Algebraic Surfaces

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Contents

INTRODUCTION

In these notes, we will study some fundamental aspects of the theory of complex projective algebraic surfaces. Let us comment on the terminology:

- *surfaces:* two degrees of freedom.
- *complex:* the two degrees of freedom are given by *complex* coordinates.
- *algebraic:* the surfaces will be given as the *zero locus of polynomials*

$$F(x, y, z) = 0.$$

- *projective:* the polynomials will be taken to be *homogeneous*.

For short, a complex projective algebraic surface will be just called a *surface*, most often. To be precise, in what follows we will not attempt to discuss the case of *singular* surfaces; our attention will be restricted to *smooth* ones only. We will encounter several examples of surfaces; for example:

- The (smooth) quadric Q in projective space $\mathbb{P}^3$, defined by the equation

$$x_0x_3 - x_1x_2 = 0.$$

  There is a double "ruling" of Q (i.e. two families of lines), and in fact

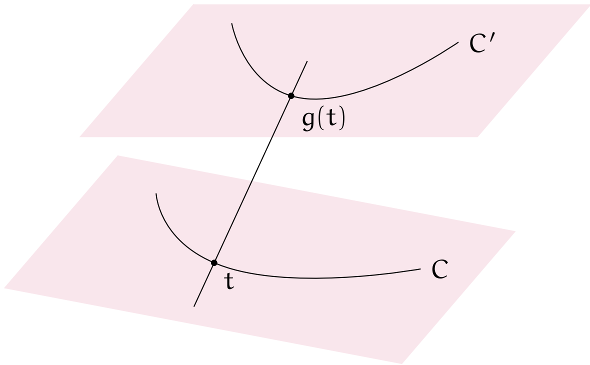
$$Q \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

  Indeed, there exists a natural embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

- Given two conics C and C' lying on two skew planes in $\mathbb{P}^4$, consider a morphism $g : C \to C'$. We define a surface Σ by

$$\Sigma := \bigcup_{t \in C} \langle t, g(t) \rangle,$$

  where $\langle p, q \rangle$ (linear span) denotes the line through the points p and q.



The 1-dimensional analogous of surfaces are algebraic curves. For example, a plane curve $C \subset \mathbb{P}^2$ given by the zero locus of a (homogeneous) polynomial $f(x_0, x_1, x_2)$. Topologically, C is a compact orientable surface – a "donut with g holes", where g is the *genus*; and in fact C is also called a *Riemann surface*. To study the geometry of C, there are two approaches:

- ⋄ Intrinsic geometry: we study C locally over its open subsets.
- ⋄ Extrinsic geometry: we study the properties of C which depend on its embedding $C \hookrightarrow \mathbb{P}^N$. For example, when $N = 2$ we may study the intersections of C with the lines of $\mathbb{P}^2$.

Similarly, these two main points of view are valid for algebraic varieties of any dimension. However, already in dimension 2, which is the case of our interest, some complications and new phenomena arise.

For example, let $\varphi : C \to \mathbb{P}^1$, where C is a curve and φ is locally represented as a ratio $h(z)/k(z)$, where h and k are holomorphic functions $\mathbb{C} \to \mathbb{C}$, with k not identically zero. Suppose that $h(z_0) = k(z_0) = 0$ at some point z_0. At first sight it may seem that φ cannot be defined at z_0. However, we may always set

$$\varphi(z_0) := \lim_{z \to z_0} \frac{h(z)}{k(z)} \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}.$$

On the other hand, when S is a surface, let $\varphi : S \to \mathbb{P}^1$ be given locally as $\varphi = h(z)/k(z)$, this time $z = (z_1, z_2) \in \mathbb{C}^2$. Now, if h and k vanish simultaneously, we cannot define uniquely a limit value. For example,

$$\lim_{z \to (0,0)} \frac{z_1}{z_2}$$

cannot be defined, since it clearly depends on the choice of the path towards the origin. Therefore, φ cannot be defined at all points of S.

For this reason, it is useful to introduce a new concept, namely that of *rational map*, which are usually denoted by a dashed arrow $\varphi : S \dashrightarrow \mathbb{P}^1$.

With the terminology of category theory, we will work in the category whose objects are algebraic surfaces, whose arrows are rational maps, and whose equivalences are *birational maps* between surfaces.

An example of birational mapping between surfaces is given by the stereographic projection $\pi : Q \dashrightarrow \mathbb{P}^2$, where Q is the quadric in space. Let $a \in Q$ be the point of projection and let $\ell, \ell' \subset Q$ be the two lines of the double ruling of Q passing through a. Then $\pi(\ell \setminus \{a\}) = p \in \mathbb{P}^2$ and $\pi(\ell' \setminus \{a\}) = q \in \mathbb{P}^2$. The map π restricts to a bijection $Q \setminus \{\ell, \ell'\} \simeq \mathbb{P}^2 \setminus \{p, q\}$. In fact, π is a birational equivalence between $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Notice that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are not even homeomorphic as topological spaces. Indeed, one can show that $e(\mathbb{P}^1 \times \mathbb{P}^1) = 4$ and $e(\mathbb{P}^2) = 3$, where e denotes the Euler characteristic (a topological invariant). This simple example is important to keep in mind: birational equivalence is somewhat of a tricky concept. The right way to think about it is: two birational equivalent surfaces are isomorphic (i.e. biholomorphic) outside of some (Zariski) closed subset (on both sides).

It is worth mentioning that in the 1-dimensional case two (smooth) algebraic curves are birationally equivalent if and only if they are isomorphic.

One of the main tools that we will introduce for the study of algebraic surfaces is that of *intersection theory*. The idea is quite simple. Given two curves C and C', lying on a surface S, we aim to define an integer $C \cdot C'$, which counts the intersection number (with multiplicities) of the two curves. In order to do so, we have to consider the ring $\mathcal{O}_p$ of (germs of) holomorphic functions at $p \in C \cap C'$, and its ideal I generated by the (germs of) functions which define C and C' as subvarieties of S.

The construction of the "intersection form" (a bilinear form which exists canonically on each surface) will be general and abstract enough so that we will be able to talk about the *self-intersection* of a curve C lying on a surface, as the integer $C^2 = C \cdot C$. The geometrical meaning of this concept is quite subtle; it is related to the possibility of "deforming" the curve C *within* the surface S. When this is possible, one has $C^2 \geq 0$, but there can be curves embedded "rigidly" in S in this sense, for which $C^2 < 0$. For example, given a smooth cubic S in space, containing a line ℓ, one has $\ell^2 = -1$.

The aim of these notes is to present some of the main technical tools of the theory and to eventually sketch the idea of how one can proceed to the birational classification of (non-singular) algebraic surfaces.

Part I

Tools from analytic and algebraic geometry

Lecture 1

A glimpse of complex analysis

A holomorphic function is the complex analogue of a differentiable analytic function. Let Ω be an open subset of $\mathbb{C}^n$ and

$$f : \Omega \to \mathbb{C}$$

be a continuous function. By definition, f is *holomorphic* in Ω if for any $t \in \Omega$ there is some disk $\Delta = \{z \in \mathbb{C}^n : |z_j - t_j| < \varepsilon\} \subset \Omega$ such that one can write f as a convergent power series. In other words, for any $z \in \Delta$,

$$f(z) = \sum_{i_1 \ldots i_n \geq 0} a_{i_1 \ldots i_n} (z_1 - t_1)^{i_1} \cdots (z_n - t_n)^{i_n}.$$

We denote by $\mathcal{O}(\Omega)$ the ring of holomorphic functions defined on Ω. For example, each polynomial function is holomorphic on the whole $\mathbb{C}^n$, so we have $\mathbb{C}[z_1, \ldots, z_n] \subset \mathcal{O}(\mathbb{C}^n)$. The functions in $\mathcal{O}(\mathbb{C}^n)$ are called *entire functions*.

We denote by $\mathcal{C}(\Omega)$ the ring of continuous functions on Ω. By $\mathcal{C}^1(\Omega)$ we denote the set of continuously differentiable functions on Ω. That is, $f \in \mathcal{C}^1(\Omega)$ if and only if, for $z_j = x_j + iy_j$, all the partial derivatives

$$\frac{\partial f}{\partial x_j} \text{ and } \frac{\partial f}{\partial y_j}$$

exist and are continuous.

Let us now list the fundamental facts concerning holomorphic functions.

THEOREM 1.1 (Osgood). *Let $f \in \mathcal{C}(\Omega)$. Then $f \in \mathcal{O}(\Omega)$ if and only if f is holomorphic in each coordinate z_j.*

THEOREM 1.2 (Cauchy-Riemann equations). *Let $f \in \mathcal{C}^1(\Omega)$. Then*

$$f \in \mathcal{O}(\Omega) \iff \frac{\partial f}{\partial x_j}(z) + i\frac{\partial f}{\partial y_j}(z) = 0, \ \forall z \in \Omega, \ \forall j \tag{1.1}$$

The equations in (1.1) are called *Cauchy-Riemann equations*. If we define

$$\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$
$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

then the Cauchy-Riemann equations can be written as

$$f \in \mathcal{O}(\Omega) \iff \frac{\partial f}{\partial \bar{z}_j}(z) = 0, \ \forall z \in \Omega, \ \forall j$$

We define $dz_j = dx_j + i dy_j$ and $d\bar{z}_j = dx_j - i dy_j$. Also,

$$\partial f := \sum \frac{\partial f}{\partial z_j} dz_j, \quad \bar{\partial} f := \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j, \quad df := \partial f + \bar{\partial} f$$

Then, we have $f \in \mathcal{O}(\Omega) \iff \bar{\partial} f = 0$ for all $z \in \Omega$. By (1.1), one shows:

- ◇ The ring $\mathcal{O}(\Omega)$ is a $\mathbb{C}$-vector space[1].
- ◇ If Ω is connected, then $\mathcal{O}(\Omega)$ is an integral domain (i.e. no zero-divisors).
- ◇ If $f \in \mathcal{O}(\Omega)$ is nowhere vanishing, then $1/f \in \mathcal{O}(\Omega)$.

THEOREM 1.3 (Maximum principle). *Let Ω be connected, $f \in \mathcal{O}(\Omega)$. If f has maximum in Ω, then f is constant.*

THEOREM 1.4 (Identity principle). *Let Ω be connected, $f, g \in \mathcal{O}(\Omega)$. Suppose there exists an open subset $A \subset \Omega$ such that $f|_A = g|_A$. Then $f = g$.*

In dimension $n = 1$, the identity principle holds under the weaker assumption that $A \subset \Omega$ is a set with infinitely many accumulation points. This fails to be true when $n \geq 2$. For example, in $\mathbb{C}^2$ we take $f = z_1$ and $g = z_1 z_2$ and we observe that $f = g = 0$ along the z_2-axis.

THEOREM 1.5 (Open mapping). *Let Ω be connected, and $f \in \mathcal{O}(\Omega)$ non-constant. Then f is an* open map, *i.e. $f(U)$ is open for any open subset $U \subset \Omega$.*

When $n \geq 2$ there exist very strong properties of holomorphic extension. For example: given two disks $\Delta' \subset \Delta$ and $f \in \mathcal{O}(\Delta \backslash \bar{\Delta}')$, then f can be extended uniquely to $f \in \mathcal{O}(\Delta)$ (cf. *Hartogs extension theorem*).

HOLOMORPHIC MAPS

Let Ω be an open subset of $\mathbb{C}^n$ and $f = (f_1, \dots, f_m) : \Omega \to \mathbb{C}^m$. By definition, f is *holomorphic* in Ω if each component $f_j : \Omega \to \mathbb{C}$ is holomorphic.

At any point $z \in \Omega$, we define the matrix of partial derivatives

$$J_f(z) := \frac{\partial(f_1, \dots, f_m)}{\partial(z_1, \dots, z_n)} = \left[\frac{\partial f_i}{\partial z_j}\right],$$

which is called the *complex Jacobian* of f at z. If $g : f(\Omega) \to \mathbb{C}^p$ is a holomorphic function, then $g \circ f$ is also holomorphic and one has the usual chain rule:

$$J_{g \circ f}(z) = J_g(f(z)) \cdot J_f(z).$$

When $m = n$, if f is holomorphic on Ω, bijective onto $f(\Omega)$ and f^{-1} is holomorphic on $f(\Omega)$, we say that f is a *biholomorphism*. One has

$$\text{biholomorphism} \Rightarrow \text{diffeomorphism} \Rightarrow \text{homeomorphism}$$

[1] in fact, a $\mathbb{C}$-algebra since functions can be multiplied.

Clearly, if f is a biholomorphism, then $\det J_f(z) \neq 0$ for any $z \in \Omega$. Through the identification $\mathbb{C}^n = \mathbb{R}^{2n}$, we could consider f as a real differentiable function. Then, one finds that the determinant of the real Jacobian associated to f is positive; hence, f is an orientation-preserving diffeomorphism.

THEOREM 1.6 (Inverse map). *Let $f : \Omega \to \mathbb{C}^n$ be holomorphic. Suppose there exists $z^0 \in \Omega$ such that $\det J_f(z^0) \neq 0$. Then there exist open neighbourhoods $U \subset \Omega$ around z^0, and $V \subset f(\Omega)$ around $f(z^0)$, such that f restricts to a biholomorphism $f : U \to V$.*

THEOREM 1.7 (Implicit map). *Let $f : \Omega \to \mathbb{C}^m$ be holomorphic. Suppose there exists $p \in \Omega$ such that $f(p) = 0$ and $\operatorname{rk} J_f(p) = k$. Assume that the square $k \times k$ submatrix of $J_f(p)$ defined by*

$$\frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_k)}$$

is of rank k. Then, in a local neighbourhood around p, there exist holomorphic functions $g_1, \dots, g_k$ in the remaining variables $z_{k+1}, \dots, z_n$ such that

$$f(z) = 0 \iff z = (g_1(z_{k+1}, \dots, z_n), \dots, g_k(z_{k+1}, \dots, z_n), z_{k+1}, \dots, z_n).$$

PROPERTIES OF THE LOCAL RING

Let U, V be open neighbourhoods of $0 \in \mathbb{C}^n$. We say that two holomorphic functions $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$ represent the same *germ* at 0 if they coincide locally, i.e. there is a small neighbourhood $W \subset U \cap V$ where $f|_W = g|_W$. In this case we write $f \sim g$. In fact, $\sim$ is an equivalence relation. We define the *ring of germs of holomorphic functions* at 0 as

$$\mathcal{O}_0 = \bigsqcup_{U \ni 0} \mathcal{O}(U)/\sim$$

where the disjoint union runs over all open neighbourhoods of $0 \in \mathbb{C}^n$. Given $f \in \mathcal{O}(U)$, we call its class $f_0 \in \mathcal{O}_0$ the *germ* of f at 0. One can show that $\mathcal{O}_0$ is isomorphic to the ring of convergent power series about 0,

$$\mathcal{O}_0 \simeq \mathbb{C}\{z_1, \dots, z_n\}.$$

THEOREM 1.8. *The ring $\mathcal{O}_0$ has the following properties:*

(i) *$\mathcal{O}_0$ is a* local ring, *i.e. it admits a unique maximal ideal $\mathfrak{m}_0$, and*

$$\mathfrak{m}_0 = \{f_0 \in \mathcal{O}_0 \mid f(0) = 0\}.$$

Thus, the invertible elements are precisely the elements in $\mathcal{O}_0 \setminus \mathfrak{m}_0$.

(ii) *$\mathcal{O}_0$ is a* UFD, *i.e. any $f_0 \in \mathcal{O}_0$ has a unique factorization*

$$f_0 = u g_1 \cdots g_r,$$

where u is invertible and the g_i's are irreducible elements of $\mathcal{O}_0$.

(iii) *$\mathcal{O}_0$ is a* Noetherian ring, *i.e. every ideal is finitely generated.*

LECTURE 2

COMPLEX MANIFOLDS

Let X be a Haussdorff connected topological space. An *analytic complex atlas* on X is a collection $\mathcal{A} = \{U_i, \varphi_i\}_{i \in I}$ where

- ◇ $\{U_i\}$ is an open covering of X.
- ◇ $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ are homeomorphisms, called *local charts*.
- ◇ Whenever U_i and U_j overlap, the so-called *transition maps*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

  are biholomorphisms[1].

For $p \in U_i$, we say that $z = (z_1, \dots, z_n) = \varphi_i(p)$ are the *coordinates* of p with respect to the chart φ_i. We consider equivalent two atlases if their union is an atlas. A *complex structure* on X is an equivalence class of atlases, and is represented by the *universal atlas:* the union of all equivalent atlases.

Definition. A Haussdorff connected topological space X, together with a complex structure is called *(analytic) complex manifold.* The integer n appearing above is the *(complex) dimension* of X. It is denoted by $\dim_{\mathbb{C}} X$, or simply $\dim X$. When $\dim X = 1$, we say that X is a Riemann surface.

By convention, any chart on a complex manifold will always be a chart belonging to the universal atlas. It is worth to mention:

(i) A complex manifold is, in particular, a (canonically oriented) differentiable manifold, of topological dimension $\dim_{\mathbb{R}} X = 2 \dim_{\mathbb{C}} X$.

(ii) There can be several non-equivalent complex structures on a fixed differentiable manifold X. In general, the set of all complex structures on a given complex manifold forms a continuous space (*moduli space*).

EXAMPLES 2.1.

(i) Each open subset $\Omega \subset \mathbb{C}^n$ is a complex manifold, with a unique chart, given by the identity map $\mathrm{id} : \Omega \to \Omega$.

[1] the condition is empty, and thus automatically satisfied whenever $U_i \cap U_j = \varnothing$

(ii) Given two complex manifolds X and Y, one can naturally give a complex structure to the Cartesian product $X \times Y$.

(iii) The *complex projective space* is by definition the quotient

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

where the action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on the non-zero vectors z in $\mathbb{C}^{n+1}$ is defined by scalar proportionality $z \mapsto \lambda z$.

For $x \in \mathbb{P}^n$, we write $x = (x_0 : \ldots : x_n)$ for the *homogeneous coordinates* of x, that is, the equivalence class of a vector $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$.

The topology of $\mathbb{P}^n$ is by definition the quotient topology: the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is a continuous action.

There is a *standard open covering* of $\mathbb{P}^n$ given by the *standard open subsets*

$$U_j = \{x = (x_0 : \ldots : x_n) \in \mathbb{P}^n \mid x_j \neq 0\}.$$

The local charts $\varphi_j : U_j \to \mathbb{C}^n$ are given by

$$\varphi_j(x) = (\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}),$$

and they are called *affine charts* on $\mathbb{P}^n$. This yields a complex structure on $\mathbb{P}^n$. It is not hard to check that the projective space is compact: it can be described topologically as the quotient of the $(2n+1)$-dimensional sphere by the antipodal map.

(iv) A *complex torus* X is by definition a quotient of $\mathbb{C}^n$ by a discrete subgroup $\Gamma \subset \mathbb{C}^n$ of rank $2n$ (as a group), i.e. $\Gamma \simeq \mathbb{Z}^{2n}$. One can naturally give the topological space $X = \mathbb{C}^n/\Gamma$ a complex structure of dimension n, where the transition maps are given by translations by elements of Γ.

§1. Structure sheaf and holomorphic maps

Let X be a complex manifold, $U \subset X$ an open subset. By definition, a function $f : U \to \mathbb{C}$ is *holomorphic* on U if for any point $p \in U$ there exists a local chart (U_i, φ_i) around p, such that $f \circ \varphi_i^{-1} : \varphi_i(U \cap U_i) \to \mathbb{C}$ is holomorphic at $\varphi_i(p)$. It is immediate to check that the definition is independent on the choice of the charts φ_i. We denote by $\mathcal{O}(U)$ the vector space (in fact, the $\mathbb{C}$-algebra) of holomorphic functions on U. One can check that $\mathcal{O}$ is a sheaf (called *structure sheaf* of X), with the natural restrictions of functions (cf. Appendix A). It is also denoted by $\mathcal{O}_X$ sometimes. For any $x \in X$, the stalk of $\mathcal{O}$ at x consists of locally convergent power series, i.e.

$$\mathcal{O}_x \simeq \mathbb{C}\{z_1, \ldots, z_n\},$$

where $z_1, \ldots, z_n$ are local charts centred at x. On the other extreme, we have $\mathcal{O}(X)$, the algebra of holomorphic functions defined on X.

THEOREM 2.2. *If X is compact, then $\mathcal{O}(X) \simeq \mathbb{C}$, i.e. the only global holomorphic functions defined on X are the constant functions.*

Sketch of the proof. Let $f \in \mathcal{O}(X)$. Since $|f| : X \to \mathbb{R}$ is continuous on X compact, it has maximum $x_0 \in X$. In other words, $|f| \leq |f(x_0)|$ on X. Let (U, φ) be a chart around x_0. Then, $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has maximum norm precisely at $\varphi(x_0)$. We can choose U to be connected, so $f \circ \varphi^{-1}$ is constant, i.e. $f|_U \equiv k \in \mathbb{C}$. Let V be another neighbourhood of x_0. Then $f \in \mathcal{O}(U \cap V)$, so $f|_{U \cap V} \equiv k$. By the identity principle, $f|_V \equiv k$. Since X is compact, with a finite number of such V we get $f \equiv k$ on all X. □

Let X and Y be complex manifolds, with atlases $\{U_i, \varphi_i\}$ and $\{V_j, \psi_j\}$ respectively. By definition, a map $F : X \to Y$ is *holomorphic* if each

$$\psi_j \circ F \circ \varphi_i^{-1} : \varphi_i(U_i \cap F^{-1}(V_j)) \to \psi_j(V_j \cap F(U_i))$$

is holomorphic. Clearly, the definition is independent on the choice of charts.

There exists a category, usually called *complex analytic category*, whose objects are complex manifolds and whose arrows are holomorphic maps. The isomorphisms are biholomorphic maps (or *biholomorphisms*). In particular, for each complex manifold X we can define its *automorphism* group

$$\mathrm{Aut}(X) = \{\text{biholomorphic maps } F : X \to X\}.$$

For example, one can show that

$$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C}) = \{f(z) = \frac{az + b}{cz + d},\ \ ad - bc \neq 0\}.$$

§2. SUBMANIFOLDS AND IDEAL SHEAF

Let X be a complex manifold. $Y \subset X$ is called *analytic subset* if any $y \in Y$ admits a neighbourhood $U \subset X$ such that we can write

$$Y \cap U = \{x \in U :\ f_1(x) = \ldots = f_r(x) = 0\},$$

for some holomorphic functions $f_j \in \mathcal{O}(U)$. If $r = 1$, we say that Y is an *analytic hypersurface*. Therefore, an analytic subset is a subset locally described as the intersection of finitely many analytic hypersurfaces.

Notice that, given an analytic subset Y as above, we get two ideals of $\mathcal{O}(U)$: the generated ideal $(f_1, \ldots, f_r) \subset \mathcal{O}(U)$ and the *defining ideal* of Y,

$$\mathcal{I}_Y(U) := \{f \in \mathcal{O}(U) :\ f(y) = 0,\ \forall y \in Y \cap U\}.$$

Clearly, the latter contains the former. However, they can differ in general.

EXAMPLE 2.3. Let $U = \mathbb{C}^2$ and $f = z_2^2$. Let $Y = \{f = 0\}$. Then $(f) = (z_2^2)$ is strictly contained in $\mathcal{I}_Y(U) = (z_2)$.

It is easy to check that $\mathcal{I}_Y$ defines a sheaf (of ideals) on X, with the natural restrictions of functions. It is called the *ideal sheaf* of Y.

Consider now the structure sheaf $\mathcal{O}_X$. On the stalks, one has the following:

◇ $\mathcal{I}_{Y,x} \subset \mathcal{O}_{X,x}$ is an ideal. Moreover,

◇ if $x \notin Y$, then $\mathcal{I}_{Y,x}$ is the whole ring $\mathcal{O}_{X,x}$.

◇ if $x \in Y$, then $\mathcal{I}_{Y,x} \subsetneq \mathcal{O}_{X,x}$ is a *radical ideal* of $\mathcal{O}_{X,x}$. In other words, whenever $f^p \in \mathcal{I}_{Y,x}$ for some $p > 0$, then $f \in \mathcal{I}_{Y,x}$.

Definition. An analytic subset $Y \subset X$ is *irreducible* if it is not possible to write $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are analytic subsets of X, different from Y.

EXAMPLE 2.4. In $\mathbb{C}^2$, the analytic subset $Y : z_1 z_2 = 0$ is reducible. On the other hand, $Z : z_2^2 = 0$ is irreducible.

Definition. Let Y be an analytic subset of X. We say that

(i) Y is a *analytic subvariety* of X if it is irreducible.

(ii) Y is a *submanifold* of X, if it is a *smooth analytic subvariety*. That is, Y is irreducible and satisfies the *Jacobian* (or *smoothness*) *condition*: locally around any $y \in Y$ we *may find* some holomorphic functions $f_j \in \mathcal{O}(U)$, such that

◇ $Y \cap U = \{x \in U : \; f_1(x) = \ldots = f_r(x) = 0\}$, and

◇ if we denote by f the map $f = (f_1, \ldots, f_r) : U \to \mathbb{C}^r$, then

$$\operatorname{rk} J_f(y) = r.$$

For an analytic subvariety Y, the smoothness condition can fail at some point $y \in Y$. These are called *singular points* of Y. The set

$$Y_{\mathrm{Sing}} := \{y \in Y : \; y \text{ is singular}\}$$

is itself an analytic subset of X. We say that Y is *singular* if $Y_{\mathrm{Sing}} \neq \varnothing$.

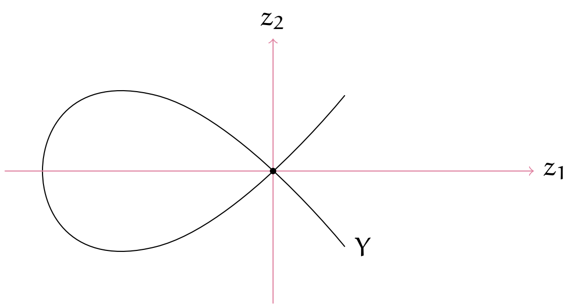
FACT (cf. GH p. 21). *Let Y be an analytic subset of X. Then*

$$Y \textit{ is irreducible} \iff Y \setminus Y_{\mathrm{Sing}} \textit{ is connected}.$$

EXAMPLES 2.5.

(i) In $\mathbb{C}^2$, let $Y : z_1 z_2 = 0$. Then $Y \setminus Y_{\mathrm{Sing}}$ is the disjoint union of two copies of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Hence, Y is reducible.

(ii) The *nodal cubic*. In $\mathbb{C}^2$, let $Y : f = 0$, where $f = z_1^2(1+z_1) - z_2^2$. The complex Jacobian $J_f = [3z_1^2 + 2z_1, -2z_2]$ has rank 1 at all points except at the origin, the only singular point of Y.

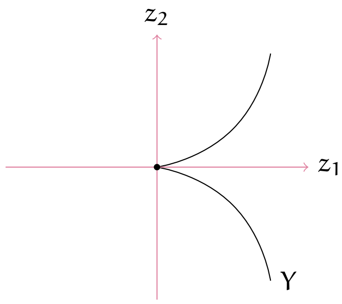


We have a parametrization of Y, given by $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$, whose restriction yields a homeomorphism $\mathbb{C} \setminus \{1, -1\} \approx Y \setminus Y_{\text{Sing}}$. Hence, Y is irreducible. Indeed, we have a homeomorphism

$$Y \approx \mathbb{C}/\{\pm 1\}$$

(the complex plane with the points 1 and -1 glued together). Finally, Y is an irreducible singular analytic subvariety of $\mathbb{C}^2$.

(iii) The *cuspidal cubic*. In $\mathbb{C}^2$, let $Y : g = 0$, where $g = z_1^3 - z_2^2$. The complex Jacobian $J_g = [3z_1^2, -2z_2]$ has rank 1 at all points except at the origin, the only singular point of Y.



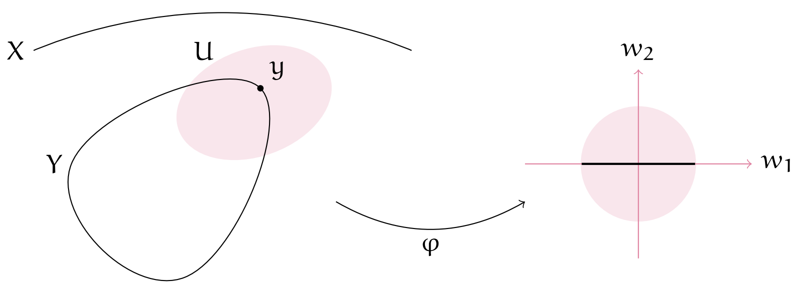
Topologically, the situation is slightly simpler than above: the parametrization $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (t^2, t^3)$ is a homeomorphism. Thus, $Y \setminus Y_{\text{Sing}} \approx \mathbb{C}^*$ is connected. Therefore Y is an irreducible, singular subvariety of $\mathbb{C}^2$.

In all the above examples, Y is an affine slice of some *algebraic curve* in the complex plane $\mathbb{P}^2$. One can show that any plane algebraic curve is an analytic subvariety of $\mathbb{P}^2$ and that it is smooth (as a subvariety) precisely when it is *smooth* as an algebraic curve.

A smooth subvariety can always be linearized locally.

PROPOSITION 2.6. *Let Y be a smooth analytic subvariety of a complex manifold X. Around any $y \in Y$ there exists a chart (U, φ) of X, with coordinates $w = (w_1, \ldots, w_n)$ centred at y, such that*

$$\varphi(Y \cap U) = \{w \in \mathbb{C}^n : w_1 = \ldots = w_r = 0\}.$$



Proof. Let $y \in Y$. By the smoothness condition, there is some local chart $z = (z_1, \ldots, z_n)$ centred at y, such that the subvariety Y is locally given by the zero locus of some holomorphic functions $f_1, \ldots, f_r$ with $\operatorname{rk} J_f = r$. We can assume that the first r coordinates $z_1, \ldots, z_r$ are such that the matrix $\frac{\partial(f_1,\ldots,f_r)}{\partial(z_1,\ldots,z_r)}$ has rank r around 0. Then, we let $w_1 := f_1(z), \ldots, w_r := f_r(z)$ and $w_j := z_j$ for $j > r$. Let $\varphi = (w_1, \ldots, w_n)$. By construction, J_φ has rank n around 0. Hence, φ yields a local biholomorphism onto a neighbourhood of $0 \in \mathbb{C}^n$, and Y is locally given as in the statement. $\square$

COROLLARY 2.7. *Let Y be as in the above proposition. Assume Y is connected. Then Y is a complex manifold of dimension $n - r$. Notice that $w_{r+1}, \ldots, w_n$ as in the proof above are local charts for Y.*

COROLLARY 2.8. *Let Y be an analytic subvariety of a complex manifold X. Then Y is locally smooth (i.e. around any smooth point $y \in Y \setminus Y_{\mathrm{Sing}}$).*

§3. IMMERSIONS AND EMBEDDINGS

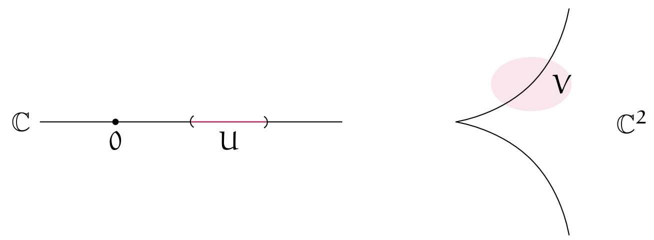
Let X and Y be compact complex manifolds of dimension n and m respectively.

Definition. Let $\varphi : X \to Y$ be a holomorphic map. We say that φ is an *immersion at* $x \in X$ if $\operatorname{rk} J_\varphi(x) = n$. In other words, the tangent map $d\varphi_x : T_xX \to T_{\varphi(x)}Y$ is injective. In particular $n \le m$. We say that φ is an *immersion* if the property holds at each point $x \in X$.

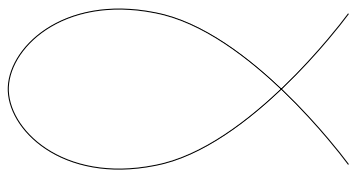
FACT. *Let φ be an immersion at $x \in X$. There exist neighbourhoods U of x, and V of $\varphi(x)$, such that $\varphi(U)$ is a smooth subvariety of V.*

EXAMPLES 2.9.

(i) $\psi : \mathbb{C} \to \mathbb{C}^2$, $z \mapsto (z^2, z^3)$. Then $\operatorname{rk} J_\varphi = 1$ at all points except at $z = 0$.



(ii) $\psi : \mathbb{C} \to \mathbb{C}^2$, $z \mapsto (z^2 - 1, z(z^2 - 1))$. Then $\operatorname{rk} J_\varphi = 1$ at all points.



Therefore ψ is an immersion. However, $\psi(\mathbb{C})$ is not a smooth subvariety of $\mathbb{C}^2$. This holds only locally, on each open subset $U \subset \mathbb{C}$ which does not contain both 1 and -1 (the points where φ fails to be injective).

Definition. An *embedding* is an injective immersion[2] $\varphi : X \to Y$.

For example, the inclusion of a smooth subvariety $Z \hookrightarrow X$ is an embedding.

PROPOSITION 2.10. *Let Z and X be complex manifolds and $\varphi : Z \to X$ an embedding. If Z is compact, then $\varphi(Z)$ is a smooth subvariety of X.*

Sketch of the proof. We know that locally, around any $x \in Z$, there are neighbourhoods U and V such that $\varphi(U)$ is a smooth subvariety of V. If, by chance, $\varphi(U) = \varphi(Z) \cap X$ we are done. Assume by contradiction that there exists $p \notin U$ such that $\varphi(p) \in V$. We can construct a sequence V_n of neighbourhoods of $\varphi(x)$, such that there exists $p_n \notin U$ with $\varphi(p_n) \in V_n$. In particular, $\lim \varphi(p_n) = \varphi(x)$. Since $Z \setminus U$ is closed and Z is compact, also $Z \setminus U$ is compact. Since $p_n \in Z \setminus U$, we have some convergence (passing to a subsequence if needed) $p_n \to p \in Z \setminus U$. However φ is continuous, being holomorphic. Therefore $\lim \varphi(p_n) = \varphi(p)$. Hence $\varphi(p) = \varphi(x)$, contradicting the fact that φ is injective. □

§4. EXPONENTIAL SEQUENCE

Let X be a complex manifold. We consider three sheaves on X, namely:

- $\mathbb{Z}$, sheaf of locally constant functions with integer values.
- $\mathcal{O}$, structure sheaf (holomorphic functions on X).
- $\mathcal{O}^*$, multiplicative sheaf of nowhere vanishing holomorphic functions.

Let U be a simply connected open subset of X isomorphic to a polydisk. Then we have an exact sequence

$$0 \longrightarrow \mathbb{Z}(U) \longrightarrow \mathcal{O}(U) \xrightarrow{e_U} \mathcal{O}^*(U) \longrightarrow 0$$

where the map $\mathbb{Z}(U) \to \mathcal{O}(U)$ is the obvious injection, while $e_U(f) := e^{2i\pi f}$. Since U is a polydisk, the exponential e_U is surjective. Therefore, we have an induced exact sequence on the stalks: for each $x \in X$,

$$0 \longrightarrow \mathbb{Z}_x \longrightarrow \mathcal{O}_x \xrightarrow{e_x} \mathcal{O}_x^* \longrightarrow 0$$

REMARK 2.11. *What does it mean that e_x is surjective?*

Let $g_x \in \mathcal{O}_x^*$. Then there is a open neighbourhood U of x and a representative $g \in \mathcal{O}(U)$ of g_x. The surjectivity of e_x does not mean *that there exists $f \in \mathcal{O}(U)$ such that $e^{2i\pi f} = g$. What it means is that there exists a small open subset $V \subset U$ and $f \in \mathcal{O}(V)$ such that $e^{2i\pi f} = g|_V$. Indeed, one can take V to be homeomorphic to a polydisk and define f on V as the complex logarithm of $g|_V$.*

[2]in the non-compact case, by *embedding* one usually means an injective immersion *which is also* a topological embedding, so that X is homeomorphic to its image. However, in the compact case, that of our interest, an injective immersion automatically has this property.

Therefore, we have an exact sequence of sheaves on X (cf. appendix A)

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

It is called the *exponential sequence* of X.

§5. Fundamental sequence of a submanifold

Let X be a complex manifold, Y a submanifold, i.e. a smooth subvariety of X. Let $\mathcal{I}_Y$ denote the ideal sheaf of Y. For any open subset $U \subset X$, we have an inclusion $\mathcal{I}_Y(U) \subset \mathcal{O}(U)$, which induces an inclusion on the stalks. For any $x \in X$, by taking the quotient $\mathcal{O}_x/\mathcal{I}_{Y,x}$, we then have an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y,x} \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_x/\mathcal{I}_{Y,x} \longrightarrow 0.$$

This yields an exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{I}_Y \longrightarrow 0.$$

We want to better understand this quotient. First, we have (cf. appendix A.17)

$$(\mathcal{O}/\mathcal{I}_Y)_x \simeq \mathcal{O}_x/\mathcal{I}_{Y,x}$$

Thus, we get nothing fancy when $x \notin Y$, since in this case $\mathcal{I}_{Y,x} = \mathcal{O}_x$, hence the quotient is trivial. Let $x \in Y$. Locally around x, there exist charts $z_1, \ldots, z_n$ such that Y is described by $\{z_1 = 0, \ldots, z_r = 0\}$ and the remaining charts $z_{r+1}, \ldots, z_n$ are local coordinates on Y as a manifold. By the identification of $\mathcal{O}_x$ with the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series about the origin, we interpret the ideal $\mathcal{I}_{Y,x} \subset \mathcal{O}_x$ as the ideal generated by $z_1, \ldots, z_r$. Then, an element in the quotient $f_x \in \mathcal{O}_x/\mathcal{I}_{Y,x}$, is described as

$$f_x = \sum a_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n} \mod (z_1, \ldots, z_r).$$

It is clear that when we quotient out the first r coordinates, what remains is a convergent power series about the origin in the remaining coordinates $z_{r+1}, \ldots, z_n$. But this is an element of $\mathcal{O}_{Y,x}$, where $\mathcal{O}_Y$ is the structure sheaf of Y. Therefore, when $x \in Y$, we have the isomorphism

$$(\mathcal{O}/\mathcal{I}_Y)_x \simeq \mathcal{O}_{Y,x}.$$

We would like to use this fact and rewrite the fundamental sequence of Y, putting $\mathcal{O}_Y$ in place of $\mathcal{O}/\mathcal{I}_Y$. However, this would not make sense, since $\mathcal{O}_Y$ is a sheaf on Y, not on X. We thus introduce the *trivial extension* of $\mathcal{O}_Y$ to all X. Let $i \colon Y \to X$ be the inclusion morphism. For $U \subset X$, we define[3]

$$i_*\mathcal{O}_Y(U) := \begin{cases} \mathcal{O}_Y(Y \cap U) & \text{if } Y \cap U \neq \varnothing \\ 0 & \text{if } Y \cap U = \varnothing \end{cases}$$

[3]the symbol i_* is just some piece of notation: this definition we are giving here is a particular case of a more general construction called *direct image sheaf*, which we do not need in the following.

