a \in X$ is $T_a^{1,0}$.

As mentioned above, we have a canonical $\mathbb{C}$-basis of $T^{1,0}$ given by $\{\frac{\partial}{\partial z_j}\}$, $j = 1, \ldots, n$. The dual vector space $(T^{1,0})^*$ is called the *holomorphic cotangent space*. Its canonical basis is given by $\{dz_j\}$, the dual basis of $\{\frac{\partial}{\partial z_j}\}$.

Similarly, one refers to $T_a^{0,1}$ and $(T_a^{0,1})^*$ as the *anti-holomorphic tangent space* and *anti-holomorphic cotangent space at* $a$, respectively.

Roughly speaking, when $a$ varies on $X$ we would like to "smoothly" glue together all the tangent spaces to obtain some sort of global object. This leads to the construction of the *complex tangent bundle* $TX_{\mathbb{C}}$, whose elements are (smooth complex) *vector fields*. This is just a special example of a more general construction: that of *vector bundles* on $X$, which will be discussed later. To get a picture of this idea in the case of the tangent bundle, think of what this should look like when working in local coordinates: a vector field $w \in TX_{\mathbb{C}}$, defined over some open subset $U \subset X$, is an expression of the form

$$w = f_1 \frac{\partial}{\partial z_1} + \cdots + f_n \frac{\partial}{\partial z_n} + g_1 \frac{\partial}{\partial \bar{z}_1} + \cdots + g_n \frac{\partial}{\partial \bar{z}_n}$$

where $f_i, g_i \in \mathcal{A}^0(U)$. A philosophical observation here is that the existence of such an object becomes interesting when the above expression remains genuinely *local* in nature, i.e. we are not able to extend it to the "trivial" case when $U = X$, since then the kind of information carried by $w$ simply amounts to that of the functions $f_i, g_i \in \mathcal{A}^0(X)$, and we would not have done anything new. The point is that these kind of non-trivial objects exist in general, i.e. we actually get local expressions as above which, though, smoothly glue together in a non-trivial way.

One proceeds in a similar fashion for the cotangent space, whose elements are called differential 1-forms and (locally) look like

$$\omega = f_1 dz_1 + \cdots + f_n z_n + g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n$$

where $\{dz_j\}$ and $\{d\bar{z}_j\}$ are the dual basis of $\{\frac{\partial}{\partial z_j}\}$ and $\{\frac{\partial}{\partial \bar{z}_j}\}$.

The study of differential forms on a complex manifold $X$ is preferred to that of vector fields because of some nice decomposition that the space of forms admits in many interesting cases. Differential forms are one of the major tools in the study of Kähler manifolds and Hodge theory.

A special and fundamental case of differential 1-form is that of *holomorphic 1-form*, when the local expressions are given by

$$\omega = f_1 dz_1 + \cdots + f_n dz_n$$

with $f_i$ *holomorphic* functions on $U$. The properties of holomorphic forms on some particular complex manifold $X$ can give a lot of information on the geometry of $X$. The above examples are just special cases of differential forms: that of 1-forms, or forms of *degree one*. In order to fully obtain a rich enough structure one has to generalize this definition to that of differential forms of higher degree. Let us sketch the basic definitions and construction in the following section.

## Differential forms on complex manifolds

Let $X$ be a complex manifold. Denote by $\mathcal{E}^r$ the space[3] of smooth real $r$-forms, as it is usually defined for any smooth real manifold. Concretely, let us recall that an $r$-form $\omega$ in $\mathcal{E}^r(U)$ ($U \subset X$ open) is locally an expression like

$$\sum_{i_1<\ldots<i_r} f_{i_1\ldots i_r} du_{i_1} \wedge \ldots \wedge du_{i_r}$$

where $u = (u_1, \ldots, u_{2n})$ is a choice of local (real) coordinates on $U$, and the restriction of the sum to $i_1 < \ldots < i_r$ is made to avoid trivial repetitions due to the skew-symmetry of the wegde product:

$$du_i \wedge du_j = -du_j \wedge du_i,$$

which can be taken to be a formal rule.

Since $X$ is a complex manifold we consider smooth *complex* $r$-forms

$$\mathcal{A}^r = \mathcal{E}^r \otimes_{\mathbb{R}} \mathbb{C}.$$

Concretely, these are forms $\omega = \omega_1 + i\omega_2$, where $\omega_1$ and $\omega_2$ are in $\mathcal{E}^r$.

Here is the key-point: the space $\mathcal{A}^r$ admits a nice decomposition which goes as follows. Given an $r$-form $\omega \in \mathcal{A}^r$, one could uniquely write $\omega$ as a sum

$$\omega = \omega_{r,0} + \omega_{r-1,1} + \cdots + \omega_{1,r-1} + \omega_{0,r},$$

where each of the factors $\omega_{p,q}$ with $p + q = r$ is a form of a special kind: they are called $r$-forms of type $(p, q)$, or just $(p, q)$-forms, and in terms of some local holomorphic chart $z = (z_1, \ldots, z_n)$ they are expressed as

$$\omega_{p,q} = \sum f_{i_1\ldots i_p j_1\ldots j_q} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$$

with the sum running over all set of indices $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q$. Since the notation becomes relatively heavy, one usually abbreviates this expression to something like

$$\omega_{p,q} = \sum f_{I,J} dz_J \wedge d\bar{z}_J$$

where the sum runs over all sets $I, J$ of ordered indices as above and we set $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$. Hence, if we let $\mathcal{A}^{p,q}$ denote the space of $(p, q)$-forms what we get is a decomposition of the kind

$$\mathcal{A}^r = \bigoplus_{p+q=r} \mathcal{A}^{p,q}.$$

The attentive reader should notice at this point that in order for this to make perfect sense one should, of course, actually *prove* that this kind of decomposition *does not depend* on the choice of the local coordinates.

Notice that for $r = 1$ what we get here is basically the decomposition of the complex tangent space in its $(1, 0)$ and $(0, 1)$ part, which we discussed above.

[3] more precisely: $\mathcal{E}^r$ is a *sheaf*. But this is to be understood later in the notes.

Among all $r$-forms of type $(p, q)$ on $X$ we particularly like those in $\mathcal{A}^{r,0}$, whose local expressions have no $d\bar{z}$'s. And among these ones, those with only holomorphic coefficients are so special which deserve a special notation:

$$\Omega^r = \{\omega \in \mathcal{A}^{r,0} : \ \omega = \sum f_I dz_I \ \text{ with } f_I \text{ holomorphic}\}.$$

These are called *holomorphic $r$-forms*.

Let us now briefly introduce some technical machinery which will be useful later on. Recall that to any smooth real function in $\mathcal{E}^0$ we can associate its differential, and this yields a map $d\colon \mathcal{E}^0 \to \mathcal{E}^1$ which sends $f$ to $df$. Moreover, we have the following linear operators

$$\begin{aligned} \partial : \mathcal{A}^0 &\longrightarrow \mathcal{A}^{1,0} \qquad & \partial f &:= \sum \frac{\partial f}{\partial z_j} dz_j \\ \bar{\partial} : \mathcal{A}^0 &\longrightarrow \mathcal{A}^{0,1} \qquad & \bar{\partial} f &:= \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \end{aligned}$$

By our discussion on the holomorphic tangent space:

$$f \text{ is holomorphic} \iff \bar{\partial} f = 0 \iff df = \partial f$$

Recall that one can extend the differential map to any degree to get

$$d\colon \mathcal{E}^r \to \mathcal{E}^{r+1}.$$

One then extends the other operators in the following way

$$\begin{aligned} d : \mathcal{A}^r &\longrightarrow \mathcal{A}^{r+1} \qquad & d(\omega_1 + i\omega_2) &= d\omega_1 + i d\omega_2 \\ \partial : \mathcal{A}^{p,q} &\longrightarrow \mathcal{A}^{p+1,q} \qquad & \partial \omega_{p,q} &= (d\omega_{p,q})_{p+1,q} \\ \bar{\partial} : \mathcal{A}^{p,q} &\longrightarrow \mathcal{A}^{p,q+1} \qquad & \bar{\partial} \omega_{p,q} &= (d\omega_{p,q})_{p,q+1} \end{aligned}$$

where $(d\omega_{p,q})_{p+1,q}$ denotes the projection of $(d\omega_{p,q})$ to its $(p+1, q)$-factor of the above decomposition and similarly for $(d\omega_{p,q})_{p,q+1}$. The key feature is that the differential map $d$ is also compatible with the decomposition, in the sense that

$$d = \partial + \bar{\partial},$$

as one can check in local coordinates.

Clearly, for a form $\omega_{r,0}$ in $\mathcal{A}^{r,0}$, one has

$$\omega_{r,0} \in \Omega^r \iff \bar{\partial}\omega_{r,0} = 0 \iff d\omega_{r,0} = \partial\omega_{r,0}.$$

When considering the possible compositions of these operators in sequences

$$\mathcal{A}^0 \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \mathcal{A}^3 \longrightarrow \mathcal{A}^4 \longrightarrow \cdots$$

one finds the following fundamental properties: $d^2 = 0$ and therefore $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and moreover $\partial\bar{\partial} = -\bar{\partial}\partial$. Finally, when one has a holomorphic map of complex manifolds $\varphi\colon X \to Y$, which naturally induces pull-back maps

$$\varphi^*\colon \mathcal{A}^r_Y \longrightarrow \mathcal{A}^r_X,$$

between the differential forms, one finds that $\varphi^*$ commutes with all the differentials: $d \circ \varphi^* = \varphi^* \circ d$ and $\partial \circ \varphi^* = \varphi^* \circ \partial$ and $\bar{\partial} \circ \varphi^* = \varphi^* \circ \bar{\partial}$.

# Isomorphisms of complex manifolds

These are biholomorphism maps of complex manifold, but let us start with some basic observations. If $F\colon V \subset \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic map, then the matrix of partial derivatives

$$J_{\mathbb{C}}F = \left[\frac{\partial F_j}{\partial z_k}\right]$$

where $F = (F_1, \dots, F_n)$, is called *complex Jacobian* of $F$. If $m = n$ then

$$|\det J_{\mathbb{C}}F|^2 > 0$$

and it is easy to see that $|\det J_{\mathbb{C}}F|^2 = \det J_{\mathbb{R}}\tilde{F}$, where $J_{\mathbb{R}}\tilde{F}$ is the usual Jacobian of the differentiable map $\tilde{F}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ naturally induced by $F$. If $F$ is a biholomorphic map then $\tilde{F}$ is a diffeomorphism. In particular, this is the case for the transition maps of a complex manifold $X$. The positivity of the real Jacobian therefore implies that any complex manifold $X$ is topologically orientable.

Let us now recall the following basic fact: if $F\colon V \subset \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic map and at some point $a \in V$ the complex Jacobian of $F$ has maximal rank $n$, then $F$ is locally a biholomorphism. In other words there exists a neighbourhood $W$ of $a$ such that $F\colon W \to F(W)$ is biholomorphic.

What does this give for a holomorphic map $f\colon X \to Y$ of complex manifolds? It can be used to show that $f$ can be locally linearised as follows: if the rank of the complex Jacobian of $f$ is $k$ at $a \in X$, then we can locally around $a$ write

$$f\colon (u_1, \dots, u_n) \longmapsto (u_1, \dots, u_k, 0, \dots, 0).$$

# Submanifolds

Looking at the above local structure of a holomorphic map of complex manifold, the definition of submanifold is the most natural one could think of: a subset $Y \subset X$ is a *submanifold* of dimension $k$ if locally around each point of $Y$ we can find a chart $(U, z)$ such that we have

$$z(U \cap Y) = \{u \in z(U)\colon\ u_{k+1} = 0, \dots, u_n = 0\}.$$

Now, when we have a holomorphic map $f$ of complex manifolds, there are two fundamental operations that we can study: the image and the fibres of $f$. The natural question is then under which circumstances these two operations preserve the manifold structure. Assume $\dim X \geq \dim Y$ and let

$$f\colon X \longrightarrow Y.$$

If $f$ has maximal rank $\operatorname{rk} f = \dim Y$ at $b \in f(X)$ then $f^{-1}(b)$ turns out to be a submanifold of $X$ of the expected dimension $\dim X - \dim Y$.

As for the other case, assume still that $\dim X \geq \dim Y$ and let

$$f\colon Y \longrightarrow X$$

be a holomorphic map. Assume moreover that $f$ is injective and $Y$ is compact. If $f$ has maximal rank $\operatorname{rk} f = \dim Y$, then $f(Y)$ turns out to be a submanifold of $X$ and $f$ is an embedding (therefore an isomorphism $Y \to f(Y)$).

# Chapter 1

# Sheaf theory

In complex geometry one frequently has to deal with functions which have various domains of definition. The notion of a sheaf gives a suitable formal setting to handle this situation. In exchange of their rather abstract and technical nature, sheaves will provide us the framework of a very general cohomology theory, which encompasses also the “usual” topological cohomology theories such as singular cohomology. Sheaf theory is a powerful tool, which allows us to unveil the links between topological and geometric properties of complex manifolds.

## 1.1 Sheaves and presheaves of abelian groups

Throughout this chapter we generally denote by $X$ a topological space and by the letters $U, V, W$ its open subsets. Moreover, if $\{U_{j_1}, \dots, U_{j_k}\}$ is a collection of open sets in $X$ we denote by $U_{j_1 \dots j_k}$ the intersection

$$U_{j_1 \dots j_k} = U_{j_1} \cap \dots \cap U_{j_k}$$

**Definition.** We say that $\mathcal{F}$ is a *presheaf of abelian groups* on $X$ if

(a) to each open subset $U \subset X$ there corresponds an abelian group $\mathcal{F}(U)$.

(b) for each inclusion of open sets $V \subset U$ there corresponds a homomorphism of groups $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ called *restriction*, such that

  (i) $\rho_U^U = \mathrm{id}$ for all $U$.

  (ii) $\rho_W^V \circ \rho_V^U = \rho_W^U$ for $W \subset V \subset U$.

By definition $\mathcal{F}(\emptyset) := 0$, the trivial group. One also writes $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$. Elements $s \in \mathcal{F}(U)$ are called *sections*. We often write $s|_V$ instead of $\rho_V^U(s)$.

Thus, in order to define a presheaf, one has to define the groups and the restrictions. In an analogous way one could talk about presheaves of vector spaces, rings, sets, etc. In the following we will simply talk of *presheaves*, always meaning presheaves of abelian groups.

**Example.** *Let $\mathcal{C}^0(U)$ be the vector space of all continous maps $f : U \to \mathbb{R}$. Then $\mathcal{F} = \mathcal{C}^0$ is a presheaf with the* natural restrictions *of maps $\rho_V^U(f) = f|_V$.*

**Example.** *For $\mathcal{F} = \mathcal{C}^0$ define the following restrictions: $\rho_U^U(f) = f$ and $\rho_V^U(f) = 0$ if $V \subsetneq U$. Then $(\mathcal{C}^0, \rho)$ is clearly a presheaf on $X$.*

The nastiness of the last example suggests us to require some more:

**Definition.** A presheaf $\mathcal{F}$ is called a *sheaf* (of abelian groups) on $X$ if it satifies the following conditions, which we will call the *sheaf axioms*:

(I) *Local identity:* If $\{U_j\}$ is a collection of open sets in $X$ and $U = \bigcup U_j$ then $s, t \in \mathcal{F}(U)$ and $s|_{U_j} = t|_{U_j}$ for all $j$ implies $s = t$.

(II) *Glueing:* If $\{U_j\}$ is a collection of open sets in $X$ and $U = \bigcup U_j$ then for any collection of sections $s_j \in \mathcal{F}(U_j)$ with $s_j|_{U_{ij}} = s_i|_{U_{ij}}$ for all $i, j$ there always exists a global section $s \in \mathcal{F}(U)$ such that $s|_{U_j} = s_j$ for all $j$.

The sections $s_j$ as in (II) are called *compatible* and $s$ is called the *glueing* of the sections $s_j$. By (I) $s$ is unique. Thus we can summarize (I) and (II) as:

In a sheaf there exists a unique glueing for all compatible sections

*Remark* 1.1.1. By linearity of the restrictions we get the following equivalence

$$\text{(I)} \iff s \in \mathcal{F}(U) \text{ with } s|_{U_j} = 0 \text{ for all } j \text{ implies } s = 0$$

which we'll use more often to check that a presheaf satisfies the first sheaf axiom.

**Example.** *$\mathcal{C}^0$ with the natural restrictions is a sheaf on $X$. In fact it clearly satisfies (I). As for the glueing axiom, suppose $f_j : U_j \to \mathbb{R}$ are continous and $f_i|_{U_{ij}} = f_j|_{U_{ij}}$. On $U = \bigcup U_j$ define the glueing $f(x) := f_j(x)$ for $x \in U_j$. We only need to check that $f \in \mathcal{C}^0(U)$. Let $B \subset \mathbb{R}$ be open. Then, since $f|_{U_j} = f_j$,*

$$f^{-1}(B) = \bigcup_j f^{-1}(B) \cap U_j = \bigcup_j f_j^{-1}(B)$$

*so $f$ is continous.*

**Example.** *In a similar fashion we see that if $X$ is a smooth manifold one has the sheaf of smooth functions $\mathcal{C}^\infty$, where*

$$\mathcal{C}^\infty(U) = \{f : U \to \mathbb{R} : \ f \text{ smooth}\}$$

*the sheaf of smooth real-valued p-forms $\mathcal{E}^p$, where*

$$\mathcal{E}^p(U) = \{\text{smooth real p-forms on } U\}$$

*If $X$ is also a complex manifold we have the sheaf of holomorphic functions $\mathcal{O}$,*

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} : \ f \text{ holomorphic}\}$$

*and the sheaf of holomorphic p-forms $\Omega^p$, and so on. We can also consider the sheaf $\mathcal{O}^*$ of non vanishing (or "invertible") holomorphic functions*

$$\mathcal{O}^*(U) = \{f : U \to \mathbb{C}^* : \ f \text{ holomorphic}\}$$

*where we are considering $\mathcal{O}^*(U)$ as a group under multiplication of functions.*

**Example.** *A presheaf that doesn't satisfy the local identity axiom: for a presheaf* $\mathcal{F}$*, redefine* $\rho_V^U = 0$ *for all* $V \subsetneq U$*. Then the local identity axiom does not hold*[1]*.*

**Example.** *A presheaf that doesn't satisfy the glueing axiom: Let* $X$ *be a topological space for which there exist two open disjoint subsets* $U_1, U_2$*. Let* $G$ *be a non trivial abelian group. Consider the presheaf* $\mathcal{G}$ *of constant maps*

$$\mathcal{G}(U) = \{f : U \to G : \ f \ \textit{constant}\}$$

*with the natural restrictions. Let* $a_1, a_2 \in G$ *be two distinct elements and define, for* $i = 1, 2$ *the maps* $f_i \in \mathcal{G}(U_i)$ *as* $f_i(x) = a_i$ *for all* $x \in U_i$*. Since* $U_1 \cap U_2 = \emptyset$ *we have* $f_i|_{U_{12}} = 0$ *(as* $\mathcal{G}(\emptyset) = 0$*). Then on* $U = U_1 \cup U_2$ *there can be no glueing* $f \in \mathcal{G}(U)$ *of* $f_1, f_2$ *because* $f$ *must be constant and* $a_1 \neq a_2$*.*

## Stalk of a presheaf

The stalk of a sheaf is a useful construction capturing the behaviour of a sheaf around a given point. Although sheaves are defined on open sets, the underlying topological space $X$ consists of points. It is reasonable to attempt to isolate the behavior of a sheaf at a single fixed point $a \in X$. Conceptually speaking, we do this by looking at small neighborhoods of the point. If we look at a sufficiently small neighborhood of $a$, the behavior of a sheaf $\mathcal{F}$ on that small neighborhood should be the same as the behavior of $\mathcal{F}$ at that point. Of course, no single neighborhood will be small enough, so we will have to take a limit of some sort. This construction is general and it is called *direct limit.* It goes as follows.

Let $\mathcal{F}$ be a presheaf on a topological space $X$. For $a \in X$ we consider the family of groups $\mathcal{F}(U)$ for which $U \ni a$. On the disjoint union of this groups we introduce an equivalence relation: for $s \in \mathcal{F}(U), t \in \mathcal{F}(V)$ we let

$$s \sim t \iff \exists W \subset (U \cap V) \text{ such that } \ s|_W = t|_W$$

In other words we consider equivalent those sections that coincide *locally.*

**Definition.** The *stalk* of the presheaf $\mathcal{F}$ at $a \in X$ is the group

$$\mathcal{F}_a := \varinjlim_{U \ni a} \mathcal{F}(U) := \bigsqcup_{U \ni a} \mathcal{F}(U) / \sim$$

An element in $\mathcal{F}_a$ is called the *germ* of a section of $\mathcal{F}$. The germ of $s \in \mathcal{F}(U)$ will be denoted by $s_a$. The germ of a section is represented by a pair $(U, s)$. For this reason when we want to keep track of $U$ for a germ we also write

$$s_a = \langle U, s \rangle \in \mathcal{F}_a$$

*Remark* 1.1.2. $\mathcal{F}_a$ is actually a group: let $s_a = \langle U, s \rangle, t_a = \langle V, t \rangle$. We define

$$s_a + t_a := \langle U \cap V, s|_{U \cap V} + t|_{U \cap V} \rangle$$

Let's check it's well defined: let $s_a = \langle U', s' \rangle, t_a = \langle V', t' \rangle$. Then there exists $W \subset (U \cap U' \cap V \cap V')$ such that $s|_W = s'|_W$ and $t|_W = t'|_W$ since $s \sim s'$, $t \sim t'$. Thus $\langle U \cap V, s|_{U \cap V} + t|_{U \cap V} \rangle = \langle W, s|_W + t|_W \rangle = \langle U' \cap V', s'|_{U' \cap V'} + t'|_{U' \cap V'} \rangle$.

[1] unless $\mathcal{F}$ is the trivial sheaf: $\mathcal{F}(U) = 0$ for all $U$

**Example** (germs of holomorphic functions)**.** *Let $X$ be a complex manifold and consider its sheaf $\mathcal{O}$ of holomorphic functions. Let $a \in U \subset X$ with a local chart $z : U \to \mathbb{C}^n$, with $z(a) = 0$. Let $f_a \in \mathcal{O}_a$ be the germ of a holomorphic function $f \in \mathcal{O}(V)$. Then $f_a = \langle V, f \rangle = \langle V \cap U, f|_{V \cap U} \rangle$. Moreover $f$ has a convergent power series expansion about $a$: for all $x$ in some $W \subset V \cap U$ we can write*

$$f(x) = \sum_{\nu_1,\dots,\nu_n=0}^{\infty} c_{\nu_1\dots\nu_n} z_1(x)^{\nu_1} \cdots z_n(x)^{\nu_n}$$

*In particular $f_a = \langle W, f|_W \rangle$, so $f_a$ is represented by a convergent power series. Conversely, two holomorphic functions on neighborhoods of $a$ determine the same germ at $a$ precisely if their series expansion about $a$ coincide. Thus there is an isomorphism of groups (in fact, of rings)*

$$\mathcal{O}_a \simeq \{\textit{convergent power series about } 0 \in \mathbb{C}^n\}$$

*Remark* 1.1.3. For any $U$ the map $\mathcal{F}(U) \to \mathcal{F}_a$, $s \mapsto s_a$ which assigns to each section its equivalence class is a homomorphism of abelian groups. It is written

$$\rho_a^U(s) := s_a$$

In fact if $s, t \in \mathcal{F}(U)$ then $s_a + t_a = (s + t)_a$.

**Proposition 1.1.1.** *Let $\mathcal{F}$ be a sheaf on $X$ and $s \in \mathcal{F}(U)$. Then*

$$s = 0 \iff s_a = 0 \textit{ for all } a \in U$$

*Proof.* Suppose $s_a = 0$ for all $a \in U$. Then $s_a = \langle U, s \rangle = \langle U, 0 \rangle$. So there is a small neighborhood $W_a$ of $a$ such that $s|_{W_a} = 0|_{W_a} = 0$. By the local identity axiom (I) it follows $s = 0$, as $U = \bigcup W_a$ and $s|_{W_a} = 0$ for all $a$. □

## 1.2 Homomorphisms and sheafification

Let $\mathcal{F}, \mathcal{G}$ be two presheaves (of abelian groups) on $X$.

**Definition.** A *homomorphism of (pre)sheaves* (or just *morphism*) is a collection of homomorphisms of groups $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any $U \subset X$ open subset, which are *compatible* with the restrictions: the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V) \end{tikzcd}$$

for all $V \subset U$ open (the vertical maps are the restriction maps of $\mathcal{F}$ and $\mathcal{G}$). In other words, we can always write

$$\alpha_U(s)|_V = \alpha_V(s|_V)$$

If $\alpha_U$ is injective for all $U$ we say that $\mathcal{F}$ is a sub(pre)sheaf of $\mathcal{G}$.

*Remark* 1.2.1. A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ induces a homomorphism on the stalks

$$\alpha_a : \mathcal{F}_a \longrightarrow \mathcal{G}_a \quad \langle U, s\rangle \mapsto \langle U, \alpha_U(s)\rangle \quad \text{(i.e. } \alpha_a(s_a) = \alpha_U(s)_a)$$

In fact we only need to check that this is well defined: if $\langle U, s\rangle = \langle V, t\rangle$ then there exists $W \subset (U \cap V)$ such that $s|_W = t|_W$. Thus

$$\langle U, \alpha_U(s)\rangle = \langle W, \alpha_U(s)|_W\rangle = \langle W, \alpha_W(s|_W)\rangle = \langle W, \alpha_W(t|_W)\rangle = \langle V, \alpha_V(t)\rangle$$

Thus we get a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}_a \arrow[r, "\alpha_a"] & \mathcal{G}_a
\end{tikzcd}$$

**Definition.** A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is called *isomorphism* of presheaves if there exists a morphism $\beta : \mathcal{G} \to \mathcal{F}$ such that $\beta \circ \alpha = \mathrm{id}_{\mathcal{F}}$ and $\alpha \circ \beta = \mathrm{id}_{\mathcal{G}}$. In other words $\alpha_U$ is an isomorphism of groups for all $U$.

*Remark* 1.2.2. The stalks $\mathcal{F}_a$ and $\mathcal{G}_a$ of two sheaves can be isomorphic for all $a \in X$ without having $\mathcal{F}$ and $\mathcal{G}$ isomorphic, of course. In other words two locally isomorphic sheaves are not isomorphic (think of vector bundles!). A conceptual explanation for why this is untrue is as follows: a sheaf consists of local data plus some global data specifying how the local data fit together. Even if all of the local data of two sheaves are isomorphic, there is no reason to believe that those isomorphisms can be fit together in a compatible way. This is why we require that the isomorphisms on stalks arise from a map that is already a morphism of sheaves: this exactly says that the data fit together in the proper way. Simply having isomorphisms "pointwise" is not enough. The isomorphisms must also commute with the restriction maps.

