Exercise sheet 6

In the following exercises, whenever it makes sense, we encourage using a computer algebra system, such as Macaulay2, to help with the computations.

Exercise 1

Let k be a field and $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Show that

$$J = \langle y_1 - f_1, \ldots, y_r - f_r \rangle$$

is a prime ideal of the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_r]$.

Exercise 2

Let $I_1, \ldots, I_N$ be ideals in $k[x_1, \ldots, x_n]$. Consider the ideal

$$J = y_1 I_1 + \cdots + y_N I_N + \langle 1 - y_1 - \cdots - y_N \rangle$$

in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_N]$. Show the following identity

$$I_1 \cap \ldots \cap I_N = J \cap k[x_1, \ldots, x_N].$$

Exercise 3

Given the polynomials $f_1 = x^2 + y^2$, $f_2 = x^3 y - x y^3$, $f_3 = x^2 y^2$ in $\mathbb{Q}[x, y]$ and

$$g = x^8 + 2x^6 y^2 - x^5 y^3 + 2x^4 y^4 + x^3 y^5 + 2x^2 y^6 + y^8$$

find $h \in \mathbb{Q}[u, v, w]$ such that $g = h(f_1, f_2, f_3)$.

Exercise 4

In $\mathbb{Q}[x, y, z]$ with Grlex ordering, consider the ideal

$$I = \langle xy - y^2, x^3 - z^2 \rangle.$$

For each of the following polynomials, determine if $f \in I$ and compute the division with respect to a Groebner basis $\{g_i\}$ of I, i.e. determine explicitly the expression

$$f = \sum_i q_i g_i + r.$$

- $f = -4x^2y^2z^2 + y^6 + 3z^5$
- $f = x^4y - 2xy^2z + 3z^3 - x$
- $f = 2y^8 - 4y^5z^2 + x^6 + 2y^2z^4 - 2x^3z^2 + z^4$

Exercise 5

In the affine space $\mathbb{A}^3$ over $\mathbb{F}_{101}$, consider the *surface* S defined by

$$S = \{(x, y, z) \in \mathbb{A}^3 \ : \ x^3 + y^3 + z^3 - 1 = 0\}$$

and the *curve* C defined by the equations

$$\begin{aligned} xy^2 &= 1 \\ y^2 - x^4 &= 0 \\ x^2 - y^3 &= 0. \end{aligned}$$

Describe the intersection $S \cap C$.

Exercise 6

Find the solutions of the following system of equations in $\mathbb{C}[x, y, z]$

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + z^2 &= y \\ x &= z. \end{aligned}$$

Exercise 7

Over $k = \mathbb{F}_{101}$, find the local optima of the function $f = x^3 + 2xyz - z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

(Hint: use the method of Lagrange multipliers and solve algebraically)

Exercise 8

In the affine space $\mathbb{A}^3$ over $\mathbb{C}$, consider the curve C described parametrically by

$$\begin{aligned} x &= t^4 \\ y &= t^3 \\ z &= t^2 \end{aligned}$$

where $t \in \mathbb{C}$. Determine $f, g \in \mathbb{C}[x, y, z]$ such that $C = V(f, g)$. Can you find the ideal defined by C as a variety, i.e. the ideal $I(C) = \{f \in \mathbb{C}[x, y, z] \colon f(x, y, z) = 0 \text{ for all } (x, y, z) \in C\}$?

Exercise 9

In $\mathbb{R}^3$, consider two lines ℓ, ℓ' given parametrically as $\ell = \{p_t \colon t \in \mathbb{R}\}$ and $\ell' = \{p'_t \colon t \in \mathbb{R}\}$ where $p_t = (t, 0, 1)$ and $p'_t = (0, 1, t)$. Consider the surface S formed by taking the *union* of all lines joining pairs of points on the two lines,

$$S = \bigcup_{t \in \mathbb{R}} L_t$$

where $L_t = \langle p_t, p'_t \rangle$ denotes the line joining p_t and p'_t.

1. Show that S can be described parametrically, as

   $$\begin{aligned} x &= ut \\ y &= 1 - u \\ z &= t + u(1 - t). \end{aligned}$$

   where $t, u \in \mathbb{R}$.

2. Find a polynomial $f \in \mathbb{R}[x, y, z]$ such that $S \subset V(f)$.

3. Can you show $S = V(f)$?

Exercise 10

Suppose we have numbers a, b, c which satify

$$\begin{aligned} a + b + c &= 3 \\ a^2 + b^2 + c^2 &= 5 \\ a^3 + b^3 + c^3 &= 7. \end{aligned}$$

Prove that $a^4 + b^4 + c^4 = 9$ and $a^5 + b^5 + c^5 \neq 11$. Can you compute $a^5 + b^5 + c^5$?