We can now rewrite the exact sequence as

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O} \longrightarrow i_* \mathcal{O}_Y \longrightarrow 0.$$

This is called the *fundamental* exact sequence of Y. Warning: with some abuse of notation, one usually just writes $\mathcal{O}_Y$ in place of $i_* \mathcal{O}_Y$ in this sequence.

LECTURE 3

ALGEBRAIC VARIETIES

We will denote by S the ring of complex polynomials in the $n+1$ variables $x_0, \ldots, x_n$. We have a graduated-ring decomposition

$$S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d,$$

where S_d denotes the ring (in fact, a $\mathbb{C}$-vector space) of homogeneous polynomials of degree d (by definition, we let $0 \in S_d$).

Definition. An ideal $\mathfrak{a} \subset S$ is said *homogeneous* if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (S_d \cap \mathfrak{a}),$$

or, equivalently, if it is generated by homogeneous polynomials.

Given ideals $\mathfrak{a}, \mathfrak{b} \subset S$, we can construct:

- The *sum ideal* $\mathfrak{a} + \mathfrak{b}$, as the ideal generated by $\mathfrak{a}$ and $\mathfrak{b}$, i.e. the smallest ideal of S containing both $\mathfrak{a}$ and $\mathfrak{b}$.
- The *product ideal* $\mathfrak{a}\mathfrak{b}$, as the ideal generated by $\{fg \mid f \in \mathfrak{a}, g \in \mathfrak{b}\}$.
- The intersection $\mathfrak{a} \cap \mathfrak{b}$. Notice $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$.
- The *radical* $\sqrt{\mathfrak{a}} = \{f \mid f^p \in \mathfrak{a}, \text{ for some } p > 0\}$. One has $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}}$.

All these ideals are homogeneous whenever $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous.

If $d \geq 1$, notice that a polynomial $f \in S_d$ does not define a function on $\mathbb{P}^n$, since $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$. However, the set of zeroes, or *zero locus* of f, denoted by $Z(f) = \{x \in \mathbb{P}^n : f(x) = 0\}$, is a well-defined subset of $\mathbb{P}^n$.

Definition. An *algebraic subset* of $\mathbb{P}^n$, is a subset of the form

$$Z(\mathfrak{a}) = \{x \in \mathbb{P}^n : f(x) = 0, \ \forall f \in \mathfrak{a}\},$$

where $\mathfrak{a}$ is a homogeneous ideal of S.

Since S is a Noetherian ring, every homogeneous ideal $\mathfrak{a}$ is generated by finitely many homogeneous polynomials $f_1, \ldots, f_m$, so that

$$Z(\mathfrak{a}) = \{x \in \mathbb{P}^n : f_1(x) = \ldots = f_m(x) = 0\}.$$

REMARK 3.1.

(i) $Z(\mathfrak{a}) = \varnothing$ *if and only if* $\mathfrak{a}$ *contains the* "irrelevant ideal" $(x_0, \ldots, x_n)$.

(ii) $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

For $Y \subset \mathbb{P}^n$, we denote by $I(Y) \subset S$ the ideal generated by the homogeneous polynomials which vanish on Y. We then have a correspondence between the homogeneous ideals of S and the subsets of $\mathbb{P}^n$, given by $I(Y) \longmapsto Y$ on one side, and by $\mathfrak{a} \longmapsto Z(\mathfrak{a})$ on the other. One can check the following:

- $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- $Y \subset Z(I(Y))$.
- $I(Y) = \sqrt{I(Y)}$. That is, $I(Y)$ is a radical ideal.

The fundamental result here is the *Hilbert theorem of zeroes*, or *Nullstellensatz*:

THEOREM 3.2 (Nullstellensatz). $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

We define the *Zariski topology* on $\mathbb{P}^n$: we call *(Zariski) closed* set any algebraic subset $Z(\mathfrak{a}) \subset \mathbb{P}^n$. In fact, one can show that the algebraic subsets of $\mathbb{P}^n$ satisfy the axioms for the closed sets of a topology, namely

- $\mathbb{P}^n = Z(0)$ and $\varnothing = Z(x_0, \ldots, x_n)$.
- $Z(\mathfrak{a}_1) \cup Z(\mathfrak{a}_2) = Z(\mathfrak{a}_1\mathfrak{a}_2)$ $(= Z(\mathfrak{a}_1 \cap \mathfrak{a}_2))$.
- $\bigcap Z(\mathfrak{a}_i) = Z(\sum \mathfrak{a}_i)$ $(= Z(\bigcup \mathfrak{a}_i))$.

The Zariski topology is very weak (not even Haussdorff!), but often useful.

Let us now revisit the correspondence mentioned above:

$$\mathfrak{a} \longmapsto Z(\mathfrak{a}) \longmapsto I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$
$$Y \longmapsto I(Y) \longmapsto Z(I(Y)) = \overline{Y}$$

where $\overline{Y}$ denotes the topological closure, with respect to the Zariski topology. Therefore, we have a bijection between the set of homogeneous radical ideals of S (irrelevant ideal excluded) and the (non-empty) algebraic subsets of $\mathbb{P}^n$. This correspondence inverts the inclusions, and switches the union $\cup$ with the intersection $\cap$ and vice-versa.

Let now $V = Z(\mathfrak{a})$ be an algebraic subset defined by homogeneous ideal $\mathfrak{a} = (f_1, \ldots, f_m)$, which we can assume to be radical. Let $\{U_i\}$, $i = 0, \ldots, n$ be the standard open covering of $\mathbb{P}^n$. Over U_i, let $z_1, \ldots, z_n$ denote affine coordinates. We can de-homogenize the polynomials f_j with respect to the variable x_i and get polynomials $F_j(z) := f_j(z_1, \ldots, 1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$. The F_j's are holomorphic functions on U_i and clearly $V \cap U_i$ is given by $\{F_1 = \ldots = F_m = 0\}$. Thus V is an analytic subset of $\mathbb{P}^n$.

FACT (Chow, 1949). *Each analytic subset of $\mathbb{P}^n$ is an algebraic subset.*

Therefore, from now on we will identify algebraic and analytic subsets of $\mathbb{P}^n$. In particular, we can transfer some of the terminology: let V be an algebraic subset of $\mathbb{P}^n$. We shall say,

- ◇ V is *irreducible* if it is irreducible as analytic subset.
- ◇ V is *smooth* if it is smooth as analytic subvariety.
- ◇ V has *dimension* k if locally, around a non-singular point, V is a smooth subvariety of dimension k (i.e. the Jacobian condition holds).

FACT. V *is irreducible if and only if* $I(V)$ *is a prime ideal.*

THEOREM 3.3 (Lasker-Noether). *Any radical ideal* $\mathfrak{a}$ *of* S *is the intersection of finitely many prime ideals.*

The immediate geometrical consequence is that any algebraic subset can be written as the union of finitely many irreducible components,

$$V = V_1 \cup \ldots \cup V_l.$$

Definition. A *projective algebraic variety* (or *algebraic subvariety of* $\mathbb{P}^n$) is by definition an irreducible algebraic subset V.

REMARK 3.4. *If* V *is a* smooth *algebraic variety, then* V *is a complex manifold. Moreover, it is compact. Indeed,* V *is closed in the Zariski topology of* $\mathbb{P}^n$*, hence it is closed also in the usual (or analytic) topology.*

REMARK 3.5. *The Jacobian condition can also be verified in homogeneous coordinates. Indeed, let* $f = (f_1, \ldots, f_m)$ *and* $x = (x_0, \ldots, x_n)$*. One has*

$$J_f(\rho x) = J_f(x)\,\mathrm{diag}(\rho^{d_1 - 1}, \ldots, \rho^{d_n - 1})$$

where $\rho \in \mathbb{C}^*$, $d_i = \deg f_i$ *and* $\mathrm{diag}(\cdots)$ *is a diagonal matrix. Therefore, the rank of* J_f *is well defined at each point* $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n$*. Over the standard open subset* U_0*, with affine coordinates* $z_i = x_i/x_0$*, let* F_j *denote the de-homogenized polynomial* f_j *with respect to* x_0*. Denote by*

$$J := \frac{\partial(f_1, \ldots, f_m)}{\partial(x_0, \ldots, x_n)}, \quad A := \frac{\partial(F_1, \ldots, F_m)}{\partial(z_1, \ldots, z_n)}.$$

$V = Z(f_1, \ldots, f_m)$*. Over* $V \cap U_0$ *we have* $\frac{\partial f_j}{\partial x_i} = \frac{\partial F_j}{\partial z_i}$*, hence*

$$J = \left(\frac{\partial f}{\partial x_0} \,\middle|\, A \right)$$

where we denote by $\frac{\partial f}{\partial x_0}$ *the column vector with entries* $\frac{\partial f_j}{\partial x_0}$*. Now, by Euler's theorem on homogeneous functions, we have* $\sum_{i=0}^n x_i \frac{\partial f_j}{\partial x_i} = d_j f_j$*. However, over* $V \cap U_0$ *we have* $f_j = 0$ *for all* j*. Thus* $\frac{\partial f_j}{\partial x_0} = -\sum_{i=1}^n \frac{x_i}{x_0} \frac{\partial f_j}{\partial x_i}$*, i.e. the first column of* J *is a linear combination of the rows of* A*. In conclusion,*

$$\mathrm{rk}\, J = \mathrm{rk}\, A \quad (\textit{over } V \cap U_0).$$

EXAMPLE 3.6. The cubic surface in $\mathbb{P}^3$ with equation

$$f := x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0,$$

is smooth. Indeed, the Jacobian $J_f(x) = (3x_0^2, 3x_1^2, 3x_2^2, 3x_3^2)$ has rank zero only at $(0:0:0:0) \notin \mathbb{P}^3$.

We now present some classical examples of algebraic varieties.

HYPERSURFACES IN $\mathbb{P}^N$

These are the algebraic subsets Σ defined as the zero locus of a single polynomial $f \in S_d$. We shall say that d is the *degree* of Σ. Thus,

$$\Sigma = Z(f) = \{x \in \mathbb{P}^N : f(x) = 0\}$$

and $I(\Sigma) = \sqrt{(f)}$. If (f) is a radical ideal, so that $I(\Sigma) = (f)$, we say that Σ is *reduced*. Note that Σ is irreducible if and only if (f) is a prime ideal.

Σ has codimension 1 in $\mathbb{P}^N$, i.e. $\dim \Sigma = N - 1$. If $N = 2$, Σ is a plane algebraic curve. If $N = 3$, Σ is an algebraic surface in space, and so on.

A hypersurface of degree $d = 1$ is called a *hyperplane*. When $d = 2$, Σ is called a *quadric hypersurface* (or *hyperquadric*). In this case

$$f = \sum a_{ij} x_i x_j,$$

where the coefficients $a_{ij} = a_{ji}$ are not all zero. Let $A = [a_{ij}]$ be the (symmetric) matrix of the coefficients. Then,

$$J_f = 2(x_0, \ldots, x_N)A.$$

In particular, Σ is smooth (i.e. $\operatorname{rk} J_f = 1$ everywhere) if and only if $\det A \neq 0$. Moreover, one can show that Σ is irreducible if and only if $\operatorname{rk} A \geq 3$.

COMPLETE INTERSECTIONS

Let $V \subset \mathbb{P}^N$ be an algebraic variety of dimension n (i.e. codimension $N - n$). We say that V is a *complete intersection* if

$$I(V) = (f_1, \ldots, f_{N-n}),$$

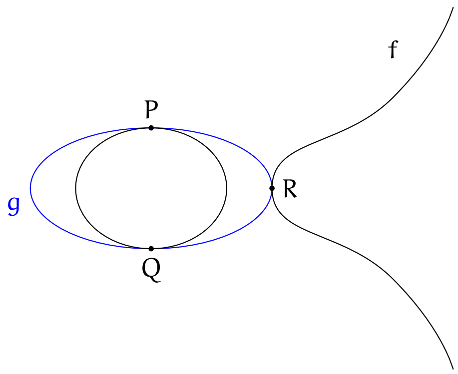
for some homogeneous polynomials f_i, for $i = 1, \ldots, N - n$.

REMARK 3.7. *Let V be a complete intersection as above and denote by $\Sigma_i := Z(f_i)$. Then V is the intersection of these hypersurfaces,*

$$V = \Sigma_1 \cap \ldots \cap \Sigma_{N-n}.$$

However, it is worth to notice that the vice-versa does not hold, i.e. we can have a variety $V = Z(\mathfrak{a})$, with $\mathfrak{a} = (g_1, \ldots, g_{N-n})$ which is not a complete intersection, i.e. $I(V) = \sqrt{\mathfrak{a}} = (f_1, \ldots, f_M)$ with $M > N - n$. For example, let $P, Q, R \in \mathbb{P}^2$ be three non-aligned points lying on a cubic $\{f = 0\}$, and let $\{g = 0\}$ be a conic tangent to the cubic in these three points. Then,

$$V := Z(f, g) = \{P, Q, R\},$$



and notice that we have $I(V) = (g, h, k)$, *where* $h = 0$ *defines another conic through* P, Q, R *cutting the first conic in a fourth point* T, *and* $k = 0$ *defines a conic through* P, Q, R *not passing through* T.

EXAMPLES 3.8.

(i) Let Q_1, Q_2 be hyperquadrics in $\mathbb{P}^N$ defined by f_1 and f_2 respectively. Then $Q_1 \cap Q_2$ is a complete intersection, $I(Q_1 \cap Q_2) = (f_1, f_2)$.

(ii) A *linear subspace* $\mathbb{P}^n \subset \mathbb{P}^N$ is a complete intersection of hyperplanes. $I(\mathbb{P}^n) = (f_1, \dots, f_{N-n})$, where each f_i defines a hyperplane.

(iii) A hyperquadric $Q \subset \mathbb{P}^{n+1} \subset \mathbb{P}^N$ is a complete intersection of $N-(n+1)$ hyperplanes and one hypersurface.

(iv) Any smooth algebraic variety is a complete intersection, *locally*.

RATIONAL NORMAL CURVES

Consider the *twisted cubic* in $\mathbb{P}^3$, defined as a *determinantal variety* by

$$\Gamma = \{x \in \mathbb{P}^3 : \operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} < 2\}.$$

In other words, $I(\Gamma) = (f_1, f_2, f_3)$, where

$$f_1 = x_0x_2 - x_1^2, \quad f_2 = x_0x_3 - x_1x_2, \quad f_3 = x_1x_3 - x_2^2.$$

In particular, Γ is not a complete intersection. Precisely, one has to show that 3 is indeed the minimum number of generators of $I(\Gamma)$. It is simpler to see this geometrically: let Q_i be the quadric defined by f_i. One has

$$Q_i \cap Q_j = \Gamma \cup \ell_{ij},$$

where ℓ_{ij} is a line[1]. Hence, Γ is not a complete intersection.

Let us see how Γ can be locally obtained as a complete intersection. Consider the standard open subset $U_0 = \{x \in \mathbb{P}^3 : x_0 \neq 0\}$ with affine charts $x = x_1/x_0$, $y = x_2/x_0$ and $z = x_3/x_0$. Then, $\Gamma \cap U_0$ is defined by

$$\operatorname{rk}\begin{pmatrix} 1 & x & y \\ x & y & z \end{pmatrix} < 2,$$

[1] for example, when $i = 1$ and $j = 2$ one has the line $x_0 = x_1 = 0$.

i.e. by the polynomials $F_1 = y - x^2$, $F_2 = z - xy$ and $F_3 = xz - y^2$. However, observe that $F_3 = xF_2 - yF_1 \in (F_1, F_2)$. Therefore, $\Gamma \cap U_0$ is obtained as the complete intersection of $F_1 = 0$ and $F_2 = 0$. Notice that

$$J_{(F_1,F_2)} = \begin{pmatrix} -2x & 1 & 0 \\ -y & -x & 1 \end{pmatrix}$$

has rank 2 everywhere (similarly on the other U_j's). Therefore Γ is smooth and $\dim \Gamma = 3 - \mathrm{rk} = 1$, i.e. Γ is a smooth curve.

We observe that Γ is the image of $\mathbb{P}^1$ under the (holomorphic) map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad (t_0 : t_1) \longmapsto (t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3).$$

Definition. The *rational normal curve of degree* d is the image of the map

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^d, \quad (t_0 : t_1) \longmapsto (t_0^d : t_0^{d-1} t_1 : \ldots : t_0 t_1^{d-1} : t_1^d).$$

PROPOSITION 3.9. *ϕ is an embedding.*

Proof. It is obviously injective. With local coordinates $t := t_1/t_0$ on $\mathbb{P}^1$ and $w_i = x_i/x_0$ on $\mathbb{P}^d$, the map ϕ is locally described by

$$t \mapsto (t, t^2, \ldots, t^d).$$

Therefore $J_\phi = (1, 2t, \ldots, dt)$ has rank 1 everywhere. Similarly, the same holds in the other affine charts. $\square$

Thus, $\Gamma_d := \phi(\mathbb{P}^1)$ is a smooth subvariety of $\mathbb{P}^d$ of dimension 1, i.e. a smooth algebraic curve. Γ_1 is a line, Γ_2 a conic, Γ_3 is the twisted cubic.

EXERCISE. Find $I(\Gamma_d)$.

SEGRE EMBEDDINGS

Let $\sigma : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$, where $N = (r+1)(s+1) - 1$ be defined as

$$\sigma((t_0 : \ldots : t_r), (\tau_0 : \ldots : \tau_s)) = (t_0\tau_0 : \ldots : t_i\tau_j : \ldots : t_r\tau_s).$$

One can check that σ is an embedding and $\sigma(\mathbb{P}^r \times \mathbb{P}^s)$, called *Segre embedding*[2] of $\mathbb{P}^r \times \mathbb{P}^s$, is a smooth algebraic variety of dimension $r + s$.

EXAMPLE 3.10. When $r = s = 1$, we have

$$\sigma((t_0 : t_1), (\tau_0 : \tau_1)) = (t_0\tau_0 : t_0\tau_1 : t_1\tau_0 : t_1\tau_1).$$

Then, $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ is described by $x_0x_3 - x_1x_2$, the smooth quadric in $\mathbb{P}^3$.

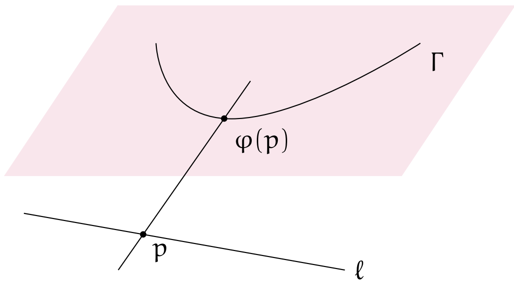
EXAMPLE 3.11. When $r = 2, s = 1$, the embedding σ is locally described as $((t, s), \tau) \mapsto (\tau, t, s, t\tau, s\tau)$. Hence, we notice that

$$J_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tau & 0 & t \\ 0 & \tau & s \end{pmatrix}$$

has rank 3 everywhere.

[2]Corrado Segre (1863-1924).

Let us see, for example, the so called *cubic scroll* of $\mathbb{P}^4$. In some projective space, we consider a line ℓ which is skew with respect to a plane π. We then let $\mathbb{P}^4$ be the linear space generated by ℓ and π. In symbols, $\mathbb{P}^4 = \langle \ell, \pi \rangle$.



Let Γ be an irreducible conic lying over π. Let φ be an isomorphism $\ell \simeq \Gamma$, i.e. an embedding $\varphi : \ell = \mathbb{P}^1 \longrightarrow \Gamma \subset \mathbb{P}^2 = \pi$. We define the scroll

$$S := \bigcup_{p \in \ell} \langle p, \varphi(p) \rangle.$$

Let us find some equations for S. Let, for example

$$\ell : \begin{cases} x_0 = t_0 \\ x_1 = t_1 \\ x_2 = x_3 = x_4 = 0 \end{cases} \qquad \Gamma : \begin{cases} x_0 = x_1 = 0 \\ x_2 = t_0^2 \\ x_3 = t_0 t_1 \\ x_4 = t_1^2 \end{cases}$$

for parameters $(t_0 : t_1) \in \mathbb{P}^1$. We get the parametric equations for S,

$$S : \begin{cases} x_0 = \lambda t_0 \\ x_1 = \lambda t_1 \\ x_2 = \mu t_0^2 \\ x_3 = \mu t_0 t_1 \\ x_4 = \mu t_1^2 \end{cases}$$

for parameters $(t_0 : t_1) \in \mathbb{P}^1$ and $(\lambda : \mu) \in \mathbb{P}^1$. Eliminating the parameters,

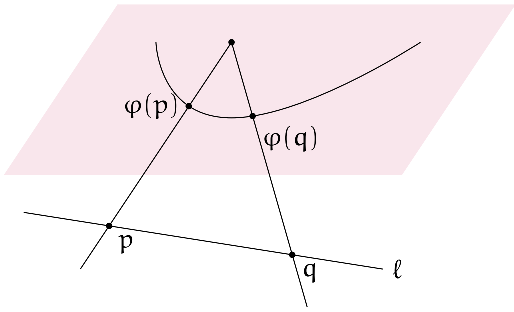
$$S : \begin{cases} f_1 := x_0 x_3 - x_1 x_2 = 0 \\ f_2 := x_0 x_4 - x_1 x_3 = 0 \\ f_3 := x_2 x_4 - x_3 x_3 = 0 \end{cases}$$

and one has $I(S) = (f_1, f_2, f_3)$. Thus, as a determinantal variety,

$$S = \{x \in \mathbb{P}^4 : \operatorname{rk} \begin{pmatrix} x_0 & x_2 & x_3 \\ x_1 & x_3 & x_4 \end{pmatrix} < 2\}.$$

One finds that $J_{(f_1, f_2, f_3)}$ has rank 2 everywhere. Therefore, S is a smooth variety of dimension $4 - \text{rk} = 2$, i.e. a smooth algebraic surface in $\mathbb{P}^4$.

REMARK 3.12. *The lines* $\langle p, \varphi(p)\rangle$ *and* $\langle q, \varphi(q)\rangle$ *do not intersect if* $p \neq q$. *Indeed, assume by contradiction* $\langle p, \varphi(p)\rangle \cap \langle q, \varphi(q)\rangle \neq \varnothing$. *Make a drawing:*



Then $p, q, \varphi(p), \varphi(q)$ *must lie on the same plane. It follows that*

$$\langle p, q\rangle \cap \langle \varphi(p), \varphi(q)\rangle \neq \varnothing,$$

a contradiction: $\langle \varphi(p), \varphi(q)\rangle \subset \pi$ *and* $\langle p, q\rangle = \ell$ *is skew with* π.

This means that there exists a well defined fibration $S \to \mathbb{P}^1$ whose fibres are precisely the lines $\langle p, \varphi(p)\rangle$ which generate the surface S.

§1. DEGREE OF A PROJECTIVE VARIETY

Let $V \subset \mathbb{P}^N$ be an algebraic variety[3]. We shall say:

- $x \in V$ is a *generic point* if it lies in the complement of some proper algebraic subset of V (i.e. out of a Zariski closed subset). A hyperplane $H \subset \mathbb{P}^N$ is *generic* if it is generic as a point in the dual $(\mathbb{P}^N)^*$.
- $\dim_{\mathbb{C}} V = n$ if in some small neighbourhood of a generic point, V is locally a smooth analytic subvariety of dimension n.
- V is *degenerate* if it is contained in some hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$.
- We denote by $\langle V\rangle$ the *linear span* of V, i.e.

  $$\langle V\rangle := \bigcup_{p,q\in V} \langle p, q\rangle.$$

  Then, V is non-degenerate if and only if $\langle V\rangle = \mathbb{P}^N$.

LEMMA 3.13. *Let* $C \subset \mathbb{P}^N$ *be a non-degenerate irreducible algebraic curve. For any pair of generic hyperplanes* $H_1, H_2 \subset \mathbb{P}^N$ *we have*

$$\#(C \cap H_1) = \#(C \cap H_2).$$

Proof. We assume that C is smooth (i.e. a compact Riemann surface). Let

$$H_i : \quad \sum_{j=0}^{N} a_j^{(i)} x_j = 0.$$

[3] recall that V is irreducible by definition.

Then, we define the following rational function on $\mathbb{P}^N$,

$$F = \frac{\sum a_j^{(1)} x_j = 0}{\sum a_j^{(2)} x_j = 0}.$$

The restriction $f = F|_C$ defines a *meromorphic function* on C, i.e. f has at most poles as singularities. By the basic properties of meromorphic functions one has #{zeroes of f} = #{poles of f}. However, notice that

$$\{\text{zeroes of } f\} = (C \cap H_1), \quad \text{and} \quad \{\text{poles of } f\} = (C \cap H_2).$$

When C is a singular curve, one considers the *normalization* of C: one can show that there exists a smooth curve $\widetilde{C}$ and a morphism $\nu : \widetilde{C} \to C$, which is an isomorphism onto $C \setminus C_{\text{Sing}}$. □

Definition. Let $C \subset \mathbb{P}^N$ be a non-degenerate irreducible algebraic curve. We define the *degree* of C by

$$\deg C = \#(C \cap H),$$

where H is a generic hyperplane of $\mathbb{P}^N$.

REMARK 3.14. *The definition is still valid when* H *is non-generic, but we have to take multiplicities into account. For example, consider the twisted cubic curve, in affine charts*

$$C : \begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$$

Let $H : ax + by + cz + d = 0$ *be a generic hyperplane. Then* H *and* C *intersect at three distinct points. On the other hand, if we consider* $H : z = 0$*, the intersection yields* $t^3 = 0$*, i.e. there is a single point in the intersection, but the multiplicity is* 3*.*

Now we prove that there is a lower bound for the degree of an embedded curve and classify those curves of minimal degree: they are rational curves.

THEOREM 3.15. *Let* $C \subset \mathbb{P}^N$ *be a non-degenerate irreducible algebraic curve. Then* $\deg C \geq N$ *and, if the equality holds,* C *is smooth and* $g(C) = 0$*.*

Proof. Since $\langle C \rangle = \mathbb{P}^N$, a choice of N points of C which are linearly independent determines a hyperplane H which cuts out on C (at least) these N points. Thus, by definition of degree, we get $\deg C = H \cap C \geq N$.

Assume $\deg C = N$. Let $p_1, \dots, p_{N-1} \in C$ be linearly independent. Consider the family of hyperplanes $\{H_t\}_{t \in \mathbb{P}^1}$ with base[4]

$$\Lambda = \langle p_1, \dots, p_{N-1} \rangle = \mathbb{P}^{N-2}.$$

Cutting C with H_t, we get a family of points $p_t \in C$, i.e.

$$H_t \cap C = \{p_1, \dots, p_{N-1}, p_t\}.$$

[4]mental picture: think of a family of planes in space rotating about an axis $\Lambda = \langle p, q \rangle$.

The idea is the following: we can assume C to be smooth (if not, we replace C with its normalization). Then, one can show that the map

$$\mathbb{P}^1 \longrightarrow C, \quad t \longmapsto p_t$$

is holomorphic and surjective. By the Riemann-Hurwitz theorem this implies that $g(C) \leq g(\mathbb{P}^1)$. However, one has $g(\mathbb{P}^1) = 0$. □

Some easy consequences of the Theorem:

- ◇ A curve of degree 1 is a line ($C = \mathbb{P}^1$).
- ◇ If C is a curve of degree 2, then $\langle C \rangle = \mathbb{P}^2$, i.e. C is a plane conic. In particular, there exist no skew conics[5].

Now, let $V \subset \mathbb{P}^N$ be an algebraic variety of dimension $n \geq 2$.

FACT. *For any generic hypersurface $\Sigma \subset \mathbb{P}^N$, the algebraic subset $W := V \cap \Sigma$ is irreducible (i.e. W itself is a variety) and $\dim W = n - 1$.*

Then what? Iterating this precisely $n-1$ times we see that the generic linear subspace of $\mathbb{P}^N$ of codimension $n-1$ cuts V along an irreducible curve[6]. By induction on $n = \dim V$, it is easy to see that two such curves have the same degree. When $n = 2$, take two generic hyperplanes H_1, H_2. Let $C_i = V \cap H_i$.

$$\deg C_1 = \#(C_1 \cap H_2) = \#(V \cap H_1 \cap H_2) = \#(C_2 \cap H_1) = \deg C_2.$$

Definition. Let $V \subset \mathbb{P}^N$ be a non-degenerate variety. The *degree* of V is

$$\deg V := \deg C,$$

where C is the curve cut out on V by the generic linear subspace of $\mathbb{P}^N$ of codimension $n-1$, where $n = \dim V$. Equivalently,

$$\deg V = \#(V \cap L),$$

where $L = \mathbb{P}^{N-n}$ is a generic linear subspace of $\mathbb{P}^N$ of codimension n.

REMARK 3.16. *Let H be a generic hyperplane and consider the codimension 1 (in V) subvariety of V given by $W = V \cap H$. Then*

$$\deg V = \deg W.$$

THEOREM 3.17. *Let $V \subset \mathbb{P}^N$ be a non-degenerate variety. Then*

$$\deg V \geq \operatorname{codim} V + 1.$$

Proof. By induction on $n = \dim V$. When $n = 1$, we have seen it above. We therefore assume the Theorem to be true up to dimension $n-1$. Let V be a non-degenerate variety of dimension n. Consider a generic hyperplane section $W = V \cap H$. Then $\deg V = \deg W \geq \operatorname{codim}_H W + 1$ by the inductive hypothesis, where W is considered non-degenerate as a variety in $H = \mathbb{P}^{N-1}$. Hence $\operatorname{codim}_H W = \dim H - \dim W = (N-1) - (n-1) = \operatorname{codim} V$. □

[5] when referring to curves, the term *skew* stands for *not contained in a plane*.

[6] equivalently: the intersection of V with the generic linear subspace L of complementary dimension (i.e. of codimension n) consists of finitely many points! Now guess how we define $\deg(V)$.

Some easy consequences of the Theorem:

- $\diamond$ A variety V of degree 1 is a linear subspace ($V = \mathbb{P}^n$).
- $\diamond$ A variety V of degree 2 is a quadric hypersurface in $\langle V \rangle$.

Definition. A non-degenerate variety $V \subset \mathbb{P}^N$ with $\deg V = \operatorname{codim} V + 1$ is called *variety of minimal degree*. For example, the hypersurfaces of minimal degree are the hyperquadrics, but the family is much larger. However, there exists a complete classification (Bertini, Del Pezzo).

EXAMPLES OF DEGREE COMPUTATION

(i) *Rational normal curve of degree* d. Let $C = \phi(\mathbb{P}^1)$, where

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^d, \quad (t_0 : t_1) \longmapsto (t_0^d : t_0^{d-1} t_1 : \ldots : t_0 t_1^{d-1} : t_1^d).$$

Let $H \subset \mathbb{P}^d$ be the hyperplane with equation

$$H : \quad \sum_{j=0}^{d} a_j x_j = 0.$$

Since ϕ is injective, the number $\#(C \cap H)$ is equal to the number of solutions to the equation

$$\sum_{j=0}^{d} a_j t_0^{d-j} t_1^j = 0,$$

where $(t_0 : t_1) \in \mathbb{P}^1$, i.e. the number of zeroes (on the line) of a homogeneous polynomial of degree d, which are precisely d. Thus $\deg C = d$. In particular, C is a variety of minimal degree.

(ii) *Veronese surface*. Consider the map

$$\nu : \mathbb{P}^2 \longrightarrow \mathbb{P}^5, \quad (t_0 : t_1 : t_2) \longmapsto (t_0^2 : t_0 t_1 : t_0 t_2 : t_1^2 : t_1 t_2 : t_2^2).$$

It is not hard to show that ν is an embedding. The image $S := \nu(\mathbb{P}^2)$ is called *Veronese surface*[7]. As a determinantal variety we have

$$S = \{x \in \mathbb{P}^5 : \operatorname{rk} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix} < 2\}.$$

Let us compute $\deg S$. Consider two generic hyperplanes of $\mathbb{P}^5$,

$$\begin{aligned} H_1 : &\quad a_0 x_0 + \ldots + a_5 x_5 = 0 \\ H_2 : &\quad b_0 x_0 + \ldots + b_5 x_5 = 0 \end{aligned}$$

Since ν is injective, the number $\#(S \cap H_1 \cap H_2)$ is equal to the number of solutions to the system of equations

$$\begin{cases} a_0 t_0^2 + a_1 t_0 t_1 + a_2 t_0 t_2 + a_3 t_1^2 + a_4 t_1 t_2 + a_5 t_2^2 \\ a_0 t_0^2 + a_1 t_0 t_1 + a_2 t_0 t_2 + a_3 t_1^2 + a_4 t_1 t_2 + a_5 t_2^2 \end{cases}$$

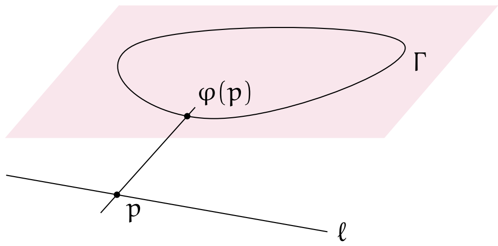
[7] Giuseppe Veronese (1854-1917)

where $(t_0 : t_1 : t_2) \in \mathbb{P}^2$. This corresponds to the intersection number of two conics in $\mathbb{P}^2$. Thus[8] $\deg S = 4$, and S is a variety of minimal degree.

(iii) *Cubic scroll of* $\mathbb{P}^4$. Consider the surface

$$S = \bigcup_{p \in \ell} \langle p, \varphi(p) \rangle$$

obtained as the cubic scroll of $\mathbb{P}^4$ described above. We give a heuristic yet instructive argument for the computation of $\deg S$.



Let $H \subset \mathbb{P}^3$ be a hyperplane containing the plane $\pi = \langle \Gamma \rangle$. Let q be the point of intersection of H and ℓ. Then

$$H \cap S = \Gamma \cup \langle q, \varphi(q) \rangle =: C$$

i.e. C is the union of a conic and a line. Thus $\deg S = \deg C = 3$.

[8]the choice of H_1, H_2 generic corresponds to the fact that the conics intersect at 4 distinct points. A non-generic choice would result in points of tangency of the two conics.

LECTURE 4

DIVISORS AND LINE BUNDLES

§1. MEROMORPHIC FUNCTIONS AND CARTIER DIVISORS

Let X be a complex manifold of dimension n and $U \subset X$ an open subset.

Definition. A *meromorphic function* on U is a function which is locally represented as a quotient of holomorphic functions. We write $f \in \mathfrak{M}(U)$. We denote by $Z(f)$ and $P(f)$ the sets of *zeroes* and *poles* of f, respectively.

$\mathfrak{M}(U)$ is a ring containing the ring of holomorphic functions $\mathcal{O}(U)$. One can check that $\mathfrak{M}(U)$ is a field whenever U is connected. Moreover, if U is sufficiently small, $\mathfrak{M}(U)$ is precisely the *quotient field* of $\mathcal{O}(U)$. This is false when $U = X$ in the compact case[1] where we always have a proper inclusion $\mathcal{O}(X) = \mathbb{C} \subsetneq \mathfrak{M}(X)$. Indeed, a deep result (Riemann Existence Theorem) states that $\mathfrak{M}(X)$ is a finitely generated field over $\mathbb{C}$ of transcendence degree 1. For example, for the Riemann sphere we have $\mathfrak{M}(\mathbb{P}^1) = \mathbb{C}(z)$.

REMARK 4.1.

(i) *The elements $f \in \mathfrak{M}(U)$ are not "functions" in classical terms (neither can they be interpreted as holomorphic functions $U \to \mathbb{P}^1$), in the sense that there can exist points of U where f just cannot be defined. This happens precisely at those points which give zero at both numerator and denominator (for any representation of f as a quotient). It is called the set of* indeterminacies *of f, and denoted by $I(f)$. In other words,*

$$I(f) = Z(f) \cap P(f).$$

e.g. $f : \mathbb{C}^2 \to \mathbb{C}$, $f(z) = z_1/z_2$ has an indeterminacy at the origin.

(ii) *Let $f \in \mathcal{O}(U)$, $f \neq 0$. Then $Z(f)$, in a neighbourhood of $p \in U$, is the union of finitely many irreducible hypersurfaces of X. Indeed, $f_p \in \mathcal{O}_p$, which is a UFD, hence we can write $f_p = u \cdot h_1 \cdots h_r$.*

Similarly, given $f \in \mathfrak{M}(U)$ non-zero, we have that $Z(f)$ and $P(f)$ are locally given as the union of finitely many irreducible hypersurfaces.

CONSEQUENCE. *If X is compact, the set of zeroes $Z(f)$ and the set of poles $P(f)$ of any non-zero meromorphic function $f \in \mathfrak{M}(X)$ are given as the union of* finitely many *irreducible hypersurfaces of X.*

[1] while it holds in the non-compact case by the Behnke-Stein theorem (1948)

It is easy to check that $\mathfrak{M}$, with the natural restrictions of functions, yields a sheaf of abelian groups on X, the *sheaf of (germs of) meromorphic functions*. We get also two (multiplicative) sheaves of abelian groups on X:

- ⋄ $\mathfrak{M}^*$ = sheaf of (germs of) non-zero meromorphic functions.

$$\mathfrak{M}^*(U) = \mathfrak{M}(U) \setminus \{0\}$$

is an abelian group under multiplication of functions.

- ⋄ $\mathcal{O}^*$ = sheaf of (germs of) nowhere-vanishing holomorphic functions.

$$\mathcal{O}^*(U) = \{\text{holomorphic functions } f : U \to \mathbb{C}^*\}$$

is an abelian group under multiplication of functions.

Despite the similar notation, we point out a distinction: in $\mathcal{O}^*$ we require the functions to have no zeroes at all, while the functions in $\mathfrak{M}^*$ are allowed to have zeroes; we only remove the constant zero function.