**Example** (exterior derivative)**.** *If $X$ is a complex manifold let*

$$\mathcal{A}^p = \{\textit{smooth complex p-forms}\}$$

*Then $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$, $\omega \mapsto d\omega$ is a morphism of sheaves.*

**Example.** $\mathcal{C}^\infty \hookrightarrow \mathcal{C}^0$ *on a complex manifold $X$ is a sheaf morphism.*

**Example.** *Let $X = \mathbb{R}$. Fixing $h \in \mathcal{C}^\infty(\mathbb{R})$ we can define $\alpha : \mathcal{C}^\infty \to \mathcal{C}^\infty$ as the sheaf morphism $\alpha_U(f) = hf$ for all $U$. Another one: $\beta : \mathcal{C}^\infty \to \mathcal{C}^\infty$, $\beta_U(f) = f'$.*

*Remark* 1.2.3. Suppose $s_a$ has representative $\tilde{s} \in \mathcal{F}(U)$ and $t_a = \alpha_a(s_a)$ has representative $\tilde{t} \in \mathcal{G}(V)$. Then we can always find a small open set $W$ such that $s_a$ and $t_a$ have representatives $s \in \mathcal{F}(W), t \in \mathcal{G}(W)$ satisfying

$$t = \alpha_W(s)$$

In fact $t_a = \langle V, \tilde{t}\rangle = \langle U, \alpha_U(\tilde{s})\rangle$ so there is $W \subset U \cap V$ such that $(s := \tilde{s}|_W)$

$$t := \tilde{t}|_W = \alpha_U(\tilde{s})|_W = \alpha_U(\tilde{s}|_W) = \alpha_U(s)$$

**Definition.** Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves on $X$. We say that a sequence of morphisms

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is *exact* if it is exact on the stalks: for all $a \in X$ the following sequence is exact

$$\mathcal{F}_a \xrightarrow{\alpha_a} \mathcal{G}_a \xrightarrow{\beta_a} \mathcal{H}_a$$

In particular we call $\alpha : \mathcal{F} \to \mathcal{G}$ *injective/surjective* if it is such on the stalks.

*Remark* 1.2.4. We *do not* require $\mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ to be exact.

**Example.** *It is possible to have a sheaf morphism $\alpha : \mathcal{F} \to \mathcal{G}$ surjective on the stalks (so $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \to 0$ exact) but with $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$ not surjective. Consider the punctured plane $X = \mathbb{C}^*$ and the sheaves $\mathcal{O}$ and $\mathcal{O}^*$ on $X$. Let*

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*, \quad f \longmapsto e^f$$

*Now $\exp_X : \mathcal{O}(X) \to \mathcal{O}^*(X)$ is not surjective: $\mathrm{id} : \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z$ does not admit a global logarithm (no $f = \log(z)$ as a single-valued function on $\mathbb{C}^*$). On the other hand $\exp_a : \mathcal{O}_a \to \mathcal{O}_a^*$ is surjective for any $a \in X$: let $g_a \in \mathcal{O}_a^*$. Then $g_a = \langle U, g\rangle$ with $U$ a small open ball around $a$ and $g : U \to \mathbb{C}^*$ holomorphic. As $U$ is simply connected and $g$ is non vanishing we get a well defined[2] holomorphic function $f := \log(g)$ on $U$. Hence $\exp_U(f) = g$, so $\exp_a(f_a) = g_a$.*

**Proposition 1.2.1.** *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves on $X$. Then*

*(i) Let $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ be an exact sequence. For any $U \subset X$ open,*

$$\mathrm{Im}(\alpha_U) \subset \ker(\beta_U).$$

*(ii) Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ be an exact sequence. For any $U \subset X$ open,*

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$$

*is an exact sequence.*

*Proof.* (i) Let $f \in \mathcal{F}(U)$, some $U$. Since for all $a \in X$ we have $\mathrm{Im}(\alpha_a) = \ker(\beta_a)$ we get $(\beta_U(\alpha_U(f)))_a = \beta_a(\alpha_U(f)_a) = \beta_a \circ \alpha_a(f_a) = 0$. In other words, the germ of $\beta_U(\alpha_U(f)) \in \mathcal{H}(U)$ at $a$ is zero for all $a \in U$. Since $\mathcal{H}$ is a sheaf, by the local identity axiom (I) it follows $\beta_U(\alpha_U(f)) = 0$.
(ii) Exactness in $\mathcal{F}(U)$: let $f \in \mathcal{F}(U)$ with $\alpha_U(f) = 0$. Then on all the stalks $\alpha_a(f_a) = 0$, thus $f_a = 0$ for all $a$ since $\alpha_a$ is injective. Hence $f = 0$ by (I).
Exactness in $\mathcal{G}(U)$: by the first part of the proposition we know that $\alpha_U \circ \beta_U = 0$, thus $\mathrm{Im}(\alpha_U) \subset \ker(\beta_U)$. Let's prove the other inclusion. Let $g \in \mathcal{G}(U)$ with $\beta_U(g) = 0$. Then $0 = \beta_U(g)_a = \beta_a(g_a)$. Thus $g_a \in \ker(\beta_a) = \mathrm{Im}(\alpha_a)$. Hence $g|_{W_a} = \alpha_{W_a}(f_{W_a})$ for some $W_a \ni a$ and $f_{W_a} \in \mathcal{F}(W_a)$. The collection $\{W_a\}_{a \in U}$ is a covering of $U$ and on $W_{ab} := W_a \cap W_b$ we get symmetrically

$$g|_{W_{ab}} = (g|_{W_a})|_{W_{ab}} = \alpha_{W_a}(f_{W_a})|_{W_{ab}} = \alpha_{W_{ab}}(f_{W_a}|_{W_{ab}})$$
$$g|_{W_{ab}} = (g|_{W_b})|_{W_{ab}} = \alpha_{W_b}(f_{W_b})|_{W_{ab}} = \alpha_{W_{ab}}(f_{W_b}|_{W_{ab}})$$

[2] fixing $\log(z) := \log|z| + i\arg(z)$

Hence $f_{W_a}|_{W_{ab}} = f_{W_b}|_{W_{ab}}$ as $\alpha_U$'s injective. By the glueing axiom on $\mathcal{F}$ there must be $f \in \mathcal{F}(U)$ such that $f|_{W_a} = f_{W_a}$. On the other hand, for all $W_a$

$$\alpha_U(f)|_{W_a} = \alpha_{W_a}(f|_{W_a}) = g|_{W_a}$$

which implies $\alpha_U(f) = g$ by the local identity axiom on the sheaf $\mathcal{G}$. $\square$

**Proposition 1.2.2.** *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then $\alpha$ is an isomorphism of sheaves if and only if $\alpha_a$ is an isomorphism for all $a \in X$.*

*Proof.* Let $\alpha_a : \mathcal{F}_a \to \mathcal{G}_a$ be an isomorphism for all $a$. This is equivalent to

$$0 \to \mathcal{F}_a \xrightarrow{\alpha_a} \mathcal{G}_a \to 0$$

exact for all $a \in X$. Thus $0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \to 0$ is exact for all $U$ by the previous proposition. In other words $\alpha_U$ is an isomorphism for all $U$. $\square$

## Kernel, Image and Quotient sheaves

Let $\mathcal{F}, \mathcal{G}$ be sheaves on $X$ and $\alpha : \mathcal{F} \to \mathcal{G}$ a morphism. Then we get a subsheaf of $\mathcal{F}$: for each group $\mathcal{F}(U)$ we can consider its subgroup

$$\ker(\alpha)(U) := \ker\{\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)\}$$

and using the same restrictions existing on $\mathcal{F}$ we get the *kernel sheaf* $\ker(\alpha)$.

*Remark* 1.2.5. $\ker(\alpha)$ is actually a sheaf. In fact let $U = \bigcup U_j$ be a covering of an open subset $U \subset X$. The first axiom is obviously satisfied. The gluing axiom is also valid: let $s_j \in \ker(\alpha)(U_j) = \ker(\alpha_{U_j})$ be such that $s_j|_{U_{ij}} = s_i|_{U_{ij}}$ for all $i, j$. As $\ker(\alpha_{U_j}) \subset \mathcal{F}(U_j)$ and $\mathcal{F}$ is a sheaf we know that there exists a (unique) section $s \in \mathcal{F}(U)$ such that $s|_{U_j} = s_j$. We only need to prove $s \in \ker(\alpha_U)$. This holds because $\mathcal{G}$ is a sheaf and thus $\alpha_U(s)|_{U_j} = \alpha_U(s_j) = 0$ implies $\alpha_U(s) = 0$.

*Remark* 1.2.6. Note that we get an exact sequence of sheaves by inclusion

$$0 \to \ker(\alpha) \hookrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$$

Analogously, one can consider the subpresheaf of $\mathcal{G}$ given by the family of subgroups $\mathrm{Im}(\alpha)(U) := \mathrm{Im}(\alpha_U) \subset \mathcal{G}(U)$. However $\mathrm{Im}(\alpha)$ is not a sheaf!

**Example.** *Let $X = \mathbb{C}^*$ and consider $\exp : \mathcal{O} \to \mathcal{O}^*$. As we have already seen $\mathcal{O}^*(\mathbb{C}) \ni \mathrm{id} \notin \mathrm{Im}(\exp_{\mathbb{C}})$. However, using the* log *on a family of disks $U_j$ of radius $j$ around the origin we see that the sections $t_j := \mathrm{id}\,|_{U_j} \in \mathcal{O}^*(U_j)$ are such that there exist $s_j \in \mathcal{O}(U_j)$ with $t_j = \exp(s_j)$. Since $t_i|_{U_{ij}} = t_j|_{U_{ij}}$ and $\mathcal{O}^*$ is a sheaf there is a unique gluing $s \in \mathcal{O}^*(\mathbb{C})$. But $s = \mathrm{id} \notin \mathrm{Im}(\exp_{\mathbb{C}})$. Thus the presheaf* $\mathrm{Im}(\exp)$ *fails to satisfy the glueing axiom.*

Analogously, one can consider $\mathrm{coker}(\alpha)$ as the subpresheaf of $\mathcal{G}$ given by

$$\mathrm{coker}(\alpha)(U) := \mathrm{coker}(\alpha_U) = \frac{\mathcal{G}(U)}{\mathrm{Im}(\alpha_U)}$$

which also fails to be a sheaf in general. The cokernel sheaf is important as it is the starting point for the construction of the *quotient sheaf* $\mathcal{G}/\alpha(\mathcal{F})$. Precisely, the quotient sheaf is defined as the sheaf generated by the presheaf $\mathrm{coker}(\alpha)$. In the same manner, the *image sheaf* is conveniently defined to be the sheaf generated by the presheaf $\mathrm{Im}(\alpha)$. But, first of all, we need to know what a "sheaf generated by a presheaf" actually is. This leads to *sheafification.*

## Sheafification

Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space $X$. We want to show that there exists a sheaf $\mathcal{F}^+$ on $X$ and a morphism $\tau : \mathcal{F} \to \mathcal{F}^+$ with the following universal property: if $\mathcal{G}$ is a sheaf on $X$ and $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism then there exists a unique morphism $\alpha^+ : \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\tau"] \arrow[dr, "\alpha"'] & & \mathcal{F}^+ \arrow[dl, dashed, "\exists!"] \\
& \mathcal{G} &
\end{tikzcd}$$

In particular, the pair $(\mathcal{F}^+, \tau)$ is unique (up to isomorphism). Hence we just get $\mathcal{F}^+ = \mathcal{F}$, if $\mathcal{F}$ is a sheaf.

$$\mathcal{F}^+(U) := \left\{ s : U \longrightarrow \bigcup_{a \in U} \mathcal{F}_a \quad \text{satifying (i) and (ii)} \right\}$$

(i) $s$ preserves the stalks: $s(a) \in \mathcal{F}_a$ for all $a \in U$.

(ii) $s$ is locally a section of $\mathcal{F}$: any point $a \in U$ has a neighborhood $V_a \subset U$ and a section $t \in \mathcal{F}(V_a)$ such that $s(b) = t_b$ for all $b \in V_a$.

We define restrictions on $\mathcal{F}^+$ as the natural restrictions of maps: $s \mapsto s|_V$ for all $V \subset U$. First of all we claim that this is a sheaf on $X$. In fact:

(I) Local identity holds: let $s \in \mathcal{F}^+(U)$ and $s|_{U_j} = 0$ on a covering of $U$. Then $s(x) = 0$ for all $x \in U$. Thus $s = 0$.

(II) Gluing axiom holds: let $s_j \in \mathcal{F}^+(U_j)$ with $s_j|_{U_{ij}} = s_i|_{U_{ij}}$ on all the intersections $U_{ij}$ of a covering of $U \subset X$. Define a section $s \in \mathcal{F}^+(U)$ as the most reasonable one: $s(x) := s_j(x)$ for $x \in U_j$. The local conditions of the $s_j$'s make it well defined. Moreover it is the glueing by definition. We have to show that $s$ is actually a section of $\mathcal{F}(U)$. Condition (i) is obvious. Also (ii) is clearly satisfied as $s$ is locally equal to some $s_j \in \mathcal{F}^+(U_j)$.

There is a natural morphism $\tau : \mathcal{F} \to \mathcal{F}^+$. Let $f \in \mathcal{F}(U)$. Define

$$f^+ : U \longrightarrow \bigcup \mathcal{F}_a \quad a \longmapsto f_a$$

Then $f^+ \in \mathcal{F}^+(U)$ for (i) is obvious and (ii) holds with $V_a = U$ and $t = f$. Put $\tau_U(f) = f^+$. Then $\tau$ is clearly a morphism of presheaves. Now let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism. Define $\alpha^+ : \mathcal{F}^+ \to \mathcal{G}$ as follows. Let $s \in \mathcal{F}^+(U)$. Then by definition there exist neighborhoods $V_a \subset U$ for all $a \in U$ as in (ii), i.e. with some sections $t = t(a) \in \mathcal{F}(V_a)$ such that $s(b) = t(a)_b$ for all $b \in V_a$. Define $\alpha^+_U(s) = g \in \mathcal{G}(U)$ where $g|_{V_a} = \alpha_{V_a}(t)$.

*Remark* 1.2.7. The induced $\tau_a : \mathcal{F}_a \to \mathcal{F}^+_a$ is an isomorphism of groups. So by proposition 1.2.2 if $\mathcal{F}$ is a sheaf then $\tau$ is an isomorphism of sheaves.

**Definition.** Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define the *quotient sheaf* $\mathcal{G}/\alpha(\mathcal{F})$ as the sheafification of $\operatorname{coker}(\alpha)$.

*Remark* 1.2.8. We have an exact sequence of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \arrow[r, "q^+"] \arrow[d, "q"'] & \mathcal{G}/\alpha(\mathcal{F}) \arrow[r] & 0 \\ & \operatorname{coker}(\alpha) \arrow[ur, "\tau"'] & & \end{tikzcd}$$

where $q$ is the projection on the quotient. In particular we note that $q_a^+$ is surjective and induces an isomorphism

$$\frac{\mathcal{G}_a}{\alpha_a(\mathcal{F}_a)} \simeq (\mathcal{G}/\alpha(\mathcal{F}))_a$$

## 1.3 Sheaf Cohomology

In this section we develop the basic constructions of the cohomological theory of sheaves. We do so by means of the notion of soft sheaves.

### Soft sheaves

Here we assume the topological space $X$ to be Haussdorff and *paracompact*. The latter means that every open covering $\{U_j\}$ of $X$ has a subcovering $\{V_j\}$ which is *locally finite*: every point $x \in X$ admits a neighborhood $W$ which intersects only finitely many $V_j$'s.

*Remark* 1.3.1. If $X$ is a smooth manifold (thus Haussdorff and paracompact) then every open covering $\{U_j\}$ admits a *partition of unity*. This is a collection of smooth maps $\varphi_j : X \to [0, 1]$ such that[3]

1. $\mathrm{Supp}(\varphi_j) \subset U_j$.
2. $\{\mathrm{Supp}(\varphi_j)\}$ is a locally finite (closed) cover of $X$.
3. $\sum_j \varphi_j(x) = 1$ for all $x \in X$.

*Remark* 1.3.2. If $\{S_i\}_{i \in I}$ is a closed, locally finite cover of $X$ and $J \subset I$, then

$$S_J := \bigcup_{j \in J} S_j$$

is closed. In fact if $x \in X \setminus S_J$ let $W$ be a neighborhood of $x$ such that $W \cap S_j \neq \emptyset$ only for $j = i_1, \ldots, j_N \in J$. Then $S_W = S_{j_1} \cup \cdots \cup S_{j_N}$ is closed and we have $W \cap S_J = W \cap S_W$.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. For any $K \subset X$ closed we want to define a group $\mathcal{F}(K)$ as a direct limit over the open subsets $U \supset K$. Thus we need to set an equivalence relation as follows.
If $K \subset U_1 \cap U_2$ with $U_i$ open and $f_i \in \mathcal{F}(U_i)$ we put $f_1 \sim f_2$ if and only if there is $W$ open such that $K \subset W \subset U_1 \cap U_2$ and $f_1|_W = f_2|_W$. We thus define

$$\mathcal{F}(K) := \varinjlim_{U \supset K} \mathcal{F}(U) = \bigsqcup_{U \supset K} \mathcal{F}(U) \big/ \sim$$

[3] note that the sum in 3. is always finite by 2.

If $U \supset K$ and $f \in \mathcal{F}(U)$ we write $f|_K \in \mathcal{F}(K)$ for the equivalence class of $f$.

*Remark* 1.3.3. If $K = \{a\}$ this is nothing new: $\mathcal{F}(\{a\}) = \mathcal{F}_a$. So what we have done here is a generalization of the stalk construction to *all* closed sets.

**Definition.** A sheaf $\mathcal{F}$ on $X$ is called *soft* if any section over any closed subset of $X$ can be extended to a global section. In other words for any $K \subset X$ closed the *restriction* $\mathcal{F}(X) \to \mathcal{F}(K)$, $f \mapsto f|_K$ is surjective. Thus, given $K$ and a section $g|_K \in \mathcal{F}(K)$ with representative $g \in \mathcal{F}(U)$ where $K \subset U$, there is $W$ open, $K \subset W \subset U$ and there exists $f \in \mathcal{F}(X)$ with $f|_W = g|_W$.

**Proposition 1.3.1.** *Let $X$ be a smooth manifold and let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

*be an exact sequence of sheaves on $X$. If $A$ is soft then we get an exact sequence:*

$$0 \to A(X) \xrightarrow{\alpha_X} B(X) \xrightarrow{\beta_X} C(X) \to 0$$

*Sketch of the proof.* By proposition 1.2.1 what remains to be proved is that $\beta_X$ is surjective. Let $c \in C(X)$. Since $\beta_a : B_a \to C_a$ is surjective for all $a \in X$ we have $c_a \in \mathrm{Im}(\beta_a)$. Hence there is an open cover $\{U_i\}$ of $X$ and local sections $b_i \in B(U_i)$ such that $\beta_{U_i}(b_i) = c|_{U_i}$. Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$ and let $S_i = \mathrm{Supp}(\varphi_i) \subset U_i$. Locally $\sum \varphi_i = 1$, thus $\{S_i\}$ is a covering of $X$ (closed and locally finite). So $b_i|_{S_i} \in B(S_i)$ are such that

$$\beta_{S_i}(b_i|_{S_i}) = c|_{S_i}$$

where we have naturally set $\beta_{S_i}(b_i|_{S_i}) := \beta_{U_i}(b_i)|_{S_i}$. By Zorn's lemma we can pick the *maximal* $S$ of the $S_i$'s on which there is $b \in B(S)$ with $\beta_S(b) = c|_S$. What remains to be proved is that $S = X$ and this is left to the reader. □

**Corollary 1.3.1.** $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *exact with $A, B$ soft implies $C$ soft.*

## The Canonical resolution

From now on, we assume $X$ to be a smooth manifold.

**Definition.** Let $\mathcal{F}$ be a sheaf on $X$. A family $\{\mathcal{F}^q\}_{q\in\mathbb{N}}$ of sheaves on $X$ together with a family of morphisms $d^q : \mathcal{F}^q \to \mathcal{F}^{q+1}$, is called *resolution* of $\mathcal{F}$ if there exists an injection $\gamma : \mathcal{F} \to \mathcal{F}^0$ and an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \mathcal{F}^2 \longrightarrow \dots$$

Let $\mathcal{F}$ be a sheaf on $X$. We define a soft sheaf $\mathcal{D}(\mathcal{F})$ on $X$ as

$$\mathcal{D}(\mathcal{F})(U) := \left\{ t : U \longrightarrow \bigcup_{a\in U} \mathcal{F}_a \mid t(a) \in \mathcal{F}_a \right\}$$

The restrictions are the natural restrictions of maps. $\mathcal{D}(\mathcal{F})$ is called the *sheaf of discontinous sections of* $\mathcal{F}$. Note how its construction is very similar to the sheafification, except that now we start from a sheaf and we do not require the sections of $\mathcal{D}(\mathcal{F})$ to be locally equal to those of $\mathcal{F}$. Let's check that any section on a closed set has a global extension. Let $K \subset X$ be closed and let $s|_K \in \mathcal{D}(\mathcal{F})(K)$ have representative $s \in \mathcal{D}(\mathcal{F})(U)$, some $U \supset K$. Put

$$f(a) := \begin{cases} s(a) & a \in U \\ 0 & a \in X \setminus U \end{cases}$$

thus $f \in \mathcal{D}(\mathcal{F})(X)$ and $f|_K = s|_K$. Hence $\mathcal{D}(\mathcal{F})$ is a soft sheaf, indeed.

*Remark* 1.3.4. We have an injection

$$\gamma : \mathcal{F} \longrightarrow \mathcal{D}(\mathcal{F})$$

by $\gamma_U : \mathcal{F}(U) \to \mathcal{D}(\mathcal{F})(U)$, $s \mapsto \gamma_U(s)$ where $\gamma_U(s)(a) := s_a \in \mathcal{F}_a$.

Now we can construct the so called Canonical resolution of the sheaf $\mathcal{F}$. Let

$$\mathcal{C}^0 := \mathcal{D}(\mathcal{F})$$

and $\gamma : \mathcal{F} \to \mathcal{C}^0$ the above injection. Let $\tilde{\mathcal{C}}^1 := \mathcal{C}^0/\gamma(\mathcal{F})$ be the quotient sheaf and $\alpha_0 : \mathcal{C}^0 \to \tilde{\mathcal{C}}^1$ the quotient map. Let $\mathcal{C}^1 := \mathcal{D}(\tilde{\mathcal{C}}^1)$ be the (soft) sheaf of discontinous sections of $\tilde{\mathcal{C}}^1$. Then we get an injection $\beta_0 : \tilde{\mathcal{C}}^1 \to \mathcal{C}^1$ as above. As $\gamma$ and $\beta$ are injective and $\alpha$ surjective[4] we get the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r, "\gamma"] & \mathcal{C}^0 \arrow[rr, "\beta_0 \circ \alpha_0"] \arrow[dr, "\alpha_0"'] & & \mathcal{C}^1 \\ & & & \tilde{\mathcal{C}}^1 \arrow[ur, "\beta_0"'] & \end{tikzcd}$$

By inductively repeating this construction we get exact sequences of the form

$$\begin{tikzcd} \mathcal{C}^{q-1} \arrow[rr, "d^{q-1}"] \arrow[dr, "\alpha_{q-1}"'] & & \mathcal{C}^q \arrow[rr, "d^q"] \arrow[dr, "\alpha_q"'] & & \mathcal{C}^{q+1} \\ & \tilde{\mathcal{C}}^q \arrow[ur, "\beta_{q-1}"'] & & \tilde{\mathcal{C}}^{q+1} \arrow[ur, "\beta_q"'] & \end{tikzcd}$$

where we have set $\tilde{\mathcal{C}}^{q+1} = \mathcal{C}^q/\tilde{\mathcal{C}}^q$, $\mathcal{C}^{q+1} = \mathcal{D}(\tilde{\mathcal{C}}^{q+1})$ with the corresponding projections on the quotient $\alpha_q$ and injections $\beta_q$ and we have put $d^q = \beta_q \circ \alpha_q$.

*Remark* 1.3.5. Suppose $\mathcal{F}$ is soft. Then each of the $\tilde{\mathcal{C}}^q$ is soft. In fact, as

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{C}^0 \xrightarrow{\alpha_0} \tilde{\mathcal{C}}^1 \longrightarrow 0$$

is exact then by corollary 1.3.1 follows $\tilde{\mathcal{C}}^1$ soft. Then, the exactness of

$$0 \longrightarrow \tilde{\mathcal{C}}^{q-1} \xrightarrow{\beta} \mathcal{C}^{q-1} \xrightarrow{\alpha} \tilde{\mathcal{C}}^q \longrightarrow 0$$

with $\mathcal{C}^q$ soft and $\tilde{\mathcal{C}}^{q-1}$ soft by induction, implies $\tilde{\mathcal{C}}^q$ soft.

[4]remember: this means that they are injective/surjective *on the stalks*!

**Definition.** The resolution of $\mathcal{F}$ given by

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2 \longrightarrow \dots$$

obtained as above is called the *Canonical (soft) resolution* of $\mathcal{F}$.