$\mathcal{O}^*$ is sometimes called sheaf of *invertible* holomorphic functions, since

$$f \in \mathcal{O}^*(U) \iff 1/f \in \mathcal{O}^*(U).$$

Notice that $\mathcal{O}^*(U) \subset \mathfrak{M}^*(U)$, which induces an inclusion on the stalks. Thus, the inclusion of sheaves $\mathcal{O}^* \hookrightarrow \mathfrak{M}^*$ yields a short exact sequence[2]

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathfrak{M}^* \longrightarrow \mathfrak{M}^*/\mathcal{O}^* \longrightarrow 0$$

Definition. The quotient sheaf $\mathfrak{D} = \mathfrak{M}^*/\mathcal{O}^*$ is the *sheaf of divisors* on X. An element $D \in \mathfrak{D}(U)$ is called *divisor* on U. In particular, we will call the elements $D \in \mathfrak{D}(X)$ (i.e. the global sections of $\mathfrak{D}$) *Cartier divisors*[3].

Here is the way to think of a divisor. We identify $D \in \mathfrak{D}(U)$ with a collection of meromorphic functions $f_\alpha \in \mathfrak{M}^*(U_\alpha)$, where $\{U_\alpha\}$ is some open covering of U, satisfying the following compatibility condition: whenever $U_\alpha \cap U_\beta \neq \varnothing$, the ratio of f_α and f_β is an invertible holomorphic function:

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

We write $D = (f_\alpha)$. The parenthesis ( ) are used to denote the collection. Despite $\mathcal{O}^*(U)$ and $\mathfrak{M}^*(U)$ are multiplicative groups, we will use additive notations for the group of divisors $\mathfrak{D}(U)$. Therefore, if $D_1 = (f_\alpha)$ and $D_2 = (g_\alpha)$, we have $D_1 + D_2 = (f_\alpha) + (g_\alpha) = (h_\alpha)$, where $h_\alpha = f_\alpha g_\alpha$.

The above short exact sequence of sheaves is the beginning of a long story. At this point, however, the powerful tool of sheaf cohomology becomes an unavoidable necessity. For the sake of the reader not familiar with sheaves, we briefly sketch the fundamental facts about this theory. A somewhat more detailed discussion can be found in the appendix A.

[2]one might put 1's in place of the zeroes in the sequence, since we are talking about multiplicative groups. However, we will often mix additive and multiplicative notation.

[3]Pierre Cartier (born 1932)

§2. Sheaf cohomology in a nutshell

Given a sheaf of abelian groups $\mathcal{F}$ on X, one can define, for all $q \in \mathbb{N}_0$, the so called q-*th cohomology group of* $\mathcal{F}$, an abelian group denoted by

$$H^q(X, \mathcal{F}).$$

The 0-th cohomology group is the group to the global sections of $\mathcal{F}$, i.e.

$$H^0(X, \mathcal{F}) \simeq \mathcal{F}(X).$$

Any sheaf morphism $f : \mathcal{F} \to \mathcal{G}$ induces homomorphisms

$$f_q : H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{G}),$$

which have the following (functorial) properties:

(a) $f_0 = f_X : \mathcal{F}(X) \to \mathcal{G}(X)$ (by the identification $H^0(X, \mathcal{F}) = \mathcal{F}(X)$).

(b) If $\mathcal{F} = \mathcal{G}$ and $f = \mathrm{id}_{\mathcal{F}}$ then $f_q = \mathrm{id}$

(c) For another sheaf morphism $g : \mathcal{G} \to \mathcal{H}$ one has $(g \circ f)_q = g_q \circ f_q$

Finally, for each short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ f\ } \mathcal{G} \xrightarrow{\ g\ } \mathcal{H} \longrightarrow 0$$

there exist homomorphisms $\delta^q : H^q(X, \mathcal{H}) \longrightarrow H^{q+1}(X, \mathcal{F})$ which induce the following long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "f_0"] & H^0(X,\mathcal{G}) \arrow[r, "g_0"] & H^0(X,\mathcal{H}) \arrow[dll, "\delta^0"] \\
 & H^1(X,\mathcal{F}) \arrow[r, "f_1"] & H^1(X,\mathcal{G}) \arrow[r, "g_1"] & H^1(X,\mathcal{H}) \arrow[dll, "\delta^1"] \\
 & H^2(X,\mathcal{F}) \arrow[r, "f_2"] & H^2(X,\mathcal{G}) \arrow[r, "g_2"] & H^2(X,\mathcal{H}) \cdots
\end{tikzcd}$$

§3. The exact sequence of divisors

Back to our story. The short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathfrak{M}^* \longrightarrow \mathfrak{D} \longrightarrow 0$$

induces a long exact sequence of cohomology, the beginning of which reads

$$0 \longrightarrow \mathcal{O}^*(X) \longrightarrow \mathfrak{M}^*(X) \longrightarrow \mathfrak{D}(X) \longrightarrow H^1(X, \mathcal{O}^*)$$

We will refer to this sequence as *the exact sequence of divisors* on X. The maps and the groups appearing in it are the characters of a beautiful play, full of surprises and dramatic turns of events. We now introduce the characters one at a time, and gradually discover how they interact with each other.

CARTIER DIVISORS

$\mathfrak{D}(X)$ is the group of Cartier divisors, already introduced above. Let $\{U_\alpha\}$ be some open covering of X. Then, a Cartier divisor $D \in \mathfrak{D}(X)$ is represented by $D = (f_\alpha)$, where $f_\alpha \in \mathfrak{M}^*(U_\alpha)$ are such that

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$$

which we will generally call *compatibility condition* for Cartiers divisors.

THE MAP $\mathfrak{M}^*(X) \longrightarrow \mathfrak{D}(X)$

This homomorphism of groups is denoted by $(\quad)$. It takes a very simple form: let $f \in \mathfrak{M}^*(X)$ and $\{U_\alpha\}$ be an open covering of X. Then one has

$$(f) = (f|_{U_\alpha}) \in \mathfrak{D}(X).$$

Note that the compatibility condition is trivially verified: $f|_{U_\alpha}/f|_{U_\beta} = 1$.

LINEAR EQUIVALENCE AND THE PICARD GROUP

A Cartier divisor $D \in \mathfrak{D}(X)$ lying in the image of the above map, i.e. such that $D = (f)$ for some $f \in \mathfrak{M}^*(X)$ is called *principal divisor*. We say that two divisors $D_1, D_2 \in \mathfrak{D}(X)$ are *linearly equivalent*, and write $D_1 \sim D_2$, if their difference is a principal divisor $(f) = D_1 - D_2$. The *Picard group* of X is the quotient of Cartier divisors modulo principal divisors or, equivalently, modulo linear equivalence:

$$\mathrm{Pic}(X) := \frac{\mathfrak{D}(X)}{\mathrm{Im}\,\mathfrak{M}^*(X)} = \mathfrak{D}(X)/\sim$$

WEIL DIVISORS

Cartier divisors are somewhat tricky to deal with. We now introduce another notion of divisor, of a much stronger geometrical flavour.

We assume X compact. A *Weil divisor*[4] on X is a finite formal sum

$$D = \sum m_i V_i$$

where $m_i \in \mathbb{Z}$ and V_i are irreducible hypersurfaces (possibly singular) of X. The group of Weil divisors will be denoted by $\mathrm{Div}(X)$. It is thus the free abelian group generated by the set of irreducible hypersurfaces of X.

Let $V \subset X$ be an irreducible analytic hypersurface. In a neighbourhood U of a given point $p \in V$, let $\nu \in \mathcal{O}(U)$ be the function defining $V \cap U$. Given $f \in \mathcal{O}(U)$ non-zero, we define the *order of* f *along* V *at* p by

$$\mathrm{ord}_{V,p}(f) := \max\{k \text{ such that } f_p = \nu_p^k \cdot g_p \text{ for some } g_p \in \mathcal{O}_p\}.$$

FACT (GH p. 10 or Ha p. 21). $\mathrm{ord}_{V,p}(f)$ *is independent of* $p \in V$.

[4] André Weil (1906-1998). Cartier was one of Weil's students.

Therefore, it is well defined $\mathrm{ord}_V(f)$, the order of f along V. One has

$$\mathrm{ord}_V(fg) = \mathrm{ord}_V(f) + \mathrm{ord}_V(g).$$

Moreover, for $f \in \mathcal{O}(U)$ non-zero, $\mathrm{ord}_V(f) = 0$ if and only if $V \cap U$ is not entirely contained in $Z(f)$, i.e. there exists $x \in V \cap U$ such that $f(x) \neq 0$.

Let us extend this to meromorphic functions. Let $f \in \mathfrak{M}^*(U)$ be locally represented by g/h, with g and h holomorphic (and coprime). We let

$$\mathrm{ord}_V(f) := \mathrm{ord}_V(g) - \mathrm{ord}_V(h).$$

PROPERTIES. *Let* $f, g \in \mathfrak{M}^*(U)$. *Then*

(a) $\mathrm{ord}_V(fg) = \mathrm{ord}_V(f) + \mathrm{ord}_V(g)$.

(b) $\mathrm{ord}_V(f) = 0$ *if and only if* $V \cap U$ *is not entirely contained in* $Z(f) \cup P(f)$.

PROPOSITION 4.2. *Let* X *be a compact complex manifold. Then*

$$\mathrm{Div}(X) \simeq \mathfrak{D}(X).$$

Notice that in the algebraic setting the result holds for *smooth* varieties.

Sketch of the proof. Let $\{U_\alpha\}$ be an open covering of X. Let $(f_\alpha) \in \mathfrak{D}(X)$. Whenever $U_\alpha \cap U_\beta \neq \varnothing$, notice that the meromorphic functions f_α and f_β have the same zeroes and poles over $U_\alpha \cap U_\beta$, by the compatibility condition. Therefore, given an irreducible hypersurface V, we always have, wherever it makes sense, $\mathrm{ord}_V(f_\alpha) = \mathrm{ord}_V(f_\beta)$. Fix an index α. Since X is compact $\mathrm{ord}_V(f_\alpha) \neq 0$ only for finitely many hypersurfaces V (by property (b) above and the consequences of remark 4.1). Therefore we can define

$$D = \sum_{V \subset X} \mathrm{ord}_V(f_\alpha) V \in \mathrm{Div}(X)$$

where the sum runs over all the analytic hypersurfaces of X. This yields a map $\mathfrak{D}(X) \to \mathrm{Div}(X)$, and one proves that it is:

- ◇ A homomorphism (by property (a) above)
- ◇ Injective: if $(f_\alpha) \mapsto D = 0$ it means that $\mathrm{ord}_V(f_\alpha) = 0$ for all V. Thus f_α has neither zeroes nor poles on U_α, i.e. $f_\alpha \in \mathcal{O}^*(U_\alpha)$, i.e. $(f_\alpha) = 0$.
- ◇ Surjective: let $D = \sum m_i V_i \in \mathrm{Div}(X)$. Let $g_{i\alpha} \in \mathcal{O}(U_\alpha)$ the function which defines V_i over U_α. Let $f_\alpha := \prod g_{i\alpha}^{m_i} \in \mathfrak{M}^*(U_\alpha)$. The collection (f_α) defines a Cartier divisor such that $(f_\alpha) \mapsto D$. □

Cartier and Weil divisors behave differently when we admit singularities. An example of a surface on which the two concepts differ is a quadric cone. At the singular point, the vertex of the cone, a single line ℓ drawn on the cone is a Weil divisor but not a Cartier divisor.

From now on, whenever we consider any compact complex manifold (in particular, any smooth projective variety), we identify $\mathfrak{D}(X) = \mathrm{Div}(X)$, and simply talk about "divisors". Given a divisor $D \in \mathrm{Div}(X)$, we will use its Cartier or Weil description, depending on the specific situation.

DESCRIPTION OF $H^1(X, \mathcal{O}^*)$ IN TERMS OF COCYCLES

The first cohomology group of $\mathcal{O}^*$ can be given a useful description in terms of cocycles[5], which we briefly sketch as follows. Once we fix an open covering $\{U_\alpha\}$ of X, an element $\xi \in H^1(X, \mathcal{O}^*)$ (a "cocycle") is given by a collection $(\xi_{\alpha\beta})$, where $\xi_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfy the *cocycle condition*

$$\xi_{\alpha\beta}\xi_{\beta\gamma} = \xi_{\alpha\gamma}$$

over $U_\alpha \cap U_\beta \cap U_\gamma$. Two cocycles $\xi = (\xi_{\alpha\beta})$ and $\xi' = (\xi'_{\alpha\beta})$ are *homologous* (and we identify them $\xi = \xi'$) if and only if there exists a collection $(\eta_{\alpha\beta})$, where $\eta_\alpha \in \mathcal{O}^*(U_\alpha)$, such that over $U_\alpha \cap U_\beta$ one can write

$$\xi'_{\alpha\beta} = \eta_\alpha^{-1}\xi_{\alpha\beta}\eta_\beta.$$

In particular, the neutral element $1 \in H^1(X, \mathcal{O}^*)$ is represented by any collection of the form $\xi_{\alpha\beta} = \eta_\beta/\eta_\alpha$, with $\eta_\alpha \in \mathcal{O}^*(U_\alpha)$.

THE MAP $\mathfrak{D}(X) \longrightarrow H^1(X, \mathcal{O}^*)$

The homomorphism $\delta : \mathfrak{D}(X) \to H^1(X, \mathcal{O}^*)$ appearing in the exact sequence of divisors is the "boundary map", constructed as in the *Snake lemma*.

Let $D \in \mathfrak{D}(X)$ be represented by $D = (f_\alpha)$ where $f_\alpha \in \mathfrak{M}^*(U_\alpha)$. Denote by $\xi_{\alpha\beta} := f_\alpha/f_\beta$. By the compatibility condition prescribed for Cartier divisors we know $\xi_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Notice that $\xi_{\alpha\beta}\xi_{\beta\gamma} = \xi_{\alpha\gamma}$. In other words, we have a well-defined cocycle $\xi = (\xi_{\alpha\beta}) \in H^1(X, \mathcal{O}^*)$. We will write $\xi = [D]$. Clearly $[\ \ ]$ is linear, i.e. we have a homomorphism

$$[\ \ ] : \mathfrak{D}(X) \longrightarrow H^1(X, \mathcal{O}^*), \quad (f_\alpha) \longmapsto (f_\alpha/f_\beta).$$

Is this simple homomorphism the same as the boundary map? Almost: if one computes δ as in the Snake lemma, will find that it is the "opposite", i.e. we have $\delta(D) = [-D]$. In terms of cocycles $\delta : (\xi_{\alpha\beta}) \longmapsto (f_\beta/f_\alpha)$.

By abuse of notation, we still denote by $[\ \ ]$ the induced injective homomorphism $\mathfrak{D}(X)/\ker \to H^1(X, \mathcal{O}^*)$, where ker is the kernel of $[\ \ ]$. Clearly $\ker = \ker(\delta)$. Hence, by the exactness of the sequence of divisors, ker is precisely the image of $(\ \ )$. Finally, we have an injective homomorphism[6]

$$[\ \ ] : \mathrm{Pic}(X) = \frac{\mathfrak{D}(X)}{\mathrm{Im}\,\mathfrak{M}^*(X)} \longrightarrow H^1(X, \mathcal{O}^*).$$

In other words, two divisors $D_1, D_2 \in \mathfrak{D}(X)$ define the same cocycle in $H^1(X, \mathcal{O}^*)$ if an only if they are linearly equivalent, i.e.

$$[D_1] = [D_2] \iff D_1 \sim D_2$$

where $\sim$ denotes linear equivalence.

THEOREM 4.3. *If X is smooth and projective, then $[\ \ ]$ is surjective, i.e.*

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^*).$$

This is not the end of the story. Another fundamental character needs to be introduced. Namely, the concept of holomorphic line bundle.

[5] cf. Čech cohomology.

[6] we remark that in the literature [D] is usually denoted by $\mathcal{O}_X(D)$.

§4. LINE BUNDLES

Let X be a complex manifold.

Definition. A *(holomorphic) vector bundle of rank* r on X is given by a surjective holomorphic map $\pi : E \to X$, such that:

- E is a complex manifold.
- Each *fiber* $E_x := \pi^{-1}(x)$ is a complex vector space of dimension r.
- There exists a *trivialization* of E, i.e. an open covering $\{U_\alpha\}$ of X and a family of bioholomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}^r$ such that:

  (i) The following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\simeq", "\psi_\alpha"'] \arrow[drr, "\pi"'] & & U_\alpha \times \mathbb{C}^r \arrow[d] \\ & & U_\alpha \end{tikzcd}$$

  where the vertical map is the projection on the 1^{st} factor.

  (ii) Whenever two open subsets U_α and U_β overlap, there exists a holomorphic map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Gl}(r, \mathbb{C})$ such that[7]

$$\begin{aligned} \psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r &\longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r \\ (x, v) &\longmapsto (x, g_{\alpha\beta}(x) \cdot v) \end{aligned}$$

  In other words, we have a family of isomorphisms of vector spaces $g_{\alpha\beta}(x) : \mathbb{C}^r \to \mathbb{C}^r$ which depends holomorphically on $x \in U_\alpha \cap U_\beta$.

The collection of holomorphic maps $g_{\alpha\beta}$ is called *cocycle* of the vector bundle. A holomorphic vector bundle of rank $r = 1$ is called *line bundle*.

Notice that the cocycle $g_{\alpha\beta}$ of a vector bundle satisfies

$$g_{\alpha\alpha}(x) \equiv I \in \mathrm{Gl}(r, \mathbb{C}), \quad g_{\alpha\beta}(x) = g_{\beta\alpha}^{-1}(x), \quad g_{\alpha\gamma}(x) = g_{\alpha\beta}(x) g_{\beta\gamma}(x).$$

REMARK 4.4. *By the above definition, the cocycle of a line bundle is a family of holomorphic maps* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Gl}(1, \mathbb{C}) = \mathbb{C}^*$. *That is,* $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. *Therefore, the collection* $g_{\alpha\beta}$ *defines a cocycle in cohomology*

$$g_{\alpha\beta} \rightsquigarrow (g_{\alpha\beta}) = \xi \in H^1(X, \mathcal{O}^*).$$

Viceversa, one can show that each $(\xi_{\alpha\beta}) \in H^1(X, \mathcal{O}^*)$ *defines a line bundle.*

[7] we consider $\mathrm{Gl}(r, \mathbb{C})$ as a submanifold of $\mathbb{C}^{r^2}$.

Definition. Two vector bundles (E, π) and (E', π') on X are *isomorphic* if there is a biholomorphic map $\phi : E \to E'$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

FACT. *E and E' are isomorphic (if and) only if there exists a family of holomorphic functions $h_\alpha : U_\alpha \to Gl(r, \mathbb{C})$ such that over $U_\alpha \cap U_\beta$*

$$g'_{\alpha\beta} = h_\alpha^{-1} g_{\alpha\beta} h_\beta,$$

where $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are the cocycles of E and E' respectively.

In terms of line bundles this means that the two cocycles are homologous.

CONSEQUENCE. *There exists a bijection between the group of cocycles $H^1(X, \mathcal{O}^*)$ and the set of holomorphic line bundles modulo isomorphism.*

For this reason, we will call $H^1(X, \mathcal{O}^*)$ the *group of (holomorphic) line bundles* on X. The product $L \otimes L'$ of two line bundles L and L' is defined as the line bundle whose cocycle is given the product of the cocycles $(g_{\alpha\beta} \cdot g'_{\alpha\beta})$.

§5. SHEAF OF HOLOMORPHIC SECTIONS OF A LINE BUNDLE

Let U be an open subset of X and E a vector bundle. A holomorphic map $s : U \to E$ that preserves the fibers of the vector bundle (i.e. $\pi(s(x)) = x$) is called *section* on U. Note that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U \cap U_\alpha) \arrow[r, "\psi_\alpha"] \arrow[dr, bend left, "\pi"] & (U \cap U_\alpha) \times \mathbb{C}^r \arrow[d] \\ & U \cap U_\alpha \arrow[ul, bend left, "s"] \end{tikzcd}$$

In other words, for any $x \in U \cap U_\alpha$ we have $\psi_\alpha(s(x)) = (x, s_\alpha(x))$, where

$$s_\alpha : U \cap U_\alpha \to \mathbb{C}^r$$

is holomorphic. Since $\psi_\beta(s(x)) = (x, s_\beta(x))$, on the overlaps one has

$$\begin{cases} \psi_\alpha \circ \psi_\beta^{-1}(x, s_\beta(x)) = (x, s_\alpha(x)) \\ \psi_\alpha \circ \psi_\beta^{-1}(x, s_\beta(x)) = (x, g_{\alpha\beta}(x) s_\beta(x)). \end{cases}$$

Hence, the collection $s_\alpha \in \mathcal{O}(U \cap U_\alpha)$ satisfies the following condition

$$s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x),$$

where $g_{\alpha\beta}$ is the cocycle of E. Conversely, any collection as such determines uniquely a section s on U (for any $x \in U \cap U_\alpha$ let $s(x) := \psi_\alpha^{-1}(x, s_\alpha(x))$).

$$\begin{tikzcd} \mathbb{C}^r & \pi^{-1}(U \cap U_\alpha \cap U_\beta) \arrow[l] \arrow[r] & \mathbb{C}^r \\ & U \cap U_\alpha \cap U_\beta \arrow[u] \arrow[ul, bend left, "s_\alpha"] \arrow[ur, bend right, "s_\beta"'] & \end{tikzcd}$$

Consequently, a section s of $E = (g_{\alpha\beta})$ on U will always be interpreted as a collection $s = (s_\alpha)$ of holomorphic maps $s_\alpha : U \cap U_\alpha \to \mathbb{C}^r$ satisfying

$$s_\alpha = g_{\alpha\beta} s_\beta.$$

Of particular interest is the case of line bundles, for which $s_\alpha \in \mathcal{O}(U \cap U_\alpha)$. Given a cocycle $\xi = (\xi_{\alpha\beta}) \in H^1(X, \mathcal{O}^*)$ we denote by $\Gamma(U, \mathcal{O}(\xi))$ the set of holomorphic local sections of ξ on U. Clearly, $\Gamma(U, \mathcal{O}(\xi))$ is an abelian group (in fact, a $\mathbb{C}$-vector space). It is easy to see that this defines a sheaf on X (with the natural restrictions of maps).

Definition. We denote by $\mathcal{O}(\xi)$ or $\mathcal{O}_X(\xi)$ the *sheaf of (germs of) holomorphic sections* of a line bundle $\xi \in H^1(X, \mathcal{O}^*)$. In particular, the space of global sections of $\mathcal{O}(\xi)$ is identified with cohomology in degree zero:

$$\Gamma(X, \mathcal{O}(\xi)) \simeq H^0(X, \mathcal{O}(\xi)).$$

Similarly, one introduces $\mathfrak{M}(\xi)$, the sheaf of meromorphic sections of ξ.

NOTATION. *For the cohomology groups of $\mathcal{O}(\xi)$, sometimes we will write shortly $H^q(\xi) := H^q(X, \mathcal{O}(\xi))$. For a divisor $D \in \mathfrak{D}(X)$, we will denote by $\mathcal{O}(D)$ the sheaf of sections of the line bundle which corresponds to D, which we have denoted by $[D]$. Sometimes we write $H^q(D) := H^q(X, \mathcal{O}(D))$. Since we are talking about $\mathbb{C}$-vector spaces, whenever it makes sense we denote by*

$$h^q(\xi) := \dim H^q(\xi), \quad h^q(D) := \dim H^q(D).$$

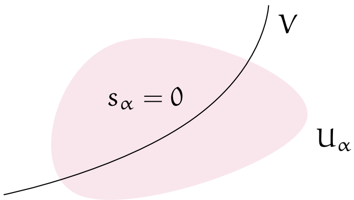
As a matter of fact, a line bundle is the same thing as a special kind of sheaf, called invertible sheaf. *The sheaf of sections $\mathcal{O}(\xi)$ is precisely the correspondence in one direction. This is why, with some abuse of notation, we will often identify a line bundle with its corresponding sheaf of sections and write $\xi = \mathcal{O}(\xi) = \mathcal{O}(D)$, where $\xi = [D]$.*

§6. LINE BUNDLE ASSOCIATED TO A HYPERSURFACE

Let X be a compact complex manifold and $\{U_\alpha\}$ an open covering of X. Let V be an irreducible hypersurface of X. Then, V is a particular case of a Weil divisor. What is the cocycle of V as a Cartier divisor? If we let $s_\alpha \in \mathcal{O}(U_\alpha)$ be the function defining V in U_α (i.e. $V \cap U_\alpha : s_\alpha = 0$), we see that $V = (s_\alpha)$ as a Cartier divisor.

Let $[V] \in H^1(X, \mathcal{O}^*)$ be the line bundle associated to V. By definition of the homomorphism $[\ \ ]$, its cocycle is given by $[V] = \xi := (s_\alpha / s_\beta)$. Tautologically we see that $s_\alpha = (s_\alpha/s_\beta)s_\beta$ on the overlaps. Hence the collection $s = (s_\alpha)$ defines a global section of [V], called the *tautological section* of [V],

$$s = (s_\alpha) \in H^0(X, \mathcal{O}(\xi)).$$



REMARK 4.5.

(i) *Let $\xi \in H^1(X, \mathcal{O}^*)$ be a line bundle. For each $x \in X$,*

$$\mathcal{O}(\xi)_x \simeq \mathcal{O}_x$$

since a section of ξ is locally a holomorphic function.

(ii) *Let $V = (s_\alpha)$ be an irreducible hypersurface as above and consider the associated line bundle $\xi = [V] \in H^1(X, \mathcal{O}^*)$. Its cocycle is given by $\xi_{\alpha\beta} = s_\alpha/s_\beta$. Let U be an open subset of X and let $f \in \mathcal{I}_V(U)$, i.e. $f \in \mathcal{O}(U)$ such that $f|_V = 0$. For each $f_\alpha := f|_{U_\alpha}$, we can write $f_\alpha = g_\alpha \cdot s_\alpha$, for some $g_\alpha \in \mathcal{O}(U_\alpha)$. On the overlaps we have $f_\alpha = f_\beta$, which yields*

$$g_\alpha = \left(\frac{s_\beta}{s_\alpha}\right) g_\beta.$$

Notice that (s_β/s_α) is the cocycle of $\xi^{-1} = [-V]$. Therefore, the collection (g_α) defines a section of $[-V]$, i.e. $(g_\alpha) \in \mathcal{O}(-V)(U)$. Thus, multiplication with s_α induces a morphism of sheaves $\mathcal{O}(-V) \to \mathcal{I}_V$. It is not hard to check that this is in fact an isomorphism,

$$\mathcal{O}(-V) \simeq \mathcal{I}_V.$$

In particular, when V is smooth, its fundamental sequence reads

$$0 \longrightarrow \mathcal{O}(-V) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_V \longrightarrow 0.$$

(iii) *The same thing holds for any line bundle $\eta = (\eta_{\alpha\beta}) \in H^1(X, \mathcal{O}^*)$ when, instead of a function, we consider a section $f \in \mathcal{O}(\eta)(U)$ such that $f|_V = 0$. We have $f_\alpha = \eta_{\alpha\beta} f_\beta$, where $f_\alpha = f|_{U_\alpha}$ and we can write $f_\alpha = g_\alpha \cdot s_\alpha$, for some $g_\alpha \in \mathcal{O}(U_\alpha)$. Then we get*

$$g_\alpha = \left(\eta_{\alpha\beta}\frac{s_\beta}{s_\alpha}\right) g_\beta.$$

Notice that $(\eta_{\alpha\beta} s_\beta/s_\alpha)$ is the cocycle of $\eta \otimes [-V]$. Therefore,

$$(g_\alpha) \in \mathcal{O}(\eta \otimes [-V])(U).$$

In particular, when V is smooth we get the short exact sequence

$$0 \longrightarrow \mathcal{O}(\eta \otimes [-V]) \longrightarrow \mathcal{O}(\eta) \longrightarrow \mathcal{O}_V(\eta) \longrightarrow 0$$

which we regard as the fundamental sequence of V "tensored with η".

§7. EXAMPLE: THE PROJECTIVE SPACE

In this section we study the special case $X = \mathbb{P}^n$. Being an algebraic variety,

$$\mathrm{Pic}(\mathbb{P}^n) \simeq H^1(\mathbb{P}^n, \mathcal{O}^*).$$

The exponential sequence induces a long exact sequence in cohomology which, around $H^1(\mathbb{P}^n, \mathcal{O}^*)$ reads

$$\cdots \to H^1(\mathbb{P}^n, \mathcal{O}) \to H^1(\mathbb{P}^n, \mathcal{O}^*) \to H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathcal{O}) \to \cdots$$

We have to invoke without proof the following non-trivial result:

FACT. *For any $q > 0$ one has*

$$H^q(\mathbb{P}^n, \mathcal{O}) = 0$$

(on the other hand $H^0(\mathbb{P}^n, \mathcal{O}) \simeq \mathcal{O}(\mathbb{P}^n) \simeq \mathbb{C}$, since $\mathbb{P}^n$ is compact)

Therefore, the above exact sequence yields an isomorphism

$$H^1(\mathbb{P}^n, \mathcal{O}^*) \simeq H^2(\mathbb{P}^n, \mathbb{Z}),$$

where $H^2(\mathbb{P}^n, \mathbb{Z})$ is the 2^{nd} singular cohomology group with integer coefficients. In particular, the Picard group (and the group of line bundles) of $\mathbb{P}^n$ is completely determined by its topology.

TOPOLOGY

One shows that $\mathbb{P}^n$ admits a CW-complex structure consisting of a single cell for each even (topological) dimension. This corresponds to a decomposition

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \ldots \subset \mathbb{P}^n,$$

where each $\mathbb{P}^k$ represents a $2k$-cell. By the theory of singular cohomology for CW-complexes this yields

$$H^q(\mathbb{P}^n, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } q = 2k,\ 0 \le q \le 2n \\ 0 & \text{otherwise} \end{cases}$$

where a generator of $H^{2k}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$ is the class of the cell $\mathbb{P}^{n-k} \subset \mathbb{P}^n$. In particular, $H^2(\mathbb{P}^n, \mathbb{Z})$ is generated by the class of a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$.

Definition. Let $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a hyperplane. By the identifications

$$\mathrm{Pic}(\mathbb{P}^n) = H^1(\mathbb{P}^n, \mathcal{O}^*) = H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}H,$$

the class of H as an element in $\mathrm{Pic}(\mathbb{P}^n)$ is called *hyperplane divisor*. As an element in $H^1(\mathbb{P}^n, \mathcal{O}^*)$, the class of H is called *hyperplane line bundle*, and it will be denoted by $\mathcal{O}(1)$ or $\mathcal{O}_{\mathbb{P}^n}(1)$. For any $d \in \mathbb{Z}$, we let

$$\mathcal{O}(d) := \begin{cases} \mathcal{O}(1)^{\otimes d} & \text{for } d \geq 0 \\ (\mathcal{O}(1)^{-1})^{\otimes -d} & \text{for } d < 0 \end{cases}$$

Finally, we have $H^1(\mathbb{P}^n, \mathcal{O}^*) = \{\mathcal{O}(d) : d \in \mathbb{Z}\}$. In particular, the neutral element $\mathcal{O} := \mathcal{O}(0)$ is the trivial line bundle[8] and $\mathcal{O}(d) \otimes \mathcal{O}(d') = \mathcal{O}(d + d')$.

Notice that, geometrically, what we have done so far shows that if V is a hypersurface of degree d in $\mathbb{P}^n$, then V, as an element in $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}H$ is equal to dH (i.e. V is linearly equivalent to dH). In particular, $\mathcal{O}(d)$ is the sheaf (or the line bundle) associated to any such hypersurface.

First, we want to find out what is the cocycle of the hyperplane line bundle $\mathcal{O}(1) \in H^1(\mathbb{P}^n, \mathcal{O}^*)$. Let H be the hyperplane of $\mathbb{P}^n$ with equation

$$F = \sum_{j=0}^{n} a_j x_j = 0.$$

Let U_i denote the standard open subset of $\mathbb{P}^n$. Then H is described by

$$\begin{aligned} g_i &:= F/x_i = a_0 \frac{x_0}{x_i} + \cdots + a_i + \cdots + a_n \frac{x_n}{x_i} = 0 \quad \text{over } U_i \\ g_j &:= F/x_j = a_0 \frac{x_0}{x_j} + \cdots + a_j + \cdots + a_n \frac{x_n}{x_j} = 0 \quad \text{over } U_j. \end{aligned}$$

Thus, over the overlaps $U_i \cap U_j$ we have

$$g_i = \frac{x_j}{x_i} g_j,$$

i.e. $\xi_{ij} := x_j/x_i$ is the cocycle of $\mathcal{O}(1)$. Notice that

$$\xi_{ij}^d = \left(\frac{x_j}{x_i}\right)^d$$

is the cocycle of $\mathcal{O}(d)$, for any $d \in \mathbb{Z}$.

SHEAF OF SECTIONS

We want to determine completely the cohomology groups $H^q(\mathbb{P}^n, \mathcal{O}(d))$. We begin with $q = 0$, i.e. the global sections of the line bundle $\mathcal{O}(d)$. Let

$$S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d,$$

where S_d is the $\mathbb{C}$-vector space of homogeneous polynomials of degree d.

[8] although this notation might cause confusion, it is natural: identifying a line bundle ξ with its sheaf of sections $\mathcal{O}(\xi)$, the trivial bundle is identified with the sheaf of holomorphic functions $\mathcal{O}$.

PROPOSITION 4.6. S_d *(*$d \geq 0$*) is a vector subspace of* $H^0(\mathbb{P}^n, \mathcal{O}(d))$ *and*

$$\dim S_d = \binom{n+d}{n}$$

Proof. A base of S_d is given by the set of monomials

$$\{f = x_0^{i_0} \cdots x_n^{i_n} \mid i_k \geq 0,\ i_0 + \cdots + i_n = d\},$$

whose cardinality is precisely $\binom{n+d}{n}$. One way to see this is that the above set is bijective to that of n-tuples of the form

$$(i_0 + 1, i_0 + i_1 + 2, \ldots, i_0 + \cdots + i_{n-1} + n)$$

which, we notice, is a strictly increasing sequence of integers varying between 1 and $n + d$. Make a drawing. There are precisely $\binom{n+d}{n}$ such sequences.

Now, let $g \in S_d$. Let g_i denote the de-homogenization of g over U_i, i.e.

$$g_i = g(x_0/x_i, \ldots, 1, \ldots, x_n/x_i).$$

It is easy to check that $g_i = (x_i/x_j)^d g_j$ over the overlaps $U_i \cap U_j$. In other words $g = (g_i)$ defines a section of the line bundle of $\mathbb{P}^n$ with cocycle $(x_i/x_j)^d$, the cocycle of $\mathcal{O}(d)$. This yields an injection $S_d \hookrightarrow H^0(X, \mathcal{O}(d))$. □

It turns out that this injection is in fact an isomorphism.

THEOREM 4.7. *There are no global sections of* $\mathcal{O}(d)$ *for* $d < 0$*; while for* $d \geq 0$*, those are precisely the homogeneous polynomials of degree* d*. In other words,*

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \simeq \begin{cases} S_d & \textit{for } d \geq 0 \\ 0 & \textit{for } d < 0 \end{cases}$$

Proof. By the above Proposition, it suffices to show that $h^0(\mathcal{O}(d))$ is equal to $\binom{n+d}{n}$ when $d \geq 0$ and vanishes for $d < 0$. Let $H = \mathbb{P}^{n-1}$ be a hyperplane of $\mathbb{P}^n$. Then $\mathcal{O}(-H) = \mathcal{O}_{\mathbb{P}^n}(-1)$, so its exact sequence (cf. Remark 4.5) reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

More generally, for any integer d, tensoring this sequence with $\mathcal{O}(d)$ yields

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow 0.$$

This sequence induces in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \arrow[dll] \\ & H^1(\mathcal{O}_{\mathbb{P}^n}(d-1)) \arrow[r] & H^1(\mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r] & H^1(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \cdots \end{tikzcd}$$

Recall that by construction, the map $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d))$ is a restriction map. Now, if we take $d = 0$, this is then an isomorphism, since $H^0(\mathcal{O}_{\mathbb{P}^n}) = \mathbb{C}$ and $H^0(\mathcal{O}_{\mathbb{P}^{n-1}}) = \mathbb{C}$. It follows that the first term $H^0(\mathcal{O}_{\mathbb{P}^n}(-1))$ of the sequence (with $d = 0$) vanishes, being the kernel of an isomorphism. Now we proceed inductively. Indeed, notice that by placing $d = -1$, the beginning of the sequence shows $0 \to H^0(\mathcal{O}(-2)) \to 0$, which means the vanishing of this term. And so on for any $d < 0$, it follows by induction that $h^0(\mathcal{O}(-d)) = 0$.

Now, the case $d \geq 0$ is just slightly more involved, but the trick is similar. Our target is the guy sitting in the middle of the first line in the above long exact sequence. Happily enough, that first line is in fact a short exact sequence (i.e. the second line is actually zero). In other words, we claim that

$$H^1(\mathbb{P}^n, \mathcal{O}(d)) = 0$$

for any n and for any d. This is the guy in the middle in the second line. Indeed, we show this by induction on both n and d. The initial steps of this double induction are then:

- ◇ $n = 0$. Trivially true since $\mathbb{P}^0$ is a point, so $h^1(\mathcal{O}_{\mathbb{P}^0}(d)) = 0$ for any d.
- ◇ $d = 0$. Then $h^1(\mathcal{O}_{\mathbb{P}^n}) = 0$ (non-trivial fact already stated above).

Then, induction on n and d gives $h^1(\mathcal{O}_{\mathbb{P}^n}(d-1)) = 0$ and $h^1(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) = 0$, so that the guy sitting in the middle of the second line has zeroes on his left and right sides, and is therefore itself zero.

The first line being a short exact sequence implies that the dimension of the middle term is just the sum of the two side terms, so that we have

$$h^0(\mathbb{P}^n, \mathcal{O}(d)) = h^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) + h^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \tag{4.1}$$

Now, with a similar double induction we show $h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{n}$.

- ◇ $n = 0$. True since $h^0(\mathcal{O}_{\mathbb{P}^0}(d)) = 1$ and indeed $\binom{0+d}{0} = 1$ for any d.
- ◇ $d = 0$. True since $h^0(\mathcal{O}_{\mathbb{P}^n}) = 1$ and $\binom{n+0}{n} = 1$ for any n.

Thus, we can assume by double induction that this is true for the right-hand side of equation (4.1), and just do the math

$$h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d-1}{n} + \binom{n-1+d}{n-1} = \binom{n+d}{n}. \qquad \square$$

By the proof of the above theorem we deduce an important fact:

COROLLARY 4.8. *The sheaf $\mathcal{O}(d)$ has no higher cohomology. That is, for any $q > 0$*

$$H^q(\mathbb{P}^n, \mathcal{O}(d)) = 0.$$

This gives a hint that $\mathbb{P}^n$ is one of the simplest varieties we can have: its only invertible sheaves (i.e. line bundles) are $\mathcal{O}(d)$, for which all cohomology is concentrated in degree zero - i.e. sections (with of course no sections for $d < 0$).

Lecture 5

Canonical bundle and differential forms

Given a complex manifold X, there is a special line bundle which turns out to be very natural to consider in order to gather a lot of informations about the geometry of X. It takes the name of *canonical bundle*, and in particular we will see how it plays a fundamental role for the classification of surfaces. Its importance however goes far beyond this. For example, the canonical bundle is the main character in one of the most powerful theorems in algebraic geometry: the Serre[1] duality Theorem, which we will later state without proof.

Recall that, given an atlas of X with transition functions $\varphi_{\alpha\beta}$, where

$$\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

we get a holomorphic map

$$J\varphi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Gl}(n, \mathbb{C})$$

defined for any $x \in U_\alpha \cap U_\beta$ by taking the Jacobian matrix of $\varphi_{\alpha\beta}$ evaluated at $\varphi_\beta(x)$. Over triple intersections, since $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$, we get the *chain rule*

$$J\varphi_{\alpha\gamma} = J\varphi_{\alpha\beta} \cdot J\varphi_{\beta\gamma}.$$

We want to get the cocycle of a line bundle out of this[2]. Let

$$\kappa_{\alpha\beta} := \frac{1}{\det J\varphi_{\alpha\beta}}.$$

Then we have $\kappa_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and, by the Binet Formula,

$$\kappa_{\alpha\gamma} = \kappa_{\alpha\beta}\kappa_{\beta\gamma}$$

over triple intersections. Therefore, the collection $(\kappa_{\alpha\beta})$ defines a cocycle of $\mathcal{O}^*$, that is, a cohomology class in $H^1(X, \mathcal{O}^*)$. In other words, it yields a line bundle on X, which is called *canonical bundle* and denoted by ω_X. We will denote by K_X a *canonical class*, or a *canonical divisor*, that is, any divisor whose class yields the canonical bundle. Thus $\omega_X = [K_X]$ or equivalently, as invertible sheaves,

$$\omega_X = \mathcal{O}(K_X).$$

[1]Jean-Pierre Serre (born 1926)

[2]as a matter of fact, the collection $J\varphi_{\alpha\beta}$ defines a vector bundle on X, which is called the (holomorphic) *tangent bundle*. The collection $(J\varphi_{\alpha\beta})^{-1}$ yields the *cotangent bundle*.

§1. Canonical bundle of the projective space

Let $X = \mathbb{P}^n$ with its standard open covering $\{U_\alpha\}$, $0 \le \alpha \le n$. We have

$$\varphi_0 \colon U_0 \longrightarrow \mathbb{C}^n \quad (x_0 : \ldots : x_n) \mapsto (w_1 = \frac{x_1}{x_0}, w_2 = \frac{x_2}{x_0}, w_3 = \frac{x_3}{x_0}, \ldots, w_n = \frac{x_n}{x_0})$$
$$\varphi_2 \colon U_2 \longrightarrow \mathbb{C}^n \quad (x_0 : \ldots : x_n) \mapsto (z_1 = \frac{x_0}{x_2}, z_2 = \frac{x_1}{x_2}, z_3 = \frac{x_3}{x_2}, \ldots, z_n = \frac{x_n}{x_2})$$

Thus, the transition map φ_{02} is given by

$$\varphi_{02} \colon \begin{cases} w_1 = \frac{z_2}{z_1} \\ w_2 = \frac{1}{z_1} \\ w_3 = \frac{z_3}{z_1} \\ \vdots \\ w_n = \frac{z_n}{z_1} \end{cases}$$

Deriving this vector by the z_i's, we get $J\varphi_{02}$ and we compute

$$\det J\varphi_{02} = \det \left(\begin{array}{cc|ccc} -\frac{z_2}{z_1^2} & \frac{1}{z_1} & 0 & \ldots & 0 \\ -\frac{1}{z_1^2} & 0 & 0 & \ldots & 0 \\ \hline -\frac{z_3}{z_1^2} & 0 & \frac{1}{z_1} & & \\ \vdots & \vdots & & \ddots & \\ -\frac{z_n}{z_1^2} & 0 & & & \frac{1}{z_1} \end{array} \right) = \left(\frac{1}{z_1} \right)^{n+1}$$

Since $z_1 = x_0/x_2$, by taking the inverse of this determinant we get

$$\kappa_{02} = (\xi_{02})^{-n-1},$$

where we set $\xi_{ij} = x_j/x_i$, which we recall being the cocycle of the hyperplane bundle $\mathcal{O}(1)$. Similarly, by computing the Jacobian matrix of $\varphi_{0\alpha}$ for any α, we find $\kappa_{0\alpha} = (\xi_{0\alpha})^{-n-1}$. Then, we notice

$$\kappa_{\alpha\beta} = \kappa_{\alpha 0}\kappa_{0\beta} = \kappa_{0\alpha}^{-1}\kappa_{0\beta} = (\xi_{\alpha\beta})^{-n-1}.$$

Therefore, we have determined the canonical bundle of $\mathbb{P}^n$,

$$\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1).$$

In terms of divisors,

$$K_{\mathbb{P}^n} \sim (-n-1)H$$

where H is a hyperplane of $\mathbb{P}^n$.

COROLLARY 5.1. *The canonical bundle of* $\mathbb{P}^n$ *has no global sections.*

This is a consequence of our computation of $h^0(\mathcal{O}(d)) = 0$ for $d < 0$.

§2. Holomorphic and meromorphic differential forms

Let X be a complex manifold and fix an open covering $\{U_\alpha\}$ with local charts $z_1^\alpha, \ldots, z_n^\alpha$ over each U_α. Let $p \geq 0$ be a fixed integer.

Definition. A *holomorphic* p*-form over* $U \subset U_\alpha$ is any expressions like

$$\omega_\alpha = \sum_{i_1 \cdots i_p} f_{i_1 \cdots i_p} dz_{i_1}^\alpha \wedge \ldots \wedge dz_{i_p}^\alpha$$

where each $f_{i_1 \cdots i_p} \in \mathcal{O}(U)$ and we have the following formal rule

$$dz_i^\alpha \wedge dz_j^\alpha = -dz_j^\alpha \wedge dz_i^\alpha.$$

In particular, $dz_i^\alpha \wedge dz_i^\alpha = 0$. (hence, any $\omega_\alpha = 0$ if $p > n$).

Let now U be any open subset of X.

Definition. A *holomorphic* p*-form over* U is a collection $\omega = (\omega_\alpha)$ where each ω_α is a p-form over $U \cap U_\alpha$ and such that

$$\omega_\alpha = \omega_\beta \quad \text{over } U \cap U_\alpha \cap U_\beta.$$

Setting $\Omega_X^p(U) = \{\text{holomorphic } p\text{-forms over } U\}$, which is a $\mathbb{C}$-vector space, we get a sheaf Ω_X^p on X (the restrictions are the natural restrictions). It is called *sheaf of holomorphic* p*-forms*, without much creativity.

The integer p is by definition the *degree* of the forms. Notice that by definition $\Omega_X^p = 0$ if $p > n$. Hence, we morally consider only $0 \leq p \leq n$. Observe:

- ◇ $\Omega_X^0 = \mathcal{O}_X$, just the sheaf of holomorphic functions.
- ◇ $\Omega_X^1 = (T_X)^\vee$, the so-called *cotangent bundle* (which is the dual of the tangent bundle T_X, whose cocycle is $J\varphi_{\alpha\beta}$).
- ◇ Ω_X^n is the sheaf of forms of "top" degree, which locally are of the form

  $$\omega_\alpha = f_\alpha dz_1^\alpha \wedge \ldots \wedge dz_n^\alpha$$

  and which are therefore completely determined by a collection of holomorphic functions $f_\alpha \in \mathcal{O}(U_\alpha)$ satisfying some glueing conditions.

§3. Sections of the canonical bundle

What are the holomorphic sections of the canonical bundle ω_X? It turns out they are nothing but the holomorphic differential forms of top degree in Ω_X^n.

Fix an open covering $\{U_\alpha\}$ of X with local charts $z_1^\alpha, \ldots, z_n^\alpha$ over each U_α, and let $\varphi_{\alpha\beta}$ be the transition charts of X with respect to this covering, so that $\kappa_{\alpha\beta} = (\det J\varphi_{\alpha\beta})^{-1}$ is the cocycle of the canonical bundle ω_X.

Recall that a section of K_X over $U \subset X$ is given by a collection (f_α), where f_α is a holomorphic function on $U \cap U_\alpha$ and $f_\alpha = \kappa_{\alpha\beta} f_\beta$ on the overlaps.

Let $\omega \in \Omega_X^n(U)$ be a holomorphic form of top degree. Locally over U_α,

$$\omega_\alpha = f_\alpha dz_1^\alpha \wedge \ldots \wedge dz_n^\alpha$$

with $f_\alpha \in \mathcal{O}(U \cap U_\alpha)$. Now, let us compute this expression over an overlap $U \cap U_\alpha \cap U_\beta$. We have $z^\alpha = \varphi_{\alpha\beta}(z^\beta)$, and so $f_\alpha(z^\alpha) dz_1^\alpha \wedge \ldots \wedge dz_n^\alpha$ becomes

$$f_\alpha(\varphi_{\alpha\beta}(z^\beta)) \left(\sum_{j_1=1}^{n} \frac{\partial z_1^\alpha}{\partial z_{j_1}^\beta} dz_{j_1}^\beta \right) \wedge \ldots \wedge \left(\sum_{j_n=1}^{n} \frac{\partial z_n^\alpha}{\partial z_{j_n}^\beta} dz_{j_n}^\beta \right)$$

By the rule for manipulating differential forms, we can reorder this expression so to collect all the functions on top, and this leads to

$$f_\alpha(\varphi_{\alpha\beta}(z^\beta)) \left(\sum_{j_1 \ldots j_n}^{n} \frac{\partial z_1^\alpha}{\partial z_{j_1}^\beta} \cdots \frac{\partial z_n^\alpha}{\partial z_{j_n}^\beta} \cdot \epsilon(j_1, \ldots, j_n) \right) dz_1^\beta \wedge \ldots \wedge dz_n^\beta$$

where ϵ denotes the sign of the permutation. In the middle of the expression, in between the big round brackets, we recognize the term

$$\det J\varphi_{\alpha\beta} = \det \left[\frac{\partial z_i^\alpha}{\partial z_j^\beta} \right] = \sum_{j_1 \ldots j_n}^{n} \frac{\partial z_1^\alpha}{\partial z_{j_1}^\beta} \cdots \frac{\partial z_n^\alpha}{\partial z_{j_n}^\beta} \cdot \epsilon(j_1, \ldots, j_n)$$

Finally, we have arrived at the expression

$$f_\alpha \det J\varphi_{\alpha\beta} \; dz_1^\beta \wedge \ldots \wedge dz_n^\beta$$

so that imposing $\omega_\alpha = \omega_\beta$ amounts to asking

$$f_\alpha \det J\varphi_{\alpha\beta} \; dz_1^\beta \wedge \ldots \wedge dz_n^\beta = f_\beta dz_1^\beta \wedge \ldots \wedge dz_n^\beta.$$

This simply means $f_\alpha = \kappa_{\alpha\beta} f_\beta$. To sum up, we showed that ω is determined by a collection (f_α) satisfying $f_\alpha = \kappa_{\alpha\beta} f_\beta$, i.e. by a section of the canonical bundle. This shows that we have an isomorphism

$$\Omega_X^n(U) \simeq \Gamma(U, \omega_X)$$

This is true for any open subset U of X. Hence, we have isomorphic sheaves

$$\Omega_X^n \simeq \mathcal{O}(K_X).$$

A similar result holds for meromorphic forms. What we just showed is only a particular case of the isomorphisms between Ω_X^p and $\mathcal{O}(\bigwedge^p T_X^*)$, where $\bigwedge^p T_X^*$, which is of rank $\binom{n}{k}$, is the p-th exterior power of the cotangent bundle T_X^*. In particular, when $p = 1$ we have

$$\Omega_X^1 \simeq \mathcal{O}(T_X^*).$$

EXAMPLE 5.2. By $\Omega_{\mathbb{P}^n}^n \simeq \mathcal{O}(K_{\mathbb{P}^n}) \simeq \mathcal{O}(-n-1)$, since $-n-1 < 0$ we deduce

$$H^0(\mathbb{P}^n, \Omega^n) = 0,$$

that is, on $\mathbb{P}^n$ there are no holomorphic n-forms.

Wait a second. On $\mathbb{P}^1$ we have a local coordinate z, and so the 1-form dz. What is more innocent than that? The point is that at infinity we get troubles: by the chart change $w = 1/z$ we have $dz = -\frac{1}{w^2} dw$ and so dz has a pole of order two at infinity. Hence it is a *meromorphic* form, and not holomorphic.

More generally, one can prove that $H^0(\mathbb{P}^n, \Omega^p) = 0$ for any $1 \leq p \leq n$.

LECTURE 6

LINEAR SYSTEMS OF DIVISORS

In this chapter we denote by X a compact complex (smooth) manifold.

§1. THE RIEMANN-ROCH SPACE

Let $D \in \mathrm{Div}(X)$ be a divisor, say $D = \sum m_i V_i$ in its Weil description.

Definition. We call D *effective* if each $m_i \geq 0$. In this case we write $D \geq 0$.

We define the *Riemann-Roch space* associated to a divisor D by

$$\mathcal{L}(D) := \{f \in \mathfrak{M}(X) : \ f \equiv 0 \text{ or } (f) + D \geq 0\}.$$

In other words, $f \in \mathcal{L}(D)$ if and only if at each irreducible component V_i of D,

$$\mathrm{ord}_{V_i} f \geq -m_i$$

What does this mean?

- If $m_i < 0$ then f *must* have a zero at V_i, of order at least m_i. That is, we must have locally $f = \nu^m g$, with $m \geq m_i$.
- If $m_i \geq 0$ it means that f *is allowed to* have a pole along V_i, but of order *at most* m_i. That is, if ν is a local equation for V_i, we can locally write $f = g/\nu^m$, with g holomorphic (and non-zero on V_i) and $m \leq m_i$. In particular, any f holomorphic will do (but we throw away any f with poles of higher multiplicity).

REMARK 6.1. *If* $D \geq 0$ *then* $\mathcal{L}(D)$ *contains in particular* $\mathcal{O}_X(X) = \mathbb{C}$.

We observe that for any divisor D, the Riemann-Roch space $\mathcal{L}(D)$ is in fact a $\mathbb{C}$-vector space. Indeed, for any $f_1, f_2 \in \mathcal{L}(D)$, either $f_1 + f_2 \equiv 0$ or

$$\mathrm{ord}_{V_i}(f_1 + f_2) \geq \min\{\mathrm{ord}_{V_i}(f_1), \mathrm{ord}_{V_i}(f_2)\} \geq -m_i$$

and therefore $f_1 + f_2 \in \mathcal{L}(D)$. Moreover $\mathrm{ord}_{V_i}(\lambda f) = \mathrm{ord}_{V_i}(f)$ for any non-zero constant λ, so that $\lambda f \in \mathcal{L}(D)$ if $f \in \mathcal{L}(D)$.

The key point is that this space of meromorphic functions is isomorphic to the space of global sections of the line bundle associated to D. Precisely:

THEOREM 6.2. *Let* X *be a compact complex manifold and* $D \in \mathrm{Div}(X)$. *Then*

$$\mathcal{L}(D) \simeq H^0(X, D).$$

Proof. Write $D = (f_\alpha)$ as a Cartier divisor with respect to a covering $\{U_\alpha\}$, where $f_\alpha \in \mathfrak{M}^*(U_\alpha)$ and $\xi_{\alpha\beta} := f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Notice $f_\alpha = \xi_{\alpha\beta} f_\beta$ and recall that $\xi = (\xi_{\alpha\beta}) \in H^1(X, \mathcal{O}^*)$ is the line bundle associated to D. Pick a section $s \in H^0(X, \mathcal{O}(\xi))$, i.e. $s = (s_\alpha)$ with $s_\alpha \in \mathcal{O}(U_\alpha)$ and $s_\alpha = \xi_{\alpha\beta} s_\beta$. Consider the meromorphic function $s_\alpha/f_\alpha \in \mathfrak{M}(U_\alpha)$. This extends to a global meromorphic function $F \in \mathfrak{M}(X)$, since we have $F|_{U_\alpha} = s_\alpha/f_\alpha = s_\beta/f_\beta$ over the intersections. Let $D = \sum m_i V_i$ as a Weil divisor, that is, $m_i := \mathrm{ord}_{V_i}(f_\alpha)$ for any α such that V_i and U_α intersect. Then

$$\mathrm{ord}_{V_i}(F) = \mathrm{ord}_{V_i}(s_\alpha) - \mathrm{ord}_{V_i}(f_\alpha) \geq -m_i$$

which means $F \in \mathcal{L}(D)$. Thus, we constructed a homomorphism

$$H^0(X, \mathcal{O}(\xi)) \longrightarrow \mathcal{L}(D), \quad s \longmapsto F = \left(\frac{s_\alpha}{f_\alpha}\right)$$

which is clearly injective: if $F \equiv 0$ then $s_\alpha \equiv 0$ for all α, that is $s \equiv 0$. Let us show it is surjective. Given $F \in \mathcal{L}(D)$, let $s_\alpha := f_\alpha F|_{U_\alpha}$. Then

$$\mathrm{ord}_{V_i}(s_\alpha) = \mathrm{ord}_{V_i}(f_\alpha) + \mathrm{ord}_{V_i}(F) \geq m_i - m_i = 0$$

which implies $s_\alpha \in \mathcal{O}(U_\alpha)$. Since $s_\alpha = \xi_{\alpha\beta} s_\beta$, this shows that $s = (s_\alpha)$ is a global holomorphic section of $\mathcal{O}(\xi)$, and by construction $s \mapsto F$. □

Assume we have a non-zero section $s = (s_\alpha)$ in $H^0(X, D)$. We define

$$\mathrm{div}(s) := \sum \mathrm{ord}_{V_i}(s_\alpha) V_i \in \mathrm{Div}(X)$$

where $D = \sum m_i V_i$. By the proof of the Theorem, we have

$$\mathrm{div}(s) = \sum (m_i + \mathrm{ord}_{V_i}(F)) V_i = D + (F)$$

where $(F) = \sum \mathrm{ord}_{V_i}(F) V_i$ is the divisor associated to the meromorphic function $F \in \mathfrak{M}(X)$ constructed as in the proof of the Theorem. Since we have $m_i + \mathrm{ord}_{V_i}(F) \geq 0$, we see that $D' := \mathrm{div}(s)$ is an effective divisor. Moreover, we have $D' - D = (F)$, which means that D' and D are linearly equivalent. Finally, we have showed that if a divisor D admits a non-zero section, then D is linearly equivalent to an *effective* divisor!

Definition. The *complete linear system* associated to a divisor D is the set

$$|D| := \{D' \in \mathrm{Div}(X) : D' \geq 0 \text{ and } D' \sim D\}$$

Of course, this set could be empty. However, we just showed that this is not the case provided that D has at least a non-zero section. Observe that the construction of the divisor $\mathrm{div}(s)$ associated to a section s of D yields a map

$H^0(X, D) \to |D|$. Notice that $\operatorname{div}(s) = \operatorname{div}(\lambda s)$ for any non-zero scalar λ. Moreover, it is an easy exercise to show that any divisor in $|D|$ arises as the divisor of some section of D. Therefore we have a bijection

$$|D| \leftrightarrow \mathbb{P}(H^0(X, D)).$$

Is the vector space $H^0(X, D)$ always finite-dimensional? Let us recall the following non-trivial results.

FACT. *Let X be a complex manifold and $\xi \in H^1(X, \mathcal{O}^*)$ a line bundle.*

(i) Dolbeault Theorem*: $H^q(X, \mathcal{O}(\xi)) = 0$ for any $q > \dim X$.*

(ii) Cartan-Serre Finiteness Theorem*: if X is compact, then all the vector spaces $H^q(X, \mathcal{O}(\xi))$ are finite-dimensional.*

As usual, we will denote by $h^q(\xi)$ or $h^q(D)$ the dimensions of the vector spaces $H^q(X, \mathcal{O}(\xi))$ and $H^q(X, D) := H^q(X, \mathcal{O}(D))$ respectively.

COROLLARY 6.3. *On a compact complex manifold X, for any divisor $D \in \operatorname{Div}(X)$, the Riemann-Roch space $\mathcal{L}(D)$ is a finite-dimensional vector space, of dimension $h^0(D)$ (which therefore only depends on the linear equivalence class of D).*

Finally, by the above discussion, we can associate to the complete linear system $|D|$ the structure of a projective space

$$|D| = \mathbb{P}^N,$$

where $N = h^0(D) - 1$. By convention, we set $|D| = \{\varnothing\}$ if $h^0(D) = 0$.

REMARK 6.4. *$h^0(D) > 0$ if and only if D is linearly equivalent to an effective divisor. If $h^0(D) = 1$, the complete linear system $|D| = \mathbb{P}^0$ consists of a single point: only D itself. Roughly speaking, we say D is* fixed*, in the sense that we cannot "deform" it within the manifold X by linear equivalence.*

EXAMPLE 6.5. Let $X = \mathbb{P}^n$ and $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ a hyperplane. We have seen that the associated line bundle $\mathcal{O}(1)$ has for sections the space of linear forms

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \simeq S_1$$

where S_d = forms of degree d. Therefore we find the identification

$$|H| = \{\text{hyperplanes of } \mathbb{P}^n\} = (\mathbb{P}^n)^\vee.$$

More generally, for any $d > 0$ we have

$$|dH| = \{\text{hypersurfaces of degree } d \text{ in } \mathbb{P}^n\} = \mathbb{P}(S_d).$$

In this case we think of the projective space $\mathbb{P}(S_d) \simeq \mathbb{P}^N$ with $N = \binom{n+d}{n} - 1$ as the *parameter space* of degree d hypersurfaces in $\mathbb{P}^n$.

It should be clear that determining the number $h^0(D)$ of linearly independent sections of a divisor D on a compact complex manifold X, becomes a very important issue. It is called *Riemann-Roch Problem*, since $h^0(D) = \dim \mathcal{L}(D)$.

One of the two most striking results of Algebraic Geometry is the Hirzebruch-Riemann-Roch Theorem. It gives a formula which expresses the quantity

$$\chi(D) := \sum_{i=0}^{n} (-1)^i h^i(D)$$

in terms of some numerical characters (*Chern Classes*) depending on X and D.

Therefore, in order to solve the Riemann-Roch problem, all sort of results which ensure the vanishing of $h^q(D)$ for $q > 0$, become very useful. This type of theorems are called Vanishing Theorems.

Before going on and discussing the example of curves, we need to recall some easy properties of short exact sequences. Suppose

$$0 \longrightarrow A \longrightarrow V \longrightarrow B \longrightarrow 0$$

is a short exact sequence of (finite-dimensional) vector spaces. Then

$$\dim V = \dim A + \dim B$$

as one learns in a first course in linear algebra (*Rank-Nullity Theorem*). This easy generalizes to the following fact: *for any exact sequence of vector spaces the alternating sum of the dimensions is zero*. Indeed, if

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

is exact then $\dim V_i = \dim(\operatorname{Im} f_i) + \dim(\ker f_i)$ by the Rank-Nullity Theorem, where we denote by f_i the map $V_i \to V_{i+1}$ in the sequence. On the other hand $\operatorname{Im}(f_i) = \ker(f_{i+1})$ since the sequence is exact. Therefore

$$\sum (-1)^i \dim V_i = 0.$$

Now, suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0$$

on some manifold X. If the cohomology groups are finite dimensional vector spaces, applying the above observation on the long exact sequence in cohomology induced by this short exact sequence yields

$$\chi(\mathcal{F}) = \chi(\mathcal{G}) + \chi(\mathcal{H}).$$

One says that the Euler characteristic χ is *additive* on short exact sequences.

§2. Riemann-Roch theorem for curves

Let X be a (smooth) curve, that is, a compact Riemann surface of genus g. In this setting, the easiest choice for the definition of genus is to take

$$g := h^1(\mathcal{O}_X).$$

It is possible to show that this is indeed equal to the topological genus of X,

$$g = \frac{1}{2} b_1(X) = \frac{1}{2} \operatorname{rk} H^1(X, \mathbb{Z}).$$

By definition, a divisor on X is a finite sum $D = \sum m_i p_i$, where $p_i \in X$. There is a natural homomorphism of groups, called *degree*, defined by

$$\deg\colon \operatorname{Div}(X) \longrightarrow \mathbb{Z}, \quad \sum m_i p_i \longmapsto \sum m_i.$$

For a divisor $D = \sum m_i p_i$, its Riemann-Roch space is given by

$$\mathcal{L}(D) = \{f \in \mathfrak{M}(X) : \ f \equiv 0 \text{ or } \operatorname{ord}_{p_i}(f) \geq -m_i\}.$$

Its dimension is $h^0(D)$. The following result is a particular case of the Hirzebruch-Riemann-Roch theorem, when $\dim(X) = 1$. In the sequel, we will refer to it as the *Riemann-Roch theorem for curves*.

THEOREM 6.6. *For* $D \in \operatorname{Div}(X)$ *we have the following formula:*

$$\chi(D) = \chi(\mathcal{O}_X) + \deg(D).$$

Proof. If $D = 0$ the formula is true. Any effective divisor D is obtained by adding finitely many points from the zero divisor. Thus to prove the formula it suffices to show that if it holds for D it must also hold for $D + p$, where $p \in X$. To do this one considers the short exact sequence of sheaves

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D + p) \to \mathcal{O}_p \to 0,$$

and gets $\chi(D+p) = \chi(D) + \chi(\mathcal{O}_p) = \chi(D) + 1$, which yields the assertion since by assumption the formula holds for D and $\deg D + 1 = \deg(D + p)$. □

Let us rewrite the formula in a more explicit version. Since X is one-dimensional, we have $\chi(D) = h^0(D) - h^1(D)$. Moreover $\chi(\mathcal{O}_X) := h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - g$. Therefore the Riemann-Roch formula states:

$$h^0(D) - h^1(D) = 1 - g + \deg(D).$$

As a corollary we get an inequality, which is very useful in practical situations:

$$h^0(D) \geq 1 - g + \deg(D).$$

EXAMPLE 6.7. This inequality is sharp. Indeed, consider the divisor $D = dp$ on $\mathbb{P}^1$, where $d > 0$ and $p \in \mathbb{P}^1$. Recall $g(\mathbb{P}^1) = 0$. We already know

$$h^0(D) = h^0(\mathbb{P}^1, \mathcal{O}(d)) = \binom{1+d}{1} = d + 1 = \deg(D) + 1.$$

(Exercise: knowing $\mathfrak{M}(\mathbb{P}^1) = \mathbb{C}(z)$, give an explicit description of $\mathcal{L}(D)$)

Let us show the most important property of the degree homomorphism: it is invariant under linear equivalence. Therefore it factors through $\operatorname{Pic}(X) \to \mathbb{Z}$.

PROPOSITION 6.8. *Let* $D \in \operatorname{Div}(X)$. *If* $D \sim D'$ *then* $\deg(D) = \deg(D') = 0$.

Proof. Let $D - D' = (F)$ where $F \in \mathfrak{M}(X)$. Since a meromorphic function has an equal number of zeroes and poles, the divisor (F) has degree zero. $\square$

The induced homomorphism $\mathrm{Pic}(X) \to \mathbb{Z}$ will be still denoted by deg.

REMARK 6.9. *If* $h^0(D) > 0$ *then* $\deg(D) \geq 0$. *Indeed, as we have seen above, if* D *has a non-zero section then it is linearly equivalent to an effective divisor* $D' \geq 0$, *and so* $\deg(D) = \deg(D') \geq 0$.

A straightforward consequence of this simple observation is the following:

$$\deg(D) < 0 \;\Rightarrow\; h^0(D) = 0.$$

§3. FIRST APPLICATIONS OF SERRE DUALITY

At this point, we are ready to state the second most important theorem of Algebraic Geometry: the duality Theorem of Serre (1954). We shall not prove it, but rather show some of its power in applications. Together with the Theorem of Riemann-Roch, Serre duality will be used nearly anywhere from now on.

THEOREM 6.10 (Serre duality). *Let* X *be a compact complex manifold of (complex) dimension* n. *Let* E *be a vector bundle on* X. *The vector spaces*

$$H^q(X, E) \quad \textit{and} \quad H^{n-q}(X, \omega_X \otimes E^*)$$

are dual of each other, for any $q = 0, \ldots, n$.

In the statement, ω_X denotes the canonical line bundle on X, and E^* the dual vector bundle of E. In the following, we will apply Serre duality only in the case when E is a line bundle. If the complex manifold X is projective, so that each line bundle arises from a divisor on X, the Theorem can be rephrased as a duality between $H^q(D)$ and $H^{n-q}(K_X - D)$, where K_X is a canonical divisor.

COROLLARY 6.11. *If* X *is a smooth projective variety of dimension* n,

$$h^q(D) = h^{n-q}(K_X - D)$$

for any divisor $D \in \mathrm{Div}(X)$.

Let X be a compact Riemann surface of genus g. As a first application of Serre duality, let us solve the Riemann-Roch problem for the canonical divisor K_X. Since X is 1-dimensional, $\mathcal{O}(K_X) = \Omega^1_X$. So the number $h^0(K_X)$ of independent global sections of K_X is equal to the number $\dim \Omega^1(X)$ of independent global holomorphic 1-forms on X. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0$$

where the map $\mathcal{O}_X \to \Omega^1_X$ is given by the differential

$$d\colon \mathcal{O}_X \longrightarrow \Omega^1_X, \quad f \longmapsto df.$$

Let us show that this is indeed a short exact sequence. Firstly, for any $x \in X$, there is a suitably small neighbourhood U of x such that $\ker(d_U) = \mathbb{C}$. Secondly, given a 1-form $\omega \in \Omega^1(U)$, write $\omega = g(z)dz$ where $g \in \mathcal{O}_X(U)$ and z is a local coordinate centred at x. Thus $g(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ is integrable, up to restricting to a smaller neighbourhood $V \subset U$. Namely, the function $f(z) = c + c_0 z + \frac{1}{2} c_1 z^2 + \cdots$, for any constant $c \in \mathbb{C}$, is such that $f \in \mathcal{O}_X(V)$ and $f'(z) = g(z)$, that is $df = g dz$. We have thus showed that the above sequence is exact on the stalks, and is therefore an exact sequence of sheaves.

THEOREM 6.12. *Let* X *be a compact Riemann Surface of genus* g*. Then*

$$h^0(K_X) = g.$$

Proof. Consider the long exact sequence of cohomology induced by the above short exact sequence of sheaves. Since $\dim X = 1$ we have $H^2(X, \mathcal{O}_X) = 0$, so that the exact sequence goes between $H^0(X, \mathbb{C})$ and $H^2(X, \mathbb{C})$. Then, by taking the alternating sum of the dimensions of the vector spaces in this sequence we get $h^0(K_X) = h^0(\Omega^1_X) = g$, by observing:

- ⋄ $h^1(\mathcal{O}_X) = h^0(K_X)$ and $h^1(\Omega^1_X) = h^0(\mathcal{O}_X)$ by Serre duality.
- ⋄ $h^0(X, \mathbb{C}) = 1$ and $h^0(\mathcal{O}_X) = 1$ since X is compact.
- ⋄ $h^1(X, \mathbb{C}) = \dim(H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}) = \operatorname{rk} H^1(X, \mathbb{Z}) = 2g$.
- ⋄ $h^2(X, \mathbb{C}) = \operatorname{rk} H^2(X, \mathbb{Z}) = 1$ since X is compact and oriented. □

In particular, the degree of a canonical divisor is determined by the genus:

COROLLARY 6.13. *Let* X *be a compact Riemann Surface of genus* g*. Then*

$$\deg(K_X) = 2g - 2.$$

Proof. By Riemann-Roch $h^0(K) - h^1(K) = 1 - g + \deg(K)$. By the above computation $h^0(K) = g$. By Serre duality $h^1(K) = h^0(\mathcal{O}_X) = 1$. □

This yields a first example of vanishing theorem:

COROLLARY 6.14. *Let* X *be a compact Riemann Surface of genus* g*. If* $D \in \mathrm{Div}(X)$ *is a divisor on* X *which is such that* $\deg(D) > 2g - 2$*, then*

$$h^1(D) = 0.$$

Proof. By Serre duality $h^1(D) = h^0(K - D)$. By linearity of the degree homomorphism, $\deg(K - D) = \deg(K) - \deg(D) = 2g - 2 - \deg(D) < 0$ by assumption. As we have observed above, a divisor with negative degree has no global sections. Thus $h^0(K - D) = 0$. □

On a curve X, divisors with non-vanishing h^1 are called *special divisors*. The reason for this terminology can be guessed by the above Corollary: if one takes any divisor on X of positive degree, then its multiples will eventually have degree greater than $2g - 2$, and will therefore be non-special.

§4. LINEAR SYSTEMS AND MORPHISMS

In this section we develop the basic theory of linear systems of divisors in the algebraic setting, and we explain the relation linear systems and morphisms of a variety to projective spaces. Let X be a smooth projective variety. We define:

- The *support* of an effective divisor $D = \sum m_i V_i \geq 0$, to be
$$\mathrm{Supp}(D) = \bigcup_{m_i > 0} V_i.$$
- A *linear system of divisors on* X is a projective linear subspace $\mathcal{S}$ of some complete linear system $|D|$. In other words, we have
$$\mathcal{S} = \mathbb{P}(V) \subset \mathbb{P}(H^0(X, D)) = |D|,$$
where V is some vector subspace of $H^0(X, D)$. Some classical terminology: if $\dim(V) = 2$, so that $\dim \mathcal{S} = 1$, i.e. it is a line in $|V|$, we say that $\mathcal{S}$ is a *pencil*. A linear system of dimension 2 is a *net*. Of dimension 3 a *web*.
- Given a linear system $\mathcal{S}$, its *base locus* is the set of points of X which are contained in every divisor $D \in \mathcal{S}$, i.e.
$$\mathrm{Bs}(\mathcal{S}) = \bigcap_{D \in \mathcal{S}} \mathrm{Supp}(D).$$
Observe that, given some linearly independent divisors $D_0, \ldots, D_k$ in $\mathcal{S}$, where $\dim \mathcal{S} = k$, the base locus is
$$\mathrm{Bs}(\mathcal{S}) = \mathrm{Supp}(D_0) \cap \ldots \cap \mathrm{Supp}(D_k).$$
Thus, it consists of a (possibly empty) union of finitely many subvarieties of X of dimension $0, 1, 2, \ldots, n-1$, where $\dim(X) = n$. The part of dimension $n-1$ consists of a divisor F, called *fixed part* of $\mathcal{S}$.

Slightly more explicitly, let D be a divisor with at least one section, and choose a basis $s_0, \ldots, s_k$ of $H^0(D)$. If we denote by $D_0, \ldots, D_k$ the corresponding divisors, then we may write $|D| = \{D_\lambda\}_{\lambda \in \mathbb{P}^k}$, where
$$D_\lambda = \lambda_0 D_0 + \cdots + \lambda_k D_k.$$

It is important to have some concrete example in mind before proceeding.

EXAMPLE 6.15. Consider the system of conics in $\mathbb{P}^2$. That is, any curve C of degree 2, in our notation $C \in |\mathcal{O}_{\mathbb{P}^2}(2)|$, is given by the zero locus of a degree two polynomial $f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. We write
$$f = a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2,$$
so that, by choosing the monomials $x_0^2, x_0x_1, \ldots$ as a basis for $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, the curve C is uniquely determined by the coefficients a_{ij} up to multiples. In other words we consider $|\mathcal{O}_{\mathbb{P}^2}(2)|$ as the projective space $\mathbb{P}^5$ where the point

corresponding to C is $(a_{00} : \cdots : a_{22}) \in \mathbb{P}^5$. This $\mathbb{P}^5$ is thus the complete linear system of conics[1]. Now fix a point $p \in \mathbb{P}^2$ and consider the linear system

$$\mathcal{S} = \{\text{conics passing through } p\} \subset |\mathcal{O}_{\mathbb{P}^2}(2)|.$$

It is easy to see that $\mathcal{S}$ is a hyperplane $\mathcal{S} \simeq \mathbb{P}^4 \subset \mathbb{P}^5$. For example, if we take the point $p = (1 : 0 : 0)$ then the equation of $\mathcal{S}$, in our $\mathbb{P}^5$ with the above coordinates $(a_{00} : \cdots : a_{22})$, is given by $\{a_{00} = 0\}$. One says that *passing through a point imposes one linear condition*. Notice that the base locus of $|\mathcal{O}_{\mathbb{P}^2}(2)|$ is empty, while $\mathrm{Bs}(\mathcal{S}) = \{p\}$, coincides with its fixed part.

Given a general pencil of conics, will there be a member of this pencil containing the point p? The answer is yes: since a line ℓ in $\mathbb{P}^5$ if general enough (i.e. not contained in $\mathcal{S}$) will intersect the hyperplane $\mathcal{S}$ at one point, there will be precisely one conic of the pencil passing through p.

Once we are given a (possibly singular) effective divisor D on some variety X, a very natural question is whether we can move D in its (complete) linear system so that we find a *smooth* member $D' \sim D$. More generally we may want to perform this sort of deformation in a linear system $\mathcal{S} \subset |D|$. It turns out that, for "most" divisors $D' \in \mathcal{S}$, its singularities must be contained in $\mathrm{Bs}(\mathcal{S})$. This is the content of the following fundamental result:

THEOREM 6.16 (Bertini[2]). *The general member of a linear system is smooth away from the base locus.*

Here *general* means that the property holds for an open dense subset of $\mathcal{S} \simeq \mathbb{P}^k$. To put it the other way around: the locus of divisors $D' \in \mathcal{S}$ which have singularities outside $\mathrm{Bs}(\mathcal{S})$ is a closed subvariety of $\mathcal{S} \simeq \mathbb{P}^k$.