The Canonical resolution defines a complex[5] of abelian groups $\mathscr{C}_X$ as

$$0 \longrightarrow \mathcal{C}^0(X) \xrightarrow{d_X^0} \mathcal{C}^1(X) \xrightarrow{d_X^1} \mathcal{C}^2(X) \longrightarrow \dots$$

**Definition.** The *q-th cohomology group* of the sheaf $\mathcal{F}$ is the abelian group

$$H^q(X, \mathcal{F}) := H^q(\mathscr{C}_X) = \frac{\ker(d_X^q)}{\mathrm{Im}(d_X^{q-1})}$$

also called *q-th cohomology group of $X$ with coefficients in $\mathcal{F}$*. In particular

$$H^0(X, \mathcal{F}) := \ker(d_X^0)$$

## Properties of the cohomology groups

**Theorem 1.3.1.** *Let $\mathcal{F}$ be a sheaf on $X$. Then*[6]

$$H^0(X, \mathcal{F}) = \mathcal{F}(X)$$

*Moreover, if $\mathcal{F}$ is soft then for all $q > 0$*

$$H^q(X, \mathcal{F}) = 0$$

*Proof.* By Canonical resolution we have $0 \to \mathcal{F} \xrightarrow{\gamma} \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1$ exact, thus

$$0 \to \mathcal{F}(X) \xrightarrow{\gamma_X} \mathcal{C}^0(X) \xrightarrow{d_X^0} \mathcal{C}^1(X)$$

is exact by proposition 1.2.1. Hence $\gamma_X$ is injective and

$$\mathcal{F}(X) \simeq \gamma_X(\mathcal{F}(X)) = \ker(d_X^0) = H^0(X, \mathcal{F})$$

Suppose $\mathcal{F}$ is soft. By remark 1.3.5 we know that each $\tilde{\mathcal{C}}^q$ is soft and thus

$$0 \longrightarrow \tilde{\mathcal{C}}^q(X) \xrightarrow{(\beta_{q-1})_X} \mathcal{C}^q(X) \xrightarrow{(\alpha_q)_X} \tilde{\mathcal{C}}^{q+1}(X) \longrightarrow 0$$

is exact by proposition 1.3.1. Therefore

$$\mathrm{Im}(d_X^{q-1}) = \mathrm{Im}(\beta_{q-1})_X = \ker(\alpha_q)_X = \mathrm{Im}(d_X^q)$$ □

[5]In fact $d_X \circ d_X = 0$ follows from proposition 1.2.1 and exactness of

$$\mathcal{C}^{q-1} \longrightarrow \mathcal{C}^q \longrightarrow \mathcal{C}^{q+1}$$

[6]especially in this context people often write $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$

**Theorem 1.3.2.** *Any sheaf morphism* $f : \mathcal{F} \to \mathcal{G}$ *induces homomorphisms*

$$f_q : H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{G}),$$

*which have the following (functorial) properties:*

*(a)* $f_0 = f_X : \mathcal{F}(X) \to \mathcal{G}(X)$

*(b)* $f_q = \mathrm{id}$ *if* $\mathcal{F} = \mathcal{G}$ *and* $f = \mathrm{id}_{\mathcal{F}}$

*(c)* $(g \circ f)_q = g_q \circ f_q$ *for a sheaf morphism* $g : \mathcal{G} \to \mathcal{H}$

*Sketch of the proof.* Let's "align" the two Canonical resolutions as follows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "f"] & \mathcal{C}^0_{\mathcal{F}} \arrow[r, "d^0_{\mathcal{F}}"] \arrow[d, dotted] & \mathcal{C}^1_{\mathcal{F}} \arrow[r, "d^1_{\mathcal{F}}"] \arrow[d, dotted] & \mathcal{C}^2_{\mathcal{F}} \arrow[r] \arrow[d, dotted] & \dots \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{C}^0_{\mathcal{G}} \arrow[r, "d^0_{\mathcal{G}}"] & \mathcal{C}^1_{\mathcal{G}} \arrow[r, "d^1_{\mathcal{G}}"] & \mathcal{C}^2_{\mathcal{G}} \arrow[r] & \dots
\end{tikzcd}$$

We first construct sheaf morphisms $f^q : \mathcal{C}^q_{\mathcal{F}} \longrightarrow \mathcal{C}^q_{\mathcal{G}}$. Consider the case $q = 0$.

$$\begin{tikzcd}
\mathcal{C}^0_{\mathcal{F}}(U) \arrow[d, dotted, "f^0_U"'] & = \{s : U \longrightarrow \bigcup \mathcal{F}_a \mid s(a) \in \mathcal{F}_a\} \\
\mathcal{C}^0_{\mathcal{F}}(U) & = \{t : U \longrightarrow \bigcup \mathcal{G}_a \mid t(a) \in \mathcal{G}_a\}
\end{tikzcd}$$

As $f$ induces a homomorphism $f_a : \mathcal{F}_a \to \mathcal{G}_a$ on each stalk we set

$$f^0_U(s) := t, \quad t(a) = f_a(s(a))$$

Therefore the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\gamma_{\mathcal{F}}"] \arrow[d, "f"'] & \mathcal{C}^0_{\mathcal{F}} \arrow[d, bend left, "f^0"] \\
0 \arrow[r] & \mathcal{G} \arrow[r, "\gamma_{\mathcal{G}}"] & \mathcal{C}^0_{\mathcal{G}}
\end{tikzcd}$$

Thus $f^0$ induces a morphism of sheaves on the quotients

$$\tilde{f}^0 : \tilde{\mathcal{C}}^1_{\mathcal{F}} = \mathcal{C}^0_{\mathcal{F}} / \operatorname{Im}(\gamma_{\mathcal{F}}) \longrightarrow \mathcal{C}^0_{\mathcal{G}} / \operatorname{Im}(\gamma_{\mathcal{G}}) = \tilde{\mathcal{C}}^1_{\mathcal{G}}$$

Similarly, we get that $\tilde{f}^0$ induces a morphism $f^1 : \mathcal{D}(\tilde{\mathcal{C}}^1_{\mathcal{F}}) \to \mathcal{D}(\tilde{\mathcal{C}}^1_{\mathcal{G}})$ which, again induces a homomorphism $\tilde{f}^1$ on the quotients... and so on. The morphisms $f^q$ are such that $f^{q+1} \circ d^q_{\mathcal{F}} = d^q_{\mathcal{G}} \circ f^q$. Hence they induce homomorphisms $f_q$ on the cohomology groups[7] and these satisfy the above functorial properties. □

[7] by $f_q : H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{G}), \ [a] \longmapsto [f^q_X(a)]$

**Theorem 1.3.3.** *For each short exact sequence of sheaves*

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

*there exist homomorphisms* $\delta^q : H^q(X, \mathcal{H}) \longrightarrow H^{q+1}(X, \mathcal{F})$ *which induce the following exact sequence in cohomology*

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "f_0"] & H^0(X,\mathcal{G}) \arrow[r, "g_0"] & H^0(X,\mathcal{H}) \arrow[dll, "\delta^0"] \\
& H^1(X,\mathcal{F}) \arrow[r, "f_1"] & H^1(X,\mathcal{G}) \arrow[r, "g_1"] & H^1(X,\mathcal{H}) \arrow[dll, "\delta^1"] \\
& H^2(X,\mathcal{F}) \arrow[r, "f_2"] & H^2(X,\mathcal{G}) \arrow[r, "g_2"] & H^2(X,\mathcal{H}) \cdots
\end{tikzcd}$$

*Moreover, for each commutative diagram of sheaves with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "f"] \arrow[d, "a"] & \mathcal{G} \arrow[r, "g"] \arrow[d, "b"] & \mathcal{H} \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r, "f'"] & \mathcal{B} \arrow[r, "g'"] & \mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

*then also the following diagram in cohomology is commutative*

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "f_0"] \arrow[d, "a_0"] & H^0(X,\mathcal{G}) \arrow[r, "g_0"] \arrow[d, "b_0"] & H^0(X,\mathcal{H}) \arrow[r, "\delta^0"] \arrow[d, "c_0"] & H^1(X,\mathcal{F}) \arrow[r] \arrow[d, "a_1"] & \cdots \\
0 \arrow[r] & H^0(X,\mathcal{A}) \arrow[r, "f_0'"] & H^0(X,\mathcal{B}) \arrow[r, "g_0'"] & H^0(X,\mathcal{C}) \arrow[r, "\delta^1"] & H^1(X,\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}$$

The proof makes use of the Snake's Lemma and some usual diagram chasing. We shall see in the next section how the properties from the above three theorems characterize uniquely the cohomology groups $H^q(X, \mathcal{F})$.

## Acyclic resolutions: abstract de Rham Theorem

Soft sheaves have no cohomology (cf. theorem 1.3.1). This property is crucial, in order to compute the cohomology, as shown in the following theorem.

**Definition.** A resolution

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \longrightarrow \dots$$

is called *acyclic* if $H^q(X, \mathcal{A}^i) = 0$ for all $i \geq 0$ and $q \geq 1$.

**Example.** *The Canonical resolution is acyclic, as each* $\mathcal{C}^i$ *is soft.*

For a resolution of $\mathcal{F}$ as above let $\mathscr{A}_X$ denote the complex of global sections

$$\mathcal{F}(X) \longrightarrow \mathcal{A}^0(X) \xrightarrow{d_X^0} \mathcal{A}^1(X) \xrightarrow{d_X^1} \mathcal{A}^2(X) \xrightarrow{d_X^2} \dots$$

**Theorem 1.3.4** (abstract de Rham Theorem)**.** *Each acyclic resolution of $\mathcal{F}$ computes the sheaf cohomology. Precisely,*

$$H^q(X,\mathcal{F}) \simeq H^q(\mathscr{A}_X).$$

*Proof.* As $0 \to \mathcal{F} \xrightarrow{\gamma} \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1$ is exact then by proposition 1.2.1 we get that

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\gamma_X} \mathcal{A}^0(X) \xrightarrow{d_X^0} \mathcal{A}^1(X) \longrightarrow 0$$

is exact. Therefore $\gamma_X(\mathcal{F}(X)) = \ker(d_X^0) = H^0(\mathscr{A}_X)$. On the other hand we know that $H^0(X,\mathcal{F}) = \mathcal{F}(X)$ and also $\mathcal{F}(X) = \gamma_X(\mathcal{F}(X))$ as $\gamma$ is injective. Thus the theorem is proved for $q = 0$. We define, for $p \geq 0$, the kernel sheaves

$$\mathcal{K}^p := \ker(d^p)$$

In particular $\mathcal{K}^0 \simeq \mathcal{F}$. Moreover we get short exact sequences for all $p$

$$0 \longrightarrow \mathcal{K}^p \longrightarrow \mathcal{A}^p \xrightarrow{d^p} \mathcal{K}^{p+1} \longrightarrow 0$$

By theorem 1.3.3 we get homomorphisms $\delta^q : H^q(X,\mathcal{K}^{p+1}) \longrightarrow H^{q+1}(X,\mathcal{K}^p)$ and the induced long exact sequence in cohomology is full of zeroes:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{K}^p) \arrow[r] & H^0(X,\mathcal{A}^p) \arrow[r, "d_X^p"] & H^0(X,\mathcal{K}^{p+1}) \arrow[dll, "\delta^0"'] \\
 & H^1(X,\mathcal{K}^p) \arrow[r] & 0 \arrow[r] & H^1(X,\mathcal{K}^{p+1}) \arrow[dll, "\delta^1"'] \\
 & H^2(X,\mathcal{K}^p) \arrow[r] & 0 \arrow[r] & H^2(X,\mathcal{K}^{p+1}) \cdots
\end{tikzcd}$$

Then $\delta^q$ is an isomorphism for $q \geq 1$. As $\mathcal{F} \simeq \mathcal{K}^0$, for any $q > 1$ we get

$$H^q(X,\mathcal{F}) \simeq H^q(X,\mathcal{K}^0) \simeq H^{q-1}(X,\mathcal{K}^1) \simeq \dots \simeq H^1(X,\mathcal{K}^{q-1})$$

So it all comes down to compute $H^1(X,\mathcal{K}^p)$ for any $p$. From the long exact cohomology sequence we see that $\delta_0$ is surjective. Hence, by algebra

$$H^1(X,\mathcal{K}^p) \simeq H^0(X,\mathcal{K}^{p+1})/\ker\delta_0$$

But we know $H^0(X,\mathcal{K}^{p+1}) \simeq \mathcal{K}^{p+1}(X)$ and by exactness $\ker\delta_0 = \operatorname{Im} d_X^p$. So

$$H^1(X,\mathcal{K}^p) \simeq \mathcal{K}^{p+1}(X)/\operatorname{Im} d_X^p = \ker d_X^{p+1}/\operatorname{Im} d_X^p = H^{p+1}(\mathscr{A}_X)$$

For $p = 0$ this yields us $H^1(X,\mathcal{F}) \simeq H^1(\mathscr{A}_X)$. For $q > 1$

$$H^q(X,\mathcal{F}) \simeq H^1(X,\mathcal{K}^{q-1}) \simeq H^q(\mathscr{A}_X)$$ □

*Remark* 1.3.6. In the proof we used only the properties of the groups $H^q(X,\mathcal{F})$ and never their explicit definition via Canonical resolution. As a consequence we see that those properties determine the cohomology groups uniquely.

## 1.4 De Rham and Dolbeault theorems

Let $X$ be a smooth manifold and $\mathcal{E}^p$ be the sheaf of smooth real $p$-forms on $X$.

$$0 \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \dots$$

where $d$ is the exterior derivative is then an exact sequence of sheaves[8]. Let $\mathscr{E}_X$ denote the associated complex of global sections. We have the de Rham groups

$$H^q_{\mathrm{dR}}(X) := H^q(\mathscr{E}_X)$$

Let $\mathbb{R}$ denote the sheaf of locally constant functions on $X$. That is, the sheaf[9]

$$\ker(d : \mathcal{E}^0 \longrightarrow \mathcal{E}^1) \simeq \mathbb{R}$$

Then we have a resolution of the sheaf $\mathbb{R}$ of locally constant functions by

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \dots$$

We claim that each $\mathcal{E}^p$ is soft. Hence the resolution is acyclic and we find

$$H^q(X, \mathbb{R}) \simeq H^q_{\mathrm{dR}}(X)$$

This result is generally known as the *de Rham theorem.*

*Proof.* We show that $\mathcal{E}^p$ is soft, i.e. any section on a closed subset $K \subset X$ admits a global extension. Let $\omega|_K \in \mathcal{E}^p(K)$ with representative $\omega \in \mathcal{E}^p(U)$, where $U \supset K$ is open. Let $\{\psi_U, \psi\}$ be a partition of unity subordinate to the covering $X = U \cup (X \setminus K)$. Thus $\psi_U, \psi : X \to [0, 1]$ are smooth, $\mathrm{Supp}(\psi_U) \subset U$, $\mathrm{Supp}(\psi) \subset (X \setminus K)$ and $\psi_U(a) + \psi(a) = 1$ for all $a \in X$. Then $\psi_U \omega \in \mathcal{E}^p(U)$.

$$\mathcal{E}^p(X) \ni \tilde{\omega}(a) := \begin{cases} \psi_U(a)\omega(a) & a \in U \\ 0 & a \notin U \end{cases}$$

We note that $K \subset X \setminus \mathrm{Supp}(\psi) =: V$ which is open and such that $\psi_U(a) = 1$ for all $a \in V$. Hence $K \subset W := U \cap V$ is open and $\tilde{\omega}|_W = \omega|_W$. $\square$

In a similar fashion we can consider the sheaf $\Omega^p$ of holomorphic $p$-forms on a complex manifold $X$. Let $\mathcal{A}^{p,q}$ be the sheaf of smooth $(p, q)$-forms on $X$. As for the case of $\mathcal{E}^p$ one proves that the $\mathcal{A}^{p,q}$ are soft. The "$\bar{\partial}$-Poincaré lemma" guarantees that the following is an exact sequence of sheaves

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots$$

Which is therefore an acyclic resolution of $\Omega^p$. Hence the groups $H^q(X, \Omega^p)$ can be computed in terms of $(p, q)$-forms $\bar{\partial}$-closed modulo $(p, q)$-forms $\bar{\partial}$-exact[10]. This result is generally known as the *Dolbeault theorem.*

[8] in fact on the stalks clearly $\mathrm{Im}(d^p_a) \subset \ker(d^{p+1}_a)$. Conversely, given $\omega_a \in \ker(d^{p+1}_a)$ with representative $\omega \in \mathcal{E}^{p+1}(U)$, by Poincaré lemma there is $a \in V \subset U$ with $V$ diffeomorphic to a ball such that $\omega|_V = d\eta$, some $\eta \in \mathcal{E}^p(V)$. So $\omega_a = d^p_a \eta_a \in \mathrm{Im}(d^p_a)$

[9] $f : X \to \mathbb{R}$ with $0 = df = \sum \frac{\partial f}{\partial x_i} dx_i$ implies $\frac{\partial f}{\partial x_i} = 0$ for all $i$. So locally $f \equiv \lambda \in \mathbb{R}$

[10] in particular $H^q(X, \Omega^p) = 0$ if $q > \dim_{\mathbb{C}} X$ since in this case $\mathcal{A}^{p,q} = 0$

## 1.5 Sheaves and Algebraic Topology

We denote by $\Delta_q \subset \mathbb{R}^{q+1}$ the standard *q-simplex*

$$\Delta_q = \{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} : \ t_i \geq 0, \ t_0 + \cdots + t_q = 1\}$$

A *singular q-simplex* of a topological space $X$ is a continous map

$$\sigma : \Delta_q \longrightarrow X$$

If $X$ is also a smooth manifold then $\sigma$ is said to be *smooth* if it extends smoothly on an open neighborhood of $\Delta_q$. One then defines the group $C_q(X)$ of *singular q-chains* as the free abelian group generated by the set of $q$-simplices of $X$. In other words a $q$-chain $c \in C_q(X)$ is a finite formal sum $c = \sum n_i \sigma_i$ where the $n_i$ are integers and the $\sigma_i$ are $q$-simplices. Similarly one defines the group $C_q(X)_\infty$ of *smooth q-chains*. Note that if $\sigma$ is a $q$-simplex then its restriction $\sigma|_{t_i=0}$ is a $(q-1)$-simplex[11]. Thus one defines a homomorphism $d : C_q(X) \to C_{q-1}(X)$ as

$$d(\sigma) = \sum_{i=0}^{q} (-1)^i \sigma|_{t_i=0}$$

for each $q$-simplex $\sigma \in C_q(X)$ and then extending linearly to all of $C_q(X)$. One easily checks that $d \circ d = 0$ and so we define the *q-th singular homology group*

$$H_q(X, \mathbb{Z}) = \frac{\ker(d : C_q(X) \to C_{q-1}(X))}{\operatorname{Im}(d : C_{q+1}(X) \to C_q(X))}$$

Since each open subset $U \subset X$ is a topological space, with the above construction we similarly get the groups $C_q(U)$, by considering $q$-simplices $\sigma : \Delta_q \to U$.

Let now $G$ be an abelian group. We define the groups[12]

$$C^q(U) := \hom_{\mathbb{Z}}(C_q(U), G)$$

Note that if $V \subset U$ is open then $C_q(V) \subset C_q(U)$ as $\sigma : \Delta_q \to V \subset U$. Therefore if $f \in C^q(U)$ the inclusion $C_q(V) \hookrightarrow C_q(U)$ defines a restriction

$$\begin{tikzcd}
C_q(V) \arrow[rr, hook] \arrow[drr, dotted, "\rho^U_V(f)"'] & & C_q(U) \arrow[d, "f"] \\
& & G
\end{tikzcd}$$

In other words $U \mapsto C_q(U)$ is a presheaf of abelian groups on $X$. Let $C^q$ denote the sheaf generated by this presheaf, called the *sheaf of singular q-chains with coefficients in G*. There is a sheaf morphism $\delta : C^q \to C^{q+1}$ defined as

$$\begin{tikzcd}
C_{q+1}(U) \arrow[rr, "d_U"] \arrow[drr, dotted, "\delta_U(f)"'] & & C_q(U) \arrow[d, "f"] \\
& & G
\end{tikzcd}$$

[11] as $\Delta_q \cap \{t_i = 0\} \simeq \Delta_{q-1}$

[12] and similarly $C^q(U)_\infty$ in the case of a smooth manifold

Thus $\delta \circ \delta = 0$. Let $G$ also denote the sheaf of locally constant functions on $X$ with values in the group $G$. If $X$ is a manifold then

$$0 \longrightarrow G \longrightarrow C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} \dots$$

is an acyclic resolution. The associated sheaf cohomology groups $H^q(X, G)$ are thus isomorphic to the *singular cohomology groups of $X$ with coefficients in $G$*

$$H^q(X, G) \simeq H^q_{Sing}(X, G) = \frac{\ker(\delta_X : C^q(X) \to C^{q+1}(X))}{\mathrm{Im}(\delta_X : C^{q-1}(X) \to C^q(X))}$$

Suppose now $X$ is a smooth manifold. In the special case $G = \mathbb{R}$ we have already seen $H^q(X, \mathbb{R}) \simeq H^q_{\mathrm{dR}}(X)$. What happens is that there another acyclic resolution for $\mathbb{R}$, namely

$$0 \longrightarrow \mathbb{R} \longrightarrow C^0_\infty \xrightarrow{\delta} C^1_\infty \xrightarrow{\delta} C^2_\infty \xrightarrow{\delta} \dots$$

Therefore we obtain the following isomorphism

$$H^q_{\mathrm{dR}}(X) \simeq H^q_{Sing}(X, \mathbb{R})$$

which we explain as follows. By definition $H^q_{\mathrm{dR}}(X) = H^q(\mathscr{E}_X)$ where $\mathscr{E}_X$ is the complex of global differential $q$-forms on $X$. So the question is: *how do we get a smooth q-cochain from a differential q-form on X?* There we go:

$$I^q_U : \mathcal{E}^q(U) \longrightarrow C^q_\infty(U), \quad \omega \longmapsto I^q_U(\omega)$$

is the homomorphism defined by the linear map[13]

$$I^q_U(\omega) = \left[\sigma \longmapsto \int_{\Delta_q} \sigma^*\omega\right] \in \hom\left(C_q(U), \mathbb{R}\right)$$

so we get a morphism $I^q : \mathcal{E}^q \to C^q_\infty$ which induces the isomorphism above.

[13] note that the integral is well defined as $\Delta_q$ is compact!

# Chapter 2

# Holomorphic vector bundles

**Definition.** Let $X$ be a complex manifold. A *holomorphic vector bundle of rank* $r$ on $X$ is a complex manifold $E$ together with a surjective holomorphic map $p : E \to X$, such that:

- Each *fiber* $E_x = p^{-1}(x)$ is a complex vector space of dimension $r$.
- There is an open covering $X = \bigcup U_\alpha$ and a family of bioholomorphisms

  $$\psi_\alpha : p^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}^r$$

  which commute with the projections on $U_\alpha$, and such that the induced restrictions on the fibers $E_x \simeq \mathbb{C}^r$ are linear (hence isomorphisms).

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\simeq", "\psi_\alpha"'] \arrow[drr, "p"'] & & U_\alpha \times \mathbb{C}^r \arrow[d] \\ & & U_\alpha \end{tikzcd}$$

We sum up this situation by saying that $E$ is *locally trivial.* The maps $\psi_\alpha$ are called *trivializations* of the bundle. A *trivial bundle* is a globally trivial one, $E \simeq X \times \mathbb{C}^r$. A holomorphic vector bundle of rank $r = 1$ is called a *line bundle.*

Let $E$ be a (holomorphic) vector bundle of rank $r$ on $X$. Fixing $x \in X$ the family of trivializations defines a family of isomorphisms $g_{\alpha\beta}(x) : \mathbb{C}^r \to \mathbb{C}^r$ by

$$\begin{aligned} \psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r &\longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r \\ (x, v) &\longmapsto (x, g_{\alpha\beta}(x) \cdot v) \end{aligned}$$

called *transition functions.* The map $x \mapsto g_{\alpha\beta}(x)$ is holomorphic. The transition functions are thus invertible matrices $g_{\alpha\beta}(x) \in \mathrm{Gl}(r, \mathbb{C})$. We have

$$g_{\alpha\alpha} = I, \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \quad g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$$

A family of transition functions as above, together with the open cover $\{U_\alpha\}$ of $X$ determines uniquely[1] the vector bundle $E$. Hence we'll write

$$E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$$

[1] idea: set $E := \bigsqcup (U_\alpha \times \mathbb{C}^r)/\sim$ where $(x, v) \sim (x, w) \iff v = g_{\alpha\beta}(x) w$

## 2.1 Holomorphic sections

**Definition.** A holomorphic map $s : X \to E$ that preserves the fibers of the vector bundle (i.e. $p(s(x)) = x$) is called *section*. Sometimes a section is only defined locally $s : U \to E$ and is then called a *local section*.

For any vector bundle $E$ there always exists a global section: the so called *zero section* $x \mapsto 0 \in E_x$. For each open $U \subset X$, the set of sections $U \to E$ is naturally a complex vector space, and we denote it by

$$\Gamma(U, E) = \{\text{holomorphic sections } s : U \to E\}.$$

In particular we have the space of global sections $\Gamma(X, E)$.

Let $E \longleftrightarrow (U_\alpha, g_{\alpha\beta})$ and $s \in \Gamma(X, E)$. For any $x \in U_\alpha$ we must have

$$\psi_\alpha(s(x)) = (x, s_\alpha(x)),$$

where $s_\alpha : U_\alpha \to \mathbb{C}^r$ is holomorphic. If $x \in U_\alpha \cap U_\beta$ then $\psi_\beta(s(x)) = (x, s_\beta(x))$,

$$\begin{cases} \psi_\alpha \circ \psi_\beta^{-1}(x, s_\beta(x)) = & (x, s_\alpha(x)) \\ \psi_\alpha \circ \psi_\beta^{-1}(x, s_\beta(x)) = & (x, g_{\alpha\beta}(x) s_\beta(x)). \end{cases}$$

Hence any section $s \in \Gamma(X, E)$ defines a collection $\{U_\alpha, s_\alpha : U_\alpha \to \mathbb{C}^r\}$, where $s_\alpha$ are holomorphic maps which under a change of charts satisfy

$$s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x).$$

Conversely, a collection as such, determines uniquely a section $s \in \Gamma(X, E)$. Indeed, for any $x \in U_\alpha$ we can define

$$s(x) := \psi_\alpha^{-1}(x, s_\alpha(x)),$$

which is independent of the choice of charts: if $x \in U_\alpha \cap U_\beta$ then

$$\begin{aligned} \psi_\alpha^{-1}(x, s_\alpha(x)) &= \psi_\alpha^{-1}(x, g_{\alpha\beta}(x) s_\beta(x)) \\ &= \psi_\alpha^{-1}(\psi_\alpha \circ \psi_\beta^{-1}(x, s_\beta(x))) \\ &= \psi_\beta^{-1}(x, s_\beta(x)). \end{aligned}$$

Therefore, for a section $s \in \Gamma(X, E)$ we'll write

$$s \longleftrightarrow \{U_\alpha,\ s_\alpha : U_\alpha \to \mathbb{C}^r,\ s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x)\}$$

and call this the *local description* of $s$. This is very useful. More often, instead of working with a global section $s$ it is easier to use its local description.

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] & & U_\alpha \times \mathbb{C}^r \\ U_\alpha \arrow[u, "s"] \arrow[urr, dashed, bend right, "\mathrm{id} \times s_\alpha"'] & & \end{tikzcd}$$

## 2.2 Line bundles

We first construct two line bundles on $\mathbb{P}^n$. Let's begin with $L(1)$.

$$L(1) := \mathbb{P}^{n+1} \setminus \{(0 : \ldots : 0 : 1)\}$$

The map $p : L(1) \to \mathbb{P}^n$, $(x_0 : \ldots : x_{n+1}) \mapsto (x_0 : \ldots : x_n)$ is then well defined and holomorphic. Suppose $x, y \in L(1)$ belong to the same fiber, that is $p(x) = p(y)$. Then $x = (x_0 : \ldots : x_n : t)$ and $y = (x_0 : \ldots : x_n : s)$ for some $t, s \in \mathbb{C}$. Hence $x, y$ belong to the same line in $\mathbb{P}^{n+1}$, namely

$$\lambda x + \mu y = (x_0 : \ldots : x_n : \lambda s + \mu t)$$

and this gives the structure of a one-dimensional complex vector space on each fiber. We see the geometrical meaning of the map $p$ as the projection of this lines in $\mathbb{P}^{n+1}$ on points in $\mathbb{P}^n$. Let $\{U_j\}$ be the standard cover of $\mathbb{P}^n$. Define

$$\psi_j : p^{-1}(U_j) \longrightarrow U_j \times \mathbb{C}, \quad x \longmapsto (p(x), \tfrac{x_{n+1}}{x_j})$$

then $\psi_j \circ \psi_k^{-1}((x_0 : \ldots : x_n), z) = ((x_0 : \ldots : x_n), \frac{x_k}{x_j} z)$. Hence we have found

$$L(1) \longleftrightarrow \{U_j,\ g_{jk}(x) = \tfrac{x_k}{x_j}\}$$

We now construct the so called *tautological line bundle* $L(-1)$ on $\mathbb{P}^n$. By viewing $\mathbb{P}^n$ as the set of lines $\ell$ through the origin of $\mathbb{C}^{n+1}$ we naturally obtain a line bundle: it suffices to identify each point $\ell \in \mathbb{P}^n$ with the 1-dimensional complex vector space that the line $\ell$ itself represents when viewed as a linear subspace of $\mathbb{C}^{n+1}$. More precisely

$$L(-1) := \{(\ell, z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} :\ z \in \ell\}$$

and the natural projection on the first factor $\pi : L(-1) \to \mathbb{P}^n$, $(\ell, z) \mapsto \ell$ defines this line bundle. Again on the standard cover $\{U_j\}$ of $\mathbb{P}^n$ we have trivializations

$$\phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{C}, \quad (\ell, z) \longmapsto (\ell, z_j)$$

which also provide $L(-1)$ with local charts[2]. The tautological bundle $L(-1)$ is thus endowed with a complex structure of dimension $n + 1$. Let's find out its transition functions. Let $\ell = (x_0 : \ldots : x_n) \in U_j \cap U_k$, so $x_j, x_k \neq 0$. Then

$$\phi_j \circ \phi_k^{-1}(\ell, z_k) = (\ell, z_j)$$

and $z_j, z_k$ are coordinates on a common line $\ell$, so that $z_j = \lambda x_j$ and $z_k = \lambda x_k$ for some $\lambda \in \mathbb{C}$. Hence $z_j = \lambda \frac{x_k}{x_k} x_j = \frac{x_j}{x_k} z_k = h_{jk}(\ell) z_k$. We have found

$$L(-1) \longleftrightarrow \{U_j,\ h_{jk}(\ell) = \tfrac{x_j}{x_k}\}$$

*Remark* 2.2.1. In a trivial bundle there always exists a global section with no zeros. By this we mean a section $s : X \to E$ such that $s(x) \neq 0$ for all $x \in X$. For, if $E \simeq X \times \mathbb{C}^r$ then for any non zero vector $w \in \mathbb{C}^r$ there always exists the constant section $x \mapsto w \in E_x$. This simple remark can become very useful to show that a vector bundle is not the trivial bundle: if we are able to show that each global section admits a zero we're done!

[2] by composing $\phi_j$ with the charts of $\mathbb{P}^n$ on the first component $U_j$

Let's begin with $L(-1)$. Its *only* global section is the zero section.

**Fact.** *The only global section on $L(-1)$ is the zero section. In symbols*

$$\Gamma(\mathbb{P}^n, L(-1)) = 0$$

*Proof.* Let $s : \mathbb{P}^n \to L(-1)$ be a holomorphic section. For any $\ell \in \mathbb{P}^n$ we have $s(\ell) = (\ell, z_\ell)$ for some $z_\ell \in \mathbb{C}^{n+1}$ lying on the line $\ell$. Thus $\ell \mapsto z_\ell$ is a holomorphic map $\mathbb{P}^n \to \mathbb{C}^{n+1}$. By the maximum principle this map must be constant, so $z_\ell \equiv w \in \mathbb{C}^{n+1}$. On the other hand $s$ is fiber preserving, so $w \in \ell$ for *each* line $\ell$ through the origin of $\mathbb{C}^{n+1}$. Hence $w = 0$. □

*Remark* 2.2.2. Let's motivate the notation $L(-1)$ for the tautological bundle. Take a local section $s \in \Gamma(U, L(-1))$. Then $s(\ell) = (\ell, z_\ell)$ for each $\ell \in U$. Let

$$\varpi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$$

denote the usual projection. Say $\ell = [x]$ for some $x \in \varpi^{-1}(U)$. Since $z_\ell \in \ell$ we get $z_\ell = \lambda x$ for some $\lambda = \lambda_s(x) \in \mathbb{C}$. So each local section $s$ determines a holomorphic function

$$\lambda_s : \varpi^{-1}(U) \longrightarrow \mathbb{C}$$

which must be homogeneus of degree $-1$. In fact our construction has to be independent of the choice of $x$, so $\lambda_s(\mu x)\mu x = \lambda_s(x)x$ for any $\mu \in \mathbb{C}^*$, that is

$$\lambda_s(\mu x) = \mu^{-1}\lambda_s(x)$$

Let's now take a look at the global sections of $L(1)$, that is, the vector space

$$\Gamma(\mathbb{P}^n, L(1)) = \{s : \mathbb{P}^n \longrightarrow L(1) : s \text{ holomorphic}, \ \ p \circ s = \mathrm{id}_{\mathbb{P}^n}\}$$

First note that each homogeneus coordinate $x_j$ on $\mathbb{P}^n$ defines a section: the map

$$x_j : \ (x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n : x_j)$$

It can be showed that these form a basis for $\Gamma(\mathbb{P}^n, L(1))$, so that any global section $s$ of $L(1)$ is of the form

$$s : \ (x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n : a_0x_0 + \cdots + a_nx_n)$$

for some coefficients $a_j \in \mathbb{C}$. Note that the image of $s$ is a hyperplane in $\mathbb{P}^{n+1}$

$$s(\mathbb{P}^n) = \{(x_0 : \ldots : x_{n+1}) \in \mathbb{P}^{n+1} : \ a_0x_0 + \cdots + a_nx_n - x_{n+1} = 0\}$$

which doesn't contain the point $(0 : \ldots : 0 : 1)$. We now want to find out the local description of $s$. We see that

$$\psi_j(s(x)) = \left(x, \frac{\sum a_ix_i}{x_j}\right)$$

with the given trivializations $\psi_j$ on the standard covering $\{U_j\}$ of $\mathbb{P}^n$. Hence

$$s_j(x) = \sum_{i=0}^{n} a_i \frac{x_i}{x_j}$$

and one can check that the desired relation $s_j(x) = g_{jk}(x)s_k(x)$ actually holds.

**Definition.** Let $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ be a vector bundle of rank $r$ on $X$. We define the *dual bundle* $E^*$ as the vector bundle of rank $r$ given by

$$E^* \longleftrightarrow \{U_\alpha, {}^t g_{\alpha\beta}^{-1}\}$$

In the case of a line bundle $L$, since ${}^t g_{\alpha\beta}^{-1} = g_{\alpha\beta}^{-1}$ we use the notation $L^{-1} = L^*$.

*Remark* 2.2.3. We take transpose inverses instead of just inverses because

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{and} \quad {}^t(AB) = {}^tB\,{}^tA$$

As we have seen before $L(1)^{-1} = L(-1)$. Also, we noted that the tautological bundle $L(-1)$ is not trivial, since its only global section is the zero section. More generally, for any line bundle $L \to X$ on a compact manifold $X$ there is a strong result which characterizes global sections of $L$ and of its dual.

**Proposition 2.2.1.** *Let $L \to X$ be a holomorphic line bundle on a compact manifold $X$. Suppose $s \in \Gamma(X, L)$ is not the zero section. Then either one of the following holds:*

*(i) $L \simeq X \times \mathbb{C}$ is the trivial bundle and $s$ has no zeros.*

*(ii) $\Gamma(X, L^{-1}) = 0$ and the section $s$ admits at least one zero.*

*Proof.* Let $L \leftrightarrow \{U_\alpha, g_{\alpha\beta}\}$, $L^{-1} \leftrightarrow \{U_\alpha, g_{\alpha\beta}^{-1}\}$ and $s \leftrightarrow \{U_\alpha, s_\alpha\}$. Suppose $t \in \Gamma(X, L^{-1})$ have local description $t \leftrightarrow \{U_\alpha, t_\alpha\}$. Then

$$\begin{cases} s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x) \\ t_\alpha(x) = g_{\alpha\beta}^{-1}(x)t_\beta(x) \end{cases}$$

Hence $s_\alpha(x)t_\alpha(x) = s_\beta(x)t_\beta(x)$ for all $x \in U_\alpha \cap U_\beta$ and for all $\alpha, \beta$. We glue all this pieces and get a holomorphic function $f : X \to \mathbb{C}$ such that $f|_{U_\alpha} = s_\alpha t_\alpha$. Since $X$ is compact, by the maximum principle $f \equiv c \in \mathbb{C}$.

(i) $c \neq 0$. Then $s_\alpha(x) \neq 0$ for all $x \in U_\alpha$ for all $\alpha$. Hence $s(x) \neq 0$ on all $X$. Then $(x, \lambda) \mapsto \lambda s(x)$ is a holomorphic and invertible map $X \times \mathbb{C} \to L$.

(ii) $c = 0$. By hypothesis there is some point in $X$ where $s$ is not zero. Hence there is an open subset of $X$ where $s$ is not zero. Then it must be $t \equiv 0$ on this open subset. Since $t$ is holomorphic it follows $t \equiv 0$ everywhere. Hence $\Gamma(X, L^{-1}) = 0$, which implies that $L$ is cannot be the trivial bundle[3]. Thus $s$ must have a zero: if it were nowhere vanishing then $(x, \lambda) \mapsto \lambda s(x)$ would be a biolomorphism $X \times \mathbb{C} \to L$ as in (i). Absurd. □

We have already noted that the rank $r$ trivial bundle always admits a nowhere vanishing section. From the proof of the preceding proposition we see that in the case of line bundles the viceversa holds as well: if $s$ is a nowhere vanishing section then $(x, \lambda) \mapsto \lambda s(x)$ is a trivialization $X \times \mathbb{C} \to L$. The compactness of $X$ in this argument plays no role. Let's summarize this.

**Fact.** *A line bundle is trivial if and only if it has a nowhere vanishing section.*

[3]because the dual of the trivial bundle is a trivial bundle

There are many ways to construct new vector bundles from a given one. The dual bundle is an example. Virtually, any canonical construction in linear algebra gives rise to a geometric version for holomorphic vector bundles. We'll see more examples in the following. Now, let $L \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ be a line bundle. An idea is to take powers of $g$: for any integer $k$ define

$$L^{\otimes k} \longleftrightarrow \{U_\alpha, g^k_{\alpha\beta}\}$$

In particular $L^{\otimes -1} = L^{-1}$ and $L^0 \simeq X \times \mathbb{C}$. Suppose $s \in \Gamma(X, L)$ with local description $s \longleftrightarrow \{U_\alpha, s_\alpha : U_\alpha \to \mathbb{C}\}$. Then for any $k > 0$

$$s^k \longleftrightarrow \{U_\alpha,\ s^k_\alpha : U_\alpha \to \mathbb{C}\}$$

is a section of $L^{\otimes k}$. In fact $(s_\alpha)^k = (g_{\alpha\beta}s_\beta)^k = g^k_{\alpha\beta}s^k_\beta$

**Example.** *Let $L(k) := L(1)^{\otimes k}$. If $k > 0$ the $k$-th power of the first homogeneus coordinate $x^k_0$ defines a section of $L(k)$. Each point of the form $(0 : x_1 : \ldots : x_n)$ is a zero of $x^k_0$. More generally,*

$$(x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n : f(x_0, \ldots, x_n))$$

*where $f$ is a homogeneus polynomial of degree $k$ is a global section of $L(k)$.*

## Projective embeddings and line bundles

Let $X$ be a complex manifold that admits an embedding $i : X \hookrightarrow \mathbb{P}^n$. Identifying $X = i(X)$ for simplicity of notation, we get a line bundle on $X$

$$\begin{tikzcd} L(1)|_X \arrow[rr] \arrow[d] & & L(1) \arrow[d, "p"'] \\ X \arrow[rr, hook, "i"] \arrow[u, bend left, "s_j"] & & \mathbb{P}^n \arrow[u, bend right, "x_j"'] \end{tikzcd}$$

by restriction of $L(1)$ to $X$. Precisely $L(1)|_X = p^{-1}(i(X))$. Each section $x_j$ of $L(1)$ gives a section $s_j$ of $L(1)|_X$ by restriction: $s_j = x_{j|X}$. By construction, these $s_j$'s define in fact the embedding $i$: for $x \in X$ we have

$$i(x) = (x_0 : \ldots : x_n) = (s_0(x) : \ldots : s_n(x)).$$

Viceversa, let $p : L \to X$ be a line bundle on a complex manifold $X$ with some sections $s_0, \ldots, s_n \in \Gamma(X, L)$ such that their *base locus*

$$\{x \in X : s_0(x) = 0, \ldots, s_n(x) = 0\}$$

is empty. Then we can define a map $X \to \mathbb{P}^n$, simply by setting

$$\phi(x) := (s_0(x) : \ldots : s_n(x)).$$

Notice the abuse of notation: here $s_j(x) \in p^{-1}(x)$ and not in $\mathbb{C}$. To be precise, the map $\phi$ is defined locally: let $L \leftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ and $s_j \leftrightarrow \{U_\alpha,\ s_{j\alpha} : U_\alpha \to \mathbb{C}\}$. Then $\phi(x) := (s_{0\alpha}(x) : \ldots : s_{n\alpha}(x))$ for $x \in U_\alpha$. Since $s_{j\alpha}(x) = g_{\alpha\beta}(x)s_{j\beta}(x)$, this yields a well defined holomorphic map $\phi$ from $X$ to $\mathbb{P}^n$. When $\phi$ happens to be an embedding (this is not always the case) $L$ is called a *very ample* line bundle. It is a key concept in the study of complex manifolds.

## 2.3 More examples

Higher rank vector bundles are in general much more difficult objects to construct than line bundles. One of the most important ones that exists on any complex manifold $X$ is the holomorphic tangent bundle $TX$, of rank $r = \dim X$.

### The holomorphic tangent bundle $TX$

Let $X$ be a $n$-dimensional complex manifold. We define

$$TX := \bigsqcup_{a \in X} T_a^{1,0}$$

the projection maps the fiber $T_a^{1,0}$ to $a \in X$. Let $(U_\alpha, z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n))$ be an atlas on $X$. The bundle trivializations are easily defined as follows. Let $\theta \in T_a^{1,0}$. Then there is a vector $v \in \mathbb{C}^n$ which represents the coefficients of $\theta$ on the basis of $T_a^{1,0}$. Hence we set

$$\begin{aligned} \psi_\alpha : TU_\alpha &\longrightarrow U_\alpha \times \mathbb{C}^n \\ \theta = \sum_{j=1}^n v_j \frac{\partial}{\partial z_\alpha^j}(a) &\longmapsto (a, v) \qquad (\theta \in T_a^{1,0}) \end{aligned}$$

Let's find $\psi_\beta(\psi_\alpha^{-1}(a, v)) = \psi_\beta(\theta)$ on $U_\alpha \cap U_\beta$ in terms of $z_\alpha$,$z_\beta$. Let

$$\frac{\partial}{\partial z_\alpha^j} = \sum_{k=1}^n c_k \frac{\partial}{\partial z_\beta^k}$$

so that $\frac{\partial z_\beta^k}{\partial z_\alpha^j} = c_k$. Then $\theta = \sum w_k \frac{\partial}{\partial z_\beta^k}$ where $w_k = \sum_j v_j \frac{\partial z_\beta^k}{\partial z_\alpha^j}$. So $\psi_\beta(\theta) = (a, w)$,

$$w = \left[\frac{\partial z_\beta^k}{\partial z_\alpha^j}\right] \cdot v = g_{\beta\alpha}(a) \cdot v$$

Hence the transition functions for the tangent bundle are given by the complex Jacobian matrix of the chart change $G_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}$, that is

$$g_{\alpha\beta}(a) = \left[\frac{\partial z_\alpha^k}{\partial z_\beta^j}\right] = \mathrm{J}_{\mathbb{C}} G_{\alpha\beta}(z_\beta(a)) \in \mathrm{Gl}(n, \mathbb{C})$$

**Example** (tangent bundle on the torus)**.** *Let $X = \mathbb{C}^n/\Lambda$ be a $n$-dimensional torus. Then $G_{\alpha\beta}(u) = z_\alpha \circ z_\beta^{-1}(u) = u + \omega$ for some $\omega = \omega_{\alpha\beta} \in \Lambda$. Thus*

$$\mathrm{J}_{\mathbb{C}} G_{\alpha\beta} \equiv \frac{\partial(u_j + \omega_j)}{\partial u_k} \equiv I$$

*Hence $TX \simeq X \times \mathbb{C}^n$ is the trivial bundle!*

**Example** (tangent bundle on $\mathbb{P}^1$)**.** *On $X = \mathbb{P}^1$ let $z_0(x_0 : x_1) = x_1/x_0 = u$ and $z_1(x_0 : x_1) = -x_0/x_1 = -1/u$. Then $G_{10} : u \mapsto -1/u$ and $\mathrm{J}_{\mathbb{C}} G_{10} = (x_0/x_1)^2$,*

$$\textit{i.e.}\quad T\mathbb{P}^1 \simeq L(2)$$

*This is not surprising: it can in fact be shown that any line bundle on $\mathbb{P}^n$ is isomorphic to some $L(k)$.*

## The holomorphic cotangent bundle $T^*X$

We define the holomorphic cotangent bundle $T^*X$, also denoted $\Omega_X$, as

$$T^*X := \bigsqcup_{a\in X} (T_a^{1,0})^*$$

the projection maps the fiber $(T_a^{1,0})^*$ to $a \in X$. Let $(U_\alpha, z_\alpha = (z_\alpha^1, \dots, z_\alpha^n))$ be an atlas on $X$. The bundle trivializations are easily defined as follows. Let $\omega_a \in (T_a^{1,0})^*$. Then there is a vector $v \in \mathbb{C}^n$ which represents the coefficients of $\omega_a$ on the basis of $(T_a^{1,0})^*$. Hence we set

$$\begin{aligned} \psi_\alpha : T^*U_\alpha &\longrightarrow U_\alpha \times \mathbb{C}^n \\ \omega_a = \sum_{j=1}^n v_j (dz_\alpha^j)_a &\longmapsto (a, v) \end{aligned}$$

Let's find $\psi_\beta(\psi_\alpha^{-1}(a,v)) = \psi_\beta(\omega_a)$ on $U_\alpha \cap U_\beta$ in terms of $z_\alpha$,$z_\beta$. Let

$$dz_\alpha^j = \sum_{k=1}^n c_k dz_\beta^k \qquad \text{on } U_\alpha \cap U_\beta$$

The coefficients $c_k$ are then obtained by applying $dz_\alpha^j$ to $\frac{\partial}{\partial z_\beta^k}$. Thus

$$c_k = dz_\alpha^j(\partial/\partial z_\beta^k) = \frac{\partial z_\alpha^j}{\partial z_\beta^k}$$

Then $\omega_a = \sum w_k dz_\beta^k$ where $w_k = \sum_j v_j \frac{\partial z_\alpha^j}{\partial z_\beta^k}$. So $\psi_\beta(\omega_a) = (a, w)$, where

$$w = \left[\frac{\partial z_\alpha^j}{\partial z_\beta^k}\right] \cdot v = {}^t g_{\beta\alpha}^{-1} \cdot v$$

where $g_{\alpha\beta}(a) = \mathrm{J}_\mathbb{C} G_{\alpha\beta}(z_\beta(a))$ and $G_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}$. Therefore

$$TX \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\} \iff T^*X \longleftrightarrow \{U_\alpha, {}^t g_{\alpha\beta}^{-1}\}$$

In other words, the cotangent bundle is the dual bundle of the tangent bundle:

$$T^*X = (TX)^*$$

## The canonical line bundle $\omega_X$

Let's denote with $\Omega_X = T^*X$ the holomorphic cotangent bundle on $X$. For any $0 \le p \le n$ we can consider the vector bundle on $X$ given by the $p$-th exterior power of the cotangent bundle. That is, the bundle of holomorphic $p$-forms $\Omega_X^p := \bigwedge^p \Omega_X$. By this we mean the bundle on $X$ whose fibers are canonically isomorphic to the $p$-th exterior power of the cotangent space. More precisely,

$$\Omega_X^p := \bigsqcup_{a\in X} \bigwedge^p (T_a^{1,0})^*$$

In the case $p = n$ this is denoted by $\omega_X = \Omega_X^n$ and it is called the *canonical line bundle* on $X$. The canonical bundle is described in a particularly nice form.

**Definition.** Let $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ be a holomorphic vector bundle on $X$. The *determinant bundle* is the line bundle defined as

$$\det E \longleftrightarrow \{U_\alpha, \det g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*\}$$

The canonical line bundle is the determinant bundle of the cotangent bundle

$$\omega_X = \det \Omega_X$$

Its transition functions are thus the complex Jacobian of the chart changes. The bundle trivializations in local coordinates take the form

$$\begin{aligned} \psi_\alpha : \omega_X|_{U_\alpha} &\longrightarrow U_\alpha \times \mathbb{C} \\ \lambda(dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n)_a &\longmapsto (a, \lambda) \end{aligned}$$

**Example** (torus)**.** *Let $X$ be a $n$-dimensional torus. Then*

- ⋄ $TX \simeq X \times \mathbb{C}^n$ *as we've seen. So* $TX \longleftrightarrow \{X, I = \mathrm{id} \in \mathrm{Gl}(n, \mathbb{C})\}$
- ⋄ $\Omega_X \simeq X \times \mathbb{C}^n$ *since* ${}^tI^{-1} = I$
- ⋄ $\omega_X \simeq X \times \mathbb{C}$ *since* $\det I = 1$

**Proposition 2.3.1.** *The canonical line bundle of $\mathbb{P}^n$ is given by*

$$\omega_{\mathbb{P}^n} \simeq L(-n-1)$$

*Proof.* Let $X = \mathbb{P}^n$ with the standard covering $\{U_j\}$. Consider a $n$-form $\eta$ on $U_\alpha \cap U_0$. We write $\eta$ in affine coordinates on $U_\alpha \simeq \mathbb{C}^n$ as

$$\eta_x = (-1)^\alpha \lambda(x) du_0 \wedge \cdots \wedge \widehat{du_\alpha} \wedge \cdots \wedge du_n \xmapsto{\psi_\alpha} (x, \lambda(x))$$

where $u_0 = x_0/x_\alpha, \ldots, \hat{u}_\alpha, \ldots, u_n = x_n/x_\alpha$ on $U_\alpha$. Let's compute $g_{\alpha\beta}$ for $\beta = 0$. Let $s_1 = x_1/x_0, \ldots, s_n = x_n/x_0$ be affine coordinates on $U_0$. Then

$$\begin{cases} s_1 = \frac{u_1}{u_0} \\ \vdots \\ s_\alpha = \frac{1}{u_0} \\ \vdots \\ s_n = \frac{u_n}{u_0} \end{cases} \implies \begin{cases} ds_1 = d\left(\frac{u_1}{u_0}\right) = \frac{1}{u_0} du_1 - \frac{u_1}{u_0^2} du_0 \\ \vdots \\ ds_\alpha = -\frac{1}{u_0^2} du_0 \\ \vdots \\ ds_n = d\left(\frac{u_n}{u_0}\right) = \frac{1}{u_0} du_n - \frac{u_n}{u_0^2} du_0 \end{cases}$$

Therefore

$$\begin{aligned} \eta_x &= (-1)^0 \lambda(x) ds_1 \wedge \cdots \wedge \cdots \wedge ds_n \\ &= \lambda(x) \left(\tfrac{1}{u_0} du_1 - \tfrac{u_1}{u_0^2} du_0\right) \wedge \cdots \wedge \left(-\tfrac{1}{u_0^2} du_0\right) \wedge \cdots \wedge \left(\tfrac{1}{u_0} du_n - \tfrac{u_n}{u_0^2} du_0\right) \\ &= (-1)^\alpha \lambda(x) \tfrac{1}{u_0^{n+1}} du_0 \wedge \cdots \wedge \widehat{du_\alpha} \wedge \cdots \wedge du_n \xmapsto{\psi_\alpha} \left(x, \tfrac{1}{u_0^{n+1}} \lambda(x)\right) \end{aligned}$$

Hence we see that

$$g_{\alpha 0}(x_0 : \ldots : x_n) = \frac{1}{u_0^{n+1}} = \left(\frac{x_\alpha}{x_0}\right)^{n+1}$$

and $g_{\alpha\beta} = g_{\alpha 0} g_{0\beta} = g_{\alpha 0} g_{\beta 0}^{-1} = (x_\alpha/x_\beta)^{n+1}$, the same of $L(-n-1)$. □

## 2.4 Morphisms and Quotient bundles

Let $p_E : E \to X$ and $p_F : F \to X$ be vector bundles of rank $e$ and $f$ on $X$.

**Definition.** A holomorphic map $\Phi : E \to F$ such that

(i) it commutes with the projections: $p_F \circ \Phi = p_E$

(ii) it is linear on the fibers: $\Phi_x : E_x \to F_x$

(iii) it has constant rank: $\mathrm{rk}(\Phi_x)$ is independent of $x \in X$

is called *vector bundle morphism*, or simply *morphism*.

*Remark* 2.4.1. We require $\Phi$ to have constant rank to avoid situations like the following: $X = \mathbb{C}$ and $E = F = \mathbb{C}^2$ are both the trivial bundle, $\Phi(z, v) = (z, zv)$. Then $\Phi_z : v \mapsto zv$ and $\mathrm{rk}(\Phi_0) = 0,\ \mathrm{rk}(\Phi_z) = 1$ if $z \neq 0$.

*Remark* 2.4.2. We can always assume that two vector bundles $E$ and $F$ on $X$ are defined with two families of trivializations on the *same* covering $X = \bigcup U_\alpha$. This is done by restriction: for any $x \in X$ we can find two open neighborhoods $U_x, V_x$ of $x$ and trivializations $E|_{U_x} \simeq U_x \times \mathbb{C}^e$ and $F|_{V_x} \simeq V_x \times \mathbb{C}^f$. Hence we can restrict both trivializations to $W_x = U_x \cap V_x$ and get

$$E|_{W_x} \simeq W_x \times \mathbb{C}^e, \quad F|_{W_x} \simeq W_x \times \mathbb{C}^f$$

Suppose $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ and $F \longleftrightarrow \{U_\alpha, h_{\alpha\beta}\}$ and $\Phi : E \to F$ is a morphism. Let $\psi_\alpha$ and $\vartheta_\alpha$ be the trivializations of $E$ and $F$ respectively. Then there is a map that makes the following diagram commute

$$\begin{tikzcd}
E|_{U_\alpha} \arrow[r, "\Phi"] \arrow[d, "\psi_\alpha"'] & F|_{U_\alpha} \arrow[d, "\vartheta_\alpha"] \\
U_\alpha \times \mathbb{C}^e \arrow[r, dashed] & U_\alpha \times \mathbb{C}^f
\end{tikzcd}$$

This map $U_\alpha \times \mathbb{C}^e \to U_\alpha \times \mathbb{C}^f$ has to be the identity on the first component and a linear map on the second one, that is $(x, v) \mapsto (x, \Phi_\alpha(x) \cdot v)$ where

$$\Phi_\alpha : U_\alpha \longrightarrow \mathcal{M}_{f \times e}(\mathbb{C})$$

is a holomorphic map (since the diagram above commutes) and satisfies

$$\Phi_\alpha(x) = h_{\alpha\beta}(x)\Phi_\beta(x)g_{\beta\alpha}(x)$$

for all $x \in U_\alpha \cap U_\beta$. In fact, since the diagram above commutes we get

$$\begin{aligned}
(x, \Phi_\alpha(x)v) &= \vartheta_\alpha \circ \Phi \circ \psi_\alpha^{-1}(x, v) \\
&= \vartheta_\alpha \circ \vartheta_\beta^{-1} \circ \vartheta_\beta \circ \Phi \circ \psi_\beta^{-1} \circ \psi_\beta \circ \psi_\alpha^{-1}(x, v) \\
&= \vartheta_\alpha \circ \vartheta_\beta^{-1}(x, \Phi_\beta(x)g_{\beta\alpha}(x)v) \\
&= (x, h_{\alpha\beta}(x)\Phi_\beta(x)g_{\beta\alpha}(x)v)
\end{aligned}$$

Hence for a vector bundle morphism $\Phi : E \to F$ we'll write

$$\Phi \longleftrightarrow \{\Phi_\alpha : U_\alpha \longrightarrow \mathcal{M}_{f\times e}(\mathbb{C}) :\ \Phi_\alpha(x) = h_{\alpha\beta}(x)\Phi_\beta(x)g_{\beta\alpha}(x)\}$$

since a morphism $\Phi$ is uniquely determined by this collection of matrix-valued holomorphic maps $\Phi_\alpha$ satisfying the above gluing conditions.

Suppose now $e = \operatorname{rk} E \leq \operatorname{rk} F = f$. It turns out that an injective vector bundle morphism $\Phi : E \to F$ behaves like an inclusion in the following sense.