The first observation is that if the Theorem fails for some linear system $\mathcal{S}$, then it also fails for a pencil. In fact, suppose that the locus in $\mathcal{S}$ of divisors which are either smooth or whose singularities lie in $\mathrm{Bs}(\mathcal{S})$, is a closed subvariety Z of $\mathcal{S} \simeq \mathbb{P}^k$. Then any general line in $\mathcal{S}$, where general means not contained in Z, will intersect Z at a finite number of points, so that a general point of this line will correspond to a member of $\mathcal{S}$ having a singularity outside $\mathrm{Bs}(\mathcal{S})$. But a line in $\mathcal{S}$ is just a pencil. Therefore it suffices to prove the Theorem for a pencil.

Proof. Assume by contradiction that we have a pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ with base locus $B \subset X$, such that for all but finitely many $\lambda \in \mathbb{P}^1$ the divisor $D_\lambda = D_0 + \lambda D_\infty$ has a singular point x_λ which is not in B. We can work locally in the analytic topology of X, by taking an open (analytic) subset $U \subset X$ with local charts $z = (z_1, \ldots, z_n)$ where D_0 and D_∞ are defined by the analytic equations $f(z) = 0$ and $g(z) = 0$ respectively. The conditions $x_\lambda \in D_\lambda$ and $x_\lambda \in \mathrm{Sing}(D_\lambda)$ yield

$$f(x_\lambda) + \lambda g(x_\lambda) = 0 \quad \text{and} \quad \frac{\partial f}{\partial z_i}(x_\lambda) + \lambda \frac{\partial g}{\partial z_i}(x_\lambda) = 0 \ (\forall i)$$

For $\lambda \neq 0, \infty$ clearly x_λ does not belong to D_0 nor D_∞. Indeed, if it belonged to one of them, then it would have to belong to the other one too by the

[1] where the word *conic* includes also the degenerate ones.
[2] Eugenio Bertini (1846-1933)

first equation above, hence $x_\lambda \in \mathrm{Supp}(D_0) \cap \mathrm{Supp}(D_\infty) = B$ which is impossible. Thus we can write $\lambda = -f(x_\lambda)/g(x_\lambda)$, substitute in the equation for the derivatives and get

$$g(x_\lambda)\frac{\partial f}{\partial z_i}(x_\lambda) - f(x_\lambda)\frac{\partial g}{\partial z_i}(x_\lambda) = 0$$

The left-hand side of this equation is just $g(x_\lambda)^2$ times the derivative of f/g with respect to z_i, evaluated at x_λ. Its vanishing means that f/g is locally constant.

Via the projection $U \times \mathbb{P}^1 \to U$ (which is holomorphic and proper), the analytic subset

$$(x, \lambda) \in U \times \mathbb{P}^1 : \quad \begin{cases} f(x) + \lambda g(x) = 0 \\ \frac{\partial f}{\partial z_i}(x) + \lambda \frac{\partial g}{\partial z_i}(x) = 0 \ (\forall i) \end{cases}$$

is projected onto the locus $\Sigma \subset U$ of singular points of the pencil. A Theorem of Remmert and Stein guarantees that Σ is analytic, therefore on each connected component of $\Sigma \setminus B$ the function f/g is constant. Consequently, $\lambda = -f/g$ can attain only finitely many values on $\Sigma \setminus B$. This contradicts our assumption that λ attains infinitely many values on U. □

An obvious ideal situation in which the Theorem applies is when the linear system has no base locus at all: then the general member is smooth. Such linear systems are called *basepoint free*. They are extremely important because it turns out that they define a morphism from the variety to some projective space.

Definition. A line bundle ξ is called *spanned* if for every $x \in X$ there exists a global section $s \in H^0(\xi)$ such that $s(x) \neq 0$. Other common terminology: ξ is *generated by its global sections*, or also *globally generated*.

Let us observe that the condition $s(x) \neq 0$ is well defined: if $s = (s_\alpha)$ with $s_\alpha \in \mathcal{O}(U_\alpha)$ and $\xi = (\xi_{\alpha\beta})$, on the overlaps we may have $s_\alpha = \xi_{\alpha\beta} s_\beta$. However $\xi_{\alpha\beta}$ is nowhere vanishing, hence $s_\alpha(x) \neq 0$ if and only if $s_\beta(x) \neq 0$.

A globally generated line bundle $\xi = \mathcal{O}(D)$ is required by definition to have some global sections, so ξ yields a (non-empty) linear system $|D|$. The basic observation is that to each $s \in H^0(\xi)$ such that $s(x) \neq 0$ there corresponds a divisor $D' \in |D|$ such that $x \notin \mathrm{Supp}(D')$ and vice versa. Thus we obtain:

REMARK 6.17. *Let $\xi = \mathcal{O}(D)$ be a non-trivial line bundle (i.e. $\mathcal{O}(D) \neq \mathcal{O}_X$). Then ξ is spanned if and only if $|D|$ (is not empty and) has empty base locus (basepoint free). In particular, in this case the general member of $|D|$ is smooth by Bertini.*

Given two line bundles $\xi, \eta \in H^1(X, \mathcal{O}_X^*)$ we have a canonical map

$$H^0(\xi) \otimes H^0(\eta) \to H^0(\xi \otimes \eta)$$

at the level of global section, defined by multiplication: $s \otimes \sigma \mapsto s \cdot \sigma$ (and then linearly extended). In other words for $s = (s_\alpha)$ and $\sigma = (\sigma_\alpha)$ we have $s \cdot \sigma = (s_\alpha \sigma_\alpha) \in H^0(\xi \otimes \eta)$. In particular, given a global section s of ξ, its powers s^k, with $k \geq 1$, are global sections of $\xi^{\otimes k}$,

$$s^k \in H^0(\xi^{\otimes k}).$$

More generally, any homogeneous degree k polynomial expression in the sections of ξ yields a section of $\xi^{\otimes k}$.

If ξ is spanned then so is $\xi^{\otimes k}$ for $k \geq 1$ and more generally the tensor product $\xi \otimes \eta$ of spanned line bundle is again spanned. In fact for any $x \in X$ we just pick the section $s \cdot \sigma$ with $s \in H^0(\xi)$ and $\sigma \in H^0(\eta)$ not vanishing at x.

Let now ξ be a globally generated line bundle and choose a basis $s_0, \dots, s_N$ for the vector space $H^0(\xi)$ of global sections. For $i = 1, \dots, N$, let $s_i = (s_{i\alpha})$ with $s_{i\alpha} \in \mathcal{O}(U_\alpha)$ and $s_{i\alpha} = \xi_{\alpha\beta} s_{i\beta}$ on the overlaps. Let, for $x \in U_\alpha$,

$$\varphi_\xi(x) = (s_{1\alpha}(x) : \dots : s_{N\alpha}(x)).$$

Since X is covered by the U_α's, this defines a holomorphic map, or *morphism*,

$$\varphi_\xi \colon X \longrightarrow \mathbb{P}^N.$$

On overlaps $U_\alpha \cap U_\beta$ the vectors $(s_{1\alpha}(x), \dots, s_{N\alpha}(x))$ and $(s_{1\beta}(x), \dots, s_{N\beta}(x))$ differ by multiplication by $\xi_{\alpha\beta}(x) \neq 0$. Moreover $s_{1\alpha}(x), \dots, s_{N\alpha}(x)$ cannot all be zero because ξ is spanned. Thus, our map φ_ξ is well defined. To see that φ_ξ is indeed holomorphic, consider affine charts $w_i = z_i/z_0$ on the open subset $\{z \in \mathbb{P}^N \colon z_0 \neq 0\}$. Then φ_ξ is locally given on some open subset $V \subset X$ by the N functions $s_{1\alpha}(x)/s_{0\alpha}(x), \dots, s_{N\alpha}(x)/s_{0\alpha}(x)$, holomorphic on $V \setminus \{s_{0\alpha} = 0\}$.

A large part of algebraic geometry is related to the study of the properties of the morphisms φ_ξ associated to a spanned line bundle ξ.

EXAMPLE 6.18. Let $X = \mathbb{P}^1$ and pick ξ the line bundle with sheaf of sections $\mathcal{O}(d)$, with $d > 0$. Then $H^0(\mathcal{O}(d)) = \langle t_0^d, t_0^{d-1} t_1, \dots, t_1^d \rangle$, so that

$$\varphi_\xi \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$

is the embedding $(t_0 \colon t_1) \mapsto (t_0^d \colon t_0^{d-1} t_1 \colon \dots \colon t_1^d)$ which embeds $\mathbb{P}^1$ as the rational normal curve of degree d.

EXAMPLE 6.19. Let X be a compact Riemann surface of genus one. Consider the divisor $D = p + q$ with p, q two distinct points of X. First we show that $\xi = \mathcal{O}_X(D)$ is spanned. For any point $x \in X$, its fundamental sequence reads

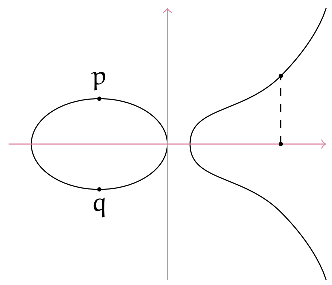
$$0 \longrightarrow \mathcal{O}_X(-x) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{S}_x \longrightarrow 0$$

where $\mathcal{O}_X(-x) \simeq \mathcal{I}_x$ is the ideal sheaf of x and $\mathcal{S}_x$ is the trivial extension to X of the structure sheaf of x. One says that $\mathcal{S}_x$ is a *skyscraper sheaf* on X supported on x. Very simply this just means that the stalk of $\mathcal{S}_x$ is $\mathcal{O}_x$ at x and zero at any other point. Tensoring the sequence with $\mathcal{O}_X(D)$ we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D - x) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D)|_x \longrightarrow 0.$$

Taking cohomology we get a map $H^0(D) \to H^0(\mathcal{O}(D)|_x) = \mathbb{C}$ which is just the evaluation at x of the sections of D, that is $s \mapsto s(x)$. By exactness, and the fact that $H^1(D - x)$ is zero, since $\deg(D - x) = 1 > 0$, this map is surjective. But the surjectivity of this map means precisely that there exists $s \in H^0(D)$ such that $s(x) \neq 0$. Since this holds for any x we have that D is globally generated.

We thus have a morphism $\varphi_\xi \colon X \to \mathbb{P}^N$, which we want to determine. First, $h^1(D) = 0$ since $\deg D = 2 > 0$, hence by the Riemann-Roch Theorem for curves $h^0(D) = \deg D + 1 - g = 2$ and therefore $N = h^0(D) - 1 = 1$. Finally $\varphi_\xi \colon X \to \mathbb{P}^1$ is a morphism of degree $2 = \deg D$, which exhibits the curve X as a double cover of $\mathbb{P}^1$. By the Hurwitz formula[3] this must ramify precisely at 4 points of X.



As a special case of this example, consider X to be the closure in $\mathbb{P}^2$ of the affine curve $y^2 = x(x^2 - 1)$. By choosing p and q lying on the same vertical line, the morphism is just the projection to the horizontal axis.

§5. Ampleness and two vanishing theorems

Let X be a compact complex manifold and $\xi \in H^1(X, \mathcal{O}^*)$ a globally generated line bundle. As we have seen, any choice of a basis $s_0, \dots, s_N$ of $H^0(\xi)$ gives rise to a morphism $\varphi_\xi \colon X \to \mathbb{P}^N$, which, with some abuse of notation we will simply write as $\varphi_\xi(x) = (s_0(x) \colon \dots \colon s_N(x))$, without worrying to specify the open covering. Notice that by construction, the images of two such morphisms obtained by choosing two distinct bases will differ by an automorphism of $\mathbb{P}^N$ (induced by the invertible matrix of the base change in $H^0(\xi) \simeq \mathbb{C}^{N+1}$).

Obviously, the same procedure makes sense if instead of taking a basis of the whole space of global sections $H^0(\xi)$ we just pick a basis for some linear subspace $V \subset H^0(\xi)$. Again, if ξ is globally generated, this gives rise to a morphism $\varphi_{\mathcal{S}} \colon X \to \mathbb{P}(V)$ associated to the linear system $\mathcal{S} = \mathbb{P}(V) \subset |\mathcal{O}(\xi)|$.

Remark 6.20. *Let $\mathcal{S} = \{D_a\}_{a \in \mathbb{P}^k}$ with $D_a := a_0 D_0 + \cdots + a_k D_k$ and pick a basis $s_0, \dots, s_k$ of V where s_i is the section defining the divisor D_i. In $\mathbb{P}(V)$ any linear equation $a_0 s_0 + \cdots + a_k s_k = 0$ defines an hyperplane. Therefore the hyperplanes of $\mathbb{P}(V)$ correspond to divisors in $\mathcal{S}$ and vice-versa.*

Definition. A line bundle ξ is called *very ample* if φ_ξ is an embedding. A divisor D such that $\xi = \mathcal{O}(D)$ is very ample is also called very ample.

We have already seen above that this is not always the case.

Example 6.21. For $X = \mathbb{P}^N$, consider the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$. Then the associated morphism is $\varphi = \mathrm{id} \colon \mathbb{P}^N \to \mathbb{P}^N$. Thus $\mathcal{O}_{\mathbb{P}^N}(1)$ is very ample.

[3] for a finite morphism of curves $f \colon X \to Y$ of degree d then $2 - 2g(X) = d(2 - 2g(Y)) + \deg R$ where R is the ramification divisor.

EXAMPLE 6.22. More generally, let $i\colon X \hookrightarrow \mathbb{P}^N$ be an embedding. With no loss of generality we assume that $X \subset \mathbb{P}^N$ is non-degenerate (i.e. not contained in a hyperplane). Let us try to construct a linear system on X which yields back the embedding i. We have natural linear maps, obtained by restriction,

$$i^*\colon \mathfrak{M}(\mathbb{P}^N) \longrightarrow \mathfrak{M}(X) \text{ and } i^*\colon \operatorname{Div}(\mathbb{P}^N) \longrightarrow \operatorname{Div}(X)$$

the latter being compatible with linear equivalence and thus inducing

$$i^*\colon \operatorname{Pic}(\mathbb{P}^N) \longrightarrow \operatorname{Pic}(X)$$

which is also a group homomorphism. In particular, by restricting the hyperplane line bundle on $\mathbb{P}^N$ we get the line bundle

$$\mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}^N}(1).$$

Given a hyperplane of $\mathbb{P}^N$, its intersection with X yields a divisor in $|\mathcal{O}_X(1)|$ which is called a *hyperplane section* of $X \subset \mathbb{P}^N$. Thus we have a linear system

$$\mathcal{S} = \{\text{hyperplane sections of } X \subset \mathbb{P}^N\} \subset |\mathcal{O}_X(1)|.$$

Since $\mathcal{S}$ is obviously basepoint free, by Bertini's Theorem it follows that *the generic hyperplane section of a smooth projective variety is smooth.*

If we let $\mathcal{S} = \mathbb{P}(V)$ and as a basis of $V \simeq H^0(\mathbb{P}^N, \mathcal{O}(1))$ we pick the obvious one $x_0, \dots, x_N$ where the x_i's are homogeneous coordinates of $\mathbb{P}^N$ (i.e. we let $\mathcal{S}$ be generated by the standard hyperplanes $x_i = 0$) then clearly $\varphi_{\mathcal{S}} = i$ (here we are really viewing $X \subset \mathbb{P}^N$ with $i = \mathrm{id}$). Thus $\mathcal{S}$ is the linear system we were looking for!

Though, it is not always the case that the linear system $\mathcal{S}$ is *complete*, i.e. $\mathcal{O}_X(1)$ might have some additional sections which do not arise from those of $\mathcal{O}_{\mathbb{P}^N}(1)$. In other words, the restriction map

$$H^0(\mathbb{P}^N, \mathcal{O}(1)) \to H^0(X, \mathcal{O}_X(1))$$

which is injective since X is non-degenerate, *might not be surjective*. This will happen precisely when $h^0(\mathcal{O}_X(1)) > h^0(\mathcal{O}_{\mathbb{P}^N}(1)) = N + 1$.

Definition. An embedding $i\colon X \hookrightarrow \mathbb{P}^N$ is called *linearly normal* if the linear system $\mathcal{S}$ of hyperplane sections is complete.

EXAMPLE 6.23. Consider the degree 4 curve $X \subset \mathbb{P}^3$ image of the embedding

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \quad (u : v) \mapsto (u^4 : u^3v : uv^3 : v^4)$$

Then $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathcal{O}_X(1))$ cannot surjective because the source has dimension 4 while $h^0(X, \mathcal{O}_X(1)) = h^0(\mathbb{P}^1, \mathcal{O}(4)) = 5$. Therefore this is not a projectively normal embedding. However, the standard embedding

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \quad (u : v) \mapsto (u^3 : u^2v : uv^2 : v^3)$$

whose image is the twisted cubic, is projectively normal.

We now focus on dimension 1 and explain how any compact Riemann surface X of genus g admits a very ample line bundle. Thus, by Chow's Theorem, compact Riemann surfaces and smooth projective curves are the same!

We have seen that $\deg K_X = 2g-2$ and any divisor on X of degree greater than this has vanishing h^1 (by Serre duality) and is called non-special. Given a line bundle $\xi = \mathcal{O}_X(D)$, for any $x \in X$, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(D-x) \longrightarrow \mathcal{O}_X(D) \longrightarrow \xi_x \longrightarrow 0$$

(just as in Example 6.19), whose long exact cohomology sequence yields the evaluation map of sections $H^0(D) \to H^0(\xi|_x) = \mathbb{C}$. If $D-x$ is non-special, i.e. if $\deg(D) > 1 + \deg K_X$, then $h^1(D-x) = 0$ and by exactness of the sequence our evaluation map at x is surjective, for any x. This means that $\xi = \mathcal{O}_X(D)$ is spanned and its sections define a morphism $X \to \mathbb{P}^N$. By pushing this kind of analysis a little forward (see [2, p. 215] or [3, p. 308]) one can show that whenever $\deg(D) > 2 + \deg K_X$ the associated morphism is indeed an embedding:

THEOREM 6.24. *Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$. If $\deg(D) \geq 2g+1$, then D is very ample.*

REMARK 6.25. *Let X and D be as in the Theorem. Then the sections of D embed X as a smooth projective non-degenerate curve of degree $d = \deg(D)$ in $\mathbb{P}^N$.*

Indeed, let $Y = \varphi(X) \subset \mathbb{P}^N$ where φ is the morphism associated to $\xi = \mathcal{O}(D)$. Recall (cf. Lemma 3.13) that the degree of Y is the number of points of intersection between Y and a generic hyperplane $H \subset \mathbb{P}^N$. Let s be the section of $\mathcal{O}(D)$ corresponding to H (cf. Remark 6.20). Now $\deg Y = \sharp\{y \in Y : y \in H\} = \sharp\{x \in X : \varphi(x) \in H\}$ since φ is injective. But $\varphi(x) \in H$ if and only if $s(x) = 0$. So what we want to count is actually the number of zeroes of s, which is $\deg(\mathrm{div}(s)) = \deg(D)$ by definition.

Theorem 6.24 is a remarkable result: for divisors on a curve, very ampleness, which is a *functional* condition (we require a map to be an embedding), actually reduces to a purely *numerical* condition, on the degree. In particular, given any divisor $D \in \mathrm{Div}(X)$ of positive degree, any multiple mD for $m >> 0$ will eventually have degree greater than $2g$, and thus be very ample.

Definition. A line bundle ξ on a compact complex manifold X is called *ample* if $\xi^{\otimes m}$, for $m >> 0$, is eventually very ample. In other words, there exists m_0 such that $\xi^{\otimes m}$ is very ample for any $m > m_0$. A divisor D such that $\xi = \mathcal{O}(D)$ is ample is also called ample.[4]

Thus, a compact complex manifold is algebraic if and only if it admits an ample line bundle. Indeed, if $X \subset \mathbb{P}^N$ then we have already observed that $\mathcal{O}_X(1)$ gives an ample (in fact, very ample) line bundle. The other direction is obvious, up to evoking Chow's lemma to conclude that the obtained embedding $X \to \mathbb{P}^N$ realizes X as an algebraic variety.

For what concerns amplitude on curves, Theorem 6.24 immediately yields:

COROLLARY 6.26. *Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$. If $\deg(D) > 0$, then D is ample.*

[4]In the following, the property of *being ample* (of a divisor or line bundle), will be referred to as *amplitude*, or *ampleness*, somewhat interchangeably.

Of course, an ample line bundle need not be very ample. In Example 6.19 we have seen this happening. Let us exhibit another similar example.

EXAMPLE 6.27. Let X be a compact Riemann surface of genus $g = 1$. Any divisor $D = p_1 + p_2 + p_3$, sum of three points on X, is very ample since $\deg D \geq 2g+1$. By Riemann-Roch $h^0(D) = 3$, so that $\varphi_\xi \colon X \to \mathbb{P}^2$ embeds X as a smooth cubic plane curve C, where $\xi = \mathcal{O}_X(D)$. Since the effective divisor $D \in |D|$ corresponds to a hyperplane, i.e. a line $\ell \subset \mathbb{P}^2$, and $D = p_1 + p_2 + p_3$, this means that $p_1, p_2, p_3 \in C$ (we identify $p \in X$ with $\varphi_\xi(p) \in C$ etc.) lie on ℓ.

It is well-known that a smooth cubic C admits a flex point p, hence the flex tangent line ℓ_p at p cuts out on C the divisor $D' = 3p$. But then $D' \sim D$, which means $\eta^{\otimes 3} = \xi$, where $\eta = \mathcal{O}_C(p)$ is the line bundle associated to p. In particular, η is ample; surely not be very ample, since $h^0(p) = 1$, so that the associated morphism $\varphi_\eta \colon C \to \mathbb{P}^0 = \{pt\}$ must be the constant map.

PROPERTIES OF AMPLENESS AND AN IMPORTANT COROLLARY

Let X be a smooth projective variety. We have already noticed that the product $\xi \otimes \eta$ of two spanned line bundles is again spanned. If ξ and η are also very ample, then so is $\xi \otimes \eta$. Indeed, we can construct the composition

$$X \to X \times X \to \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^N$$

where the first map is the diagonal, the second one is $\varphi_\xi \times \varphi_\eta$ and the latter one is the Segre embedding. Since the diagonal map is an embedding (Exercise) and so is $\varphi_\xi \times \varphi_\eta$ (product of embeddings), then the above composition is an embedding[5]. Thus, $\xi \otimes \eta$ is very ample. Of course, we immediately get also that the tensor product of ample line bundles is ample.

Now, if ξ is very ample but η is only spanned, we can still construct the same composition map as above, even though $\varphi_\xi \times \varphi_\eta$ is not necessarily an embedding. What can still be proved (Exercise) is that $\varphi_\xi \times \varphi_\eta \circ \Delta$ still is an immersion, so the whole composition is an embedding (check it is injective!). Summing up all of this we have:

PROPOSITION 6.28. *Let ξ and η be line bundles on X.*

(i) If ξ and η are both spanned, then $\xi \otimes \eta$ is spanned.

(ii) If ξ and η are both (very) ample, then $\xi \otimes \eta$ is (very) ample.

(iii) If ξ is very ample and η is spanned, then $\xi \otimes \eta$ is very ample.

As usual, the terminology for line bundles naturally transfers to divisors: we say that a divisor D is spanned, ample etc., if $\xi = \mathcal{O}(D)$ is so. Then the above proposition can be read in terms of divisors: the sum of two spanned or (very) ample divisors is again spanned or (very) ample, and so on.

Now we come to some properties of ample divisors (or line bundles), whose proofs rely on the following non-trivial result:

[5] and it is clearly equal to the map associated to a subspace of $H^0(\xi \otimes \eta)$ – which subspace?

THEOREM 6.29 (Serre vanishing). *Let H be an ample divisor on a smooth projective variety X. Then, for any divisor D, the divisor $D + mH$ eventually has no higher cohomology, i.e. there exists an integer $m_0 = m_0(D)$ such that for any $m \geq m_0$*

$$H^q(D + mH) = 0 \quad (q = 1, 2, \ldots)$$

In the proof (cf.[4, Thm.1.2.6]), one cannot avoid the use of *coherent sheaves*[6], and indeed the result holds if we replace D by any coherent sheaf. As a consequence we derive two other useful properties of ample divisors:

PROPOSITION 6.30. *Let H and D be divisors on X.*

(i) If H is ample, then $D + mH$ is eventually spanned, i.e. there exists $m_0 = m_0(D)$ such that $D + mH$ is spanned for $m \geq m_0$. Conversely, if H is such that for any divisor D, it holds that $D + mH$ is eventually spanned, then H is ample.

(ii) If H is ample, then $D + mH$ is very ample, for big enough $m > 0$.

Proof. The first equivalence is proved by the theory of coherent sheaves, which goes beyond the scope of these notes, and we refer to [4, Thm.1.2.6]. For the second statement in (i), suppose H is not ample, and let D be a divisor such that $-D$ is ample. Suppose there is $m > 0$ such that $mH + D$ is globally generated. Then $mH = (mH + D) - D$ is the sum of a spanned and an ample divisor, hence is ample, contrary to hypothesis. For part (ii), we can take m and k big enough so that mH is very ample and $D + kH$ is spanned. But then $D + (k + m)H$ is the sum of a spanned and a very ample divisor, hence very ample. □

As a result of these properties, we can show a useful fact

COROLLARY 6.31. *Let X be a smooth projective variety. Any divisor on X is linearly equivalent to a difference $A - B$, where $A, B \in \mathrm{Div}(X)$ are smooth irreducible hypersurfaces, both very ample as divisors.*

Proof. Let $D \in \mathrm{Div}(X)$ and pick an ample divisor H. For $m > 0$ sufficiently big $mH + D$ and mH are both very ample. In particular, they are both globally generated, and by Bertini's theorem it follows that the generic hypersurfaces $A \in |mH + D|$ and $B \in |mH|$ are smooth and we have $A - B \sim D$. □

There is another vanishing theorem which will be useful later on:

THEOREM 6.32 (Kodaira[7] vanishing). *Let X be a smooth projective variety and H an ample divisor. Then*

$$H^q(K_X + H) = 0 \quad (q = 1, 2, \ldots)$$

REMARK 6.33. *When X is a compact Riemann surface, we have already seen this result as an application of Serre duality: in fact in this case H is ample if and only if $\deg H > 0$, but then $\deg(K_X + H) > \deg(K_X)$, i.e. $K_X + H$ is a non-special divisor, hence the vanishing of its h^1.*

[6] a notion of sheaf which is more general to that of the sheaf associated to a line bundle.
[7] Kunihiko Kodaira (1915-1997)

REMARK 6.34. *By applying Serre duality, the statement of the Theorem becomes*

$$H^q(-H) = 0 \quad (q = 1, 2, \dots)$$

that is, the opposite $-H$ *of an ample divisor* H *(or the inverse* ξ^{-1} *of an ample line bundle* ξ*) has no higher cohomology.*

LECTURE 7

DOLBEAULT COHOMOLOGY AND HODGE THEOREM

Let X be a complex manifold of complex dimension n. We have introduced the sheaf Ω^p of holomorphic p-forms on X. The cohomology groups $H^q(X, \Omega^p)$ turn out to be a very useful invariant of X, since they depend heavily on the complex structure of X and not solely on its topology or underlying differentiable structure. The obvious problem is then: how to compute these groups?

Recall that using the underlying differentiable structure of X, one can define differential forms on X. Since X is a complex manifold, it is more convenient to allow differential forms to have complex coefficients. Let us denote by $\mathcal{A}^r$ the sheaf of such forms, where r is the degree:

$$\mathcal{A}^r = \{\text{complex differential } r\text{-forms}\}.$$

For each p and q such that $p + q = r$, we consider the so-called (p, q)-*forms*: these are r-forms which are locally expressed as sums of monomials of the type

$$\varphi_\alpha = f_\alpha dz^\alpha_{i_1} \wedge \ldots \wedge dz^\alpha_{i_p} \wedge d\bar{z}^\alpha_{j_1} \wedge \ldots \wedge d\bar{z}^\alpha_{j_q}$$

with all the allowed combinations of indices[1], and by using the Cauchy-Riemann equations, it can be shown that the type (p, q) of this expression does not depend on the local chart. It is therefore well-defined the sheaf $\mathcal{A}^{p,q}$ of differentiable r-forms of type (p, q), which is in fact a vector sub-bundle of $\mathcal{A}^r$.

It turns out that each form in $\mathcal{A}^r$ can be expressed in a unique way as a sum of forms in the various $\mathcal{A}^{p,q}$, considering all possible combinations $p + q = r$. In other words, there is a direct sum decomposition

$$\mathcal{A}^r = \mathcal{A}^{r,0} \oplus \mathcal{A}^{r-1,1} \oplus \cdots \oplus \mathcal{A}^{1,r-1} \oplus \mathcal{A}^{0,r} = \bigoplus_{p+q=r} \mathcal{A}^{p,q}$$

In particular, for all p, q there are canonical projections $\pi^{p,q} \colon \mathcal{A}^r \to \mathcal{A}^{p,q}$. We introduce the $\bar{\partial}$-operator defined as the composition

$$\bar{\partial} \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}, \quad \bar{\partial}\varphi = \pi^{p,q}(d\varphi)$$

[1] For example, a 3-form of type $(2, 1)$ might have a local expression such as

$$e^{\bar{z}^\alpha_1}(dz^\alpha_1 \wedge dz^\alpha_2 \wedge d\bar{z}^\alpha_1) + 5(dz^\alpha_1 \wedge dz^\alpha_3 \wedge d\bar{z}^\alpha_1) + (z^\alpha_2)^3(dz^\alpha_1 \wedge dz^\alpha_2 \wedge d\bar{z}^\alpha_2) + \cdots$$

where $d\colon \mathcal{A}^r \to \mathcal{A}^{r+1}$ is the usual differential operator of forms.

Notice that the holomorphic forms Ω^r are precisely those forms in $\mathcal{A}^{r,0}$ having holomorphic coefficients, i.e. in the local expression above the f_α are all holomorphic functions. In other words, $\Omega^r = \ker(\bar{\partial}\colon \Omega^r \to \mathcal{A}^{r,0})$.

One can verify that at the level of global sections $\mathcal{A}^{p,q}(X)$, the $\bar{\partial}$-operator yields a complex, that is $\bar{\partial} \circ \bar{\partial} = 0$. Thus, in complete analogy with de Rham cohomology, we can define the *Dolbeault*[2] *groups*

$$H^{p,q}(X) = \frac{\ker(\bar{\partial}\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\mathrm{Im}(\bar{\partial}\colon \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))} = \frac{(p,q)\text{-forms } \bar{\partial}\text{-closed}}{(p,q)\text{-forms } \bar{\partial}\text{-exact}}$$

which are in fact $\mathbb{C}$-vector spaces, finite-dimensional for compact X. In the latter case, the dimensions of these spaces (*Hodge numbers*) are denoted by

$$h^{p,q}(X) = \dim H^{p,q}(X).$$

By the cohomology theory of sheaves, one can prove the following result, which shows that the cohomology groups of the sheaf of holomorphic p-forms Ω^p are really just the same as the Dolbeault groups:

THEOREM 7.1 (Dolbeault). *There is an isomorphism* $H^q(\Omega^p) \simeq H^{p,q}(X)$.

For the moment, we have just shifted our ignorance about the cohomology of Ω^p to that of Dolbeault groups. However, there is a very deep result due to Hodge[3], classically achieved by the theory of elliptic operators, which gives the link between the Dolbeault groups and the singular cohomology of X with complex coefficients. It does not hold for any compact complex manifold, but only for compact manifolds admitting a so-called *Kählerstructure*[4]. We avoid introducing Kählermanifolds here: it suffices to mention that each smooth projective variety is Kähler, so the result will hold in this special case.

THEOREM 7.2 (Hodge decomposition). *For any compact Kählermanifold* X,

$$H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X).$$

Notice that there is a natural map $\mathcal{A}^{p,q}(X) \to \mathcal{A}^{q,p}(X)$ induced by complex conjugation. Obviously this operation preserves exact forms, so that we have a natural map $H^{p,q}(X) \to H^{q,p}(X)$ induced by complex conjugation.

THEOREM 7.3 (Hodge duality). *For any compact Kählermanifold* X*, the complex conjugation map* $H^{p,q}(X) \to H^{q,p}(X)$ *is an isomorphism. In particular,*

$$h^{p,q}(X) = h^{q,p}(X).$$

The two results above together are usually referred to as "Hodge Theorem". Let us now see some immediate yet very useful consequences.

Recall that by the theorem of universal coefficients, the singular cohomology with coefficients in $\mathbb{C}$ is completely determined by the integral cohomology, inasmuch as $H^r(X,\mathbb{C}) \simeq H^r(X,\mathbb{Z}) \otimes \mathbb{C}$ (even though, tensoring with $\mathbb{C}$

[2]Pierre Dolbeault (1924 - 2015)

[3]William Vallance Douglas Hodge (1903 - 1975)

[4]Eric Kähler(1906 - 2000)

kills torsion, so we loose information in this process). In particular, the *Betti numbers*[5]

$$b_r(X) := \operatorname{rk} H^r(X, \mathbb{Z}) = \dim H^r(X, \mathbb{C})$$

are important topological invariants of X.

COROLLARY 7.4. *Let X be a smooth projective variety. Then*

(i) Odd Betti numbers are even: $b_{2k+1}(X) \in 2\mathbb{Z}$.

(ii) The Hodge number $h^{1,0} = \dim \Omega^1(X)$ *is a purely topological invariant.*

Proof. Since X is smooth projective, by Hodge decomposition we have

$$b_{2k+1}(X) = h^{2k+1,0} + h^{2k,1} + \cdots + h^{k+1,k} + h^{k,k+1} + \cdots + h^{1,2k} + h^{0,2k+1}$$

and by Hodge duality the right hand side must be an even integer. In particular, $b_1(X) = h^{1,0} + h^{0,1} = 2h^{1,0}$, i.e. $h^{1,0} = b_1/2$ is purely topological. □

On the other hand, notice that $b_{2k} = h^{k,k} + 2(\cdots) = h^{k,k}$ (mod 2), so the parity of even Betti numbers depends on this $h^{k,k}$ factor.

THE HODGE DIAMOND

Let X be a smooth projective variety. We can list all the Hodge numbers $h^{p,q}$ of X in a diagram, called the *Hodge diamond*, as follows:

$$\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & \iddots & & & & \ddots & \\
h^{n,0} & \cdots & & \cdots & & \cdots & h^{0,n} \\
 & \ddots & & & & \iddots & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

The name *diamond*, is due to the fact that it possesses several symmetries:

(i) Hodge duality: symmetry about the vertical axis: $h^{p,q} = h^{q,p}$.

(ii) Poincaré duality: symmetry about the horizontal axis: $h^{p,q} = h^{n-q,n-p}$.

(iii) Serre duality yields a symmetry about the origin for Hodge numbers along the edges: $h^{0,q} = h^{n,n-q}$. Indeed,

$$h^q(\mathcal{O}_X) = h^{n-q}(K_X) = h^{n-q}(\Omega^n_X).$$

Also, $b_0(X) = 1$ since X is connected, so $h^{0,0} = h^{n,n} = 1$.

[5]Enrico Betti (1823 - 1892)

EXAMPLE 7.5. Let X be a compact Riemann surface of genus g. Since

$$g = h^0(\Omega^1_X = h^{1,0} = h^{0,1} = h^1(\mathcal{O}_X),$$

by Hodge duality (or Serre duality, since they are equivalent in dimension 1), we have that the Hodge diamond of X is simply

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

which thus depends only on the topology of X, hence not very interesting.

EXAMPLE 7.6. Let $X = \mathbb{P}^n$. Recall that for $0 \leq r \leq 2n$,

$$H^r(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Therefore all $b_{2k+1} = 0$, while $b_{2k} = 1$. But $b_{2k} = h^{k,k}$ (mod 2), so it must be $h^{k,k} = 1$ for all k, and all other hodge numbers vanish.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & \iddots & & & & & & \ddots & \\ 0 & & & & & & & & 0 \\ & \ddots & & & & & & \iddots & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

In particular, $h^{p,0} = h^0(\Omega^p) = 0$, for any $p > 0$, which tells us that there are no holomorphic p-forms on $\mathbb{P}^n$. Also, $h^{0,q} = h^q(\mathcal{O}_{\mathbb{P}^n}) = 0$ for any $q > 0$, a vanishing result that we had already seen for $q = 1$. Notice that we could also deduce it by Kodaira vanishing:

$$h^q(\mathcal{O}_{\mathbb{P}^n}) = h^q(K_{\mathbb{P}^n} + (-K_{\mathbb{P}^n})) = 0$$

since $-K_{\mathbb{P}^n}$ is ample (recall $-K_{\mathbb{P}^n} \sim (n+1)H$ where H is a hyperplane).

EXAMPLE 7.7. Let X be a projective surface, i.e. $\dim(X) = 2$. There are only three relevant Hodge numbers: the *irregularity* of X, denoted by[6]

$$q(X) := h^{1,0}(X),$$

which is the number of independent global holomorphic 1-forms, and

$$p_g(X) := h^{2,0}(X),$$

[6] a surface X with $q = 0$ is often called *regular*.

the number of independent global holomorphic 2-forms, which is called the *geometric genus* of X. Finally, we have $h^{1,1}(X)$ which does not have any special name or notation. Observe that by Hodge duality, $q(X)$ and $p_g(X)$ encapsulate respectively the first and second cohomology of the structure sheaf $\mathcal{O}_X$, i.e.

$$q(X) = h^1(\mathcal{O}_X), \quad p_g(X) = h^2(\mathcal{O}_X).$$

The Hodge diamond of X is as follows

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

Summing up the terms on each line we can read off the Betti numbers of X. Recall that the *Euler-Poincaré characteristic* $e(X)$, is a useful topological invariant defined as the alternating sum of the Betti numbers $e(X) = \sum(-1)^k b_k(X)$. Hence, $e(X)$ is expressed in terms of the Hodge numbers of X as

$$e(X) = 2 - 4q + 2p_g + h^{1,1}.$$

We might actually say a few words about $h^{1,1}$. As a matter of fact, if X is any compact Kählermanifold, its analytic submanifolds of codimension p define, by duality, non-trivial cohomology classes in $H^{2p}(X, \mathbb{C})$ which actually entirely live inside the $H^{p,p}(X)$ piece[7]. In particular, if X is a smooth projective variety, say $X \subset \mathbb{P}^N$, in $H^{p,p}(X)$ there is the class of the section of X with a codimension p linear subspace of $\mathbb{P}^N$. Therefore, for all $0 \leq p \leq n$ one has

$$h^{p,p}(X) > 0.$$

NOT ALL SURFACES ARE PROJECTIVE

We have seen that in dimension one, all compact complex manifold are projective. This is not true in higher dimension. In this section we construct an example of a complex surface which does not admit any projective embedding.