**Proposition 2.4.1.** *If $\Phi : E \to F$ is injective then there are trivializations $\psi_\alpha : E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^e$ and $\vartheta_\alpha : F|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^f$ such that*

$$\vartheta_\alpha \circ \Phi \circ \psi_\alpha^{-1} : (x, (v_1, \ldots, v_e)) \longmapsto (x, (v_1, \ldots, v_e, 0, \ldots, 0))$$

*In other words, for all $x \in U_\alpha$,*

$$\Phi_\alpha(x) = \begin{pmatrix} I_e \\ 0 \end{pmatrix}$$

*Proof.* Let $a \in V \subset X$ and let $\psi$, $\tilde{\vartheta}$ be trivializations on $E|_V$, $F|_V$. Hence

$$\tilde{\vartheta} \circ \Phi \circ \psi^{-1} : (x, v) \longmapsto (x, \Phi_V(x)v)$$

where $\Phi_V(x) : \mathbb{C}^e \to \mathbb{C}^f$ is a linear map of rank $e$, since $\Phi$ is injective by hypothesis. Up to a permutation of the basis of $\mathbb{C}^f$ we can suppose $\Phi_V(x)$ to have the first $e$ rows linearly independent, that is

$$\Phi_V(x) = \begin{pmatrix} M(x) \\ N(x) \end{pmatrix}$$

where $M(x) : \mathbb{C}^e \to \mathbb{C}^e$ and $\det M(a) \neq 0$. Thus there is a small neighborhood $U \subset V$ of $a$ where $\det M(x) \neq 0$ for all $x \in U$. We can define now a matrix-valued function $k : U \to \mathrm{Gl}(f, \mathbb{C})$ that will adjust $\Phi_V(x)$ to the desired form:

$$k(x) = \begin{pmatrix} M(x)^{-1} & 0 \\ -N(x)M(x)^{-1} & I_{f-e} \end{pmatrix}$$

so that, for all $x \in U$, the matrix $k(x)\Phi_V(x)$ is of the form we want (that one of $\Phi_\alpha(x)$ in the statement). Then we define a new trivialization $\vartheta$ of $F|_U$ as the old $\tilde{\vartheta}$ followed by multiplication by $k(x)$ on the second component, so that

$$\vartheta_\alpha \circ \Phi \circ \psi_\alpha^{-1} : (x, v) \mapsto (x, k(x)\Phi_V(x)) = (x, (v_1, \ldots, v_e, 0, \ldots, 0)) \qquad \square$$

Using this special trivializations as in the proposition we can construct transition functions of a particularly nice form.

**Corollary 2.4.1.** *If $\Phi : E \to F$ is injective then there are transition functions $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ and $F \longleftrightarrow \{U_\alpha, h_{\alpha\beta}\}$ such that*

$$h_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} & \star \\ 0 & k_{\alpha\beta} \end{pmatrix}$$

*for certain matrix-valued functions $k_{\alpha\beta}$ holomorphic on $U_\alpha$.*

*Proof.* Just use the nice trivializations $\psi_\alpha$ and $\vartheta_\alpha$ granted by the above proposition, so that $\vartheta_\alpha \circ \Phi \circ \psi_\alpha^{-1} : (x, v) \longmapsto (x, (v, 0))$. Hence

$$\begin{aligned}
(x, h_{\alpha\beta}(x) \begin{pmatrix} v \\ 0 \end{pmatrix}) &= \vartheta_\alpha \circ \vartheta_\beta^{-1}(x, (v, 0)) = \vartheta_\alpha \circ \vartheta_\beta^{-1} \circ \vartheta_\beta \circ \Phi \circ \psi_\beta^{-1}(x, v) \\
&= \vartheta_\alpha \circ \Phi \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1}(x, v) = \vartheta_\alpha \circ \Phi \circ \psi_\alpha^{-1}(x, g_{\alpha\beta}(x)v) \\
&= (x, \begin{pmatrix} g_{\alpha\beta}(x)v \\ 0 \end{pmatrix})
\end{aligned}$$

and from this follows that the matrix $h_{\alpha\beta}(x)$ is of the desired form

$$h_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & \star \\ 0 & k_{\alpha\beta}(x) \end{pmatrix}$$

□

*Remark* 2.4.3. Since the $h_{\alpha\beta}$ are transition functions of $F$ and thus must satisfy $h_{\alpha\alpha} = I,\ h_{\beta\alpha} = h_{\alpha\beta}^{-1}$ and the cocycle condition $h_{\alpha\beta}h_{\beta\gamma} = h_{\alpha\gamma}$, it follows that also the $k_{\alpha\beta}$ in the corollary satisfy the same properties. Hence they can be taken to be the transition maps of a vector bundle on $X$.

**Definition.** If $\Phi : E \to F$ is an injective morphism we define the *quotient bundle* $F/E$ as the vector bundle with transition maps

$$F/E \longleftrightarrow \{U_\alpha, k_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^{f-e}\}$$

where the $k_{\alpha\beta}$ are those found as in corollary 2.4.1 above.

*Remark* 2.4.4. Note that $\det h_{\alpha\beta} = \det g_{\alpha\beta} \det k_{\alpha\beta}$. This means that the transition maps of the line bundle $\det F$ are simply the product of the ones of $E$ and $F/E$. Hence we write

$$\det F = \det E \otimes \det F/E$$

We denote it with $\otimes$ instead of just a dot, because more generally we can define a bundle by taking tensor product of the matrices that represent transition functions. In the case of line bundles the tensor product of scalars is just the regular product. Just as an example, for $2 \times 2$ square matrices we have

$$\begin{aligned}
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} &= \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}
\end{aligned}$$

*Remark* 2.4.5. To be more precise we should write $F/\Phi(E)$ instead of $F/E$ but that'd be a heavy and useless notation. Since $\Phi : E \to F$ is an injection we think of $E$ as a *subbundle* of $F$. We can also schematize this situation by writing a *short exact sequence* of holomorphic vector bundles

$$0 \longrightarrow E \xrightarrow{\Phi} F \longrightarrow F/E \longrightarrow 0$$

where the map $F \to F/E$ is the projection on the quotient. Once we trivialize locally $F|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^f$ and $(F/E)|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^{f-e}$, this projection is given by $(x, (v_e, v_{f-e})) \mapsto (x, v_{f-e})$. It is a surjective morphism.

## 2.5 Operations between vector bundles

We can define several operations on the set of holomorphic vector bundles on a complex manifold $X$. We have already encountered some of them and we'll repeat those here for convenience. Let $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ and $F \longleftrightarrow \{U_\alpha, h_{\alpha\beta}\}$ be vector bundles on $X$ of rank $e$ and $f$ respectively. We may define the following vector bundles on $X$:

(i) The *direct sum bundle* $E \oplus F$ whose fiber over $x \in X$ is canonically isomorphic to the direct sum $E_x \oplus F_x$ as complex vector spaces. Its description with transition function is given by

$$E \oplus F \longleftrightarrow \left\{ U_\alpha, \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \right\}$$

(ii) The *tensor product bundle* $E \otimes F$ whose fiber over $x \in X$ is canonically isomorphic to the tensor product $E_x \otimes F_x$. Its transition functions

$$E \otimes F \longleftrightarrow \{U_\alpha, g_{\alpha\beta} \otimes h_{\alpha\beta}\}$$

(iii) The $k$-th *exterior power bundle* $\bigwedge^k E$ whose fiber over $x$ is canonically isomorphic to $\bigwedge^k E_x$, where $0 \le k \le e$. The special case $k = e$ is called *determinant line bundle* $\det E = \bigwedge^e E$ since it is a line bundle and its transition functions are given by

$$\det E \longleftrightarrow \{U_\alpha, \det g_{\alpha\beta}\}$$

(iv) The *dual bundle* $E^*$ whose fiber over $x$ is canonically isomorphic to the dual vector space $(E_x)^*$.

$$E^* \longleftrightarrow \{U_\alpha, {}^t g_{\alpha\beta}^{-1}\}$$

(v) The *Hom-bundle* $\mathrm{Hom}(E, F)$ whose fiber over $x$ is canonically isomorphic to $\mathrm{Hom}(E_x, F_x)$. It is an important vector bundle because there is a 1:1 correspondence between its sections and morphisms $E \to F$. In symbols

$$\{\Phi : E \to F\} \stackrel{1:1}{\longleftrightarrow} \Gamma(X, \mathrm{Hom}(E, F))$$

The dual bundle is then just a special case of hom-bundle:

$$\mathrm{Hom}(E, X \times \mathbb{C}) \simeq E^*$$

Also, there is a relation between the operations of hom-bundle, dual bundle and tensor product. There is in fact a canonical vector bundle isomorphism

$$\mathrm{Hom}(E, F) \simeq E^* \otimes F$$

(vi) If $E$ is a holomorphic *subbundle* of $F$, i.e. there is a holomorphic injection $E \hookrightarrow F$ (as discussed in the previous section) then we can define the *quotient bundle* $F/E$. Its fibers are canonically isomorphic to $F_x/E_x$.

(vii) More generally, if $\Phi : E \to F$ is a morphism, one can define $\ker(\Phi)$ and $\mathrm{coker}(\Phi)$ as the vector bundles on $X$ whose fibers on $x$ are canonically isomorphic to $\ker(\Phi_x)$ and $\mathrm{coker}(\Phi_x)$ respectively.

## 2.6 Normal bundle and Adjunction

Let $Y, X$ be complex manifold and $f : Y \to X$ a holomorphic map. Suppose $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ is a holomorphic vector bundle on $X$. Then $f$ induces a holomorphic vector bundle $f^*E$ on $Y$ by composition.

**Definition.** The *pullback bundle* $f^*E$ is by definition the vector bundle

$$f^*E \longleftrightarrow \{f^{-1}(U_\alpha), g_{\alpha\beta} \circ f\}$$

Its fiber on $y \in Y$ is isomorphic to the fiber $E_{f(y)}$. If $i : Y \hookrightarrow X$ is a complex submanifold we call $E|_Y := i^*E$ the *restriction* of the bundle $E$ to $Y$. Its transition maps are then simply the restrictions of the $g_{\alpha\beta}$,

$$E|_Y \longleftrightarrow \{Y \cap U_\alpha, g_{\alpha\beta}|_{Y\cap U_{\alpha\beta}}\}$$

Let $n = \dim X$ and $i : Y \hookrightarrow X$ be an $m$-dimensional submanifold of $X$. Let's consider the restriction of the tangent bundle $TX|_Y$. The linear tangent map $di$ is then an injective morphism of vector bundles

$$di : TY \hookrightarrow TX|_Y$$

Thus, we can take the quotient bundle which takes a special name.

**Definition.** The *normal bundle* $\mathcal{N}_{Y/X}$ is by definition the quotient

$$\mathcal{N}_{Y/X} = \frac{TX|_Y}{TY}.$$

In other words $\mathcal{N}_{Y/X}$ is the vector bundle on $Y$ defined as the cokernel of the natural injection given by $di : TY \longrightarrow TX|_Y$. The associated short exact sequence of vector bundles is referred to as the *normal bundle sequence*

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0$$

As observed in 2.4.4 we see that

$$\det(TX|_Y) = \det(TY) \otimes \det(\mathcal{N}_{Y/X})$$

On the other hand, if $TX \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ then $\det(g_{\alpha\beta}|_Y) = \det(g_{\alpha\beta})|_Y$, so that $\det(TX|_Y) = \det(TX)|_Y$. If we substitute this in the above equation and take its dual we get

$$\omega_X|_Y = \omega_Y \otimes \det(\mathcal{N}_{Y/X})^{-1}$$

We like better to isolate $\omega_Y$ in this equation. Since they are line bundles, we can cancel out the term $\det(\mathcal{N}_{Y/X})^{-1}$ simply taking the tensor product with $\det \mathcal{N}_{Y/X}$ on both sides. What we get is the so called *adjunction formula*

$$\omega_Y = \omega_X|_Y \otimes \det(\mathcal{N}_{Y/X})$$

**Example.** *Let $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$, $x \mapsto (x : 0)$. We know $\omega_{\mathbb{P}^{n+1}} = L_{\mathbb{P}^{n+1}}(-n-2)$ and $\omega_{\mathbb{P}^n} = L_{\mathbb{P}^n}(-n-1)$. The transition functions of $L_{\mathbb{P}^{n+1}}(-n-2)$ are then $g_{jk} = (x_j/x_k)^{n+2}$. Thus $\omega_{\mathbb{P}^{n+1}}|_{\mathbb{P}^n}$ is obtained by restriction of the $g_{jk}$'s. Thus*

$$\omega_{\mathbb{P}^n} = \omega_{\mathbb{P}^{n+1}}|_{\mathbb{P}^n} \otimes \det(\mathcal{N}_{\mathbb{P}^n/\mathbb{P}^{n+1}}) = \omega_{\mathbb{P}^{n+1}}|_{\mathbb{P}^n} \otimes \mathcal{N}_{\mathbb{P}^n/\mathbb{P}^{n+1}}$$

*so that the transition maps of $\mathcal{N}_{\mathbb{P}^n/\mathbb{P}^{n+1}}$ are $\frac{(x_j/x_k)^{n+1}}{(x_j/x_k)^{n+2}} = (x_k/x_j)$, that is*

$$\mathcal{N}_{\mathbb{P}^n/\mathbb{P}^{n+1}} \simeq L(1)$$

## 2.7 The line bundle of an analytic hypersurface

Throughout this section we denote by $X$ a $n$-dimensional complex manifold and by $Y \subset X$ a submanifold of $X$ of *codimension* 1, that is

$$\dim Y = \dim X - 1$$

Then around any $a \in Y$ there is a local chart $(U, z = (z_1, \ldots, z_n))$ such that

$$Y \cap U = \{x \in U : z_n(x) = 0\}$$

For this reason $Y$ is also called *analytic hypersurface* of $X$. The goal of this section is to show that such a $Y$ is always given by the zero locus of a holomorphic global section of a "unique" line bundle $L_Y$ on $X$.

**Definition.** A *local equation* for $Y$ is a pair $(U, f)$ where $U \subset X$ is open and $f : U \to \mathbb{C}$ is a holomorphic function such that

(i) $Y \cap U = \{x \in U : f(x) = 0\}$

(ii) if $g \in \mathcal{O}(U)$ and $g(Y \cap U) = 0$ then $g = hf$ for some $h \in \mathcal{O}(U)$

**Lemma.** *$(U, z_n)$ as above is a local equation for $Y$.*

*Proof.* Let $g \in \mathcal{O}(U)$ and $g(Y \cap U) = 0$. Let $z = (z_1, \ldots, z_n) : U \to \mathbb{C}^n$ be local charts of $X$ as above. Around any $a \in Y \cap U$ we can find a neighborhood $V_a \subset U$ where $g = g(z)$ can be expanded as a convergent power series and

$$\begin{aligned} g(z) &= \sum a_{j_1 \ldots j_n} (z_1 - z_1(a))^{j_1} \cdots (z_n - z_n(a))^{j_n} \\ &= g_0 + g_1 z_n + g_2 z_n^2 + g_3 z_n^3 + \ldots \end{aligned}$$

where $g_k$ is holomorphic on $V_a$ and $g_k = g_k(z_1, \ldots, z_{n-1})$ since $z_n(a) = 0$. Moreover $g_0 = g_0(z_1, \ldots, z_{n-1})$ is zero on all of $V_a$. In fact it must be zero on $V_a \cap Y$ because $g \equiv 0$ and $z_n \equiv 0$ on $Y \cap V_a$. On the other hand

$$(z_1, \ldots, z_{n-1}) : Y \cap U \longrightarrow \mathbb{C}^{n-1}$$

is a local chart of $Y$ hence it sends $Y \cap U$ onto an open subset of $\mathbb{C}^{n+1}$ on which $g_0 \equiv 0$. Thus $g_0(z_1, \ldots, z_{n-1})$ is zero on all of $V_a$. So $g = h z_n$ where

$$h = \sum_{k=1}^{\infty} g_k(z_1, \ldots, z_{n-1}) z_n^{k-1}$$

which is holomorphic on $V_a$. Now we have to somehow extend $h$ to all of $U$. First we repeat this construction on each $a \in Y \cap U$ and find $h_a$. On $V_a \cap V_b$ we then have $g = h_a z_n = h_b z_n$. Thus $(h_a - h_b) z_n \equiv 0$ on $V_a \cap V_b$. However $z_n$ is zero only on $Y \cap V_a \cap V_b$ and so it follows $h_a \equiv h_b$ on $V_a \cap V_b$. So on all of

$$V = \bigcup_{a \in Y \cap U} V_a$$

we have an holomorphic function $h$ such that $g = h z_n$. Now on $U \setminus V$ we have $z_n \neq 0$. Thus we simply *define* $h$ to be the holomorphic function $g/z_n$. $\square$

*Remark* 2.7.1. There always exists a covering $X = \bigcup U_\alpha$ with $(U_\alpha, f_\alpha)$ local equations for $Y$. In fact around any point $a \in Y$ we have $(U, z_n)$ by the lemma. Whereas for any $a \in X \setminus Y$ the pair $(X \setminus Y, f \equiv 1)$ is a local equation for $Y$.

**Lemma.** *Suppose $(U_1, f_1)$, $(U_2, f_2)$ are two local equations for $Y \subset X$. Then the ratio $f_1/f_2$ is holomorphic and has no zeros on $U_1 \cap U_2$.*

*Proof.* By definition on $U_{12} = U_1 \cap U_2$ we have $f_1 = hf_2$ and $f_2 = gf_1$ for some functions $h, g$ holomorphic on $U_{12}$. Hence $(1 - hg)f_1 \equiv 0$ on $U_{12}$. But $f_1$ is not identically zero on $U_2$, so $h(x)g(x) = 1$ for all $x \in U_{12}$. Therefore $h, g$ are non zero, i.e. $h, g \in \mathcal{O}^*(U_{12})$. This means that if $f_1 = hf_2$ has a zero on $U_{12}$ then this point is also a zero of $f_2$ and for both of the same order; thus on those points it is well defined their ratio $f_1(x)/f_2(x) = h(x) \in \mathbb{C}^*$. All the other points are not zeroes neither for $f_1$ nor $f_2$. Therefore $f_1/f_2 \in \mathcal{O}^*(U_{12})$, as claimed. □

*Remark* 2.7.2. Moreover, if $(U, f)$ is a local equation for $Y$ then from (ii) in the definition it follows that each zero of $f$ is of order one. For, suppose that $x \in U \cap Y$ is a zero of order greater than one. Then, modulo a chart change we can describe $f$ using the last coordinate $z_n$ and locally around $x$ we can write $f$ as $z_n \mapsto z_n^k$ with $k \geq 2$. But also $z_n$ is a local equation for $Y$, and there can be no holomorphic function $h$ on this neighborhood such that $z_n = hf$.

Let now $X = \bigcup U_\alpha$ be a covering of $X$ with local equations $(U_\alpha, f_\alpha)$ for $Y$. By the last lemma we then have a family of holomorphic non zero functions

$$g_{\alpha\beta} := \frac{f_\alpha}{f_\beta} : U_{\alpha\beta} \longrightarrow \mathbb{C}^*$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Moreover, this maps clearly satisfy the glueing conditions

$$g_{\alpha\alpha} = 1, \quad g_{\beta\alpha} = g_{\alpha\beta}^{-1}, \quad g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

Thus they define a holomorphic line bundle on $X$, namely

$$L_Y \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$$

*Remark* 2.7.3. As such, this definition seems to depend on the choice of the local equations for $Y$. Whereas the special form of $g_{\alpha\beta} = f_\alpha/f_\beta$ makes it a well defined bundle. For, suppose there exist two line bundles $L, M$ on $X$ such that

$$L \longleftrightarrow \{U_\alpha, g_{\alpha\beta} = f_\alpha/f_\beta\}, \quad M \longleftrightarrow \{U_\alpha, l_{\alpha\beta} = h_\alpha/h_\beta\}$$

then we can define a line bundle isomorphism $\Phi : L \to M$ by

$$\Phi \longleftrightarrow \{U_\alpha, \Phi_\alpha = h_\alpha/f_\alpha : U_\alpha \to \mathbb{C}^*\}$$

in fact $\Phi_\alpha = h_\alpha/f_\alpha = (h_\alpha/h_\beta)(h_\beta/f_\beta)(f_\beta/f_\alpha) = l_{\alpha\beta}\Phi_\beta g_{\beta\alpha}$ and $\Phi_\alpha$ is nowhere zero by the lemma. So $\Phi_\alpha$ has rank 1 everywhere, i.e. it is an isomorphism.

If we look at the line bundle $L_Y$ on $X$ and take its restriction to $Y$ we get a line bundle on $Y$. Consider now the normal bundle on $Y \hookrightarrow X$,

$$\mathcal{N}_{Y/X} = (TX|_Y)/(TY)$$

It has rank $\mathrm{rk}(\mathcal{N}_{Y/X}) = \mathrm{codim}(Y) = 1$, so it is a line bundle on $Y$ in our case. It turns out that $\mathcal{N}_{Y/X}$ is just the restriction of $L_Y$ to $Y$.

**Theorem 2.7.1.** $\mathcal{N}_{Y/X} = L_Y|_Y$

*Proof.* Recall that for a quotient bundle $F/E$ we have $\det F = \det E \otimes \det F/E$ or, equivalently, $\det F/E = \det F \otimes (\det E)^{-1}$. In our case $F/E = \mathcal{N}_{Y/X}$ is a line bundle (so $\det F/E = F/E$), $F = TX|_Y$ and $E = TY$, so we get

$$\mathcal{N}_{Y/X} = \det TX|_Y \otimes (\det TY)^{-1}$$

Let $(U_\alpha, z^\alpha)$ be local charts of $X$ such that $(U_\alpha, z_n^\alpha)$ are local equations for $Y \subset \bigcup U_\alpha$. Then $\{Y \cap U_\alpha,\ (z_1^\alpha, \ldots, z_{n-1}^\alpha)\}$ is an atlas of $Y$, so

$$\begin{aligned} TY &\longleftrightarrow \left\{ U_\alpha \cap Y,\ g_{\alpha\beta} = \left[ \frac{\partial z_k^\alpha}{\partial z_l^\beta} \right], k, l = 1, \ldots, n-1 \right\} \\ TX|_Y &\longleftrightarrow \left\{ U_\alpha \cap Y,\ G_{\alpha\beta} = \left[ \frac{\partial z_k^\alpha}{\partial z_l^\beta} \right], k, l = 1, \ldots, n \right\} \\ L_Y &\longleftrightarrow \left\{ U_\alpha,\ h_{\alpha\beta} = \frac{z_n^\alpha}{z_n^\beta} \right\} \end{aligned}$$

Let's compute the last row of $G_{\alpha\beta}$ (so $k = n$) evaluating in $y \in Y \cap U_{\alpha\beta}$

$$\frac{\partial z_n^\alpha}{\partial z_l^\beta}(y) = \frac{\partial h_{\alpha\beta}}{\partial z_l^\beta}(y) z_n^\beta(y) + h_{\alpha\beta}(y) \frac{\partial z_n^\beta}{\partial z_l^\beta}(y) = \delta_{nl} h_{\alpha\beta}(y)$$

(where $\delta_{nl}$ is the *Kronecker delta*) since $z_n^\alpha = h_{\alpha\beta} z_n^\beta$ and $z_n^\beta(y) = 0$. Thus

$$G_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} & * \\ 0 & h_{\alpha\beta} \end{pmatrix}$$

so that $\det(G_{\alpha\beta}|_Y) = (\det g_{\alpha\beta}) \cdot h_{\alpha\beta}$, that is

$$\det TX|_Y = \det TY \otimes L_Y|_Y$$

and substituting this in the above expression for $\mathcal{N}_{Y/X}$ yields the statement. $\square$

By the adjunction formula we get the following expression for $\omega_Y$.

**Corollary 2.7.1** (Adjunction in codimension 1). $\omega_Y = (\omega_X \otimes L_Y)|_Y$

We finally get to the main result of this section.

**Theorem 2.7.2.** *Let $Y \subset X$ be an analytic hypersurface. Then*

*(i) There is a global section $s_Y \in \Gamma(X, L_Y)$ such that*

$$Y = \{x \in X :\ s_Y(x) = 0\}.$$

*(ii) There is a covering $X = \bigcup U_\alpha$ where $s_Y \longleftrightarrow \{U_\alpha, s_\alpha : U_\alpha \to \mathbb{C}\}$ is such that each $(U_\alpha, s_\alpha)$ is a local equation for $Y$.*

*(iii) If $L$ is any line bundle on $X$ with a global section $s \in \Gamma(X, L)$ which gives a family of local equations for $Y$, then $L = L_Y$.*

*Proof.* Let $(U_\alpha, f_\alpha)$ be local equations for $Y$. Then $L_Y \longleftrightarrow \{U_\alpha, g_{\alpha\beta} = f_\alpha/f_\beta\}$ and any global section $s$ of $L_Y$ must have local descriptions $s_\alpha : U_\alpha \to \mathbb{C}$ such that $s_\alpha = g_{\alpha\beta}s_\beta$. This is obviously true when $s_\alpha := f_\alpha$, so that the section

$$s_Y \longleftrightarrow \{U_\alpha, f_\alpha\}$$

is a global section of $L_Y$ and clearly $Y = \{x \in X : \ s(x) = 0\}$, since the local descriptions of $s$ are local equations for $Y$.

Now, if $L \longleftrightarrow \{U_\alpha, k_{\alpha\beta}\}$ is a line bundle on $X$ with a global section given by $s \longleftrightarrow \{s_\alpha : U_\alpha \to \mathbb{C}, \ s_\alpha = k_{\alpha\beta}s_\beta\}$ such that $(U_\alpha, s_\alpha)$ are local equations for $Y$, then we can write $k_{\alpha\beta} = s_\alpha/s_\beta$ on $U_{\alpha\beta}$. Hence

$$L_Y \longleftrightarrow \{U_\alpha, k_{\alpha\beta}\}$$

□

**Example** (projective hypersurfaces)**.** *Let $X = \mathbb{P}^n$ and $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneus polynomial of degree $d$. Assume that the dehomogenized polynomials*

$$f_j(u_1, \ldots, u_n) := F(\tfrac{x_0}{x_j}, \ldots, 1, \ldots, \tfrac{x_n}{x_j})$$

*for all $j = 0, \ldots, n$ are such that* $\mathrm{rk}\ \mathrm{J}_\mathbb{C}(f_j) = 1$ *for all $u \in \mathbb{C}^n$ with $f_j(u) = 0$.*

$$Y = \{x \in \mathbb{P}^n : \ F(x) = 0\}$$

*is then a hypersurface of $\mathbb{P}^n$ with local equations $(U_j, f_j)$, on the standard covering of $\mathbb{P}^n$. As we have seen $F$ defines a global section of $L(d)$. Thus*

$$L_Y = L(d)$$

*by theorem 2.7.2 above. By corollary 2.7.1 and proposition 2.3.1 we have*

$$\begin{aligned} \omega_Y &= (\omega_{\mathbb{P}^n} \otimes L_Y)|_Y \\ &= (L(-n-1) \otimes L(d))|_Y \\ &= L(d-n-1)|_Y \end{aligned}$$

**Corollary 2.7.2.** *Elliptic curves have trivial canonical bundle.*

Indeed, take $d = 3$ and $n = 2$ in the above example, so that $Y$ is a cubic curve in $\mathbb{P}^2$. Recall that $L(0)$ is the trivial bundle on $\mathbb{P}^n$. Then

$$\omega_Y = L(0)|_Y = (\mathbb{P}^2 \times \mathbb{C})|_Y = Y \times \mathbb{C}.$$

# Chapter 3

# Line bundles

## 3.1 Picard group

Let $X$ be a complex manifold and $L \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$, $M \longleftrightarrow \{U_\alpha, h_{\alpha\beta}\}$ two holomorphic line bundles on $X$. Suppose they are isomorphic: there exists a morphism $\phi : L \to M$ of rank 1, so that $\phi \longleftrightarrow \{U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{C}^*\}$ and we can write locally $\phi_\alpha = h_{\alpha\beta}\phi_\beta g_{\beta\alpha}$ or, in this case, $h_{\alpha\beta} = (\phi_\alpha/\phi_\beta)g_{\alpha\beta}$. So

$$L \simeq M \iff h_{\alpha\beta} = \frac{\phi_\alpha}{\phi_\beta} g_{\alpha\beta}$$

for a family of holomorphic functions $\phi_\alpha : U_\alpha \to \mathbb{C}^*$. The *picard group* of a complex manifold $X$ is the set of isomorphism classes of line bundles on $X$,

$$\operatorname{Pic}(X) = \{\text{line bundles on } X\}/\simeq$$

It has a natural group structure under the tensor product operation $\otimes$ between line bundles. The *neutral element* is the trivial bundle $X \times \mathbb{C}$ and the *inverse* of a line bundle $L$ is then given by its dual $L^{-1}$. The picard group is one of the most important invariants of a complex manifold.