One of the easiest ways to construct examples of varieties is to take quotients with respect to automorphisms on some known variety. This operation is often very delicate and we cannot discuss this problem in full generality here[8]. We will only need to invoke the following basic result:

FACT. *Let X be a complex manifold and $G \subset \mathrm{Aut}(X)$ a subgroup such that*

- *(i) G acts properly discontinuously, i.e. for any pair of compact proper subsets K_1, K_2 of X, the set $\{g \in G : g(K_1) \cap K_2 \neq \varnothing\}$ is finite.*
- *(ii) G acts without fixed points, i.e. any non-trivial element of G has no fixed points.*

Then the set of orbits X/G can be naturally given a complex structure such that the natural projection $\pi \colon X \to X/G$ is holomorphic and locally a biholomorphism.

[7] this is explained for example in [2, p.162-163].

[8] indeed, this is the basic problem studied by *Geometric Invariant Theory*.

EXAMPLE 7.8. Let $X = \mathbb{C}^n$ and choose an additive subgroup $\Gamma \subset \mathbb{C}^n$ isomorphic to $\mathbb{Z}^{2n}$. We consider Γ acting on X by translation. Obviously (i) and (ii) are verified. The quotient X/Γ is the n-dimensional complex torus.

We will now construct the *Hopf surface*. Consider the punctured plane $X = \mathbb{C}^2 \setminus \{0\}$ and $G \subset \operatorname{Aut}(X)$ the subgroup acting as follows

$$g_k \cdot (z_1, z_2) = 2^k (z_1, z_2) \quad (k \in \mathbb{Z}).$$

Clearly $G \simeq \mathbb{Z}$ and properties (i) and (ii) above are verified. Thus,

$$Y = X/G$$

is a complex surface. Notice that we have a diffeomorphism

$$f \colon S^3 \times \mathbb{R} \to X, \quad ((z_1, z_2), t) \mapsto 2^t (z_1, z_2)$$

where $S^3 \subset \mathbb{C}^2$ is the standard 3-sphere defined as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \colon \ |z_1|^2 + |z_2|^2 = 1\}.$$

Moreover, G also acts on $S^3 \times \mathbb{R}$ by

$$g_k \cdot ((z_1, z_2), t) = ((z_1, z_2), t + k).$$

The map f is equivariant with respect to these two actions: applying f and then acting with G on X is the same as acting with G on $S^3 \times \mathbb{R}$ and then applying f. This means that f induces a diffeomorphism between the quotients,

$$(S^3 \times \mathbb{R})/G \simeq X/G = Y.$$

Observe that

$$(S^3 \times \mathbb{R})/G = S^3 \times (\mathbb{R}/\mathbb{Z}) = S^3 \times S^1$$

is compact, and therefore Y is compact too. Moreover,

$$\pi_1(Y) \simeq \pi_1(S^3 \times S^1) \simeq \pi_1(S^3) \times \pi_1(S^1) \simeq \pi_1(S^1) \simeq \mathbb{Z}$$

where we used the fact that S^3 is simply connected. We have a compact complex surface Y with $b_1(Y) = 1$, which is odd! Therefore Y cannot be projective.

Lecture 8

Adjunction formula

Let X be a smooth projective variety and $V \subset X$ a *smooth* hypersurface. The adjunction formula gives the relation between their canonical bundles.

Let $i\colon V \hookrightarrow X$ be the inclusion. We have an induced homomorphism

$$i^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(V)$$

defined by restriction: $i^*\xi = \xi|_V$. In terms of sheaves this is $i^*\xi = \xi \otimes \mathcal{O}_V$. Recall that we have already seen this when $V \subset \mathbb{P}^N$.

THEOREM 8.1 (Adjunction formula). *If X and V are as above,*

$$\omega_V \simeq (\omega_X \otimes \mathcal{O}_X(V))|_V.$$

In terms of divisors this becomes

$$K_V \sim (K_X + V)|_V.$$

This formula is not hard to prove, but we refer to [2, p. 146-147]. Let us instead focus on some first applications. Consider the fundamental sequence of V twisted by the line bundle $\xi = \omega_X \otimes \mathcal{O}_X(V) = \mathcal{O}_X(K_X + V)$,

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + V) \to \mathcal{O}_V(K_V) \to 0 \tag{8.1}$$

At the level of global sections, this yields a homomorphism,

$$H^0(X, K_X + V) \to H^0(V, K_V)$$

which is nothing but the restriction to V of sections of $\mathcal{O}_X(K_X + V)$.

For example, consider the case when V is a smooth hypersurface of degree d in $\mathbb{P}^N$, i.e. $V \in |\mathcal{O}_{\mathbb{P}^N}(d)|$, and $N \geq 2$. We know $\mathcal{O}_{\mathbb{P}^N}(K_{\mathbb{P}^N}) = \mathcal{O}_{\mathbb{P}^N}(-N-1)$, hence $\mathcal{O}_{\mathbb{P}^N}(K_{\mathbb{P}^N} + V) = \mathcal{O}_{\mathbb{P}^N}(d - N - 1)$. We know $H^1(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-N-1)) = 0$ (e.g. by Kodaira vanishing). Therefore, in this example the above restriction map is surjective, that is

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d - N - 1)) \to H^0(V, K_V) \to 0$$

Thus, all the global sections of the canonical bundle of V are obtained as restrictions to V of the sections of $\mathcal{O}_{\mathbb{P}^N}(d-N-1)$. Geometrically, this means that the linear system $|K_V|$ consists precisely of the divisors on V which are cut out by the hypersurfaces of degree $d - N - 1$ in the ambient space $\mathbb{P}^N$,

$$|K_V| = \{\text{divisors cut out on } V \subset \mathbb{P}^N \text{ by } |\mathcal{O}_{\mathbb{P}^N}(d - N - 1)|\}.$$

In particular, $|K_V|$ is empty whenever $d < N + 1$. Also, we see that V has trivial canonical bundle if and only if $d = N + 1$.

EXAMPLE 8.2. When $N = 2$ and $V = C$ is a smooth curve of $\mathbb{P}^2$ of degree d and genus g, we have that the canonical series $|K_C|$ consists of the divisors cut out on C by the plane curves of degree $d - 3$. What is then the degree of these divisors? This will be the intersection number between C and a curve of degree $d - 3$, which is $d(d - 3)$, by Bézout. On the other hand, we know $\deg(K_C) = 2g - 2$. Equating the two, we obtain *Clebsch formula*,

$$g = \frac{1}{2}(d - 1)(d - 2).$$

EXAMPLE 8.3. When $N = 3$ and $V = S$ is a smooth surface of degree d in space $\mathbb{P}^3$, we have that $|K_S|$ consists of divisors cut out on S by surfaces of degree $d-4$ in $\mathbb{P}^3$. This system is empty when $d < 4$ (i.e. there are no effective divisors). Let us examine in some detail these cases of low-degree.

- ⋄ $d = 1$. Then S is a projective plane $\mathbb{P}^2$ linearly embedded in $\mathbb{P}^3$.
- ⋄ $d = 2$. Then S is a smooth *quadric* $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in space. By adjunction,

  $$K_Q \sim -2H_Q$$

  where H is a plane in $\mathbb{P}^3$ and $H_Q = H \cap Q$ the restriction.
- ⋄ $d = 3$. Then S is a smooth surface in $\mathbb{P}^3$ and $K_S \sim -H_S$.

We conclude this section by examining in some more detail the case of the quadric surface Q in $\mathbb{P}^3$. All smooth quadrics in $\mathbb{P}^3$ are isomorphic to $x_0x_3 - x_1x_2 = 0$, which is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding. Topologically, $\mathbb{P}^1$ is a 2-sphere. Hence, Q is homeomorphic to $S^2 \times S^2$. This gives the Betti numbers $b_1(Q) = b_3(Q) = 0$ and $b_2(Q) = 2$. Since K_Q is negative, we also have $h^{2,0}(Q) = h^0(K_Q) = 0$. Hence the Hodge diamond of Q is as follows

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Let us now see that also the possible line bundles or divisors on Q are completely determined by the topology of Q. The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_Q \to \mathcal{O}_Q^* \to 1$$

together with $H^1(\mathcal{O}_Q) = H^2(\mathcal{O}_Q) = 0$, yields an isomorphism

$$\mathrm{Pic}(Q) \simeq H^2(Q, \mathbb{Z}).$$

Since $H^2(Q, \mathbb{Z}) = H^2(S^2 \times S^2, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and clearly the (classes of the) lines $E = \mathbb{P}^1 \times \{p\}$ and $F = \{p\} \times \mathbb{P}^1$ in Q are two generators of $H^2(Q, \mathbb{Z})$, we get

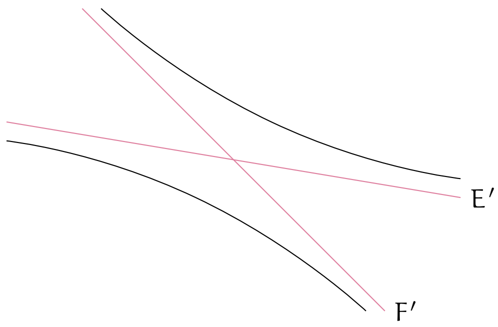
$$\mathrm{Pic}(Q) = \mathbb{Z}[E] \oplus \mathbb{Z}[F].$$

This means that for any divisor D on Q we have

$$D \sim aE + bF$$

for some $a, b \in \mathbb{Z}$. In particular, this will be true for a hyperplane section H_Q too. Since all hyperplanes of $\mathbb{P}^3$ are linearly equivalent, let us choose a special one: let H be a tangent plane to Q. Then it is well known that the intersection $H_Q = H \cap Q$ consists of two lines E' and F' (each one belonging to one of the two *rulings* of the quadric), i.e. $H_Q = E' + F'$. Since $E' \sim E$ and $F' \sim F$, we have

$$H_Q \sim E + F \quad \text{and} \quad K_Q \sim -2E - 2F.$$



Part II

Algebraic surfaces

Conventions and Notation

In the rest of these notes, we will be concerned with the theory of smooth algebraic surfaces. Therefore, we adopt the following convention: with the term *surface*, we will always mean a smooth projective surface over $\mathbb{C}$. In other words, a surface S will be a complex compact analytic manifold of dimension 2 such that there exists an ample line bundle $H \in \mathrm{Pic}(S)$. We call

$$q(S) = h^{1,0} = h^0(S, \Omega^1),$$

the *irregularity* of S. The *geometric genus* of S is by definition

$$p_g(S) = h^{2,0} = h^0(S, \Omega^2).$$

In other words, the irregularity and the geometric genus of S count the number of (independent, global, holomorphic) 1-forms and 2-forms on S, respectively. Let us now give some other interpretations of these invariants.

The first one, yet obvious but still worth to notice is that, since S is a two-dimensional manifold, $\Omega^2 = K_S$. Therefore, the geometric genus counts the sections of the canonical bundle,

$$p_g = h^0(K_S).$$

By Hodge duality, $q = h^{0,1}(S)$ and $p_g = h^{0,2}(S)$. Since $\Omega^0(S) = \mathcal{O}_S$, we get

$$q = h^1(X, \mathcal{O}_S), \quad p_g = h^2(X, \mathcal{O}_S).$$

That is, $q(S)$ and $p_g(S)$ capture the basic cohomological informations of the structure sheaf on S. In particular, $\chi(\mathcal{O}_S) = 1 - q + p_g$.

Another important (topological) invariant, is the *Euler characteristic* of S. By definition, it is the alternating sum of the Betti numbers

$$e(S) = \sum_{i=0}^{4} (-1)^i b_i(S).$$

On the other hand, by expressing the Betti numbers in terms of the Hodge numbers, we immediately find

$$e(S) = 2 - 4q + 2p_g + h^{1,1}.$$

Lecture 9

Intersection theory

Let S be a surface. In this section we construct one of the fundamental tools in the study of surfaces, which is a strong generalization of the familiar concepts regarding the intersections of smooth plane curves.

We develop this step by step. The first issue we discuss is the following: how do we count the points of intersections of two curves on S?

Let C and C' be distinct irreducible curves on S intersecting at a point p. Let U be an open neighbourhood of p where the two curves are described by the equations $f = 0$ and $g = 0$ respectively, where $f, g \in \mathcal{O}(U)$. We can then consider the generated ideal (f, g) of $\mathcal{O}(U)$. Of course, the question of intersection at p is a local property at p, so let us work on the stalk at p. Let f_p and g_p be the germs of f and g in the local ring $\mathcal{O}_{S,p}$. They generate a proper ideal:

$$\mathcal{A} := (f_p, g_p) \subsetneq \mathcal{O}_{S,p}.$$

LEMMA 9.1. *The $\mathbb{C}$-vector space $\mathcal{O}_{S,p}/\mathcal{A}$ is finite dimensional.*

Proof. The set $C \cap C'$ is discrete and S is compact, hence $C \cap C'$ is finite. Thus there exists U such that $C \cap C' \cap U = \{p\}$. Since S is algebraic, $\mathcal{O}_S$ can be thought of as the sheaf of (germs of) regular functions, and U as an open *affine* set of some embedding $S \hookrightarrow \mathbb{P}^N$, that is $U = S \cap \mathbb{C}^N \hookrightarrow \mathbb{C}^N$.

$\mathcal{O}(U)$ is the coordinate ring of the affine variety U, that is the quotient of $\mathbb{C}[x_1, \dots, x_N]$ by the defining ideal $I(U)$. If $h \in \mathcal{O}(U)$ and $h(p) = 0$ then some power h^r must fall in (f, g) (just like the Nullstellensatz).

Let I denote the ideal of p in $\mathcal{O}(U)$. As $\mathcal{O}(U)$ is a Noetherian ring, the ideal I must be generated by some $h_1, \dots, h_s$ with $h_i^{r_i} \in (f, g)$. Let $r = \max\{r_i\}$. Then $I^r \subset (f, g)$. Consider the natural homomorphism $\varphi : \mathcal{O}(U) \to \mathcal{O}_{S,p}$ given by $\varphi(I) = \mathfrak{m}_p$, where $\mathfrak{m}_p$ is the maximal ideal generated by the germs of x and y, local coordinates around p. Then $\varphi(f, g) = \mathcal{A}$, and since $I^r \subset (f, g)$ we get $\mathfrak{m}_p^r \subset \mathcal{A}$, whence the inclusion $\mathcal{O}_{S,p}/\mathcal{A} \subset \mathcal{O}_{S,p}/\mathfrak{m}_p^r$, which shows

$$\dim(\mathcal{O}_{S,p}/\mathcal{A}) \leq \dim(\mathcal{O}_{S,p}/\mathfrak{m}_p^r).$$

It is not difficult to compute the dimension of the latter quotient: identifying $\mathcal{O}_{S,p}$ with the ring $\mathbb{C}\{x, y\}$ of convergent power series around the origin, a basis of $\mathcal{O}_{S,p}/\mathfrak{m}_p^r$ is given by all homogeneous monomials in x and y of degree d, with d ranging between zero and $r-1$. Since there are exactly $d+1$ homogeneous monomials in x, y of degree d, we have that the dimension of $\mathcal{O}_{S,p}/\mathfrak{m}_p^r$ is $1+2+\cdots+r = \frac{1}{2}r(r+1)$. $\square$

Definition. If C and C' are two distinct irreducible curves on S intersecting at a point p, we call *intersection multiplicity of* C *and* C' *at* p the positive integer

$$i(C, C', p) := \dim(\mathcal{O}_{S,p}/\mathcal{A})$$

where $\mathcal{A}$ is the ideal of $\mathcal{O}_{S,p}$ generated by the local equations of C and C' at p.

The following simple examples show that this number behaves as one would naturally expect in some familiar situations.

EXAMPLE 9.2.

1. In $\mathbb{C}^2$, let C and C' be the axis $x = 0$ and $y = 0$ respectively and p the origin $(0, 0)$. Then $\mathbb{C}\{x, y\}/(x, y)$ consists of the constants only, hence $i(C, C', p) = 1$. More generally, two curves C and C' on a surface S are called *transverse* at p if they are smooth and not mutually tangent at p, i.e. the Jacobian matrix $J(f, g)$ has rank 2 at p. In this case, the map $(x, y) \mapsto (z = f(x, y), w = g(x, y))$ is a biholomorphism around p, sending p to the origin and the two curves to the axes $z = 0$ and $w = 0$. Thus, the intersection number at p is again 1.

2. Let C be $y = 0$ and C' be the parabola $x^2 - y = 0$ in $S = \mathbb{C}^2$. To compute the intersection, let us find out which monomials we have to quotient out from the basis $\{1, x, y, x^2, xy, \dots\}$ of $\mathbb{C}\{x, y\}$. Clearly 1 and x will survive, but y and its powers will not, since $y_p \in \mathcal{A}$. As for x^2, since $y_p \in \mathcal{A}$ and $(x^2 - y)_p$ is in $\mathcal{A}$ we get also $x_p^2 \in \mathcal{A}$. Thus $\mathcal{O}_{S,p}/\mathcal{A}$ is generated by 1 and x, and so $i(C, C', p) = 2$.

3. Let C be $y = 0$ and C' be the cuspidal cubic $x^3 - y^2 = 0$ in $S = \mathbb{C}^2$. Clearly, the generators $1, x$ and x^2 will survive to the quotient. There are no others, since $y = 0$ will kill all the ones containing y and it is easy to see that $x_p^3 \in \mathcal{A}$ by a similar argument as in the preceding example. So $i(C, C', p) = 3$.

Definition. By setting $i(C, C', p) := 0$ if p is not in the intersection we define the *intersection number* of the curves C and C' to be the non-negative integer

$$C \cdot C' := \sum_{p \in S} i(C, C', p).$$

We are now able to intersect pairs of distinct irreducible curves on S. We want much more. We aim at a generalization of this intersection form to *any pair of divisors* (including the case of two *equal* divisors!). How do we do this? The key observation is that the intersection multiplicity is, in fact, *cohomological information*, by the following.

REMARK 9.3. *Let* C *and* C' *be two distinct irreducible curves on* S. *We know* $\mathcal{O}_S(-C)$ *is the sheaf of holomorphic functions vanishing on* C. *We also know it injects in* $\mathcal{O}_S$

(by multiplication with a local equation for C*) and the quotient is* $\mathcal{O}_C$ *(this gives the fundamental exact sequence of* $C \subset S$*). Similarly, the same holds for* $\mathcal{O}_S(C')$*. Thus* $\mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C')$ *is the sheaf of holomorphic functions vanishing on* $C \cap C'$ *and it injects in* $\mathcal{O}_S$ *with quotient* $\mathcal{O}_{C\cap C'}$*, i.e. we have an exact sequence of sheaves*

$$0 \to \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \to \mathcal{O}_S \to \mathcal{O}_{C\cap C'} \to 0.$$

Passing to the germs at any point p *this sequence becomes*

$$0 \to \mathcal{A} = (f_p, g_p) \to \mathcal{O}_{S,p} \to \mathcal{O}_{C\cap C',p} \to 0.$$

Obviously, when p *is not in the intersection* $C \cap C'$ *then* $\mathcal{A} = \mathcal{O}_{S,p}$*, so that* $\mathcal{O}_{C\cap C',p}$ *is trivial. On the other hand, by the preceding discussion when* $p \in C \cap C'$*, we have that* $\mathcal{O}_{S,p}/\mathcal{A}$ *is a vector space of dimension* $i_p = i(C, C', p)$*. Thus,*

$$\mathcal{O}_{C\cap C',p} = \begin{cases} \mathcal{O}_{S,p}/\mathcal{A} \simeq \mathbb{C}^{i_p} & p \in C \cap C' \\ 0 & p \notin C \cap C' \end{cases}$$

One says that $\mathcal{O}_{C\cap C'}$ *is a "skyscraper sheaf", supported on the points of intersection of* C *and* C'*. Note*

$$h^0(\mathcal{O}_{C\cap C'}) = \sum_{p\in S} i_p = C \cdot C'$$

Moreover, since $C \cap C'$ *is zero-dimensional, the sheaf* $\mathcal{O}_{C\cap C'}$ *has no higher cohomology by Grothendieck vanishing. We then obtain the expression*

$$C \cdot C' = \chi(\mathcal{O}_{C\cap C'}).$$

We are now ready to give the general definition of intersection product.

Definition. We define an *intersection form* $\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$, by[1]

$$\xi \cdot \xi' := \chi(\mathcal{O}_S) - \chi(-\xi) - \chi(-\xi') + \chi(-\xi - \xi').$$

It is obviously symmetric and we will shortly prove that it is bilinear and moreover it reduces to the intersection of curves defined above when the line bundles ξ and ξ' are represented by two distinct irreducible curves.

First, we need the following lemma, which already gives some (useful) geometric insight: if ξ is any line bundle on S but ξ' is represented by a smooth irreducible curve C, that is $\xi' = \mathcal{O}_S(C)$, then $\xi \cdot \xi'$ is simply the *degree* of the line bundle on C obtained as *restriction* of ξ to the curve.

LEMMA 9.4. *Let* C *be a smooth irreducible curve on* S*. For any line bundle* ξ *on* S*,*

$$\xi \cdot \mathcal{O}_S(C) = \deg(\xi|_C).$$

Proof. Let $\xi = \mathcal{O}_S(D)$, for some divisor D. Tensoring the fundamental short exact sequence of $C \subset S$,

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

[1] we use additive notation for the operation of tensor product of line bundles in $\mathrm{Pic}(S)$.

with $\mathcal{O}_S(-D)$, yields

$$0 \to \mathcal{O}_S(-C-D) \to \mathcal{O}_S(-D) \to \mathcal{O}_C(-D|_C) \to 0.$$

By additivity of the Euler characteristic on short exact sequences[2] we get

$$\begin{cases} \chi(-C) - \chi(\mathcal{O}_S) + \chi(\mathcal{O}_C) = 0 \\ \chi(-C-D) - \chi(-D) + \chi(-D|_C) = 0. \end{cases}$$

This leads to the following equality

$$\chi(\mathcal{O}_C) - \chi(-D|_C) = \chi(\mathcal{O}_S) - \chi(-C) - \chi(-D) + \chi(-D-C),$$

whose right-hand side is by definition $\xi \cdot \mathcal{O}_S(C)$, while the left-hand side simplifies to $\deg D|_C$, by applying the Riemann-Roch theorem for curves:

$$\chi(-D|_C) = \chi(\mathcal{O}_C) + \deg(-D|_C) = \chi(\mathcal{O}_C) - \deg(D|_C).$$

Altogether this yields the assertion: $\xi \cdot \mathcal{O}_S(C) = \deg(D|_C) = \deg(\xi|_C)$. □

We are now in the position of being able to demonstrate the two most important properties of the intersection form. The first one is algebraic: this is a bi-linear form (good news for computations). The second one is geometric: this abstract non-sense cohomological product does indeed generalize our first geometric construction.

THEOREM 9.5. *The above defined intersection form is a bilinear symmetric form* $\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$. *Moreover, if* $\xi = \mathcal{O}_S(C)$ *and* $\xi' = \mathcal{O}_S(C')$ *with* C *and* C' *distinct irreducible curves, then* $\xi \cdot \xi' = C \cdot C'$.

Proof. We first prove the last assertion, using Remark 9.3. Let $s \in H^0(\xi)$ and $s' \in H^0(\xi')$ be the sections defining C and C'. We claim that

$$0 \longrightarrow \mathcal{O}_S(-C-C') \longrightarrow \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{C\cap C'} \longrightarrow 0$$

is an exact sequence, where we define the maps on the stalks as follows.

$$0 \longrightarrow \mathcal{O}_{S,p} \xrightarrow{\alpha} \mathcal{O}_{S,p} \oplus \mathcal{O}_{S.p} \xrightarrow{\beta} \mathcal{O}_{S,p} \xrightarrow{\gamma} \mathcal{O}_{C\cap C',p} \longrightarrow 0$$

Let $f, g \in \mathcal{O}_{S,p}$ be the germs of s and s' at p. We define

$$\alpha(k) = (-gk, fk), \qquad \beta(a,b) = af + bg,$$

while γ is just the natural projection on the quotient. The facts that α is injective and $\mathrm{Im}(\beta) = \ker(\gamma)$ are obvious. So we only need to show $\mathrm{Im}(\alpha) = \ker(\beta)$. The first inclusion $\subset$ is straightforward, as $\beta\alpha(k) = 0$. Thus, suppose $\beta(a,b) = 0$. It gives $af = -bg$. Note that f and g are relatively prime in $\mathcal{O}_{S,p}$ since C and C' don't have any common irreducible component by hypothesis. Hence f

[2] that is, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ then $\chi(\mathcal{F}) - \chi(\mathcal{G}) + \chi(\mathcal{H}) = 0$.

divides $-b$ and g divides a, that is $b = -hf$ and $a = kg$, for some $h, k \in \mathcal{O}_{S,p}$. Now $0 = af + bg = (k-h)fg$, and $\mathcal{O}_{S,p}$ is a UFD. It follows $k = h$. So $(a, b) = (gk, -fk)$ is clearly the image of $-k$. Finally, we conclude that the above sequence is exact. We can therefore split the above exact sequence into two short exact sequences:

$$0 \longrightarrow \mathcal{O}_S(-C - C') \longrightarrow \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \longrightarrow \mathcal{K} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{C \cap C'} \longrightarrow 0$$

by letting $\mathcal{K} = \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C')$. Since $C \cdot C' = \chi(\mathcal{O}_{C \cap C'})$, we get

$$\chi(-C - C') - \chi(-C) - \chi(-C') + \chi(\mathcal{K}) = 0,$$

$$\chi(\mathcal{K}) - \chi(\mathcal{O}_S) + C \cdot C' = 0.$$

Taking the difference of the two equations gives $C \cdot C' = \xi \cdot \xi'$, as claimed.

We now come to show the bilinearity of the intersection form. Since it is clearly symmetric, it suffices to show that is it linear in the first variable. For any triple $\xi_1, \xi_2, \xi_3 \in \mathrm{Pic}(S)$ define

$$s(\xi_1, \xi_2, \xi_3) = \xi_1 \cdot (\xi_2 + \xi_3) - \xi_1 \cdot \xi_2 - \xi_1 \cdot \xi_3.$$

Assume ξ_1 is represented by a smooth irreducible curve C. Then $s = 0$. In fact, by the above lemma and linearity of the degree,

$$s(\xi_1, \xi_2, \xi_3) = \deg(\xi_2 + \xi_3)_C - \deg(\xi_2)_C - \deg(\xi_3)_C = 0.$$

This works not only for ξ_1, but in each of the three variables, because s is symmetric in the 3 variables (as one can easily see by expanding the expression of $s(\xi_1, \xi_2, \xi_3)$ explicitly, via the definition of the intersection form).

Final step: recall that any line bundle ξ' can be written as $\xi' = \mathcal{O}_S(A - B)$, where A and B are smooth irreducible curves. Then,

$$\begin{aligned} 0 &= s(\xi, \xi', B) \\ &= \xi \cdot (\xi' + B) - \xi \cdot \xi' - \xi \cdot B \\ &= \xi \cdot A - \xi \cdot \xi' - \xi \cdot B, \end{aligned}$$

which, by the Lemma, gives us the key expression

$$\xi \cdot \xi' = \deg(\xi|_A) - \deg(\xi|_B).$$

Using this expression we show linearity in the first variable:

$$\begin{aligned} (m\xi_1 + n\xi_2) \cdot \xi' &= \deg(m\xi_1 + n\xi_2)|_A - \deg(m\xi_1 + n\xi_2)|_B \\ &= m(\deg \xi_1|_A - \deg \xi_1|_B) + n(\deg \xi_2|_A - \deg \xi_2|_B) \\ &= m\xi_1 \cdot \xi' - n\xi_2 \cdot \xi'. \end{aligned}$$

This completes the proof of the Theorem. $\square$

The intersection form of line bundles automatically defines an intersection form between divisors which is invariant under linear equivalence. Thus we will simply write $D_1 \cdot D_2$. An important number associated to a divisor (or to a line bundle), is its *self-intersection* $D^2 = D \cdot D$. The geometric meaning of self-intersection is a bit subtle. Naively, the idea is that if C is a curve on S then C^2 gives us the points of intersections between C and C', where C' is an "infinitesimal" deformation of C within S. This is true, almost: it is not always the case that we can deform C. This fact is recorded as a *negative* (!) self-intersection.

REMARK 9.6. *It can be $D^2 < 0$, even if D is a smooth irreducible curve. For example, take a smooth cubic surface S in $\mathbb{P}^3$ and a line ℓ in S. Pick a plane of $\mathbb{P}^3$ containing ℓ, which cuts on S the divisor $H_0 = \ell + R$, where R is a conic. Now let H be the generic plane section of S, so that $H \cdot \ell = 1$. Since H and H_0 are linearly equivalent we get $\ell^2 = -1$. Indeed, $1 = H \cdot \ell = H_0 \cdot \ell = (\ell + R) \cdot \ell = \ell^2 + R \cdot \ell = \ell^2 + 2$.*

REMARK 9.7. *The self-intersection of a smooth curve $C \subset S$ can be given the following geometric interpretation. Let $\mathcal{N}_{C/S}$ be the normal bundle of C embedded in S. This is by definition the quotient bundle arising from the natural injection $T_C \hookrightarrow (T_S)|_C$ of the tangent bundles (induced by the inclusion $i\colon C \hookrightarrow S$), and thus we have the* normal bundle *sequence*

$$0 \longrightarrow T_C \longrightarrow (T_S)|_C \longrightarrow \mathcal{N}_{C/S} \longrightarrow 0,$$

whose long exact sequence in cohomology gives us a map

$$H^0(\mathcal{N}_{C/S}) \longrightarrow H^1(T_C).$$

In deformation theory, one learns that the first cohomology group of the tangent bundle is in some sense what parametrizes the allowed deformations of the variety. Thus, the above map can be thought of as what parametrizes those deformations of C as an embedded curve in S. If $\xi = \mathcal{O}_S(C) \in \mathrm{Pic}(S)$ is the line bundle on the surface obtained by the curve C, one can show that $\mathcal{N}_{C/S}$ is just the restriction $\xi|_C = \mathcal{O}_C(C)$. Hence,

$$C^2 = \deg(\xi|_C) = \deg(\mathcal{N}_{C/S}).$$

In particular, if C has positive self-intersection, $\deg(\mathcal{N}_{C/S}) > 0$, hence C can be deformed in S, and one says that C is "movable" (as opposed to being "fixed").

EXAMPLE 9.8. We determine the intersection form on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

1. Let $C \in |\mathcal{O}_{\mathbb{P}^2}(n)|$ and $C' \in |\mathcal{O}_{\mathbb{P}^2}(m)|$ be two plane curves of degree n and m respectively, such that they have no common irreducible component. Take a line $\ell \in |\mathcal{O}_{\mathbb{P}^2}(1)|$. Then $C \sim n\ell$ and $C' \sim m\ell$. Since all lines are linearly equivalent in the plane, we have $\ell^2 = 1$, and so we recover the theorem of Bézout's
$$C \cdot C' = nm.$$

2. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathrm{Pic}(S) = \mathbb{Z}[E] \oplus \mathbb{Z}[F]$, where E, F are the classes of the two rulings of S. Let $\pi_i : S \to \mathbb{P}^1$ denote the projection on the i-th

factor. Then $\pi_i^* : \mathrm{Div}(\mathbb{P}^1) \to \mathrm{Div}(S)$ are the pull-backs homomorphisms. Let $F = \pi_1^* a$ for some $a \in \mathbb{P}^1$, and consider any other fibre $F' = \pi_1^* b$. Clearly $F \cdot F' = 0$ if they are distinct fibres. However $F \sim F'$, since any two points a, b in $\mathbb{P}^1$ are linearly equivalent[3]. Thus $F^2 = 0$. Similarly, one finds

$$F^2 = 0, \qquad E^2 = 0, \qquad F \cdot E = 1.$$

Let $D \in \mathrm{Div}(S)$. Then $D \sim mE + nF$. So for any other $D' \sim m'E + n'F$,

$$D \cdot D' = mn' + m'n.$$

§1. TOPOLOGICAL INTERPRETATION OF THE INTERSECTION FORM

The exponential sequence on S induces in cohomology a homomorphism

$$c : \mathrm{Pic}(S) \longrightarrow H^2(S, \mathbb{Z}).$$

Sometimes $c(\xi)$ is called the *first Chern class* of the line bundle ξ. Therefore, this map connects something which depends on the holomorphic complex structure of S (the Picard group) to the topology of the 4-manifold S (singular cohomology). On the other hand, from algebraic topology we know we have a bilinear map called *cup product*,

$$\smile : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \longrightarrow H^4(S, \mathbb{Z}) = \mathbb{Z}.$$

The key point here is that, as it turns out, the intersection form we have defined on $\mathrm{Pic}(S)$ is really just a manifestation (a restriction) of the structure and properties of the topological 2-cycles on the manifold S. Precisely:

FACT. *The intersection form is induced on* $\mathrm{Pic}(S)$ *by the cup product:*

$$\xi \cdot \xi' = c(\xi) \smile c(\xi').$$

§2. NÉRON-SEVERI GROUP AND NUMERICAL EQUIVALENCE

Let's come back to the long cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(S, \mathbb{Z}) \arrow[r] & H^1(S, \mathcal{O}_S) \arrow[r] & \mathrm{Pic}(S) \arrow[dll, "c"] & \\ & H^2(S, \mathbb{Z}) \arrow[r] & H^2(S, \mathcal{O}_S) \arrow[r] & H^2(S, \mathcal{O}_S^*) \arrow[r] & \cdots \end{tikzcd}$$

induced by the exponential sequence. We set

$$\mathrm{Pic}^0(S) = \frac{H^1(S, \mathcal{O}_S)}{H^1(S, \mathbb{Z})}.$$

[3]explicitly, $a - b = (f)$ where f is the meromorphic function $f(z) = \frac{z-a}{z-b}$.

Note that $\mathrm{Pic}^0(S)$ is a complex torus, since $H^1(S,\mathbb{Z})$ is a rank $2q$ lattice in $H^1(S,\mathcal{O}_S) \simeq \mathbb{C}^q$. Exactness of the sequence shows that $\mathrm{Pic}^0(S) = \ker(c)$, so

$$\mathrm{Pic}^0(S) = \{\xi \in \mathrm{Pic}(S) : \; c(\xi) = 0\}.$$

In particular, any $\xi \in \mathrm{Pic}^0(S)$ is *numerically trivial*: for any $\eta \in \mathrm{Pic}(S)$,

$$\xi \cdot \eta = c(\xi) \smile c(\eta) = 0 \smile c(\eta) = 0.$$

Another important group is the *Néron-Severi group*

$$\mathrm{NS}(S) = \ker\left(H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O}_S)\right).$$

By exactness, $\mathrm{NS}(S) = \mathrm{Im}(c)$. We therefore think of its elements as the cohomology classes of line bundles on S, i.e. the "algebraic cycles" in $H^2(S,\mathbb{Z})$,

$$\begin{aligned}\mathrm{NS}(S) &= \{c(\xi) : \; \xi \in \mathrm{Pic}(S)\} \\ &= \{\text{algebraic cycles}\} \subset H^2(S,\mathbb{Z}).\end{aligned}$$

Note that $\mathrm{NS}(S)$ is finitely generated, being a subgroup of $H^2(S,\mathbb{Z})$ (and S is compact!). However, *it might have torsion*. Its rank $\rho = \rho(S)$ is called the *Picard number* of S. Note that, since $\mathrm{Pic}^0(S) = \ker(c)$ and $\mathrm{NS}(S) = \mathrm{Im}(c)$, we also have

$$\mathrm{NS}(S) = \frac{\mathrm{Pic}(S)}{\mathrm{Pic}^0(S)}.$$

Therefore, the Néron-Severi group fits in the exact sequence

$$0 \to \mathrm{Pic}^0(S) \to \mathrm{Pic}(S) \to \mathrm{NS}(S) \to 0$$

This sequence is the analogue on surfaces of the following short exact sequence for curves, which we have already met,

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}(C) \to \mathbb{Z} \to 0$$

Definition. Two line bundles $\xi, \xi' \in \mathrm{Pic}(S)$ are *numerically equivalent* if they have the same intersection numbers with all other line bundles, i.e. for any $\eta \in \mathrm{Pic}(S)$ one has $\xi \cdot \eta = \xi' \cdot \eta$. We write $\xi \equiv \xi'$. The same definition holds with "divisors" in place of "line bundles".

This is an equivalence relation and it is compatible with the intersection form. In other words, numerically trivial line bundles form a subgroup of $\mathrm{Pic}(S)$ and we can define the *group of numerical equivalence classes*

$$\mathrm{Num}(S) = \mathrm{Pic}(S)/\equiv.$$

At the level of divisors, note that linear equivalence implies numerical equivalence, i.e. if $D, D' \in \mathrm{Div}(S)$ and $D \sim D'$, then $D \equiv D'$. Thus,

$$\mathrm{Num}(S) = \frac{\mathrm{Pic}(S)}{\equiv} = \frac{\mathrm{Div}(S)/\sim}{\equiv} = \frac{\mathrm{Div}(S)}{\equiv}.$$

Let us spend a couple of words on the structure of this group.

From $\mathrm{Pic}(S)$ we have two surjective homomorphisms c and π,

$$\begin{tikzcd} \mathrm{Pic}(S) \arrow[rr, "\pi"] \arrow[dr, "c"'] & & \mathrm{Num}(S) \\ & \mathrm{NS}(S) \arrow[ur, dashed, "\pi'"'] & \end{tikzcd}$$

As $\pi(\mathrm{Pic}^0(S)) = 0$ and $\mathrm{NS} = \mathrm{Pic} / \mathrm{Pic}^0$, we have that π factors through a surjective morphism π'. It follows that $\mathrm{Num}(S) = \pi'(\mathrm{NS}(S))$, is also finitely generated. A theorem by Severi guarantees that the rank ρ is preserved. Note that Num is torsion free: if $\xi \in \mathrm{Num}(S)$ is such that $n\xi = 0$ for some $n > 0$ then $0 = n\xi \cdot \eta = n(\xi \cdot \eta)$ for all $\eta \in \mathrm{Num}(S)$, hence $\xi \cdot \eta = 0$ for all η, that is $\xi = 0$ in $\mathrm{Num}(S)$. Hence Num is a finitely generated free abelian group,

$$\mathrm{Num}(S) \simeq \mathbb{Z}^\rho.$$

Note that $\rho \geq 1$, since $\mathrm{Num}(S)$ must contain the class of an ample divisor[4].

§3. RIEMANN-ROCH THEOREM AND NOETHER'S FORMULA

Recall that $\chi(\mathcal{O}_S) = 1 - q + p_g$ and $\chi(\xi) = h^0(\xi) - h^1(\xi) + h^2(\xi)$. If K_S denotes a canonical class, then Serre duality reads

$$\chi(\mathcal{O}_S) = \chi(K_S), \quad \chi(\xi) = \chi(K_S - \xi).$$

THEOREM 9.9 (Riemann-Roch for surfaces). *For any $\xi \in \mathrm{Pic}(S)$,*

$$\chi(\xi) = \chi(\mathcal{O}_S) + \tfrac{1}{2}\xi \cdot (\xi - K_S).$$

Proof. Apply the definition of intersection product to $-\xi$ and $\xi - K_S$,

$$\begin{aligned} -\xi \cdot (\xi - K_S) &= \chi(\mathcal{O}_S) - \chi(\xi) - \chi(K_S - \xi) + \chi(K_S) \\ &= \chi(\mathcal{O}_S) - \chi(\xi) - \chi(\xi) + \chi(\mathcal{O}_S) \\ &= 2(\chi(\mathcal{O}_S) - \chi(\xi)), \end{aligned}$$

where the second equality follows from Serre duality. □

Recall that the Euler characteristic is given by $e(S) = 2 - 4q + b_2(S)$. We have *Noether's formula*,

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + e(S)).$$

For a proof – which requires a considerable amount of work – we refer to [2].

[4] recall that we always assume S *projective*

§4. Genus formula and arithmetic genus

Theorem 9.10 (Genus formula). *For a smooth irreducible curve* $C \subset S$,

$$2g(C) - 2 = C^2 + C \cdot K_S.$$

Proof. $\chi(\mathcal{O}_C) = 1 - g(C)$. From the short exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

one gets $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(-C)$. The result follows by expanding $\chi(-C)$ via the Riemann-Roch theorem. □

This formula can also be obtained by the adjunction formula. In fact, we have $2g(C) - 2 = \deg(K_C) = \deg(K_S + C)|_C = (K_S + C) \cdot C$, by Lemma 9.4.

Example 9.11. The genus formula generalizes some more familiar ones.

1. For $S = \mathbb{P}^2$, this is *Clebsch formula*. Indeed, if C is a curve of degree d, then $C \sim d\ell$, where ℓ is a line. Thus $C^2 = d^2\ell^2 = d^2$. On the other hand, $K_{\mathbb{P}^2} = -3\ell$, thus $C \cdot K_{\mathbb{P}^2} = -3n$. The genus formula then yields

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

2. For $S = \mathbb{P}^1 \times \mathbb{P}^1$, this is *Segre's formula*: if $C \sim aE + bF$, with $a, b \in \mathbb{Z}$, then $C^2 = 2ab$ and $K_S \sim -2E - 2F$, so $C \cdot K_S = -2a - 2b$, and we find

$$g(C) = (a-1)(b-1).$$

So far so good, at least for *smooth* curves on S. What about *singular* ones? Riemann-Roch shows that $C^2 + C \cdot K_S = 2(\chi(-C) - \chi(\mathcal{O}_S))$ is even.

Definition. The *arithmetic genus* of a curve C is the integer $p_a(C)$ satisfying

$$2p_a(C) - 2 = C^2 + C \cdot K_S.$$

If C is smooth, then $p_a(C) = g(C)$ by the genus formula. In particular, $p_a(C)$ is the genus of the smooth curves (if any) in $|C|$. At least, it is clear that the arithmetic genus is invariant under linear equivalence.

When one wants to study singular curves, the basic tool is the construction of the *normalization* of C. This is a *smooth* curve $\hat{C}$, together with a morphism

$$\nu \colon \hat{C} \to C$$

such that ν is an *isomorphism* outside $\mathrm{Sing}(C)$. See [2], for more details.

Example 9.12. In $\mathbb{P}^2$, consider the nodal cubic

$$C\colon\ x_2^2 - x_1^2(x_0 + x_1) = 0$$

Let $\nu \colon \mathbb{P}^1 \to \mathbb{P}^2$ be the morphism given by $t \to (t^2 - 1, t(t^2 - 1))$ in affine coordinates. By a simple computation, $\nu \colon \mathbb{P}^1 \to C$ is the normalization of C.

In the example above, we notice that $g(\hat{C}) = 0$, while $p_a(C) = 1$ (since C belongs to the complete linear system of plane cubics, for which a general member is smooth of genus 1). In general, the theory shows that

$$p_a(C) = g(\hat{C}) + \sum_{x \in C} \delta(x),$$

where the integer $\delta(x) \geq 0$ depends on the *type* of singularity of C at x. For example, one has $\delta(x) = 0$ if x is smooth, $\delta(x) = 1$ is x is a node, and so on. Therefore, $g(\hat{C}) = p_a(C)$ if and only if C is smooth, and we always have

$$0 \leq g(\hat{C}) \leq p_a(C).$$

In particular, if C is an *irreducible* curve such that $p_a(C) = 0$, then C must be smooth and rational, hence $C \simeq \mathbb{P}^1$.

This whole machinery works as long as C is singular but still *irreducible*. What about the arithmetic genus of a reducible curve? Let us generalize even further this number for any divisor D on S.

Definition. Let D be any divisor on S. We define the *virtual arithmetic genus* of D to be the integer $p_a(D)$ satisfying

$$2p_a(D) - 2 = D^2 + D \cdot K_S.$$

Warning: $p_a(D)$ can be *negative* even when D is *effective*!

EXERCISE. Let S be a surface and $D \in \mathrm{Div}(S)$.

(i) Find an example of D effective with $p_a(D) < 0$.

(ii) Write $D = A + B$. Find a formula for $p_a(D)$ in terms of A and B.

(iii) Classify all divisors D such that $p_a(D) < 0$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

§5. HODGE INDEX THEOREM

In this section we discuss a fundamental result in the theory of surfaces: the index theorem. The question is simple. We have a quadratic form $\mathrm{Pic}(S) \to \mathbb{Z}$ given by self-intersection. Since $\mathrm{Pic}(S)$ has finite rank ρ, its free part is just a $\mathbb{Z}^\rho$, up to isomorphism. Up to the choice of some basis, the quadratic form is represented by a ρ by ρ matrix with integer coefficients. What is the *signature* (index) of this matrix? This question will be given a complete answer.

We begin by some basic, yet important, remark.

LEMMA 9.13. *Let* H *be an ample divisor. Then, for any non-trivial effective divisor* D*, we have* $H \cdot D > 0$*. In particular,* $H \cdot L > 0$ *for any other ample divisor* L.

Proof. Since D is an effective sum of curves, it suffices to show that $H \cdot C > 0$ for any irreducible curve C in S. This is true because for n large enough nH is very ample, so that

$$nH \cdot C = \deg(nH_C) = \deg \varphi(C) > 0,$$

where $\varphi : S \hookrightarrow \mathbb{P}^N$ is the embedding associated to nH. This shows that $H \cdot D > 0$. Of course, $|nH|$ itself contains effective curves (hyperplane sections in the above embedding). Hence, by the same argument, if L is another ample divisor, also $nH \cdot L > 0$, which proves the assertion. Note, in particular, $H^2 > 0$. $\square$

An obvious consequence of the Lemma is the following observation.

REMARK 9.14. *If D is a non-trivial divisor such that $D \cdot H \leq 0$ for some ample divisor H, then $h^0(D) = 0$. Indeed, if $h^0(D) > 0$, then D would be linearly equivalent to a non-trivial effective divisor, whence $H \cdot D > 0$, a contradiction.*

Now we are able to show that all divisors D with positive self-intersection must intersect all ample divisors H. Here, *intersect* means $D \cdot H > 0$, *up to replacing* D with $-D$ (since both will have positive square). Precisely:

PROPOSITION 9.15. *Let D be a divisor on S with $D^2 > 0$. Then, for any ample divisor H on S, it is $H \cdot D \neq 0$.*

The idea of the proof is simple: by taking $n >> 0$ big enough we will find that either $|nD|$ or $|-nD|$ will contain some curve. Hence, either $nD \cdot H > 0$, or $-nD \cdot H > 0$, by the Lemma above. In any case, we have $D \cdot H \neq 0$.

Proof. Let n be a positive integer. By Riemann-Roch

$$h^0(nD) + h^0(-nD + K_S) \geq \chi(\mathcal{O}_S) + \frac{1}{2}(n^2D^2 - nD \cdot K_S).$$

Since $D^2 > 0$, the right-hand side tends to infinity as n grows bigger. Hence, one of the two members of the left-hand side must be positive, for n big enough.

If $h^0(nD) > 0$, then $|nD| \neq \varnothing$, hence $H \cdot nD > 0$. The assertion of the Theorem follows. Therefore, we assume $h^0(K_S - nD) > 0$. We want to show that in this case $|-nD|$ contains some curve. Repeat the same game: apply Riemann-Roch to $-nD$ and find $h^0(-nD) + h^0(nD + K_S) \to +\infty$ for $n \to +\infty$. Hence, it suffices to show that $h^0(nD + K_S)$ does not diverge to infinity.

There exists a non-zero section $s \in H^0(-nD + K_S)$. Multiplication by s yields an injection

$$H^0(nD + K_S) \hookrightarrow H^0(K_S + nD + K_S - nD) = H^0(2K_S).$$

In particular, $h^0(nD+K_S) \leq h^0(2K_S)$ is bounded. Therefore, for n large enough we have $h^0(-nD) > 0$, that is $|-nD| \neq \varnothing$. $\square$

COROLLARY 9.16. *If D is a divisor such that $D \cdot H = 0$ for some ample divisor, then*

$$D^2 \leq 0.$$

The question now is the following: if D is such that $D \cdot H = 0$, in which case we have $D^2 = 0$? The answer is given by the Hodge index theorem: this holds precisely when D is numerically trivial, i.e. it is zero in the group of numerical equivalence classes

$$\mathrm{Num}(S) = \frac{\mathrm{Pic}(S)}{\equiv} = \mathbb{Z}^{\rho}.$$

In order to state the precise result, let us introduce some notation. The quadratic form on $\mathrm{Pic}(S)$ induces a quadratic form

$$q : \mathrm{Num}(S) \to \mathbb{Z}$$

which is non-degenerate, by definition of numerical equivalence. This q linearly extends to a non-degenerate quadratic form on the real vector space

$$\mathrm{Num}(S)_{\mathbb{R}} = \mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{\rho}.$$

Let H be an ample divisor on S. We denote by h its class in $\mathrm{Num}(S)_{\mathbb{R}}$. We have $q(h) = H^2 > 0$, since H is ample. Hence, q is positive definite on the subspace of $\mathrm{Num}(S)_{\mathbb{R}}$ spanned by h. Consider the orthogonal decomposition of $\mathrm{Num}(S)_{\mathbb{R}}$ determined by this subspace,

$$\mathrm{Num}(S)_{\mathbb{R}} = \mathbb{R}h \oplus h^{\perp}$$

THEOREM 9.17 (Hodge index). *q is negative definite on the hyperplane $h^{\perp}$.*

Since $h^{\perp} = \{x \in \mathrm{Num}(S)_{\mathbb{R}} \mid x \cdot h = 0\}$ what we need to show is that if $x \in h^{\perp}$ then $q(x) = x^2 < 0$ or $x = 0$. Rephrasing this in terms of divisors, we are required to show the following:

if $D \in \mathrm{Div}(S)$ is such that $D \cdot H = 0$, then either $D^2 < 0$, or $D \equiv 0$.

Proof. If $D \cdot H = 0$, by Corollary 9.16 we have $D^2 \leq 0$. We therefore assume, by contradiction, that $D^2 = 0$ and D is not numerically trivial. Then $D \cdot C \neq 0$ for some irreducible curve C. For any $m \in \mathbb{Z}$ define

$$\Gamma_m = mD + B, \quad B = (H^2)C - (C \cdot H)H.$$

Since $H^2 \neq 0$ also $D \cdot B \neq 0$. Therefore $(\Gamma_m)^2 = 2mD \cdot B + B^2$ is a polynomial of degree 1 in m, and so there is some $m \in \mathbb{Z}$ for which it is positive. However, we easily compute $\Gamma_m \cdot H = 0$. But this contradicts Corollary 9.16. $\square$

Let us re-elaborate the statement: the intersection form q on $\mathrm{Num}(S)_{\mathbb{R}}$ is such that q is positive on all lines $\mathbb{R}h$ (where h is any "ample class") and negative on the hyperplane $h^{\perp} \simeq \mathbb{R}^{\rho-1}$. In other words, the Hodge index theorem says that the signature of q is

$$(1, \rho - 1).$$

LEMMA 9.18. *Let q be a quadratic form on $\mathbb{R}^m$ with signature $(1, m-1)$. Suppose $\pi \subset \mathbb{R}^m$ is a plane containing one positive direction, i.e. there is $x \in \pi$ with $q(x) > 0$. Then, the signature of $q|_{\pi}$, the restriction of q to the plane π, is $(1, 1)$.*

Proof. Since π contains a positive direction, the signature of $q|_{\pi}$ is either $(1, 1)$ or $(1, 0)$. Thus, we only need to show that $q|_{\pi}$ is still non-degenerate. But this is clear: since $\dim(\pi) = 2$ then π must intersect any hyperplane $\Pi \simeq \mathbb{R}^{m-1}$ where q is negative-definite. Hence, π also contains a negative direction. $\square$

We now come to another reformulation of the Hodge index theorem, which is in fact the most useful one for all practical purposes.

COROLLARY 9.19. *Let* $L, D \in \mathrm{Div}(S)$ *with* $L^2 > 0$. *Then*

(i) $(L \cdot D)^2 \geq L^2 \cdot D^2$.

(ii) Equality in (i) holds if and only if L *and* D *are linearly dependent in* $\mathrm{Num}(S)_{\mathbb{Q}}$, *i.e.* $D \equiv qL$, *for some* $q \in \mathbb{Q}$ *(that is,* $rL \equiv sD$, *for some* $r, s \in \mathbb{Z}$*).*

Proof. If $D \equiv 0$ there is nothing to prove. Hence, we assume that the class d of D in $\mathrm{Num}(S)_{\mathbb{R}}$ is non-zero. Consider the subspace $\pi = \langle l, d \rangle_{\mathbb{R}}$ in $\mathrm{Num}(S)_{\mathbb{R}}$, spanned by the classes of L and D. Since $q(l) = L^2 > 0$ by assumption, π contains a positive direction.

Let $x \in \pi$. If we write $x = x_1 l + x_2 d$, we have

$$q(x) = (x_1 l + x_2 d)^2 = (L^2)x_1^2 + 2(L \cdot D)x_1 x_2 + (D^2)x_2^2$$

In other words, the matrix of the quadratic form $q|_{\pi}$ is

$$A = \begin{pmatrix} L^2 & L \cdot D \\ L \cdot D & D^2 \end{pmatrix}$$

Now, either $\dim(\pi) = 2$ or $\dim(\pi) = 1$. If $\dim(\pi) = 2$, then the signature of $q|_{\pi}$ is $(1, 1)$ by Hodge index; whence $\det A = L^2 D^2 - (L \cdot D)^2 < 0$. Obviously, $\dim(\pi) = 1$ if and only if l and d are $\mathbb{R}$-linearly dependent (in which case $\det A = 0$). Since l and d are *integral* classes (in $\mathrm{Num}(S) = \mathbb{Z}^{\rho} \subset \mathbb{R}^{\rho}$), linear dependence over $\mathbb{R}$ means linear dependence over $\mathbb{Q}$. Note that l and d are $\mathbb{Q}$-linearly dependent if and only if $rL \equiv sD$, for some $r, s \in \mathbb{Z}$. $\square$

This gives a strengthening of the Hodge index theorem: by the only assumption that $L^2 > 0$ (and not necessarily L ample!), the quadratic form q must be negative definite on the orthogonal hyperplane $(L)^{\perp}$ in $\mathrm{Num}(S)_{\mathbb{R}}$.

COROLLARY 9.20 (Strong Hodge index). *Let* L *be a divisor on* S *with* L^2. *If* D *is a divisor such that* $D \cdot L = 0$, *then either* $D^2 < 0$, *or* $D \equiv 0$.

Proof. We have $D^2 \leq 0$. Assume by contradiction $D^2 = 0$ with D non-zero in $\mathrm{Num}(S)$. By Corollary 9.19 we have $rL \equiv sD$ for some $r, s \in \mathbb{Z}$. Then $0 = s^2 D^2 = r^2 L^2$, hence $r = 0$. Therefore, $sD = 0$ in $\mathrm{Num}(S)$, that is $sD \equiv 0$. Since $\mathrm{Num}(S)$ has no torsion, we get $D \equiv 0$. $\square$

LECTURE 10

CONES

§1. AMPLE CONE AND NEF CONE

Let S be a surface. If S has Picard number $\rho = 1$ then $\mathrm{Num}(S)$ is generated by a single class, that one of some ample line bundle H. This situation (which occurs for example for $S = \mathbb{P}^2$) is then quite clear, from an intersection-theoretic point of view. We therefore assume $\rho(S) \geq 2$.

We have the quadratic form $q : \mathrm{Num}(S)_{\mathbb{R}} \to \mathbb{R}$, $q(x) = x^2$. By standard linear algebra, there is a basis of $\mathrm{Num}(S)_{\mathbb{R}}$ for which q is expressed in canonical (Sylvester) form. Since $\mathrm{sgn}(q) = (1, \rho - 1)$, the matrix of q in such a basis is

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

If $x \in \mathrm{Num}(S)_{\mathbb{R}}$ has coordinates $x_1, \dots, x_\rho$ with respect to this basis, then

$$q(x) = x_1^2 - x_2^2 - \cdots - x_\rho^2.$$

We define the set

$$\Omega = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \ q(x) > 0\}$$

which is the interior of the quadric cone $\{q = 0\}$. Notice that Ω has two connected components: the one with $x_1 > 0$ and that with $x_1 < 0$.

Definition. The connected component of Ω which contains an ample class is called the *positive cone*, and denoted by $\mathcal{C}$ or $\mathcal{C}_S$.

Of course, in order for this definition to make sense we have to show that $\mathcal{C}$ contains indeed all ample classes, which is a matter of convexity:

Proof. Suppose L, H are two ample classes in $\mathrm{Num}(S)$. Since $L^2 > 0$ and $H^2 > 0$, then $L, H \in \Omega$. Consider a class α_λ on the segment connecting the classes H and L in $\mathrm{Num}(S)_{\mathbb{R}}$, i.e. $\alpha_\lambda = \lambda H + (1-\lambda)L$, with $0 \leq \lambda \leq 1$. Note that

$$q(\alpha_\lambda) = \lambda^2 H^2 + 2\lambda(1-\lambda) H \cdot L + (1-\lambda)^2 L^2$$

must be positive, since H^2, L^2 and $H \cdot L$ are (Lemma 9.13), and because the coefficients above are all non-negative (but never simultaneously zero). Therefore, q is positive on the whole segment, which means that L and H must lie in the same connected component of Ω. □

We have the following amplitude criterion, which we state without proof:

THEOREM 10.1 (Nakai-Moishezon[1]). *A divisor D in S is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C.*

We have already observed how the two conditions on D in the Theorem are necessary for D to be ample. The remarkable fact is that they are indeed sufficient! Whence, ampleness is a well defined property in $\mathrm{Num}(S)$. In other words, each representative of an ample class is itself an ample divisor:

If H is ample and $H' \equiv H$, then H' is ample.

We derive a straightforward consequence:

COROLLARY 10.2. *The sum of two ample divisors is ample.*

Proof. Apply the criterion of Nakai-Moishezon and Lemma 9.13. □

Where do we go from here? We have a powerful numerical criterion for amplitude of divisors. Let us look at the two conditions in Nakai-Moishezon's theorem. What kind of geometrical data do they impose on the Euclidean geometry of $\mathrm{Num}(S)_{\mathbb{R}}$? The first condition $D^2 > 0$ just says $D \in \mathcal{C}_S$.

REMARK 10.3. *The set $D^{\perp} = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \ x \cdot D = 0\}$ is a hyperplane of $\mathrm{Num}(S)_{\mathbb{R}}$ through the origin. Hence $D_{\geq 0} = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \ x \cdot D \geq 0\}$ is a closed half-space of $\mathrm{Num}(S)_{\mathbb{R}}$. This is clear in the canonical basis of $\mathrm{Num}(S)_{\mathbb{R}}$, where, if D and x have coordinates m_i and x_i respectively, then $x \cdot D = m_1 x_1 - m_2 x_2 - \cdots - m_\rho x_\rho$. In particular, taking the intersection of several spaces like $D_{\geq 0}$ yields a* cone.

Definition. We define the *ample cone* $\mathrm{Amp}(S)$ as the convex cone in $\mathrm{Num}(S)_{\mathbb{R}}$ generated by the ample classes. In other words

$$\mathrm{Amp}(S) = \{\sum \lambda_i H_i : \lambda_i > 0, \ H_i \ \text{ ample}\}$$

(finite sums). Note $\mathrm{Amp}(S) \subset \mathcal{C}$. It is possible to show that $\mathrm{Amp}(S)$ is open in $\mathrm{Num}(S)_{\mathbb{R}}$. We define the *nef cone* of S as

$$\mathrm{Nef}(S) = \{x \in \overline{\mathcal{C}} : \ x \cdot C \geq 0 \text{ for all irreducible curves } C \subset S\}.$$

Divisors D for which $D \cdot C \geq 0$ for all irreducible curves are called *nef divisors*.

REMARK 10.4. *The Ample cone is a cone, by definition. It is less obvious that the nef cone is a cone as well. If D is a nef divisor then $D^2 \geq 0$ (see below). Therefore the classes of nef divisors lie in the nef cone (they are precisely the points of $\mathrm{Nef}(S)$ with coordinates in $\mathbb{Z}$). However, the nef cone is not generated by nef classes in general (counterexamples by Mumford-Ramanujam).*

[1] Yoshikazu Nakai and Boris Gershevich Moishezon

EXERCISE. Draw the nef and ample cones for the quadric $S = \mathbb{P}^1 \times \mathbb{P}^1$.

Our next goal is to show the following facts:

$$\mathrm{Amp}(S) = \{x \in \mathcal{C} : x \cdot C > 0 \text{ for all irreducible curves } C\} = \mathrm{Int}\,\mathrm{Nef}(S)$$
$$\overline{\mathrm{Amp}(S)} = \{x \in \overline{\mathcal{C}} : x \cdot C \geq 0 \text{ for all irreducible curves } C\} = \mathrm{Nef}(S)$$

(where Int stands for *topological interior* and the overline bar *topological closure*). In particular, $\mathrm{Amp}(S)$ is open and $\mathrm{Nef}(S)$ is closed. The second identity is the main point: the "nef condition" is the *limit condition* which defines amplitude. To put it more explicitly, amplitude fails right at the boundary of $\mathrm{Nef}(S)$, i.e.

COROLLARY 10.5. *For any class $x \in \partial\,\mathrm{Nef}(S)$, either $x^2 = 0$ or there exists an irreducible curve C on S such that $x \cdot C = 0$.*

Proof. If $x \in \partial\,\mathrm{Nef}(S)$ then clearly $x^2 \geq 0$. Suppose $x^2 > 0$. Then, if $x \cdot C > 0$ for all irreducible curves we have $x \in \mathrm{Amp}(S) = \mathrm{Int}\,\mathrm{Nef}(S)$, contradicting the assumption $x \in \partial\,\mathrm{Nef}(S)$. Thus $x \cdot C = 0$ for some C. $\square$

Let us proceed and prove the unproven.

(i) $\mathrm{Amp}(S)$ is open.

Proof. We prove that any point $h \in \mathrm{Amp}(S)$ belongs to the interior, of $\mathrm{Amp}(S)$, that is, we construct an open neighborhood of h contained in $\mathrm{Amp}(S)$. Fist, let us do this when h is just the class of an ample divisor $H \in \mathrm{Div}(S)$. Choose divisors $D_1, \cdots, D_\rho$ whose classes give a basis of $\mathrm{Num}(S)$. By the properties of amplitude (cf. Proposition 6.30) we can choose some integers $m_1, \dots, m_\rho$ such that both $D_i + m_i H$ and $-D_i + m_i H$ are very ample, for each i. Dividing out by m_i we therefore have $H \pm \frac{1}{m_i} D_i \in \mathrm{Amp}(S)$. We take these divisors as vertexes of a polyhedron. By convexity, this polyhedron is entirely inside $\mathrm{Amp}(S)$. This shows that H lies in the interior of $\mathrm{Amp}(S)$, as claimed.

On the other hand, if we consider multiples $h = \lambda H$, with λ it is clear that the same argument still works (up to a homothety of factor $1/\lambda$). Same if we consider $h = \lambda H + \mu H'$, with $\lambda, \mu > 0$. This shows that any point $h \in \mathrm{Amp}(S)$, which we can write as $h = \sum \lambda_i H_i$ with $\lambda_i > 0$ and H_i ample, lies in the interior of $\mathrm{Amp}(S)$, and therefore $\mathrm{Amp}(S)$ is open in $\mathrm{Num}(S)_{\mathbb{R}}$. $\square$

(ii) $\overline{\mathrm{Amp}(S)} = \mathrm{Nef}(S)$.

Proof. We obviously have $\mathrm{Amp}(S) \subset \mathrm{Nef}(S)$. Next, notice that $\mathrm{Nef}(S)$ is closed. Indeed, by definition it is the intersection of the closure of the positive cone $\overline{\mathcal{C}}$ with the intersection of all the (closed) half-spaces $C_{\geq}0 = \{x \cdot C \geq 0\}$, for all irreducible curves $C \subset S$. Therefore $\overline{\mathrm{Amp}(S)} \subset \mathrm{Nef}(S)$ and we need to show the other inclusion. We want to prove that each little guy $h \in \mathrm{Nef}(S)$ is the limit of a sequence in $\mathrm{Amp}(S)$. We may choose a basis $h_1, \dots, h_\rho$ of $\mathrm{Num}(S)_{\mathbb{R}}$ such that each h_i is the class of an ample divisor (use that $\mathrm{Amp}(S)$ is open). For each $n \geq 1$ we construct a little box containing h,

$$B_n = \{h + \sum_{i=1}^{\rho} t_i h_i : 0 < t_i < \frac{1}{n}.\}$$

We pick any point $\tilde{h}_n \in B_n$ having rational coordinates (it exists by density of $\mathbb{Q}$ in $\mathbb{R}$). Hence some integer multiple $m\tilde{h}_n$ is the class of a divisor D. Then,

$$D^2 = m^2(\tilde{h}_n)^2 = m^2(h^2 + 2\sum t_i h_i \cdot h + \sum t_i t_j h_i \cdot h_j) > 0$$

since $h^2 \geq 0$ and $h \cdot h_i \geq 0$ (because h is nef) and $h_i \cdot h_j > 0$ (ample). Also, for any irreducible curve $C \subset S$,

$$D \cdot C = m\tilde{h}_n \cdot C = m(h \cdot C + \sum t_i h_i \cdot C) > 0$$

since $h \cdot C \geq 0$ (nef) and $h_i \cdot C > 0$ (ample). By the criterion of Nakai-Moishezon, it follows that D is ample, so that

$$\tilde{h}_n = \frac{1}{m} D \in \mathrm{Amp}(S).$$

Finally, it is clear that $\tilde{h}_n$ converges to h as $n \to \infty$. □

§2. NEF DIVISORS

We have defined *nef divisors* by the condition $D \cdot C \geq 0$ for any irreducible curve C on S. Equivalently $D \cdot E \geq 0$ for any effective divisor E. Here is the crucial property of nef divisors:

THEOREM 10.6 (Kleiman). *If* D *is nef, then* $D^2 \geq 0$.

Proof. Pick an ample divisor H and define, for $t \in \mathbb{R}$, the function

$$p(t) = (D + tH)^2 = D^2 + 2tD \cdot H + t^2 H^2.$$

Since $H^2 > 0$, we have $p(0) = D^2$ and $p(t) > 0$ when $t > 0$ is big enough. Hence we only need to find a sequence $\{t_n\}$, converging to zero for $n \to \infty$ and such that $p(t_n) > 0$. The simplest choice will do the job:

Let $t_0 \in \mathbb{Q}$ be such that $p(t_0) > 0$. Then $p(t_0/2) > 0$.

Let us prove this claim. Up to clearing the denominator of t_0 we can assume it is an integer, hence $D + t_0 H$ is a divisor. Then $(D + t_0 H)^2 > 0$ implies that for $n > 0$ big enough, either $n(D + t_0 H)$ or its opposite is effective. Indeed, the former is true, because $H \cdot n(D + t_0 H) > 0$. Since D is nef, we must then have $D \cdot n(D + t_0 H) \geq 0$. This easily gives $p(t_0/2) = D^2 + t_0 D \cdot H + \frac{t_0^2}{4} H^2 > 0$. □

By Nakai-Moishezon criterion and Kleiman's theorem we may compare the numerical conditions which define amplitude and nefness of a divisor D.

AMPLITUDE	NEFNESS
$D \cdot C > 0$, and	$D \cdot C \geq 0$, hence
$D^2 > 0$	$D^2 \geq 0$

where "for any smooth irreducible curve C" is understood. It might be worth to point out that the "strictly nef" condition $D \cdot C > 0$ alone does not imply $D^2 > 0$. There exist in fact some (rare) examples of strictly nef divisors with $D^2 = 0$, thus non ample.

§3. The cone of curves

We define the *cone of curves* $\mathrm{Eff}(S)$, or *effective cone*, as the cone in $\mathrm{Num}(S)_{\mathbb{R}}$ generated by the (classes of) curves of S (or, equivalently, by the classes of the effective divisors). Concretely, its elements are finite linear combinations

$$x = \sum \lambda_i C_i$$

where C_i are irreducible curves on S and $\lambda_i \in \mathbb{R}_{>0}$. We have that the effective cone *contains* the positive cone,

$$\mathcal{C}_S \subset \mathrm{Eff}(S).$$

Proof. Since they are cones, it suffices to check on the generators. Let $D \in \mathcal{C}_S$. Then $D^2 > 0$, thus either nD or $-nD$ is effective, for n sufficiently large. Pick an ample class $H \in \mathcal{C}_S$ and consider the hyperplane $H^\perp$ in $\mathrm{Num}(S)$. By Hodge index theorem q is negative definite on $H^\perp$, thus $\mathcal{C}_S$ and $H^\perp$ do not intersect, i.e. $\mathcal{C}_S$ is contained in the half-space $H_{>0}$. Since clearly $H \in H_{>0}$ we get $D \cdot H > 0$, hence nD is effective and, therefore, D too. ☐

Definition. The closure $\overline{\mathrm{Eff}(S)}$ in $\mathrm{Num}(S)_{\mathbb{R}}$ is called the *Kleiman-Mori cone*.[2]

We have the following situation:

$$\begin{array}{ccccc} \mathrm{Amp}(S) & \subset & \mathcal{C}_S & \subset & \mathrm{Eff}(S) \\ \cap & & \cap & & \cap \\ \overline{\mathrm{Amp}(S)} & \subset & \overline{\mathcal{C}}_S & \subset & \overline{\mathrm{Eff}(S)} \end{array}$$

On the other hand, we have the following simple

REMARK 10.7. *Recall that the closure $\overline{\Omega} = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : x^2 \geq 0\}$ is the union of two components $\overline{\mathcal{C}} \cup (-\overline{\mathcal{C}})$ intersecting at zero, where $\mathcal{C}$ is the positive cone. The effective classes in $\overline{\Omega}$ are all contained in $\overline{\mathcal{C}}$. In other words,*

$$\overline{\Omega} \cap \mathrm{Eff}(S) \subset \overline{\mathcal{C}}.$$

Indeed, let $x \in \overline{\Omega} \cap \mathrm{Eff}(S)$ and assume by contradiction $x \in (-\overline{\mathcal{C}})$ (and x non-zero). For any ample class $h \in \mathcal{C}$, since x is effective, $x \cdot h > 0$. Since $x^2 \geq 0$, we have $q(\lambda x + (1-\lambda)h) > 0$ for any $0 < \lambda < 1$. Thus this segment is entirely contained in $\Omega = \mathcal{C} \sqcup (-\mathcal{C})$ and connects $x \in (-\overline{\mathcal{C}})$ and $h \in \mathcal{C}$; therefore it has to go through the origin, a contradiction.

COROLLARY 10.8. *For any two* effective *classes x and y* such that $x^2 \geq 0$ *and* $y^2 \geq 0$ *(hence $x, y \in \overline{\mathcal{C}}$), we have $x \cdot y \geq 0$.*

(this was already clear when either $x^2 > 0$ or $y^2 > 0$, by Hodge index theorem. When $x^2 = y^2 = 0$, we see that q cannot drop negative along the segment from x to y).

We end this section by mentioning some kind of description of the Kleiman-Mori cone. If D is a divisor, we denote by $\mathbb{R}_+[D]$ the *ray* generated by (the class of) D in $\mathrm{Num}(S)_{\mathbb{R}}$, that is, $\mathbb{R}_+[D] = \{\lambda[D] : \lambda > 0\}$.

Now suppose we have a curve $C \subset S$ with negative self-intersection $C^2 < 0$. Then obviously $\mathbb{R}_+[C]$ is not contained in $\mathcal{C}_S$, though $\mathbb{R}_+[C] \subset \overline{\mathrm{Eff}(S)}$.

[2] Steven Lawrence Kleiman (born 1942) and Shigefumi Mori (born 1951).

THEOREM 10.9. *The Kleiman-Mori cone is the smallest convex (closed) cone inside* $\mathrm{Num}(S)_{\mathbb{R}}$ *which contains both the closed positive cone and the rays generated by all curves* C *with negative self-intersection. In other words,*

$$\overline{\mathrm{Eff}(S)} = \overline{\mathcal{C}}_S + \sum_{C^2<0} \mathbb{R}_+[C]$$

(where the sum runs over all curves C *on* S *with* $C^2 < 0$*, if any).*

REMARK 10.10. *The Theorem is not as useful as it may seem. There exists a better description of the part of* $\overline{\mathrm{Eff}(S)}$ *which lies in the half-space* $\{x : x \cdot K_S < 0\}$*. The result is known as the Mori Cone Theorem.*

Recall the Nakai-Moishezon criterion: a divisor D is ample if and only if $D^2 > 0$ *and* $D \cdot C > 0$ for any irreducible curve C. The latter condition can be re-expressed by saying that the functional $\phi_D\colon \mathrm{Num}(S)_R \to \mathbb{R}$ given by

$$\phi_D(x) = x \cdot D$$

is *positive* on the effective cone $\mathrm{Eff}(S)$. This latter condition alone does not guarantee D to be ample. On the other hand, if we extend this condition *up to the boundary* of $\mathrm{Eff}(S)$, then D is ample. More explicitly,

THEOREM 10.11 (Kleiman). D *is ample if and only if its associated functional* ϕ_D *is positive on the Kleiman-Mori cone* $\overline{\mathrm{Eff}(S)}$.

The importance of this theorem is that it generalizes in higher dimensions. In fact, if X is a compact projective complex manifold of dimension $\dim(X) \geq 3$, we can define in a natural way an intersection form $C \cdot Y$ between curves C and hypersurfaces Y inside X. This form extends linearly to $\mathbb{R}$, to give a pairing

$$\mathrm{Num}_1(X)_{\mathbb{R}} \times \mathrm{Num}^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

where we denote by $\mathrm{Num}_1(X)$ the numerical classes of 1-dimensional cycles (as that of C) and by $\mathrm{Num}^1(X)$ the numerical classes of 1-codimensional cycles (as that of Y). This pairing is in fact a duality and the number $\rho(X) = \dim \mathrm{Num}^1(X) = \dim \mathrm{Num}_1(X)$ is called the *Picard number* of X. Now, on the one hand we have inside $\mathrm{Num}^1(X)$ the ample cone $\mathrm{Amp}(X)$ (and its closure coincides with the nef cone $\mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X)$). At the other extreme, we have inside $\mathrm{Num}_1(X)$ the cone of curves $\mathrm{Eff}(X)$, and its closure, the Kleiman-Mori cone. In this setting, the theorem of Kleiman still holds as stated above.

LECTURE 11

SURFACES WITH K_S NON-NEF

LECTURE 12

BIRATIONAL MORPHISMS

§1. RATIONAL MAPS AND LINEAR SYSTEMS

FIXED AND MOVING PART OF A LINEAR SYSTEM

Let L be a line bundle on S. Here we identify L with the linear equivalence class of divisors which represents it. We call *complete linear system* of L,

$$|L| = \mathbb{P}(H^0(L)) \simeq \{D \in \mathrm{Div}(S) : \; D \geq 0 \text{ and } D \sim L\}.$$

If $h^0(L) \geq 2$, the rational map defined by the independent sections of L,

$$\varphi_L : S \dashrightarrow \mathbb{P}^N,$$

is regular on $S \setminus \mathrm{Bs}\,|L|$. Note that the base locus of L,

$$\mathrm{Bs}\,|L| = \bigcap_{D \in |L|} \mathrm{Supp}(D),$$

consists (eventually) of isolated points and 1-dimensional components. The latter is thus an effective divisor F (eventually $F = 0$), called the *fixed part* of $|L|$. Since any member of $|L|$ can be written as $F + M$, for some effective divisor M which does not contain F as a component, one usually writes

$$|L| = F + |M|,$$

and call the linear system $|M|$ the *moving part* of $|L|$. By definition

$$|M| = |L| - F = \{D - F : \; D \in |L|\}.$$

The natural inclusion $|M| \hookrightarrow |L|$ yields an isomorphism $H^0(M) \simeq H^0(L)$. Therefore we identify $\varphi_M = \varphi_L$ on $S \setminus \mathrm{Bs}\,|L|$. Note that φ_M is regular on $S \setminus \mathrm{Bs}\,|M|$ and $\mathrm{Bs}\,|M|$ is precisely the set $\{x_i\}$ of isolated points of $\mathrm{Bs}\,|L|$. In this sense we can always extend φ_L to a regular map

$$\varphi_L : S \setminus \{x_i\} \longrightarrow \mathbb{P}^N.$$

Note that saying that $\mathrm{Bs}\,|L| = F$ means that $\{x_i\}$ is empty.