**Example.** *$L(d) \in \operatorname{Pic}(\mathbb{P}^n)$ for all $d \in \mathbb{Z}$. By definition $L(d) \otimes L(k) = L(d+k)$ and $L(d)^{-1} = L(-d)$. So there is a subgroup of $\operatorname{Pic}(\mathbb{P}^n)$ given by*

$$\{L(d) : d \in \mathbb{Z}\} \simeq \mathbb{Z}$$

*We will prove that this subgroup is actually the whole $\operatorname{Pic}(\mathbb{P}^n)$.*

Let $\mathcal{O}^*$ be the sheaf of holomorphic non vanishing functions on $X$, that is

$$\mathcal{O}^*(U) = \{\text{holomorphic } g : U \longrightarrow \mathbb{C}^*\}$$

We consider it as a sheaf of abelian groups on $X$ under multiplication. Note that for a line bundle $L \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ we have $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$.

*Notation.* Most often we just write $\mathcal{O}$ instead of $\mathcal{O}_X$ (same with $\mathcal{O}^* = \mathcal{O}^*_X$) when it is clear from the context which underlying complex manifold $X$ we are considering. In the same fashion we will often write $H^q(\mathcal{F})$ in place of $H^q(X, \mathcal{F})$ for the cohomology groups of some sheaf $\mathcal{F}$ on $X$.

**Theorem 3.1.1.** *There is a group isomorphism* $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^*)$

This can be proved in different ways. A proof by means of Čech cohomology is given in the appendix A.2. Another way is to use the canonical resolution. The latter is harder but we still give a sketch of the map:

*Sketch of the map.* Let $\mathcal{C}^0 := \mathcal{D}(\mathcal{O}^*)$ be the sheaf of discontinous sections of $\mathcal{O}^*$. Since $\mathcal{C}^0$ is soft $H^q(X, \mathcal{C}^0) = 0$ for all $q \geq 1$. So we have an exact sequence

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{\psi} \mathcal{C}^0 \xrightarrow{\phi} Q \longrightarrow 0$$

where $Q = \mathcal{C}^0/\mathcal{O}^*$ is the quotient sheaf and $\phi$ is the quotient map. This leads to a long exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}^*) \xrightarrow{\psi_X} H^0(\mathcal{C}^0) \xrightarrow{\phi_X} H^0(Q) \xrightarrow{\delta} H^1(\mathcal{O}^*) \longrightarrow H^1(\mathcal{C}^0) = 0$$

Hence $\delta$ is surjective. Let $c \in H^1(\mathcal{O}^*)$ and $c = \delta(g)$ for some $g \in H^0(Q)$. Since $\phi : \mathcal{C}^0 \to Q$ is surjective there is an open covering $X = \bigcup U_\alpha$ such that

$$g_{|U_\alpha} = \phi_{U_\alpha}(g_\alpha)$$

for some $g_\alpha \in \mathcal{C}^0(U_\alpha)$. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then

$$\begin{aligned}
\phi_{U_{\alpha\beta}}\left((g_{\alpha|U_{\alpha\beta}}) \cdot (g_{\beta|U_{\alpha\beta}})^{-1}\right) &= \phi_{U_{\alpha\beta}}(g_{\alpha|U_{\alpha\beta}}) \cdot \phi_{U_{\alpha\beta}}(g_{\beta|U_{\alpha\beta}})^{-1} \\
&= \phi_{U_\alpha}(g_\alpha)_{|U_{\alpha\beta}} \cdot \phi_{U_\beta}(g_\beta)^{-1}_{|U_{\alpha\beta}} \\
&= g_{|U_{\alpha\beta}} \cdot g^{-1}_{|U_{\alpha\beta}} = 1
\end{aligned}$$

Hence $(g_{\alpha|U_{\alpha\beta}}) \cdot (g_{\beta|U_{\alpha\beta}})^{-1} \in \ker(\phi_{U_{\alpha\beta}}) = \mathrm{Im}(\psi_{U_{\alpha\beta}})$. Thus there exist functions $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ such that $g_{\alpha\beta} = (g_{\alpha|U_{\alpha\beta}}) \cdot (g_{\beta|U_{\alpha\beta}})^{-1}$. One then checks that

$$g_{\alpha\alpha} = 1, \quad g^{-1}_{\beta\alpha} = g_{\alpha\beta}, \quad g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

thus we can define a line bundle $L \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$. One then has to check that $L$ does not depend on the choices of the covering $\{U_\alpha\}$, of $g \in H^0(Q)$ and on the choice of the $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$. So it depends only on $c \in H^1(\mathcal{O}^*)$. Now it comes the most difficult part of the proof: to show that the map

$$H^1(\mathcal{O}^*) \longrightarrow \mathrm{Pic}(X), \quad c \longmapsto L$$

is linear and bijective, so it is an isomorphism of abelian groups. □

## 3.2 Exponential sequence and Néron-Severi group

The exponential sequence is the main tool for determining $H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$. Consider $\mathbb{Z}$ as a sheaf of locally constant functions on $X$. We have an obvious injection $j : \mathbb{Z} \hookrightarrow \mathcal{O}$. Also, between the sheaves of holomorphic functions $\mathcal{O}$ and of holomorphic invertible functions $\mathcal{O}^*$ on $X$ we have a morphism given by

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*, \quad \exp_U : f \longmapsto e^{2i\pi f}$$

*Remark* 3.2.1. exp is surjective: let $a \in X$ and $g_a \in \mathcal{O}_a^*$ with representative $g \in \mathcal{O}^*(U)$. Then on a small ball around $a$ the function $f := \frac{1}{2i\pi}\log(g)$ exists and $g = \exp_U(f)$, so $g_a = \exp_a(f_a)$.

As a consequence we get the following short exact sequence of sheaves on $X$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

which is called the *exponential sequence of* $X$. By theorem 1.3.3 this induces a long exact sequence in cohomology. We will make the following assumptions:

- $X$ connected (so $H^0(X,\mathbb{Z}) = \mathbb{Z}$).
- $X$ compact (so $H^0(X,\mathcal{O}) = \mathbb{C}$ and $H^0(X,\mathcal{O}^*) = \mathbb{C}^*$).

Then we get

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "e_0"] & \mathbb{C}^* \arrow[dll, "\delta^0"] \\
& H^1(X,\mathbb{Z}) \arrow[r, "j_1"] & H^1(X,\mathcal{O}) \arrow[r, "e_1"] & \mathrm{Pic}(X) \arrow[dll, "\delta^1"] \\
& H^2(X,\mathbb{Z}) \arrow[r, "j_2"] & H^2(X,\mathcal{O}) \arrow[r, "e_2"] & H^2(X,\mathcal{O}^*)\cdots
\end{tikzcd}$$

and $e_0 : z \mapsto e^{2i\pi z}$ is surjective. Hence $\mathrm{Im}(e_0) = \mathbb{C}^* = \ker(\delta^0)$ by exactness. Then $0 = \mathrm{Im}(\delta^0) = \ker(j_1)$. So $j_1$ is an injection and we may define the quotient

$$\mathrm{Pic}^0(X) := \frac{H^1(X,\mathcal{O})}{H^1(X,\mathbb{Z})}.$$

By exactness $\mathrm{Im}(j_1) = \ker(e_1)$, which leads to

$$\mathrm{Pic}^0(X) = \frac{H^1(X,\mathcal{O})}{\mathrm{Im}(j_1)} = \frac{H^1(X,\mathcal{O})}{\ker(e_1)} \simeq \mathrm{Im}(e_1).$$

In other words we have an inclusion of groups $\mathrm{Pic}^0(X) \hookrightarrow \mathrm{Pic}(X)$. Now define

$$\mathrm{NS}(X) := \mathrm{Im}(\delta^1) \subset H^2(X,\mathbb{Z}),$$

called the *Néron-Severi group of* $X$ (note that it is discrete). Thus

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\delta^1} \mathrm{NS}(X) \longrightarrow 0$$

is a short exact sequence. Note that $\mathrm{NS}(X) = \ker(j_2)$ by exactness.

### The case of projective varieties

The group $\mathrm{Pic}^0(X)$ carries a complex structure. If $X \hookrightarrow \mathbb{P}^N$ it turns out that

- $\mathrm{Pic}^0(X)$ is a complex torus (hard to show). Therefore
- $H^1(X, \mathcal{O}) = \mathbb{C}^g$, some integer $g \geq 0$.
- $H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$.
- $H^1(X, \mathbb{Z}) \subset H^1(X, \mathcal{O})$ is a lattice (as $H^1(X, \mathcal{O}) = H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$).
- $\mathrm{NS}(X) = \mathbb{Z}^\rho \oplus T$, with $T$ a finite group (torsion).

$\rho \geq 1$ is the *Picard number of* $X$ and depends on the complex structure of $X$.

**Example.** *If $X \subset \mathbb{P}^2$ is a cubic curve then $X \simeq \mathbb{C}/\Lambda$, so $X \simeq \mathrm{Pic}^0(X)$.*

## 3.3 Basic properties of $H^q(X, \mathbb{Z})$ and $H^q(X, \mathcal{O})$

As we have seen in section 1.5 the groups $H^q(X, \mathbb{Z})$ are nothing but the singular cohomology groups $H^q_{Sing}(X, \mathbb{Z})$. Since $X$ is compact, we know that the latter decomposes into free and torsion part and so we can write

$$H^q(X, \mathbb{Z}) \simeq \mathbb{Z}^{b_q} \oplus T_q,$$

where the integer $b_q \geq 0$ is called the *$q$-th Betti number of* $X$ and $T_q$ is a finite group. By the universal coefficients theorem one gets

$$T_1 = 0.$$

Also, we get $H^q(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = H^q(X, \mathbb{R}) = H^q_{\mathrm{dR}}(X) \simeq \mathbb{R}^{b_q}$ (tensoring with $\mathbb{R}$ kills torsion; the second equality is de Rham theorem). Hence, the Betti numbers are completely determined by the de Rham cohomology of $X$, which therefore determines the free part of $H^q(X, \mathbb{Z})$. Then, a useful way to study the free part of $H^q(X, \mathbb{Z})$ is to embed it in the de Rham groups as:

$$H^q_{\mathrm{dR}}(X, \mathbb{Z}) := \mathrm{Im}\{H^q(X, \mathbb{Z}) \longrightarrow H^q_{\mathrm{dR}}(X)\} \simeq \mathbb{Z}^{b_q}.$$

Equivalently, we can give it the following explicit description:

$$H^q_{\mathrm{dR}}(X, \mathbb{Z}) = \{[\omega] \in H^q_{\mathrm{dR}}(X) : \int_Y \omega \in \mathbb{Z} \;\text{ for all } Y \in \mathcal{K}_q(X)\},$$

where $\mathcal{K}_q(X) = \{\text{compact real submanifolds } Y \subset X \text{ of dimension } q\}$.

We now investigate $H^q(X, \mathcal{O})$. As $\mathcal{O} = \Omega^0$, we have

$$H^q(X, \mathcal{O}) = H^q(X, \Omega^0) = \frac{\{(0,q)\text{-forms } \bar{\partial}\text{-closed}\}}{\{(0,q)\text{-forms } \bar{\partial}\text{-exact}\}} \simeq \mathbb{C}^{p_q},$$

for some integers $p_q \geq 0$ (the second equality follows by Dolbeault theorem; the last one because $X$ is compact by assumption). In particular, since there are no $(0,q)$-forms on $X$ when $q > \dim(X)$, we get a "vanishing" theorem:

$$H^q(X, \mathcal{O}) = 0 \quad \text{if } q > \dim(X).$$

**Example** (Riemann Surfaces). *Let $X$ be a compact connected complex manifold of dimension 1. Then $H^2(\mathcal{O}) = 0$, so $\mathrm{NS}(X) = H^2(\mathbb{Z})$. Topologically $X$ is a compact orientable surface of genus $g$. By singular cohomology $H^1(\mathbb{Z}) = \mathbb{Z}^{2g}$ and $H^2(\mathbb{Z}) = \mathbb{Z}$. So $H^2_{\mathrm{dR}}(X) = \mathbb{R}$ (also from orientability). So $b_1 = 2g$ and $b_2 = 1$. One proves that $H^1(\mathcal{O}) = \mathbb{C}^g$, so $\mathrm{Pic}^0(X) = \mathbb{C}^g/\mathbb{Z}^{2g}$ is a torus.*

$$0 \longrightarrow \mathbb{C}^g/\mathbb{Z}^{2g} \longrightarrow \mathrm{Pic}(X) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

**Example** (complex projective space). *Let $X = \mathbb{P}^n$. Then*

$$H^q(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} 0 & q \text{ odd} \\ \mathbb{Z} & q \text{ even} \end{cases} \qquad H^q_{\mathrm{dR}}(\mathbb{P}^n) = \begin{cases} 0 & q \text{ odd} \\ \mathbb{R} & q \text{ even} \end{cases}$$

*One can prove that $H^q(\mathbb{P}^n, \mathcal{O}) = 0$ for all $q \geq 1$. Therefore, by the long exact sequence in cohomology induced by the exponential sequence, we get*

$$\mathrm{Pic}(\mathbb{P}^n) \simeq H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}.$$

*From what we said above it follows that $\mathrm{Pic}(\mathbb{P}^n) = \{L(d) : d \in \mathbb{Z}\}$.*

**Consequence.** *Let $Y \subset \mathbb{P}^n$ be an analytic hypersurface (codimension 1 submanifold). Then $Y$ is algebraic. In fact, we know $L_Y \simeq L(d)$ for some $d \in \mathbb{Z}$, and there exists a global section $s_Y$ such that $Y$ is the zero locus of $s_Y$. However, one can show that the global sections of $L(d)$ are (isomorphic to) homogeneous polynomials of degree $d$ for $d \geq 0$, while there are no nontrivial global sections for $d < 0$. Thus $s_Y \leftrightarrow F$ homogeneous polynomial of degree $d$ and $Y = Z(F)$. More generally we get Chow's lemma: if $Y \subset \mathbb{P}^n$ is a compact submanifold then it is an algebraic variety, i.e. there exists homogeneous polynomials $F_i$ such that*

$$Y = \{x \in \mathbb{P}^n : F_i(x) = 0 \quad \forall i\}$$

## 3.4 The first Chern class of a line bundle

Now that we know the basic facts about $H^1(X, \mathcal{O})$, the natural next step for studying the Picard group is to investigate the boundary map $\delta := \delta^1$ in the long exact cohomology sequence induced by the exponential sequence. That is,

$$\delta : \mathrm{Pic}(X) = H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z}).$$

Essentially, the first Chern class of a line bundle will be its image under $\delta$.

### Smooth sections of a line bundle

Suppose $p : L \to X$ is a holomorphic line bundle on $X$. Once an open covering $X = \bigcup U_\alpha$ is fixed we get trivializations $\psi_\alpha : L|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}$ and

$$L \longleftrightarrow \{U_\alpha,\ g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{C}^*\}$$

Being $L$ and $X$ complex manifold, they carry, in particular, a smooth (or $\mathcal{C}^\infty$) structure. So instead of the holomorphic sections of $L$ we can consider the *smooth* sections: smooth maps $s : U \to L$ such that $p \circ s = \mathrm{id}_U$ where $U \subset X$ is open. We denote by $\mathcal{A}^0_L$ the sheaf on $X$ of the smooth sections of $L$.

$$\mathcal{A}^0_L(U) = \{s : U \longrightarrow L,\ s \in \mathcal{C}^\infty,\ p \circ s = \mathrm{id}_U\}$$

Over each $U_\alpha$ we define the holomorphic section $s_\alpha$ by

$$s_\alpha : U_\alpha \longrightarrow L|_{U_\alpha} \quad s_\alpha(x) := \psi_\alpha^{-1}(x, 1). \tag{3.1}$$

*Remark* 3.4.1. Notice that $s_\alpha(x) = \psi_\beta^{-1} \circ \psi_\beta \circ \psi_\alpha^{-1}(x, 1) = \psi_\beta^{-1}(x, g_{\beta\alpha}(x))$ and as $\psi_\beta$ is linear this equals $g_{\beta\alpha}(x)\psi_\beta^{-1}(x, 1) = g_{\beta\alpha}(x)s_\beta(x)$. We get the identity

$$s_\alpha(x) = g_{\beta\alpha}(x)s_\beta(x). \tag{3.2}$$

Warning: (can cause confusion) the $s_\alpha$ are sections, not local descriptions!

Now, if $U \subset U_\alpha$ and $s \in \mathcal{A}^0_L(U)$ then $\psi_\alpha(s(x)) = (x, f(x)) = (x, f(x) \cdot 1)$ for some smooth $f : U \to \mathbb{C}$. Therefore, we can write $s$ locally (i.e. $x \in U$) as

$$s(x) = f(x)s_\alpha(x), \tag{3.3}$$

where the scalar product is taken in $L_x$. This local form will be useful later.

## Connections of line bundles and curvature 2-forms

For $a \in X$ we consider the complexified tangent space

$$T_aX_{\mathbb{C}} := T_aX \otimes_{\mathbb{R}} \mathbb{C},$$

and denote its dual by $T^*_aX_{\mathbb{C}}$. So we have a vector bundle on $X$

$$\bigwedge^k T^*X_{\mathbb{C}} := \bigsqcup_{a \in X} \left( \bigwedge^k T^*_aX_{\mathbb{C}} \right),$$

whose smooth sections live in the sheaf $\mathcal{A}^k$, defined by

$$\mathcal{A}^k(U) = \{\text{smooth complex-valued } k\text{-forms on } U\}.$$

In particular, for $k = 0$ we get

$$\mathcal{A}^0(U) = \{f : U \longrightarrow \mathbb{C} \ \text{ smooth}\}.$$

Recall that if $E \leftrightarrow \{U_\alpha, h_{\alpha\beta}\}$ is a rank $r$ vector bundle on $X$ then

$$L \otimes E \longleftrightarrow \{U_\alpha,\ g_{\alpha\beta}h_{\alpha\beta}\}$$

is a rank $r$ vector bundle on $X$. We define the following sheaf.

**Definition.** $\mathcal{A}^k_L$ is the sheaf of smooth sections of $\left(\bigwedge^k T^*X_{\mathbb{C}}\right) \otimes L$

*Remark* 3.4.2. A section in $\mathcal{A}^k_L(U_\alpha)$ has the form

$$\omega \otimes s = \omega \otimes (f \cdot s_\alpha) = f \cdot (\omega \otimes s_\alpha) = (f \cdot \omega) \otimes s_\alpha$$

with $\omega \in \mathcal{A}^k(U_\alpha)$ and $f$ smooth on $U_\alpha$. In other words, elements in $\mathcal{A}^k_L(U_\alpha)$ can be written as $\omega_\alpha \otimes s_\alpha$ for some $\omega_\alpha \in \mathcal{A}^k(U_\alpha)$.

*Remark* 3.4.3. Note that $\bigwedge^0 T^*X_{\mathbb{C}}$ is the trivial bundle so when $k = 0$ the general definition of $\mathcal{A}_L^k$ is coherent with the one of $\mathcal{A}_L^0$ given before.

**Definition.** A *connection* on $L$ is a sheaf homomorphism

$$\nabla : \mathcal{A}_L^0 \longrightarrow \mathcal{A}_L^1,$$

which satisfies the *Leibniz rule*: for all $f \in \mathcal{A}^0(U)$ and $s \in \mathcal{A}_L^0(U)$,

$$\nabla(fs) = df \otimes s + f\nabla s,$$

where we simply denoted $\nabla = \nabla_U$.

Remarkably, every line bundle $L$ admits a connection! It is convenient to postpone the proof of this fact and just assume it for the moment.

Suppose $\nabla$ is a connection on $L$. We can extend it to $\nabla : \mathcal{A}_L^k \to \mathcal{A}_L^{k+1}$ by

$$\nabla(\omega \otimes s) := d\omega \otimes s + (-1)^k \omega \otimes \nabla s \tag{3.4}$$

The term $\omega \otimes \nabla s$ needs a bit of explanation: if $\nabla s = \eta \otimes s$ for some $\eta$, then

$$\omega \otimes \nabla s = \omega \otimes (\eta \otimes s) = (\omega \wedge \eta) \otimes s.$$

A useful remark: for $f \in \mathcal{A}^0(U)$ and $\psi \in \mathcal{A}_L^k(U)$ an easy calculation shows

$$\nabla(f\psi) = df \otimes \psi + f\nabla\psi. \tag{3.5}$$

Therefore the Leibniz rule gets extended.

A sheaf morphism $\phi : \mathcal{A}_L^0 \longrightarrow \mathcal{A}_L^k$ is called $\mathcal{A}^0$*-linear* if for all smooth functions $f \in \mathcal{A}^0(U)$ and for all smooth bundle sections $s \in \mathcal{A}_L^0(U)$,

$$\phi_U(fs) = f\phi_U(s).$$

The importance of this property is due to the following fact:

**Lemma.** *Let $\phi : \mathcal{A}_L^0 \longrightarrow \mathcal{A}_L^k$ be a $\mathcal{A}^0$-linear morphism. Then there exists a global smooth $k$-form $\omega \in \mathcal{A}^k(X)$ such that for all smooth sections $s \in \mathcal{A}_L^0(U)$,*

$$\phi_U(s) = \omega \otimes s.$$

**Definition.** The *curvature* $F_\nabla$ of a connection $\nabla$ on $L$ is by definition

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}_L^0 \longrightarrow \mathcal{A}_L^2.$$

A calculation shows that $F_\nabla$ is $\mathcal{A}^0$-linear. Hence there exists some global 2-form $\Theta_\nabla \in \mathcal{A}^2(X)$, called the *curvature* 2*-form of* $L$, with

$$F_\nabla(s) = \Theta_\nabla \otimes s.$$

We are almost there. Our initial goal was to understand the image

$$\delta(L) \in H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{Z}) \otimes \mathbb{R} = H^2_{dR}(X).$$

In other words we want an element in $H^2_{dR}(X, \mathbb{Z})$. So far, starting from $L$ we produced $\Theta_\nabla \in \mathcal{A}^2(X)$. The first step is to show that $\Theta_\nabla$ is closed, and therefore defines a class in the de Rham group of $\mathbb{C}$-valued 2-forms, that is

$$[\Theta_\nabla] \in H^2_{dR}(X)_{\mathbb{C}} = \frac{\ker\{d : \mathcal{A}^2(X) \to \mathcal{A}^3(X)\}}{\mathrm{Im}\{d : \mathcal{A}^1(X) \to \mathcal{A}^2(X)\}}.$$

Moreover, our definition seems to depend on the choice of the connection $\nabla$. This is not the case, essentially because the space of connections on $L$ is something like an affine space on the space of global 1-forms. More precisely, let $\nabla, \nabla'$ be two connections on $L$. By the Leibniz rule we see that $\nabla - \nabla'$ is $\mathcal{A}^0$-linear. Therefore, by the lemma, there exists a global 1-form $\omega \in \mathcal{A}^1(X)$ such that for all smooth bundle sections $s \in \mathcal{A}^0_L(U)$,

$$(\nabla - \nabla')(s) = \omega \otimes s. \tag{3.6}$$

**Theorem 3.4.1.** *Let $\nabla, \nabla'$ be two connections on $L$. Then*

*(i) $\Theta_\nabla$ is closed. In particular $\Theta_\nabla$ defines a class $[\Theta_\nabla] \in H^2_{dR}(X)_{\mathbb{C}}$*

*(ii) $[\Theta_\nabla] = [\Theta_{\nabla'}]$*

*Proof.* We show that $\Theta_\nabla$ is locally exact, hence closed. Fix an open covering $\{U_\alpha\}$ on $X$ and consider the sections $s_\alpha$ as in 3.1. By remark 3.4.2 we have $\nabla s_\alpha = \omega_\alpha \otimes s_\alpha$ for some $\omega_\alpha \in \mathcal{A}^1(U_\alpha)$. We show that $\Theta_\nabla|_{U\alpha} = d\omega_\alpha$. Indeed,

$$\begin{aligned}
\Theta_\nabla|_{U_\alpha} \otimes s_\alpha &= \nabla^2 s_\alpha = \nabla(\omega_\alpha \otimes s_\alpha) = d\omega_\alpha \otimes s_\alpha + (-1)^1 \omega_\alpha \otimes \nabla s_\alpha \\
&= d\omega_\alpha \otimes s_\alpha - (\omega_\alpha \otimes (\omega_\alpha \otimes s_\alpha)) \\
&= d\omega_\alpha \otimes s_\alpha - (\omega_\alpha \wedge \omega_\alpha) \otimes s_\alpha \\
&= d\omega_\alpha \otimes s_\alpha.
\end{aligned}$$

Now, let $\nabla'$ be another connection. Let $\omega \in \mathcal{A}^1(X)$ be as in 3.6. Then,

$$\begin{aligned}
\Theta_\nabla \otimes s &= \nabla^2 s = \nabla(\nabla' s + \omega \otimes s) \\
&= (\nabla' + \omega)(\nabla' s + \omega \otimes s) \\
&= (\nabla')^2 s + \nabla'(\omega \otimes s) + \omega \otimes \nabla' s + \omega \otimes (\omega \otimes s) \\
&= (\nabla')^2 s + d\omega \otimes s + (-1)^1 \omega \otimes \nabla' s + \omega \otimes (\nabla' s) \\
&= \Theta_{\nabla'} \otimes s + d\omega \otimes s \\
&= (\Theta_{\nabla'} + d\omega) \otimes s, which concludes the proof.
\end{aligned}$$

Hence $\Theta_\nabla = \Theta_{\nabla'} + d\omega$. ☐

Now we have a well defined class $[\Theta_\nabla] \in H^2_{dR}(X)_{\mathbb{C}}$. We want a real form.

**Definition.** The *first Chern class* of $L$ is defined by

$$c_1(L) := \frac{i}{2\pi} [\Theta_\nabla].$$

## Hermitian metrics

We aim to show that $c_1(L)$ is a real 2-form, that is $c_1(L) \in H^2_{dR}(X)$. We do so by means of Hermitian metrics. This is particularly nice since we also get an explicit local expression for $c_1(L)$. Moreover, we will pay off a debt: the existence of a connection on any line bundle. Let $p : L \to X$ be a line bundle.

**Definition.** A *Hermitian metric* $h$ on $L$ is a scalar product $h(a)$ on each fiber $L_a \simeq \mathbb{C}$ which depends smoothly on $a \in X$. In other words, if $s, t$ are smooth sections of $L$, the function $a \mapsto h(a)(s(a), t(a))$ is smooth.

Fix $\{U_\alpha\}$ with local nowhere vanishing sections $s_\alpha$ as in 3.1. We define

$$h_\alpha : U_\alpha \longrightarrow \mathbb{R}_+ \quad a \longmapsto h_\alpha(a) = h(a)(s_\alpha(a), s_\alpha(a)) > 0.$$

For two smooth sections of $L$, say $s = fs_\alpha$ and $t = gs_\alpha$ (cf. 3.3), we get[1]

$$h(a)(s(a), t(a)) = f(a)\overline{g(a)}h_\alpha(a).$$

Hence the collection of functions $h_\alpha$ determines the metric $h$ uniquely!

**Lemma.** *Any line bundle admits a Hermitian metric.*

*Proof.* Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to the open covering $\{U_\alpha\}$ of $X$. Locally on each $U_\alpha$ we have the metric $\tilde{h}_\alpha(x)(s(x), t(x)) = f(x)\overline{g(x)}$ where $s = fs_\alpha$ and $t = gs_\alpha$. Then $h := \sum \rho_\alpha \tilde{h}_\alpha$ defines a metric on $L$, i.e.

$$h(x)(s(x), t(x)) := \sum_\alpha \rho_\alpha(x)\tilde{h}_\alpha(x)(s(x), t(x)). \qquad \square$$

Let $\nabla$ be a connection on $L \leftrightarrow \{U_\alpha,\ g_{\alpha\beta}\}$. By remark 3.4.2, $\nabla s_\alpha = \omega_\alpha \otimes s_\alpha$, for some $\omega_\alpha \in \mathcal{A}^1(U_\alpha)$. The point is that the data $\{\omega_\alpha\}$ completely determines the connection: indeed, any smooth bundle section $s$ can be written locally as $s = fs_\alpha$, therefore knowing $\nabla s_\alpha$ determines $\nabla s$ (by the Leibniz rule). The collection $\{\omega_\alpha\}$ then has to satisfy some gluing conditions on the intersections $U_{\alpha\beta}$. As the $g_{\alpha\beta}$ are holomorphic[2], we have $\bar{\partial} g_{\alpha\beta} = 0$. We have

$$\begin{aligned}
\omega_\alpha \otimes s_\alpha = \nabla s_\alpha = \nabla(g_{\beta\alpha} s_\beta) &= dg_{\beta\alpha} \otimes s_\beta + g_{\beta\alpha} \nabla s_\beta \\
&= \partial g_{\beta\alpha} \otimes g_{\alpha\beta} s_\alpha + g_{\beta\alpha}(\omega_\beta \otimes s_\beta) \\
&= g_{\alpha\beta} \partial g_{\beta\alpha} \otimes s_\alpha + g_{\beta\alpha} g_{\alpha\beta}(\omega_\beta \otimes s_\alpha) \\
&= (g_{\alpha\beta} \partial g_{\beta\alpha} + \omega_\beta) \otimes s_\alpha.
\end{aligned}$$

Thus, the gluing condition for the $\omega_\alpha$'s is the following:

$$\omega_\alpha = g_{\beta\alpha}^{-1} \partial g_{\beta\alpha} + \omega_\beta. \tag{3.7}$$

**Theorem 3.4.2.** *Let $L$ be a line bundle on $X$ with a Hermitian metric $h$. Then*

*(a) There exists a connection $\nabla$ on $L$ induced by $h$, determined by*

$$\omega_\alpha = h_\alpha^{-1} \partial h_\alpha$$

*(b) The local expression for $c_1(L)$ is given by*

$$\frac{i}{2\pi} \Theta_\nabla|_{U_\alpha} = \frac{1}{2i\pi} \partial\bar{\partial} \log h_\alpha$$

*(c) $c_1(L) \in H^2_{\mathrm{dR}}(X)$. In other words $c_1(L)$ is a real 2-form of type $(1, 1)$.*

[1] recall that an Hermitian scalar product satisfies $h(az, bw) = a\bar{b}h(z, w)$

[2] recall the decomposition $d = (\partial + \bar{\partial}) : \mathcal{A}^0 \longrightarrow \mathcal{A}^1 = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$

*Proof.* (a) we need to show that $\omega_\alpha := h_\alpha^{-1}\partial h_\alpha$ satisfies 3.7. This is achieved by using $0 = \overline{(\bar\partial g_{\beta\alpha})} = \partial(\overline{g_{\beta\alpha}})$ and the following expression

$$h_\alpha = h(s_\alpha, s_\alpha) = h(g_{\beta\alpha}s_\beta, g_{\beta\alpha}s_\beta) = g_{\beta\alpha}\overline{g}_{\beta\alpha}h_\beta.$$

(b) recall that $\Theta_\nabla$ is locally exact and equals $d\omega_\alpha$ (cf. proof of theorem 3.4.1). Now, $d\omega_\alpha = (\partial + \bar\partial)(h_\alpha^{-1}\partial h_\alpha)$ and

$$\partial(h_\alpha^{-1}\partial h_\alpha) = -h_\alpha^{-2}\partial h_\alpha \wedge \partial h_\alpha + h_\alpha^{-1}\partial^2 h_\alpha = 0.$$

Hence $\Theta_\nabla|_{U_\alpha} = \bar\partial(h_\alpha^{-1}\partial h_\alpha) = \bar\partial(\partial \log h_\alpha)$. Recalling $\partial\bar\partial = -\bar\partial\partial$ and dividing by $-2i\pi$ leads to the statement of (b).
(c) Let $f = \log h_\alpha : U_\alpha \to \mathbb{R}$, a real function. Locally

$$c_1(L) = \frac{1}{2i\pi}\partial\bar\partial f = \frac{1}{2i\pi}\sum \frac{\partial^2 f}{\partial z_j \partial \bar z_k} dz_j \wedge d\bar z_k.$$

Since $\bar f = f$, we get $\overline{c_1(L)} = (-1)(-1)c_1(L) = c_1(L)$, which proves (c). $\square$

Finally, one can prove (using Čech cohomology)

$$\delta(L) = -c_1(L).$$

## 3.5 The fundamental class of a hypersurface

Let $Y \subset X$ be a compact submanifold of codimension 1. We have a line bundle $L_Y \in \mathrm{Pic}(X)$ (cf. section 2.7), hence a class $c_1(L_Y) \in H^2_{\mathrm{dR}}(X)$ (notice that $Y$ has *real* codimension 2). Here we want to briefly discuss another way to get the map $Y \mapsto c_1(L_Y)$. There is a well-defined $\mathbb{R}$-linear map[3] ($n = \dim_{\mathbb{C}} X$)

$$H^{2n-2}_{\mathrm{dR}}(X) \longrightarrow \mathbb{R}, \quad [\omega] \longmapsto \int_Y \omega.$$

In other words, the operator $\int_Y$ is an element in the dual $H^{2n-2}_{\mathrm{dR}}(X)^\vee$. Since $X$ is a compact orientable manifold, by Poincaré duality we have a perfect pairing

$$H^k_{\mathrm{dR}}(X) \times H^{2n-k}_{\mathrm{dR}}(X) \longrightarrow \mathbb{R}, \quad ([v],[\omega]) \longmapsto \int_X v \wedge \omega.$$

Thus, any element in $H^{2n-k}_{\mathrm{dR}}(X)^\vee$ is of the form $[\omega] \mapsto \int_X v \wedge \omega$ for some unique class $[v] \in H^k_{\mathrm{dR}}(X)$. In our particular case, $\int_Y \in H^{2n-2}_{\mathrm{dR}}(X)^\vee$ is given by

$$\int_Y \omega = \int_X \theta_Y \wedge \omega,$$

for some unique class $[\theta_Y] \in H^2_{\mathrm{dR}}(X)$, called the *fundamental class* of $Y$. As a matter of fact, one can show that

$$[\theta_Y] = c_1(L).$$

[3] it is well-defined by Stoke's theorem: $\int_Y \omega = \int_Y (\omega + d\eta)$, where $\int_Y d\eta = 0$.

# Chapter 4

# Kähler manifolds

## 4.1 The Fubini-Study 2-form on $\mathbb{P}^n$

We consider the case $X = \mathbb{P}^n$, with its standard covering $\{U_\alpha\}$.

**Definition.** The *Fubini-Study 2-form* $\omega_{FS}$ on $\mathbb{P}^n$ is the 2-form given locally by

$$\omega_{FS}|_{U_\alpha} = \frac{1}{2i\pi}\,\partial\bar\partial \log\left(\frac{x_\alpha \overline{x}_\alpha}{x_0\overline{x}_0 + \cdots + x_n\overline{x}_n}\right).$$

It is tempting to split the logarithm into a difference, but the numerator and denominator are not homogeneous functions (their ratio does)! What we can do is to pull back $\omega_{FS}$ by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and then split:[1]

$$\begin{aligned}
\pi^*(\omega_{FS}|_{U_\alpha}) &= \frac{1}{2i\pi}\,\partial\bar\partial \log\left(\frac{z_\alpha \overline{z}_\alpha}{\sum z_j \overline{z}_j}\right)\\
&= \frac{1}{2i\pi}\,\partial\bar\partial\left(\log z_\alpha + \log \overline{z}_\alpha - \log\left(\sum z_j\overline{z}_j\right)\right)\\
&= \frac{i}{2\pi}\,\partial\bar\partial \log\left(\sum z_j \overline{z}_j\right) =: \widetilde{\omega}_{FS}.
\end{aligned}$$

We call $\widetilde{\omega}_{FS}$ the *Fubini-Study* 2-form on the punctured space $\mathbb{C}^{n+1} \setminus \{0\}$.

This is not random. The Fubini-Study 2-form is related to the Chern class of a very important line bundle on $\mathbb{P}^n$: the *hyperplane bundle* $p : L(1) \to \mathbb{P}^n$ (introduced in section 2.2). The construction is as follows.

On each fiber $L(1)_a = \{(x_0 : \ldots : x_n : s_a) |\, s_a \in \mathbb{C}\} \simeq \mathbb{C}$, we define

$$h(a)\left((x_0 : \ldots : x_n : s_a), (x_0 : \ldots : x_n : t_a)\right) = \frac{s_a \bar{t}_a}{x_0\overline{x}_0 + \cdots + x_n\overline{x}_n}.$$

This yields a Hermitian metric $h$ on $L(1)$. Recall the trivializations

$$\psi_\alpha : L(1)|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}, \quad (x_0 : \ldots : x_{n+1}) \longmapsto \left((x_0 : \ldots : x_n), \tfrac{x_{n+1}}{x_\alpha}\right)$$

For $a \in U_\alpha$ we then have the nowhere vanishing section $s_\alpha$, defined by

$$s_\alpha(a) = \psi_\alpha^{-1}(a, 1) = (x_0 : \ldots : x_n : x_\alpha).$$

[1] recall: $\bar\partial \log z = 0$ (holomorphic), $\partial \log \overline{z} = 0$ (antiholomorphic) and $\partial\bar\partial = -\bar\partial\partial$

Let $\nabla$ be the connection defined by $h$. By theorem 3.4.2, we get

$$\begin{aligned}\frac{i}{2\pi}\,\Theta_\nabla|_{U_\alpha} &= \frac{1}{2i\pi}\,\partial\bar\partial\log h(s_\alpha,s_\alpha)\\ &= \frac{1}{2i\pi}\,\partial\bar\partial\log\left(\frac{x_\alpha\overline{x}_\alpha}{x_0\overline{x}_0+\cdots+x_n\overline{x}_n}\right) = \omega_{FS}|_{U_\alpha}.\end{aligned}$$

Therefore $\omega_{FS}$ is a representative of the first Chern class of $L(1)$ (in particular it is $d$-closed). However, its importance goes beyond this. The Fubini-Study 2-form brings much more informations than expected.

**Lemma.** *$\omega_{FS}$ is not $d$-exact. Hence its class is non-zero in $H^2_{\mathrm{dR}}(\mathbb{P}^n)$.*

*Proof.* Assume by contradiction $\omega_{FS} = d\eta$ with $\eta\in\mathcal{E}^1(\mathbb{P}^n)$. By Stokes' theorem

$$\int_Z \omega_{FS} = \int_Z d\eta = \int_{\partial Z}\eta = 0,$$

for any $Z\subset\mathbb{P}^n$ compact of real dimension 2 (in fact $Z$ is an *algebraic* subvariety of $\mathbb{P}^n$, therefore $\partial Z$ is empty). Let $Z$ be the line parametrized by

$$\varphi:\mathbb{C}\longrightarrow\mathbb{P}^n,\quad \varphi(z) = (1:z:0:\ldots:0)\in U_0\subset\mathbb{P}^n.$$

In other words $Z=\mathbb{P}^1=\varphi(\mathbb{C})\cup(0:1:0:\ldots:0)$. We claim that $\int_Z\omega_{FS}=1$.

$$\begin{aligned}\varphi^*(\omega_{FS}|_{U_0}) &= \frac{1}{2i\pi}\,\partial\bar\partial\log\left(\frac{1}{1+z\bar z}\right)\\ &= \frac{i}{2\pi}\,\partial\bar\partial\log(1+z\bar z)\\ &= \frac{i}{2\pi}\,\partial\left(\frac{1}{1+z\bar z}\,zd\bar z\right)\qquad(\bar\partial(z\bar z)=zd\bar z)\\ &= \frac{i}{2\pi}\left(\frac{-1}{(1+z\bar z)^2}\,\bar zdz\wedge zd\bar z+\frac{1}{1+z\bar z}\,dz\wedge d\bar z\right)\\ &= \frac{i}{2\pi}\,\frac{1}{(1+z\bar z)^2}\,dz\wedge d\bar z\end{aligned}$$

Using polar coordinates we get

$$\int_Z\omega_{FS} = \int_{\mathbb{C}}\frac{i}{2\pi}\,\frac{1}{(1+z\bar z)^2}\,dz\wedge d\bar z = \frac{1}{\pi}\int_0^\infty\int_0^{2\pi}\frac{rdrd\vartheta}{(1+r^2)^2} = 1 \qquad\square$$

## 4.2 Riemannian metrics and Kähler manifolds

Recall the usual identification of a complex manifold $X$ with its underlying real manifold, which we still denote by $X$. Given local holomorphic coordinates $z=(z_1,\ldots,z_n)$, with $z_j=x_j+iy_j$, we identify, as usual

$$(z_1,\ldots,z_n)\longleftrightarrow(x_1,\ldots,x_n,y_1,\ldots,y_n).$$

For tangent vectors $v\in TX$ we identify

$$\begin{aligned}v &= \sum a_j\frac{\partial}{\partial x_j}+b_j\frac{\partial}{\partial y_j}\quad(\in TX)\\ &= \sum(a_j+ib_j)\frac{\partial}{\partial z_j}+(a_j-ib_j)\frac{\partial}{\partial\bar z_j}\quad(\in T^{1,0}\oplus T^{0,1}).\end{aligned}$$

This yields a natural inclusion $TX \hookrightarrow TX \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$. Recall the definition of the "complex structure" endomorphism

$$J : TX \longrightarrow TX, \quad \frac{\partial}{\partial x_j} \longmapsto \frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial y_j} \longmapsto -\frac{\partial}{\partial x_j},$$

whose eigenspaces are $T^{1,0}$ (with eigenvalue $i$) and $T^{0,1}$ (with eigenvalue $-i$). Often, $J$ is simply denoted by $i$.

*Remark* 4.2.1. A 2-form $\omega$ on $X$ is such that

$$\omega(v, w) = \omega(Jv, Jw) \tag{4.1}$$

(at each point, for each $v, w$) if and only if $\omega$ is of type $(1,1)$. Indeed, if $\omega$ is of type $(1,1)$, i.e. locally $\omega = \sum f_{jk} dz_j \wedge d\bar{z}_k$, then $\omega(v,w) = \sum f_{jk}(v_j \bar{w}_k - w_j \bar{v}_k)$. Since $Jv = iv$ and $Jw = iw$, we get 4.1. Conversely, suppose 4.1 holds. We can split $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$. Let $\omega^{2,0} = \sum g_{jk} dz_j \wedge dz_k$. Therefore we get $\omega^{2,0}(v,w) = \sum g_{jk}(v_j w_k - w_j v_k)$, so $\omega^{2,0}(Jv, Jw) = i^2 \omega^{2,0}(v,w) = -\omega^{2,0}(v,w)$. Thus $\omega^{2,0} = 0$. Similarly one finds $\omega^{0,2} = 0$. Hence $\omega$ is of $(1,1)$ type.

**Definition.** A *Riemannian metric* $g$ on a differentiable manifold $X$ is a family of positive definite inner products on each tangent space

$$g_p : T_pX \times T_pX \longrightarrow \mathbb{R}, \quad (v, w) \longmapsto g_p(v, w).$$

Precisely, each $g_p$ is a symmetric, positive definite, $\mathbb{R}$-bilinear form on $T_pX$. A *Kähler manifold* is a complex manifold $X$ with a Riemannian metric $g$ such that

(i) $g$ preserves the complex structure, i.e. $g(v,w) = g(Jv, Jw)$.

(ii) $d\omega = 0$, where $\omega$ is the 2-form defined by

$$\omega(v, w) := g(v, -Jw).$$

Such a $g$ is called a *Kähler metric*, and $\omega$ a *Kähler form.* Notice that

$$g(v, w) = g(v, -J^2 w) = \omega(v, Jw).$$

**Proposition 4.2.1.** *The Fubini-Study form is* positive*, in the following sense: for each non-zero tangent vector $v \in T\mathbb{P}^n$ we have $\omega_{FS}(v, Jv) > 0$.*

One can prove this by observing that $\widetilde{\omega}_{FS}$ is invariant under the *unitary transformations* of $\mathbb{C}^{n+1}$ (transformations which preserves the inner product), and so it suffices to check positivity at one point $p \in \mathbb{P}^n$. For example, a direct computation at $p = (1 : 0 : \cdots : 0)$ is easy.

**Corollary 4.2.1.** *$\mathbb{P}^n$ has a Riemannian metric, defined by*

$$g(v, w) := \omega_{FS}(v, Jw).$$

*Moreover, $\mathbb{P}^n$ is a Kähler manifold.*

*Proof.* Linearity is obvious, and $g$ is positive definite by the proposition. We only need to show that $g$ is symmetric. Since $\omega_{FS}$ is of type $(1,1)$,

$$\begin{aligned} g(w,v) &= \omega_{FS}(w,Jv) \\ &= \omega_{FS}(Jw,J^2v) \quad \text{(remark 4.2.1)} \\ &= \omega_{FS}(Jw,-v) \quad (J^2 = -\operatorname{id}) \\ &= -\omega_{FS}(-v,Jw) \quad (\omega(a,b) = -\omega(b,a)) \\ &= \omega_{FS}(v,Jw) = g(v,w) \end{aligned}$$

Finally, $\omega_{FS}$ is $d$-closed, for it defines a class in cohomology, the first Chern class of $L(1)$, as we have seen. Also, we clearly have $g(v,w) = g(Jv,Jw)$. Hence $\mathbb{P}^n$ is a Kähler manifold, as claimed. □

## 4.3 Kodaira embedding theorem

Let $g$ be a Kähler metric on $\mathbb{P}^n$, and $\omega$ the related Kähler form. Given a complex submanifold $Y \hookrightarrow \mathbb{P}^n$ (i.e. a smooth projective variety), we get a Kähler form on $Y$ (just restrict $g$ to $TY$). Precisely, let $\omega|_Y = f^*\omega$, where $f : Y \to \mathbb{P}^n$ is an embedding[2]. Thus, any smooth projective variety is a Kähler manifold. The converse is false. We will see the example of complex tori of dimension 2, which are all Kähler, though some do not admit any projective embedding.

*Remark* 4.3.1. If $X$ is a Kähler manifold, any Kähler 2-form $\omega$ is not $d$-exact (we have seen it for $\omega_{FS}$). Hence its class is non-trivial in $H^2_{\mathrm{dR}}(X)$. More generally, one has $[\omega^k] \neq 0$ in $H^{2k}_{\mathrm{dR}}(X)$, where $\omega^k = \omega \wedge \cdots \wedge \omega$ (obviously $k \leq \dim_{\mathbb{C}} X$).

Recall

$$H^2_{\mathrm{dR}}(X,\mathbb{Z}) = \{[\omega] \in H^2_{\mathrm{dR}}(X) : \int_Y \omega \in \mathbb{Z} \ \text{ for all } Y \in \mathcal{K}_2(X)\},$$

where $\mathcal{K}_2(X)$ is the set of compact (real) 2-dimensional submanifolds of $X$.

**Definition.** An element $[\omega] \in H^2_{\mathrm{dR}}(X,\mathbb{Z})$ such that $\omega$ is a Kähler form, is called an *integral Kähler class.*

The importance of this concept is evident, in light of the following remarkable result, proved in the 60's by Kunihiko Kodaira.

**Theorem 4.3.1.** *Let $X$ be a compact complex manifold. Then $X$ is projective (i.e. $X \hookrightarrow \mathbb{P}^N$) if and only if it admits an integral Kähler class.*

Given a compact complex manifold $X$ and an integral Kähler class $[\omega]$, the idea of the proof is to show that $[\omega] = c_1(L)$, for some line bundle $L \in \operatorname{Pic}(X)$, which is *ample*, in the sense that for some $k \in \mathbb{N}$ the global sections of $L^{\otimes k}$ define an immersion $\varphi : X \hookrightarrow \mathbb{P}^N$.

[2] moreover $c_1(L(1)|_Y) = [f^*\omega_{FS}]$

## 4.4 Lefschetz $(1,1)$ theorem

Let $X$ be a compact Kähler manifold. By Hodge theory, one can prove that there exists the following orthogonal decomposition, called *Hodge decomposition*,

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

We denote by $\pi^{0,2} : H^2(X,\mathbb{C}) \to H^{0,2}(X)$ the projection on this third factor.

Moreover, one has an isomorphism (*Hodge duality*)

$$H^{p,q}(X) \simeq \overline{H^{q,p}(X)}.$$

Hence, given $\omega \in H^2_{\mathrm{dR}}(X) \subset H^2(X,\mathbb{C})$, we can split it as

$$\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$$

where $\omega^{p,q} \in H^{p,q}(X)$, with $\overline{\omega^{2,0}} = \omega^{0,2}$ and $\omega^{1,1} = \overline{\omega^{1,1}}$ (which means that $\omega^{1,1}$ is a *real form*). In particular, we get $\ker(\pi^{0,2}) = H^{1,1}(X)$. We have the commutative diagram

$$\begin{tikzcd}
 & H^2_{\mathrm{dR}}(X,\mathbb{Z}) \arrow[r, hook] & H^2_{\mathrm{dR}}(X) \arrow[r, hook] \arrow[d, dashed, "\varepsilon'"] & H^2_{\mathrm{dR}}(X,\mathbb{C}) \arrow[d, "\pi^{0,2}"] \\
\mathrm{Pic}(X) \arrow[ur, bend left, "c_1"] \arrow[r, "\delta"] & H^2(X,\mathbb{Z}) \arrow[r, "\varepsilon"] & H^2(X,\mathcal{O}_X) \arrow[r, "\simeq"] & H^{0,2}(X)
\end{tikzcd}$$

where the isomorphism on the bottom-right is a particular case of Dolbeault theorem (since $\Omega^0_X \simeq \mathcal{O}_X$), the map $c_1$ is the first Chern class and the maps $\delta$ and $\varepsilon$ arise from the exponential sequence. Therefore, $\varepsilon'$ is interpreted as the restriction of $\pi^{0,2}$ to $H^2_{\mathrm{dR}}(X)$. Thus, $\ker(\varepsilon') = H^2_{\mathrm{dR}}(X) \cap H^{1,1}(X)$. Recall that by definition $\ker(\varepsilon) = \mathrm{Im}(\delta) =: \mathrm{NS}(X)$, the Néron-Severi group of $X$. By the commutativity of the diagram we get the following fundamental result, known as the Lefschetz theorem on $(1,1)$-classes:

$$\mathrm{NS}(X) \simeq H^2_{\mathrm{dR}}(X,\mathbb{Z}) \cap H^{1,1}(X).$$

### The Hodge conjecture

Let $Y \subset X$ be a compact complex submanifold of codimension $p$. Then we can define a *fundamental class* $[\theta_Y] \in H^{2p}_{\mathrm{dR}}(X)$, in an analogous way as we did for the case $p = 1$. We have discussed the fact that, when $p = 1$, we have $[\theta_Y] = c_1(L_Y) \in H^2_{\mathrm{dR}}(Y,\mathbb{Z})$. Moreover, if $X$ is a compact Kähler manifold, the Lefschetz $(1,1)$ theorem is equivalent to the fact that

$$H^2_{\mathrm{dR}}(X,\mathbb{Q}) \cap H^{1,1}(X)$$

is generated by the classes $[\theta_Y]$ of codimension 1 submanifolds $Y \subset X$. The *Hodge conjecture* aims at a generalization of this result, with a stronger hypothesis: let $X$ be a complex *projective* manifold. Then[3]

$$H^{2p}_{\mathrm{dR}}(X,\mathbb{Q}) \cap H^{p,p}(X)$$

is generated by the classes $[\theta_Y]$ of codimension $p$ submanifolds $Y \subset X$. This is one of the major unsolved problems in mathematics.

[3]notice: we take coefficients in $\mathbb{Q}$ here. The case $p = 1$, by Lefschetz $(1,1)$, is indeed valid with integer coefficients. However, there is no hope to generalize this if we restrict to $\mathbb{Z}$

## 4.5 Complex tori and abelian varieties

A *complex torus* is by definition a quotient

$$X = V/\Gamma,$$

where $V \simeq \mathbb{C}^n$ is a complex $n$-dimensional vector space and $\Gamma \simeq \mathbb{Z}^{2n}$ is a *lattice* in $V$, i.e. a discrete subgroup of $V$ which $\mathbb{R}$-spans the whole $V \simeq \mathbb{R}^{2n}$. In other words $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} = V$, so there exists a $\mathbb{Z}$-basis $e_1, \ldots, e_{2n} \in V$ for $\Gamma$,

$$\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{2n}$$
$$V = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{2n}$$

As a differentiable manifold, we then have a diffeomorphism

$$X \simeq (\mathbb{R}/\mathbb{Z})^{2n} = (S^1)^{2n}$$

where $S^1$ is the circle. Hence, the topology of $X$ is fixed, and quite simple to understand. However, complex torus $X$ can admit several (non-equivalent) complex structures, and the situation is extremely rich from this point of view.

### Topology

We need to recall the following result (which is valid in a more general setting).

**Theorem 4.5.1** (Künneth formula). *Let $M, N$ be real compact manifolds. Then*

$$H^k(M \times N, \mathbb{Z}) \simeq \bigoplus_{p+q=k} H^p(M, \mathbb{Z}) \otimes H^q(N, \mathbb{Z}).$$

*When we take coefficients in $\mathbb{R}$, the isomorphism is given by*

$$\bigoplus_{p+q=k} H^p_{\mathrm{dR}}(M) \otimes H^q_{\mathrm{dR}}(N) \longrightarrow H^k_{\mathrm{dR}}(M \times N)$$
$$[\omega] \otimes [\eta] \longmapsto [\pi_M^* \omega \wedge \pi_N^* \eta]$$

*where $\pi_M$ and $\pi_N$ are the projections from $M \times N$ onto the two factors.*

Recall $H^0_{\mathrm{dR}}(S^1) = \mathbb{R}$ and $H^1_{\mathrm{dR}}(S^1) = \mathbb{R}dx$, where we fix the generator $dx$ to be a 1-form on $S^1$ which integrates to 1, so that $H^1_{\mathrm{dR}}(S^1, \mathbb{Z}) = \mathbb{Z}dx$.