REMARK 12.1. $h^0(F) = 1$, *since* F *is* fixed *(it is equivalent to itself only).*

REMARK 12.2. M *is nef (in particular* $M^2 \geq 0$*, by Kleiman). In fact, if* $M = 0$ *the assertion is obvious. Otherwise, let* C *be an irreducible curve. Since* M *is effective,* C *intersects properly any irreducible component of* M*, distinct from* C*. However, in case* C *coincides with some component of* M*, we can replace* M *by another member of* $|M|$ *which does not contain* C*. This is always possible since* M*, being moving, has no fixed part.*

§2. BLOW-UP

PROPERTIES OF BLOW-UPS

CASTELNUOVO'S CRITERION

EXAMPLES

§3. MINIMAL MODELS

STRUCTURE THEOREM FOR BIRATIONAL MORPHISMS

LECTURE 13

RULED AND RATIONAL SURFACES

Lecture 14

Non-ruled surfaces

Appendix A

Sheaves on complex manifolds

In complex geometry one frequently has to deal with functions which have various domains of definition. The notion of a sheaf gives a suitable formal setting to handle this situation. In exchange of their rather abstract and technical nature, sheaves provide us the framework of a very general cohomology theory, which encompasses also the "usual" topological cohomology theories such as singular cohomology.

§1. Sheaves and presheaves of abelian groups

Let X be a topological space.

Definition. We say that $\mathcal{F}$ is a *presheaf* (of abelian groups) on X if

(a) to each open subset $U \subset X$ there corresponds an abelian group $\mathcal{F}(U)$.

(b) for each inclusion of open sets $V \subset U$ there corresponds a homomorphism of groups $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ called *restriction*, such that

  (i) $\rho^U_U = \mathrm{id}$ for all U.

  (ii) $\rho^V_W \circ \rho^U_V = \rho^U_W$ for $W \subset V \subset U$.

By definition $\mathcal{F}(\varnothing) := 0$, the trivial group. One also writes $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$. Elements $s \in \mathcal{F}(U)$ are called *sections*. We often write $s|_V$ instead of $\rho^U_V(s)$.

In order to define a presheaf, one has to define the groups and the restrictions. For example, consider the vector space $\mathcal{C}^0(U)$ of all continuous maps $f : U \to \mathbb{R}$, for any $U \subset X$ open. The *natural restrictions* of maps $\rho^U_V(f) := f|_V$ define a presheaf. We can also define different restrictions: quite trivially, $\rho^U_U(f) := f$ and $\rho^U_V(f) := 0$ if $V \subsetneq U$, yields another presheaf. We wish to avoid such trivial situations. We must require some more.

Definition. A presheaf $\mathcal{F}$ a *sheaf* (of abelian groups) on X if it satifies the following conditions, called *sheaf axioms*:

(I) *Local identity:* If $\{U_j\}$ is a collection of open sets in X and $U = \bigcup U_j$ then $s, t \in \mathcal{F}(U)$ and $s|_{U_j} = t|_{U_j}$ for all j implies $s = t$.

(II) *Glueing:* If $\{U_j\}$ is a collection of open sets in X and $U = \bigcup U_j$ then for any collection of sections $s_j \in \mathcal{F}(U_j)$ with $s_j|_{U_{ij}} = s_i|_{U_{ij}}$ for all i, j there always exists a global section $s \in \mathcal{F}(U)$ such that $s|_{U_j} = s_j$.

The sections s_j as in (II) are called *compatible* and s is the *glueing* of the sections s_j. By (I) s is unique. Thus we can summarize (I) and (II) as:

In a sheaf there exists a unique glueing for all compatible sections

REMARK A.1. *By linearity of the restrictions we get*

$$\textit{(I)} \iff s \in \mathcal{F}(U) \textit{ with } s|_{U_j} = 0 \textit{ for all } j \textit{ implies } s = 0$$

EXAMPLE A.2. $\mathcal{C}^0$, with the *natural restrictions* of maps, is a sheaf on X. In fact it clearly satisfies (I). As for the glueing axiom, suppose $f_j : U_j \to \mathbb{R}$ are continuous and $f_i|_{U_{ij}} = f_j|_{U_{ij}}$. On $U = \bigcup U_j$ define the glueing function $f(x) := f_j(x)$ for $x \in U_j$. We only need to check that actually $f \in \mathcal{C}^0(U)$. Let $B \subset \mathbb{R}$ be open. Then, since $f|_{U_j} = f_j$,

$$f^{-1}(B) = \bigcup_j f^{-1}(B) \cap U_j = \bigcup_j f_j^{-1}(B)$$

so f is continuous.

EXAMPLE A.3. In a similar fashion, if X is a differentiable manifold one has the sheaf of smooth functions $\mathcal{C}^\infty$, where

$$\mathcal{C}^\infty(U) = \{f : U \to \mathbb{R} : \ f \text{ smooth}\}$$

If X is a complex manifold, we have the sheaf of holomorphic functions $\mathcal{O}$,

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} : \ f \text{ holomorphic}\}$$

and the sheaf of holomorphic p-forms Ω^p, and so on. Similarly, we have the sheaf $\mathcal{O}^*$ of nowhere-vanishing (or "invertible") holomorphic functions

$$\mathcal{O}^*(U) = \{f : U \to \mathbb{C}^* : \ f \text{ holomorphic}\}$$

where we are considering $\mathcal{O}^*(U)$ as a group under multiplication of functions.

STALK OF A PRESHEAF

The stalk of a sheaf is a useful construction capturing the behaviour of a sheaf around a given point. Although sheaves are defined on open sets, the underlying topological space X consists of points. It is reasonable to attempt to isolate the behavior of a sheaf at a single fixed point $a \in X$. Conceptually speaking, we do this by looking at small neighborhoods of the point. If we look at a sufficiently small neighborhood of a, the behavior of a sheaf $\mathcal{F}$ on that small neighborhood should be the same as the behavior of $\mathcal{F}$ at that point. Of course, no single neighborhood will be small enough, so we will have to take a limit of some sort. This construction is general and it is called *direct limit*. It goes as follows.

Let $\mathcal{F}$ be a presheaf on a topological space X. For $a \in X$ we consider the family of groups $\mathcal{F}(U)$ for which $U \ni a$. On the disjoint union of this groups we introduce an equivalence relation: for $s \in \mathcal{F}(U), t \in \mathcal{F}(V)$ we let

$$s \sim t \iff \exists W \subset (U \cap V) \text{ such that } \; s|_W = t|_W$$

In other words, we consider equivalent those sections that coincide *locally*.

Definition. The *stalk* of the presheaf $\mathcal{F}$ at $a \in X$ is the group

$$\mathcal{F}_a := \varinjlim_{U \ni a} \mathcal{F}(U) := \bigsqcup_{U \ni a} \mathcal{F}(U) / \sim$$

An element in $\mathcal{F}_a$ is called the *germ* of a section of $\mathcal{F}$. The germ of $s \in \mathcal{F}(U)$ will be denoted by s_a. The germ of a section is represented by a pair (U, s). For this reason when we want to keep track of U for a germ we also write

$$s_a = \langle U, s \rangle \in \mathcal{F}_a$$

REMARK A.4. *$\mathcal{F}_a$ is a group: let $s_a = \langle U, s \rangle, t_a = \langle V, t \rangle$. We let*

$$s_a + t_a := \langle U \cap V, s|_{U \cap V} + t|_{U \cap V} \rangle$$

Well defined: let $s_a = \langle U', s' \rangle, t_a = \langle V', t' \rangle$. Then there exists $W \subset (U \cap U' \cap V \cap V')$ such that $s|_W = s'|_W$ and $t|_W = t'|_W$ since $s \sim s'$, $t \sim t'$. Thus $\langle U \cap V, s|_{U \cap V} + t|_{U \cap V} \rangle = \langle W, s|_W + t|_W \rangle = \langle U' \cap V', s'|_{U' \cap V'} + t'|_{U' \cap V'} \rangle$.

EXAMPLE A.5 (germs of holomorphic functions). Let X be a complex manifold and consider its sheaf $\mathcal{O}$ of holomorphic functions. Let $a \in U \subset X$ with a local chart $z : U \to \mathbb{C}^n$, with $z(a) = 0$. Let $f_a \in \mathcal{O}_a$ be the germ of a holomorphic function $f \in \mathcal{O}(V)$. Then $f_a = \langle V, f \rangle = \langle V \cap U, f|_{V \cap U} \rangle$. Moreover f has a convergent power series expansion about a. In particular $f_a = \langle W, f|_W \rangle$, so f_a is represented by a convergent power series. Conversely, two holomorphic functions on neighborhoods of a determine the same germ at a precisely if their series expansion about a coincide. Thus there is an isomorphism of groups (in fact, of rings)

$$\mathcal{O}_a \simeq \mathbb{C}\{z_1, \dots, z_n\} = \{\text{convergent power series about } 0 \in \mathbb{C}^n\}$$

REMARK A.6. *The map $\mathcal{F}(U) \to \mathcal{F}_a$, $s \mapsto s_a$ which assigns to each section its equivalence class is a homomorphism of abelian groups. We also write $\rho^U_a(s) := s_a$. Indeed, if $s, t \in \mathcal{F}(U)$ then $s_a + t_a = (s + t)_a$.*

PROPOSITION A.7. *Let $\mathcal{F}$ be a sheaf on X and $s \in \mathcal{F}(U)$. Then*

$$s = 0 \iff s_a = 0 \textit{ for all } a \in U$$

Proof. Suppose $s_a = 0$ for all $a \in U$. Then $s_a = \langle U, s \rangle = \langle U, 0 \rangle$. So there is a small neighborhood W_a of a such that $s|_{W_a} = 0|_{W_a} = 0$. By the local identity axiom (I) it follows $s = 0$, as $U = \bigcup W_a$ and $s|_{W_a} = 0$ for all a. $\square$

§2. EXACT SEQUENCES

Let X be a topological space, $\mathcal{F}, \mathcal{G}$ two presheaves (of abelian groups) on X.

Definition. A *homomorphism of (pre)sheaves* (or just *morphism*) is a collection of homomorphisms of groups $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any $U \subset X$ open subset, which are *compatible* with the restrictions: the following diagram commutes (the vertical maps are the restriction maps of $\mathcal{F}$ and $\mathcal{G}$)

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V) \end{tikzcd}$$

for all $V \subset U$ open. In other words, we can always write

$$\alpha_U(s)|_V = \alpha_V(s|_V)$$

If α_U is injective for all U we say that $\mathcal{F}$ is a sub(pre)sheaf of $\mathcal{G}$.

REMARK A.8. *$\alpha : \mathcal{F} \to \mathcal{G}$ induces a homomorphism on the stalks*

$$\alpha_a : \mathcal{F}_a \longrightarrow \mathcal{G}_a \quad \langle U, s\rangle \mapsto \langle U, \alpha_U(s)\rangle \quad (\textit{i.e. } \alpha_a(s_a) = \alpha_U(s)_a)$$

Indeed, we only need to check that this is well defined: if $\langle U, s\rangle = \langle V, t\rangle$ then there exists $W \subset (U \cap V)$ such that $s|_W = t|_W$. Thus

$$\langle U, \alpha_U(s)\rangle = \langle W, \alpha_U(s)|_W\rangle = \langle W, \alpha_W(s|_W)\rangle = \langle W, \alpha_W(t|_W)\rangle = \langle V, \alpha_V(t)\rangle$$

Thus, we get a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_a \arrow[r, "\alpha_a"] & \mathcal{G}_a \end{tikzcd}$$

Definition. A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is called *isomorphism* of presheaves if there exists a morphism $\beta : \mathcal{G} \to \mathcal{F}$ such that $\beta \circ \alpha = \mathrm{id}_{\mathcal{F}}$ and $\alpha \circ \beta = \mathrm{id}_{\mathcal{G}}$. In other words α_U is an isomorphism of groups for all U.

REMARK A.9. *The stalks $\mathcal{F}_a$ and $\mathcal{G}_a$ of two sheaves can be isomorphic for all $a \in X$ without $\mathcal{F}$ and $\mathcal{G}$ being isomorphic. In other words, two locally isomorphic sheaves are not isomorphic (think of vector bundles).*

REMARK A.10. *Suppose s_a has representative $\tilde{s} \in \mathcal{F}(U)$ and $t_a = \alpha_a(s_a)$ has representative $\tilde{t} \in \mathcal{G}(V)$. Then we can always find W such that s_a and t_a have representatives $s \in \mathcal{F}(W), t \in \mathcal{G}(W)$ and, most importantly*

$$t = \alpha_W(s)$$

In fact $t_a = \langle V, \tilde{t} \rangle = \langle U, \alpha_U(\tilde{s}) \rangle$ *so there is* $W \subset U \cap V$ *such that* ($s := \tilde{s}|_W$)

$$t := \tilde{t}|_W = \alpha_U(\tilde{s})|_W = \alpha_U(\tilde{s}|_W) = \alpha_U(s)$$

Definition. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves on X. A sequence of morphisms

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$$

is *exact* if it is such on the stalks: for all $a \in X$, we require the following sequence to be exact

$$\mathcal{F}_a \xrightarrow{\alpha_a} \mathcal{G}_a \xrightarrow{\beta_a} \mathcal{H}_a$$

i.e. we have $\mathrm{Im}(\alpha_a) = \ker(\alpha_a)$. In particular, we shall say that a morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ is *injective/surjective* if it is such on the stalks.

REMARK A.11. *We* do not *require* $\mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ *to be exact.*

EXAMPLE A.12. It is possible to have $\alpha : \mathcal{F} \to \mathcal{G}$ surjective on the stalks (so $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \to 0$ exact) but with $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$ not surjective. Consider the punctured plane $X = \mathbb{C}^*$ and the sheaves $\mathcal{O}$ and $\mathcal{O}^*$ on X. Let

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*, \quad f \longmapsto e^f$$

$\exp_X : \mathcal{O}(X) \to \mathcal{O}^*(X)$ is not surjective: $\mathrm{id} : \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z$ does not admit a global logarithm (no $f = \log(z)$ as a single-valued function on $\mathbb{C}^*$). On the other hand, $\exp_a : \mathcal{O}_a \to \mathcal{O}_a^*$ is surjective for any $a \in X$: let $g_a \in \mathcal{O}_a^*$. Then $g_a = \langle U, g \rangle$, with U a small open ball around a, and $g : U \to \mathbb{C}^*$ holomorphic. As U is simply connected and g is non vanishing, we get a well defined[1] holomorphic function $f := \log(g)$ on U. Hence $\exp_U(f) = g$, so $\exp_a(f_a) = g_a$.

PROPOSITION A.13. *Let* $\mathcal{F}, \mathcal{G}, \mathcal{H}$ *be sheaves on* X. *Then*

(i) *Let* $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ *be an exact sequence. For any* $U \subset X$ *open,*

$$\mathrm{Im}(\alpha_U) \subset \ker(\beta_U).$$

(ii) *Let* $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ *be an exact sequence. For any* $U \subset X$ *open,*

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$$

is an exact sequence of groups.

PROPOSITION A.14. *A morphism of sheaves* $\alpha : \mathcal{F} \to \mathcal{G}$ *is an isomorphism if and only if for all* $a \in X$, α_a *is an isomorphism on the stalks.*

[1] fixing $\log(z) := \log|z| + i\arg(z)$

Kernel, Image and Quotient sheaves

Let $\mathcal{F}, \mathcal{G}$ be sheaves on X and $\alpha : \mathcal{F} \to \mathcal{G}$ a morphism. Then we get a subsheaf of $\mathcal{F}$: for each group $\mathcal{F}(U)$ we can consider its subgroup

$$\ker(\alpha)(U) := \ker\{\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)\}$$

and using the same restrictions existing on $\mathcal{F}$ we get the *kernel sheaf* $\ker(\alpha)$.

Analogously, one can consider the subpresheaf of $\mathcal{G}$ given by the family of subgroups $\mathrm{Im}(\alpha)(U) := \mathrm{Im}(\alpha_U) \subset \mathcal{G}(U)$. However $\mathrm{Im}(\alpha)$ is not a sheaf!

Example A.15. Let $X = \mathbb{C}^*$ and consider $\exp : \mathcal{O} \to \mathcal{O}^*$. As seen above, $\mathcal{O}^*(\mathbb{C}) \ni \mathrm{id} \notin \mathrm{Im}(\exp_{\mathbb{C}})$. However, using the logarithm on a family of disks U_j of radius j around the origin we see that the sections $t_j := \mathrm{id}|_{U_j} \in \mathcal{O}^*(U_j)$ are such that there exist $s_j \in \mathcal{O}(U_j)$ with $t_j = \exp(s_j)$. Since $t_i|_{U_{ij}} = t_j|_{U_{ij}}$ and $\mathcal{O}^*$ is a sheaf there is a unique gluing $s \in \mathcal{O}^*(\mathbb{C})$. But $s = \mathrm{id} \notin \mathrm{Im}(\exp_{\mathbb{C}})$. Thus the presheaf $\mathrm{Im}(\exp)$ fails to satisfy the glueing axiom.

Analogously, one can consider $\mathrm{coker}(\alpha)$ as the subpresheaf of $\mathcal{G}$ given by

$$\mathrm{coker}(\alpha)(U) := \mathrm{coker}(\alpha_U) = \frac{\mathcal{G}(U)}{\mathrm{Im}(\alpha_U)}$$

which also fails to be a sheaf in general. The cokernel sheaf is important as it is the starting point for the construction of the *quotient sheaf* $\mathcal{G}/\mathcal{F}$. Precisely, the quotient sheaf is defined as the sheaf generated by the presheaf $\mathrm{coker}(\alpha)$. In the same manner, the *image sheaf* is conveniently defined to be the sheaf generated by the presheaf $\mathrm{Im}(\alpha)$. But, first of all, we need to know what a "sheaf generated by a presheaf" actually is.

Sheafification

Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X. We want to show that there exists a sheaf $\mathcal{F}^+$ on X and a morphism $\tau : \mathcal{F} \to \mathcal{F}^+$ such that if $\mathcal{G}$ is a sheaf on X and $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism then there exists a unique morphism $\alpha^+ : \mathcal{F}^+ \to \mathcal{G}$ that makes the following diagram commute

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\tau"] \arrow[dr, "\alpha"'] & & \mathcal{F}^+ \arrow[dl, dashed, "\exists!"] \\ & \mathcal{G} & \end{tikzcd}$$

Moreover the pair $(\mathcal{F}^+, \tau)$ is unique. In particular $\mathcal{F}^+ = \mathcal{F}$, if $\mathcal{F}$ is a sheaf.

$$\mathcal{F}^+(U) := \left\{ s : U \longrightarrow \bigcup_{a \in U} \mathcal{F}_a \quad \text{satifying (i) and (ii)} \right\}$$

(i) s preserves the stalks: $s(a) \in \mathcal{F}_a$ for all $a \in U$.

(ii) s is locally a section of $\mathcal{F}$: any point $a \in U$ has a neighborhood $V_a \subset U$ and a section $t \in \mathcal{F}(V_a)$ such that $s(b) = t_b$ for all $b \in V_a$.

We define restrictions on $\mathcal{F}^+$ as the natural restrictions of maps: $s \mapsto s|_V$ for all $V \subset U$. First of all we claim that this is a sheaf on X. In fact:

(I) Local identity holds: let $s \in \mathcal{F}^+(U)$ and $s|_{U_j} = 0$ on a covering of U. Then $s(x) = 0$ for all $x \in U$. Thus $s = 0$.

(II) Gluing axiom holds: let $s_j \in \mathcal{F}^+(U_j)$ with $s_j|_{U_{ij}} = s_i|_{U_{ij}}$ on all the intersections U_{ij} of a covering of $U \subset X$. Define a section $s \in \mathcal{F}^+(U)$ as the most reasonable one: $s(x) := s_j(x)$ for $x \in U_j$. The local conditions of the s_j's make it well defined. Moreover it is the glueing by definition. We have to show that s is actually a section of $\mathcal{F}(U)$. Condition (i) is obvious. Also (ii) is clearly satisfied as s is locally equal to some $s_j \in \mathcal{F}^+(U_j)$.

There is a natural morphism $\tau : \mathcal{F} \to \mathcal{F}^+$. Let $f \in \mathcal{F}(U)$. Define

$$f^+ : U \longrightarrow \bigcup \mathcal{F}_a \quad a \longmapsto f_a$$

Then $f^+ \in \mathcal{F}^+(U)$ for (i) is obvious and (ii) holds with $V_a = U$ and $t = f$. Put $\tau_U(f) = f^+$. Then τ is clearly a morphism of presheaves. Now let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism. Define $\alpha^+ : \mathcal{F}^+ \to \mathcal{G}$ as follows. Let $s \in \mathcal{F}^+(U)$. Then by definition there exist neighborhoods $V_a \subset U$ for all $a \in U$ as in (ii), i.e. with some sections $t = t(a) \in \mathcal{F}(V_a)$ such that $s(b) = t(a)_b$ for all $b \in V_a$. Define $\alpha^+_U(s) = g \in \mathcal{G}(U)$ where $g|_{V_a} = \alpha_{V_a}(t)$.

REMARK A.16. *The induced $\tau_a : \mathcal{F}_a \to \mathcal{F}^+_a$ is an isomorphism of groups. So by proposition A.14 if $\mathcal{F}$ is a sheaf then τ is an isomorphism of sheaves.*

Definition. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define the *quotient sheaf* $\mathcal{G}/\alpha(\mathcal{F})$ as the sheafification of $\text{coker}(\alpha)$.

REMARK A.17. *We have an exact sequence of sheaves*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \arrow[r, "q^+"] \arrow[d, "q"'] & \mathcal{G}/\alpha(\mathcal{F}) \arrow[r] & 0 \\ & \text{coker}(\alpha) \arrow[ur, "\tau"'] & & \end{tikzcd}$$

where q is the projection on the quotient. In particular we note that q^+_a is surjective and induces an isomorphism

$$\frac{\mathcal{G}_a}{\alpha_a(\mathcal{F}_a)} \simeq (\mathcal{G}/\alpha(\mathcal{F}))_a$$

§3. SHEAF COHOMOLOGY

In this section we develop the basic constructions of the cohomological theory of sheaves. We do so by means of the notion of soft sheaves.

Here we assume the topological space X to be Haussdorff and *paracompact*. The latter means that every open covering $\{U_j\}$ of X has a subcovering $\{V_j\}$ which is *locally finite*: every point $x \in X$ admits a neighborhood W which intersects only finitely many V_j's.

REMARK A.18. *If X is a smooth manifold (thus Haussdorff and paracompact) then every open covering $\{U_j\}$ admits a* partition of unity. *This is a collection of smooth maps $\varphi_j : X \to [0,1]$ such that*[2]

1. $\mathrm{Supp}(\varphi_j) \subset U_j$.
2. $\{\mathrm{Supp}(\varphi_j)\}$ *is a locally finite (closed) cover of* X.
3. $\sum_j \varphi_j(x) = 1$ *for all* $x \in X$.

REMARK A.19. *If $\{S_i\}_{i \in I}$ is a closed, locally finite cover of X and $J \subset I$, then*

$$S_J := \bigcup_{j \in J} S_j$$

is closed. In fact if $x \in X \setminus S_J$ let W be a neighborhood of x such that $W \cap S_j \neq \varnothing$ only for $j = i_1, \ldots, j_N \in J$. Then $S_W = S_{j_1} \cup \cdots \cup S_{j_N}$ is closed and we have $W \cap S_J = W \cap S_W$.

Let $\mathcal{F}$ be a sheaf of abelian groups on X. For any $K \subset X$ closed we want to define a group $\mathcal{F}(K)$ as a direct limit over the open subsets $U \supset K$. Thus we need to set an equivalence relation as follows.
If $K \subset U_1 \cap U_2$ with U_i open and $f_i \in \mathcal{F}(U_i)$, we put $f_1 \sim f_2$ if and only if there is W open such that $K \subset W \subset U_1 \cap U_2$ and $f_1|_W = f_2|_W$. We let

$$\mathcal{F}(K) := \varinjlim_{U \supset K} \mathcal{F}(U) = \bigsqcup_{U \supset K} \mathcal{F}(U) / \sim$$

If $U \supset K$, we let $f|_K \in \mathcal{F}(K)$ denote the equivalence class of $f \in \mathcal{F}(U)$.

REMARK A.20. *If $K = \{a\}$, we have $\mathcal{F}(\{a\}) = \mathcal{F}_a$. So what we have done here is a generalization of the stalk construction to* all *closed sets.*

Definition. A sheaf $\mathcal{F}$ on X is *soft* if any section over any closed subset of X can be extended to a global section.

In other words, for any $K \subset X$ closed, the *restriction* $\mathcal{F}(X) \to \mathcal{F}(K)$, $f \mapsto f|_K$ is surjective. Thus, given K and a section $g|_K \in \mathcal{F}(K)$ with representative $g \in \mathcal{F}(U)$, where $K \subset U$, there exist some open subset W, with $K \subset W \subset U$ and some $f \in \mathcal{F}(X)$, such that $f|_W = g|_W$.

PROPOSITION A.21. *Let X be a differentiable manifold and let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be an exact sequence of sheaves on X. If A is soft, then the following is an exact sequence

$$0 \to A(X) \xrightarrow{\alpha_X} B(X) \xrightarrow{\beta_X} C(X) \to 0$$

[2] note that the sum in 3. is always finite by 2.

COROLLARY A.22. *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be an exact sequence of sheaves on a differentiable manifold* X. *If* A *and* B *are soft, then* C *is soft.*

THE CANONICAL RESOLUTION

From now on, we assume X to be a differentiable manifold.

Definition. Let $\mathcal{F}$ be a sheaf on X. A family $\{\mathcal{F}^q\}_{q\in\mathbb{N}}$ of sheaves on X together with a family of morphisms $d^q : \mathcal{F}^q \to \mathcal{F}^{q+1}$, is called *resolution* of $\mathcal{F}$ if there exists an injection $\gamma : \mathcal{F} \to \mathcal{F}^0$ and an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \mathcal{F}^2 \longrightarrow \dots$$

Let $\mathcal{F}$ be a sheaf on X. We define a soft sheaf $\mathcal{D}(\mathcal{F})$ on X as

$$\mathcal{D}(\mathcal{F})(U) := \left\{ t : U \longrightarrow \bigcup_{a\in U} \mathcal{F}_a \mid t(a) \in \mathcal{F}_a \right\}$$

The restrictions are the natural restrictions of maps. $\mathcal{D}(\mathcal{F})$ is called the *sheaf of discontinous sections of* $\mathcal{F}$. Note how its construction is very similar to the sheafification, except that now we start from a sheaf and we do not require the sections of $\mathcal{D}(\mathcal{F})$ to be locally equal to those of $\mathcal{F}$. Any section on a closed set has a global extension: indeed, let $K \subset X$ be closed and let $s|_K \in \mathcal{D}(\mathcal{F})(K)$ have representative $s \in \mathcal{D}(\mathcal{F})(U)$, some $U \supset K$. Put

$$f(a) := \begin{cases} s(a) & a \in U \\ 0 & a \in X \setminus U \end{cases}$$

thus $f \in \mathcal{D}(\mathcal{F})(X)$ and $f|_K = s|_K$. Hence $\mathcal{D}(\mathcal{F})$ is a soft sheaf, indeed.

REMARK A.23. *We have an injection*

$$\gamma : \mathcal{F} \longrightarrow \mathcal{D}(\mathcal{F})$$

by $\gamma_U : \mathcal{F}(U) \to \mathcal{D}(\mathcal{F})(U)$, $s \mapsto \gamma_U(s)$ *where* $\gamma_U(s)(a) := s_a \in \mathcal{F}_a$.

Now we can construct the Canonical resolution of $\mathcal{F}$. Let

$$\mathcal{C}^0 := \mathcal{D}(\mathcal{F})$$

and $\gamma : \mathcal{F} \to \mathcal{C}^0$ the above injection. Let $\tilde{\mathcal{C}}^1 := \mathcal{C}^0/\gamma(\mathcal{F})$ be the quotient sheaf and $\alpha_0 : \mathcal{C}^0 \to \tilde{\mathcal{C}}^1$ the quotient map. Let $\mathcal{C}^1 := \mathcal{D}(\tilde{\mathcal{C}}^1)$ be the (soft) sheaf of discontinous sections of $\tilde{\mathcal{C}}^1$. Then we get an injection $\beta_0 : \tilde{\mathcal{C}}^1 \to \mathcal{C}^1$ as above. As γ and β are injective and α surjective[3] we get the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r, "\gamma"] & \mathcal{C}^0 \arrow[rr, "\beta_0 \circ \alpha_0"] \arrow[dr, "\alpha_0"'] & & \mathcal{C}^1 \\ & & & \tilde{\mathcal{C}}^1 \arrow[ur, "\beta_0"'] & \end{tikzcd}$$

[3]remember: this means that they are injective/surjective *on the stalks*!

By induction, we get exact sequences of the form

$$\begin{tikzcd}
\mathcal{C}^{q-1} \arrow[rr, "d^{q-1}"] \arrow[dr, "\alpha_{q-1}"'] & & \mathcal{C}^{q} \arrow[rr, "d^{q}"] \arrow[dr, "\alpha_{q}"'] & & \mathcal{C}^{q+1} \\
& \tilde{\mathcal{C}}^{q} \arrow[ur, "\beta_{q-1}"'] & & \tilde{\mathcal{C}}^{q+1} \arrow[ur, "\beta_{q}"'] &
\end{tikzcd}$$

where we have set $\tilde{\mathcal{C}}^{q+1} = \mathcal{C}^{q}/\tilde{\mathcal{C}}^{q},\ \mathcal{C}^{q+1} = \mathcal{D}(\tilde{\mathcal{C}}^{q+1})$ with the corresponding projections on the quotient α_q and injections β_q and we have put $d^q = \beta_q \circ \alpha_q$.

REMARK A.24. *Suppose $\mathcal{F}$ is soft. Then each of the $\tilde{\mathcal{C}}^{q}$ is soft. In fact, as*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \gamma\ } \mathcal{C}^0 \xrightarrow{\ \alpha_0\ } \tilde{\mathcal{C}}^1 \longrightarrow 0$$

is exact, then by corollary A.22 follows $\tilde{\mathcal{C}}^1$ soft. Then, the exactness of

$$0 \longrightarrow \tilde{\mathcal{C}}^{q-1} \xrightarrow{\ \beta\ } \mathcal{C}^{q-1} \xrightarrow{\ \alpha\ } \tilde{\mathcal{C}}^{q} \longrightarrow 0$$

with $\mathcal{C}^q$ soft and $\tilde{\mathcal{C}}^{q-1}$ soft by induction, implies $\tilde{\mathcal{C}}^{q}$ soft.

Definition. The resolution of $\mathcal{F}$ obtained as above,

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \gamma\ } \mathcal{C}^0 \xrightarrow{\ d^0\ } \mathcal{C}^1 \xrightarrow{\ d^1\ } \mathcal{C}^2 \longrightarrow \dots$$

is called the *Canonical (soft) resolution* of $\mathcal{F}$.

The Canonical resolution defines a complex[4] of abelian groups $\mathcal{C}_X$ as

$$0 \longrightarrow \mathcal{C}^0(X) \xrightarrow{\ d^0_X\ } \mathcal{C}^1(X) \xrightarrow{\ d^1_X\ } \mathcal{C}^2(X) \longrightarrow \dots$$

Definition. The *q-th cohomology group* of the sheaf $\mathcal{F}$ is the abelian group

$$H^q(X, \mathcal{F}) := H^q(\mathcal{C}_X) = \frac{\ker(d^q_X)}{\mathrm{Im}(d^{q-1}_X)}$$

also called *q-th cohomology group of* X *with coefficients in* $\mathcal{F}$. In particular

$$H^0(X, \mathcal{F}) := \ker(d^0_X)$$

[4]In fact $d_X \circ d_X = 0$ follows from proposition A.13 and by exactness of

$$\mathcal{C}^{q-1} \longrightarrow \mathcal{C}^{q} \longrightarrow \mathcal{C}^{q+1}$$

Properties of the cohomology groups

Theorem A.25. *Let $\mathcal{F}$ be a sheaf on X. Then*[5]

$$H^0(X, \mathcal{F}) = \mathcal{F}(X)$$

Moreover, if $\mathcal{F}$ is soft then for all $q > 0$

$$H^q(X, \mathcal{F}) = 0$$

Theorem A.26. *Any sheaf morphism $f : \mathcal{F} \to \mathcal{G}$ induces homomorphisms*

$$f_q : H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{G}),$$

which have the following (functorial) properties:

(a) $f_0 = f_X : \mathcal{F}(X) \to \mathcal{G}(X)$

(b) $f_q = \mathrm{id}$ *if* $\mathcal{F} = \mathcal{G}$ *and* $f = \mathrm{id}_{\mathcal{F}}$

(c) $(g \circ f)_q = g_q \circ f_q$ *for a sheaf morphism* $g : \mathcal{G} \to \mathcal{H}$

Theorem A.27. *For each short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \longrightarrow 0$$

there exist homomorphisms $\delta^q : H^q(X, \mathcal{H}) \longrightarrow H^{q+1}(X, \mathcal{F})$ which induce the following exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "f_0"] & H^0(X,\mathcal{G}) \arrow[r, "g_0"] & H^0(X,\mathcal{H}) \arrow[dll, "\delta^0"] \\
& H^1(X,\mathcal{F}) \arrow[r, "f_1"] & H^1(X,\mathcal{G}) \arrow[r, "g_1"] & H^1(X,\mathcal{H}) \arrow[dll, "\delta^1"] \\
& H^2(X,\mathcal{F}) \arrow[r, "f_2"] & H^2(X,\mathcal{G}) \arrow[r, "g_2"] & H^2(X,\mathcal{H}) \cdots
\end{tikzcd}$$

Moreover, for each commutative diagram of sheaves with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "f"] \arrow[d, "a"] & \mathcal{G} \arrow[r, "g"] \arrow[d, "b"] & \mathcal{H} \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r, "f'"] & \mathcal{B} \arrow[r, "g'"] & \mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

then also the following diagram in cohomology is commutative

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "f_0"] \arrow[d, "a_0"] & H^0(X,\mathcal{G}) \arrow[r, "g_0"] \arrow[d, "b_0"] & H^0(X,\mathcal{H}) \arrow[r, "\delta^0"] \arrow[d, "c_0"] & H^1(X,\mathcal{F}) \arrow[r] \arrow[d, "a_1"] & \cdots \\
0 \arrow[r] & H^0(X,\mathcal{A}) \arrow[r, "f'_0"] & H^0(X,\mathcal{B}) \arrow[r, "g'_0"] & H^0(X,\mathcal{C}) \arrow[r, "\delta^1"] & H^1(X,\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}$$

[5] especially in this context people often write $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$

Soft sheaves have no cohomology (cf. theorem A.25). This property is crucial, in order to compute the cohomology of a sheaf $\mathcal{F}$.

Definition. A resolution

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \longrightarrow \dots$$

is called *acyclic* if $H^q(X, \mathcal{A}^i) = 0$ for all $i \geq 0$ and $q \geq 1$.

EXAMPLE A.28. The Canonical resolution is acyclic, as each $\mathcal{C}^i$ is soft.

Let $\mathcal{A}_X$ denote the complex of global sections

$$\mathcal{F}(X) \longrightarrow \mathcal{A}^0(X) \xrightarrow{d_X^0} \mathcal{A}^1(X) \xrightarrow{d_X^1} \mathcal{A}^2(X) \xrightarrow{d_X^2} \dots$$

THEOREM A.29 (abstract de Rham Theorem). *Each acyclic resolution of $\mathcal{F}$ computes the sheaf cohomology. Precisely,*

$$H^q(X, \mathcal{F}) \simeq H^q(\mathcal{A}_X).$$

§4. DE RHAM AND DOLBEAULT THEOREMS

Let X be a differentiable manifold and $\mathcal{E}^p$ the sheaf of smooth real p-forms.

$$0 \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \dots$$

where d is the exterior derivative, is an exact sequence of sheaves. Let $\mathcal{E}_X$ denote the relative complex of global sections. We have the de Rham groups

$$H^q_{dR}(X) := H^q(\mathcal{E}_X).$$

Let $\mathbb{R}$ denote the sheaf of locally constant functions on X, i.e. the sheaf[6]

$$\ker(d : \mathcal{E}^0 \longrightarrow \mathcal{E}^1) \simeq \mathbb{R}$$

Then we have a resolution of the sheaf $\mathbb{R}$ of locally constant functions by

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \dots$$

One can show that each $\mathcal{E}^p$ is soft. Hence the resolution is acyclic, and

$$H^q(X, \mathbb{R}) \simeq H^q_{dR}(X)$$

This result is generally known as the *de Rham theorem*.

[6] $f : X \to \mathbb{R}$ with $0 = df = \sum \frac{\partial f}{\partial x_i} dx_i$ implies $\frac{\partial f}{\partial x_i} = 0$ for all i. So locally $f \equiv \lambda \in \mathbb{R}$

In a similar fashion we can consider the sheaf Ω^p of holomorphic p-forms on a complex manifold X. Let $\mathcal{A}^{p,q}$ be the sheaf of smooth (p,q)-forms on X. As for the case of $\mathcal{E}^p$ one proves that the $\mathcal{A}^{p,q}$ are soft. The "$\bar{\partial}$-Poincaré lemma" guarantees that the following is an exact sequence of sheaves

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots$$

Which is therefore an acyclic resolution of Ω^p. Hence, the groups $H^q(X,\Omega^p)$ can be computed in terms of $\bar{\partial}$-closed modulo $\bar{\partial}$-exact (p,q)-forms[7]. This result is generally known as the *Dolbeault theorem*.

[7] in particular $H^q(X,\Omega^p) = 0$ if $q > \dim_{\mathbb{C}} X$ since in this case $\mathcal{A}^{p,q} = 0$

BIBLIOGRAPHY

[1] W. Barth, K. Hulek, C. Peters, and A. van de Ven. *Compact Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2014.

[2] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley Classics Library. Wiley, 2011.

[3] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.

[4] R.K. Lazarsfeld. *Positivity in Algebraic Geometry I: Classical Setting: Line Bundles and Linear Series*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics. Springer, 2004.