When $X = (S^1)^{2n}$, we get by Künneth formula

$$H^k_{\mathrm{dR}}(X) = \bigoplus_{a_1+\cdots+a_{2n}=k} H^{a_1}_{\mathrm{dR}}(S^1) \otimes \cdots \otimes H^{a_{2n}}_{\mathrm{dR}}(S^1).$$

We must simplify the notation. Given $a_1, \ldots, a_{2n} \in \{0,1\}$ such that $\sum a_i = k$, we let $I = \{i_1 < \ldots < i_k\}$ be such that $a_{i_j} = 1$, for $j = 1, \ldots, k$. Then, we denote by $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. Thus $H^{a_1}_{\mathrm{dR}}(S^1) \otimes \cdots \otimes H^{a_{2n}}_{\mathrm{dR}}(S^1) = \mathbb{R}dx_I$, where the generator $dx_I$ is the $k$-form with components $dx_i = \pi_i^* dx$, where $\pi_i$ is the projection $X \to S^1$ ($i$-th factor), and $dx$ the generator of $H^1_{\mathrm{dR}}(S^1)$. Hence,

$$H^k_{\mathrm{dR}}(X) = \bigoplus_{\#I=k} \mathbb{R}dx_I$$

so that any class $[\omega] \in H^k_{\mathrm{dR}}(X)$ has a representative $\omega = \sum c_I dx_I$, with $c_I \in \mathbb{R}$ constants (on each point of $X$). Therefore $\omega$ is completely determined by its values on the tangent space at the origin $T_0X \times \cdots \times T_0X$. However, $X$ has trivial tangent bundle $TX = X \times \mathbb{R}^{2n}$. Consequently, by $V \simeq T_0X$, we have

$$H^k_{\mathrm{dR}}(X) \simeq \mathrm{Alt}^k(V, \mathbb{R}),$$

where $\mathrm{Alt}^k(V, \mathbb{R})$ is the vector space of alternating $k$-linear maps $V \times \cdots \times V \to \mathbb{R}$.

Similarly, since $dx$ is also a generator of $H^1_{\mathrm{dR}}(S^1, \mathbb{Z})$, one has

$$H^k_{\mathrm{dR}}(X, \mathbb{Z}) = \bigoplus_{\#I=k} \mathbb{Z} dx_I \simeq \mathrm{Alt}^k(\Gamma, \mathbb{Z}).$$

where $\mathrm{Alt}^k(\Gamma, \mathbb{Z})$ is the vector space of alternating $k$-linear maps $\Gamma \times \cdots \times \Gamma \to \mathbb{Z}$.

Notice that each $f \in \mathrm{Alt}^k(\Gamma, \mathbb{Z})$ extends $\mathbb{R}$-linearly to a form $\tilde{f} \in \mathrm{Alt}^k(V, \mathbb{R})$. Hence, given $f \in \mathrm{Alt}^k(\Gamma, \mathbb{Z})$, we can define a $k$-form $\omega_f$ on $X$ as follows. Given vectors $v_i \in T_pX \simeq V$, we let $\omega_f(p)(v_1, \ldots, v_k) := \tilde{f}(v_1, \ldots, v_k)$, for each $p \in X$. In particular, $d\omega_f = 0$, as $\omega_f$ does not depend on $p$.

## Kähler structure and Néron-Severi group

We consider a complex torus $X = V/\Gamma$. Given two tangent vectors

$$\begin{aligned} v &= \sum a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} = \sum v_j \frac{\partial}{\partial z_j} + \bar{v}_j \frac{\partial}{\partial \bar{z}_j} \quad (v_j = a_j + ib_j), \\ w &= \sum c_j \frac{\partial}{\partial x_j} + d_j \frac{\partial}{\partial y_j} = \sum w_j \frac{\partial}{\partial z_j} + \bar{w}_j \frac{\partial}{\partial \bar{z}_j} \quad (w_j = c_c + id_j), \end{aligned}$$

we define $g(v, w)$ to be the standard inner product of $v$ and $w$,

$$g(v, w) := \Re(\sum v_j \bar{w}_j) = \sum_{j=1}^n (a_j c_j + b_j d_j).$$

Hence, $g$ is independent of the point of tangency of $v, w$. Moreover, we see that $g(iv, iw) = g(v, w)$. The 2-form defined by

$$\omega(v, w) = g(v, -iw) = \Re(i \sum v_j \bar{w}_j),$$

is also independent of $p \in X$. Hence $d\omega = 0$. Therefore any complex torus is Kähler. Since $X$ is also compact, we have the Hodge decomposition

$$\begin{aligned} H^1(X, \mathbb{C}) &= H^{1,0}(X) \oplus H^{0,1}(X) \\ &= (\bigoplus_{j=1}^n \mathbb{C} dz_j) \oplus (\bigoplus_{j=1}^n \mathbb{C} d\bar{z}_j). \end{aligned}$$

More generally, one has $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, where

$$H^{p,q}(X) = \bigoplus_{\substack{\#I=p \\ \#J=q}} \mathbb{C} dz_I \wedge d\bar{z}_J.$$

In particular, by Lefschetz $(1,1)$ theorem we get $\mathrm{NS}(X) = H^2_{\mathrm{dR}}(X,\mathbb{Z}) \cap H^{1,1}(X)$. Let $\omega \in \mathrm{NS}(X)$. Then, for some coefficients $m_{ij} \in \mathbb{Z}$ and $a_{kl} \in \mathbb{R}$ we have two expressions for $\omega$,

$$\omega = \sum_{1 \le i < j \le 2n} m_{ij} dx_i \wedge dx_j \quad \in H^2_{\mathrm{dR}}(X,\mathbb{Z}),$$

$$\omega = \sum_{k,l=1}^{n} a_{kl} dz_k \wedge \bar{z}_l \quad \in H^{1,1}(X).$$

The fact that these two expression must equate, suggests that we find a relation between a $\mathbb{Z}$-basis of $\Gamma$ and a $\mathbb{C}$-basis of $V$ (or the respective dual basis).

One can show that for the "generic" complex torus $X$, one has $\mathrm{NS}(X) = 0$.

## An example

Let $V = \mathbb{C}^2$ and consider $\Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$, where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ e_3 = \begin{pmatrix} a \\ b \end{pmatrix} \ e_4 = \begin{pmatrix} c \\ d \end{pmatrix}$$

Then, $\Gamma$ is a lattice (i.e. the $e_i$'s form a $\mathbb{R}$-basis) if and only if

$$\det \Im \begin{pmatrix} a & c \\ b & d \end{pmatrix} \neq 0,$$

where $\Im$ denotes the imaginary part of the matrix. We have

$$H^1(X,\mathbb{Z}) = \mathbb{Z}dx_1 \oplus \cdots \oplus \mathbb{Z}dx_4,$$

where $dx_j : T_pX = V \to \mathbb{R}$, is the dual basis $dx_j(\sum t_k e_k) = t_j$ (notice $t_k \in \mathbb{R}$). We want to find the above mentioned relation between $dx_i, dx_j$ and $dz_k, d\bar{z}_l$.

//To be finished.//

## Line bundles on a complex torus

Let $X = V/\Gamma$ be a complex torus, $\pi : V \to X$ the projection. One can show that each (holomorphic) line bundle $L \in \mathrm{Pic}(X)$ arises as a quotient of $V \times \mathbb{C}$ by a certain action of $\Gamma$. First, we sketch how to go the other way around. Given $\gamma \in \Gamma$ and $(v,t) \in V \times \mathbb{C}$, we let

$$\gamma \cdot (v,t) := (v + \gamma, A_\gamma(v)t),$$

for some holomorphic function $A_\gamma : V \to \mathbb{C}^*$. This defines an action of $\Gamma$ on $V \times \mathbb{C}$ if and only if $\gamma_1 \cdot (\gamma_2 \cdot (v,t)) = (\gamma_1 + \gamma_2) \cdot (v,t)$, for all $v, t, \gamma_1, \gamma_2$. It is straightforward to check that this is equivalent to ask $\{A_\gamma\}_{\gamma \in \Gamma}$ to satisfy

$$A_{\gamma_1+\gamma_2}(v) = A_{\gamma_1}(v + \gamma_2)A_{\gamma_2}(v) \tag{4.2}$$

which is called the *cocycle condition* for $\{A_\gamma\}$. In this case, one gets a complex line bundle $L = (V \times \mathbb{C})/\Gamma$ on $X = V/\Gamma$ by

$$\begin{tikzcd} V \times \mathbb{C} \arrow[r] \arrow[d] & L \arrow[d] \\ V \arrow[r, "\pi"] & X \end{tikzcd}$$

A real (differentiable) example, just to have a picture in mind: $\Gamma = \mathbb{Z}$ acts on $V = \mathbb{R}$, by $v \mapsto v + n$. One gets the circle $X = \mathbb{R}/\mathbb{Z} = S^1$. Consider the action of $\Gamma$ on $\mathbb{R} \times [-1, 1]$ given by $n \cdot (v, t) = (v + n, (-1)^n t)$. Taking the quotient by this action yields a differentiable line bundle $L$, the *Moebius strip*.

Let $E \in \mathrm{NS}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. In light of the identification $H^2(X, \mathbb{Z}) = \mathrm{Alt}^2(\Gamma, \mathbb{Z})$, we view $E$ as an alternating bilinear form $E : \Gamma \times \Gamma \to \mathbb{Z}$ such that its $\mathbb{R}$-linear extension to $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is of type $(1, 1)$, or equivalently, $E(v, w) = E(iv, iw)$ (then, $E$ is a Kähler form if and only if $E(v, iv) > 0$).

**Lemma** (Frobenius)**.** *Given an alternating bilinear form $E : \Gamma \times \Gamma \to \mathbb{Z}$, there exists a basis $e_1, \ldots, e_{2n}$ of $\Gamma$ such that $E$ is represented by the alternating matrix*

$$E = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix},$$

*where $\Delta = \mathrm{diag}(d_1, \ldots, d_n)$ is a diagonal matrix, $d_i \in \mathbb{Z}$ and $d_i$ divides $d_{i+1}$ (note that possibly $\Delta = \mathrm{diag}(d_1, \ldots, d_k, 0, \ldots, 0)$). Explicitly, one has*

$$E\left(\sum_{j=1}^{2n} m_j e_j, \sum_{j=1}^{2n} m'_j e_j\right) = \sum_{i=1}^{n} d_i (m_i m'_{i+n} - m_{i+n} m'_i)$$

Given $E \in \mathrm{NS}(X)$, we define a Hermitian form

$$H : V \times V \to \mathbb{C}, \quad H(v, w) := E(v, iw) + iE(v, w).$$

In fact, $H$ is $\mathbb{C}$-linear in $w$ (hence, also in $v$), for $H$ is obviously $\mathbb{R}$-bilinear and

$$H(v, iw) = -E(v, w) + iE(v, iw) = iH(v, w).$$

Also $H(w, v) = \overline{H(v, w)}$, since

$$\begin{aligned} H(w, v) &= E(w, iv) + iE(w, v) \\ &= -E(iv, w) - iE(v, w) \qquad (E \text{ is altern}) \\ &= -E(i^2 v, iw) - iE(v, w) \quad (\text{of type } (1, 1)) \\ &= \overline{H(v, w)}. \end{aligned}$$

Notice that $\Im(H) = E$, hence $H$ determines $E$. This hermitian form will be important for constructing the $A_\gamma$'s.

## Abelian varieties

To be added.

# Appendix A

# Additional topics

## A.1 Line bundles on $\mathbb{P}^n$

If we are given a vector bundle $E$ on a complex manifold $X$ then we get a sheaf $\mathcal{F}_E$ on $X$ by considering the sections of the vector bundle:

$$\mathcal{F}_E(U) := \Gamma(U, E) = \{s : U \to E \ \text{ holomorphic, } s(a) \in E_a\}$$

On $\mathbb{P}^n$ we constructed the line bundles $L(d)$ for any integer $d$. Recall

$$L(d) \longleftrightarrow \{U_i,\ g_{ij}(x) = (\tfrac{x_j}{x_i})^d\}$$

where $\{U_i\}$ is the standard covering of $\mathbb{P}^n$. Let $d > 0$. Any $f \in \mathbb{C}[x_0, \dots, x_n]_d$, homogeneous polynomial of degree $d$, defines a global section $f \in \mathcal{F}_{L(d)}(\mathbb{P}^n)$ by

$$(x_0 : \dots : x_n) \mapsto (x_0 : \dots : x_n : f(x_0, \dots, x_n))$$

Recall that a section $s \in \mathcal{F}_{L(d)}(U)$ is completely determined by its local descriptions $s_i \in \mathcal{F}_{L(d)}(U_i \cap U)$ such that

$$s_j(x) = \left(\frac{x_k}{x_j}\right)^d s_k(x)$$

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the usual projection. We define a sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n$ as follows. Let $U \subset \mathbb{P}^n$ open. We set[1]

$$\mathcal{O}(d)(U) := \{f : \pi^{-1}(U) \to \mathbb{C} \text{ holomorphic, } f(tz) = t^d f(z)\}$$

The restrictions for $V \subset U$ are the natural restrictions of functions. The local identity axiom is clearly satisfied[2]. The existence of gluings is also clear as they are holomorphic functions. We only need to check that a gluing is homogeneous. If $z \in \pi^{-1}(U)$ and $t \in \mathbb{C}^*$ then $z \in \pi^{-1}(V_j)$ for some $j$ and $tz \in \pi^{-1}(V_j)$ so

$$f(tz) = f_j(tz) = t^d f_j(z) = t^d f(z)$$

**Proposition A.1.1.** *The sheaves $\mathcal{F}_{L(d)}$ and $\mathcal{O}(d)$ are isomorphic.*

[1] notice that if $z \in \pi^{-1}(U)$ then $tz \in \pi^{-1}(U)$ for all $t \in \mathbb{C}^*$

[2] noticing that if $U = \bigcup V_i$ then $\pi^{-1}(U) = \bigcup \pi^{-1}(V_i)$

*Proof.* The morphism $\phi : \mathcal{F}_{L(d)} \to \mathcal{O}(d)$ given by $\phi_U : \mathcal{F}_{L(d)}(U) \to \mathcal{O}(d)(U)$

$$\phi_U(s) = f : \pi^{-1}(U) \to \mathbb{C}$$

$$f(z) := z_j^d s_j(\pi(z)) \quad \text{where } z = (z_1, \ldots, z_n) \in \pi^{-1}(U \cap U_j)$$

(i) $\phi_U$ is well defined: if $z \in \pi^{-1}(U \cap U_j \cap U_k)$ then

$$f(z) = z_j^d s_j(\pi(z)) = z_j^d \cdot \frac{z_k^d}{z_j^d} \cdot s_k(\pi(z)) = z_k^d s_k(\pi(z))$$

(ii) $f$ is holomorphic as it is a composition of holomorphic maps.

(iii) $f$ is homogeneous: if $z \in \pi^{-1}(U \cap U_j)$ then $tz \in \pi^{-1}(U \cap U_j)$ and

$$f(tz) = (tz_j)^d s_j(\pi(tz)) = t^d z_j^d s_j(\pi(z)) = t^d f(z)$$

Consider now $\psi : \mathcal{O}(d) \longrightarrow \mathcal{F}_{L(d)}$ given by

$$\psi_U : \ (f : \pi^{-1}(U) \to \mathbb{C}) \longmapsto s = \{U_j, s_j\}$$

$$s_j(\pi(z)) := \tfrac{1}{z_j^d} f(z) \quad \text{where } \pi(z) \in U_j$$

Does this actually define a section $s$? We have to check that the gluing conditions hold: if $z \in \pi^{-1}(U \cap U_j \cap U_k)$ then

$$z_k^d s_k(\pi(z)) = f(z) = z_j^d s_j(\pi(z))$$

hence $s_j(\pi(z)) = (z_k^d / z_j^d) \cdot s_k(\pi(z))$. Clearly $\psi$ and $\phi$ are mutual inverses. □

## A.2 Čech cohomology

Given a complex manifold $X$ and a sheaf $\mathcal{F}$ (of abelian groups, or $\mathcal{O}_X$-modules), we can talk about the cohomology groups $H^p(X, \mathcal{F})$. In particular, if $E$ is a holomorphic vector bundle on $X$, we can define the $p$-th cohomology group $H^p(X, E) := H^p(X, \mathcal{F})$, where $\mathcal{F}$ is the sheaf of sections of $E$. One way to compute this groups is by means of a good resolution for $\mathcal{F}$. However, this might be hard. Čech cohomology[3] becomes then a good tool for this purposes. In this paragraph, we use the following notation and assumptions:

(i) $X$ is a topological space with an open covering $\mathscr{U} = \{U_j\}_{j \in J}$, where $J$ is some ordered set of indices. If $\sigma \subset J$ is a finite subset, we denote by

$$U_\sigma = \bigcap_{j \in \sigma} U_j.$$

If $|\sigma| = p + 1$ and its elements are ordered as $j_0 < \cdots < j_p$ we denote by

$$\sigma_k = \sigma \setminus \{j_k\}$$

(ii) $\mathcal{F}$ is a sheaf of abelian groups on $X$.

[3]historically the first version of sheaf cohomology to be defined

For any positive integer $p$ we define

$$C^p(\mathscr{U},\mathcal{F}) := \prod_{|\sigma|=p+1} \mathcal{F}(U_\sigma).$$

For $g \in C^p(\mathscr{U},\mathcal{F})$ and $|\sigma| = p+1$ we write $g(\sigma)$ for the $\sigma^{\text{th}}$ component of $g$. We define $d : C^p(\mathscr{U},\mathcal{F}) \to C^{p+1}(\mathscr{U},\mathcal{F})$ as follows. Let $|\sigma| = p+2$. We set

$$(dg)(\sigma) = \sum_{k=0}^{p+1} (-1)^k g(\sigma_k)|_{U_\sigma}$$

It is then an easy exercise to see that $d^2 = 0$. So $d$ defines a differential.

**Definition.** The *$p$-th Čech cohomology group of $\mathcal{F}$ with respect to $\mathscr{U}$* is

$$\check{H}^p(\mathscr{U},\mathcal{F}) = \frac{\ker(d^p)}{\operatorname{Im}(d^{p-1})}$$

We are not completely happy with this construction because of the dependency of the group on the covering $\mathscr{U}$. We redress this by taking a direct limit. Suppose $\mathscr{V} = \{V_k\}$ is a refinement of $\mathscr{U}$. Then each $V_k \subset U_j$ for some $j = j(k)$ and if $g \in C^p(\mathscr{U},\mathcal{F})$ we can use restrictions of $\mathcal{F}$ on each $g(\sigma) \in \mathcal{F}(U_\sigma)$ to get a "restriction" of $g$, i.e. a map sending $g$ to an element in $C^p(\mathscr{V},\mathcal{F})$. This induces a morphism $\check{H}^p(\mathscr{U},\mathcal{F}) \longrightarrow \check{H}^p(\mathscr{V},\mathcal{F})$, called *restriction*, and the composition of such restrictions is again a restriction $\check{H}^p(\mathscr{U},\mathcal{F}) \to \check{H}^p(\mathscr{V},\mathcal{F}) \to \check{H}^p(\mathscr{W},\mathcal{F})$. Thus, we can take the direct limit of this construction. We define

$$\check{H}^p(X,\mathcal{F}) := \varinjlim \check{H}^p(\mathscr{U},\mathcal{F})$$

as the *$p$-th Čech cohomology group of $\mathcal{F}$* on the space $X$.

**Proposition A.2.1.** *For any covering $\mathscr{U}$ of $X$ there exists a natural map*[4] *$\check{H}^p(\mathscr{U},\mathcal{F}) \to H^p(X,\mathcal{F})$ for any $p \geq 0$. In other words there exists a unique morphism $\check{H}^p(X,\mathcal{F}) \to H^p(X,\mathcal{F})$ that makes the following diagram commute*

$$\begin{tikzcd}
\check{H}^p(\mathscr{U},\mathcal{F}) \arrow[rr] \arrow[dr, bend right] \arrow[ddr, bend right] & & \check{H}^p(\mathscr{V},\mathcal{F}) \arrow[dl, bend left] \arrow[ddl, bend left] \\
& \check{H}^p(X,\mathcal{F}) \arrow[d, "\exists!"] & \\
& H^p(X,\mathcal{F}) &
\end{tikzcd}$$

**Fact.** *If $X$ is a complex manifold then the map is an isomorphism for $p = 1$*

$$\check{H}^1(X,\mathcal{F}) \simeq H^1(X,\mathcal{F})$$

[4] functorial in $\mathcal{F}$

## The isomorphism $\mathrm{Pic}(X) \simeq \check{H}^1(X, \mathcal{O}^*)$

Let $X$ be a compact connected complex manifold and consider the sheaf

$$\mathcal{F} = \mathcal{O}^*$$

of invertible holomorphic functions on $X$. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of $X$. As $X$ is compact we can suppose $\alpha \in A$ where $A$ is finite and therefore orderable. We will write $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Let's look at $\check{H}^1(\mathscr{U}, \mathcal{O}^*)$. By definition

$$C^1(\mathscr{U}, \mathcal{O}^*) = \prod_{\alpha<\beta} \mathcal{O}^*(U_{\alpha\beta})$$

So an element $g \in C^1(\mathscr{U}, \mathcal{O}^*)$ is a collection of holomorphic invertible functions

$$g = \{g_{\alpha\beta}\}, \quad g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{C}^*$$

We want $g = \{g_{\alpha\beta}\}$ to represent a class in $\check{H}^1(\mathscr{U}, \mathcal{O}^*)$ so we impose $dg = 0$. This means $dg(\sigma) = 0$ for all $\sigma = \{\alpha, \beta, \gamma\}$ with $\alpha < \beta < \gamma$. So we get

$$0 = (dg)(\sigma) = \sum_{k=0}^{2} (-1)^k g(\sigma_k)|_{U_{\alpha\beta\gamma}} = (g_{\beta\gamma} - g_{\alpha\gamma} + g_{\alpha\beta})|_{U_{\alpha\beta\gamma}}$$

So $g_{\alpha\beta} + g_{\beta\gamma} = g_{\alpha\gamma}$ for all $\alpha, \beta, \gamma$. But this is in additive notation, whereas the sheaf $\mathcal{O}^*$ is multiplicative. Hence the correct notation of this condition becomes

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$$

Which is exactly the cocycle condition defining the line bundle $L \leftrightarrow \{U_\alpha, g_{\alpha\beta}\}$. This correspondence is clearly surjective: every line bundle is represented by some class in $\check{H}^1$ as above. To prove that it is an isomorphism onto $\mathrm{Pic}(X)$ we have to show the following: if $L \leftrightarrow \{g_{\alpha\beta}\} = g$ and $L \leftrightarrow \{h_{\alpha\beta}\} = h$, then[5]

$$L \simeq M \iff [g] = [h] \in \check{H}^1(\mathscr{U}, \mathcal{O}^*)$$

However we know that $L$ and $M$ are isomorphic if and only if there exists a family of invertible holomorphic functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ such that for all $\alpha, \beta$

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} = \frac{f_\beta}{f_\alpha}$$

For $f = \{f_\alpha\} \in C^0(\mathscr{U}, \mathcal{O}^*)$ we write this condition in additive notation and get

$$g_{\alpha\beta} - h_{\alpha\beta} = f_\beta - f_\alpha = df(\sigma) \quad \text{for all } \sigma = \{\alpha, \beta\},\ \alpha < \beta$$

which is equivalent to $g - h = df$, that is $[g] = [h]$.

The last remark is to note that this construction depends on the fixed open covering $\mathscr{U}$ on both sides: for the line bundles, which we have identified with their cocycles (depending on $\mathscr{U}$) and on the other side on the group $\check{H}^1(\mathscr{U}, \mathcal{O}^*)$. By taking the direct limit on the open coverings we get rid of this dependency:

$$\check{H}^1(X, \mathcal{O}^*) \simeq \mathrm{Pic}(X)$$

[5] the identification of line bundles with their cocycles is made *with respect to* the fixed $\mathscr{U}$

## A.3 Divisors and the Picard group

Let $Y \subset X$ be an analytic hypersurface. Then $Y$ defines a line bundle on $X$,

$$L_Y \longleftrightarrow \{U_\alpha, g_{\alpha\beta} = f_\alpha / f_\beta\}$$

with $(U_\alpha, f_\alpha)$ local equations for $Y$ and there exists $s_Y \in \Gamma(X, L_Y)$ such that

$$Y = \{x \in X : \ s_Y(x) = 0\}$$

Moreover, $s_Y \longleftrightarrow \{s_\alpha : U_\alpha \to \mathbb{C}, \ s_\alpha = g_{\alpha\beta} s_\beta\}$ with $s_\alpha$ local equations for $Y$.

**Definition.** A *divisor* on $X$ is a finite formal sum

$$\sum_Y n_Y Y$$

where $n_Y \in \mathbb{Z}$ and the $Y$'s are hypersurfaces of $X$. Thus the set $\mathrm{Div}(X)$ of all divisors on $X$ is the free abelian group generated by the hypersurfaces of $X$.

Let $L$ be a line bundle on $X$ and $s, t \in \Gamma(X, L)$ with $t$ not the zero section. Let $s, t$ have local descriptions $s_\alpha, t_\alpha$ on some covering $\{U_\alpha\}$ of $X$. Then

$$f(x) := \frac{s_\alpha(x)}{t_\alpha(x)}, \quad x \in U_\alpha$$

is a meromorphic function on $X$. It is well defined for, if $x \in U_{\alpha\beta}$ then

$$\frac{s_\alpha(x)}{t_\alpha(x)} = \frac{g_{\alpha\beta}(x) s_\beta(x)}{g_{\alpha\beta}(x) t_\beta(x)} = \frac{s_\beta(x)}{t_\beta(x)}$$

where the $g_{\alpha\beta}$ are the transition maps of $L$. Thus

$$f \in \mathfrak{M}(X) = \{\text{meromorphic functions } X \to \mathbb{C}\}$$

with a small abuse of notation we will write $f = s/t$.

Let now $Y, Z$ be hypersurfaces of $X$. Then we have the sections $s_Y$ and $s_Z$ like above and we can consider the meromorphic function $f = s_Y / s_Z$ on $X$. Now, $f$ vanishes at (almost all) points of $Y$, with simple zeroes. Also it has simple poles at (almost all) points of $Z$. For this reason we consider $Y, Z$ as "points" and *say* that $f$ has a zero of order one on $Y$ ($n_Y = 1$) and that $f$ has a pole of order one on $Z$ ($n_Z = -1$). We can now associate a divisor to $f$ by

$$(f) := n_Y Y + n_Z Z = Y - Z \in \mathrm{Div}(X)$$

More generally, let $f \in \mathfrak{M}(X)$. Suppose there exists a point $y \in Y$ around which there is a neighborhood where we can write $f = h \cdot g^{n_Y}$, where $g$ is a local equation for $Y$ and $h$ is a non vanishing holomorphic function on this neighborhood. One shows that $n_Y$ does not depend on $g$ nor on $y$. For all hypersurfaces $Y$ for which this happens we then have an integer $n_Y$ associated to $f$. We can thus define the *divisor* of $f$ as

$$(f) := \sum_Y n_Y Y \in \mathrm{Div}(X)$$

Let $\mathfrak{M}^*(X)$ be the set of non vanishing meromorphic functions on $X$. We define the subgroup of *principal divisors* of $\mathrm{Div}(X)$ as

$$P(X) = \{D \in \mathrm{Div}(X) : \exists f \in \mathfrak{M}^*(X) \text{ such that } D = (f)\}$$

Yes, but... what about $\mathrm{Pic}(X)$? First note that there is a group homomorphism

$$\begin{aligned} \varphi : \mathrm{Div}(X) &\longrightarrow \mathrm{Pic}(X) \\ D = \sum_Y n_Y Y &\longmapsto L_D := \bigotimes_Y (L_Y^{\otimes n_Y}) \end{aligned}$$

At this stage one should wonder whether (and when) $\varphi$ is injective/surjective.

$$\ker(\varphi) = \{D \in \mathrm{Div}(X) : L_D \simeq X \times \mathbb{C}\}.$$

Assuming $X$ compact, it is possible to show that $\ker(\varphi) = P(X)$. Thus, we have an injection $\mathrm{Div}(X)/P(X) \hookrightarrow \mathrm{Pic}(X)$. What happens is that: if $X \hookrightarrow \mathbb{P}^N$ then this is an isomorphism (i.e. $\varphi$ surjective),

$$\mathrm{Pic}(X) \simeq \frac{\mathrm{Div}(X)}{P(X)}.$$