aMC@NLO exam

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1 Day

1.1 $e^+e^- \to q\bar{q}$

Consider diagrams in fig. 1. The differential cross section of a $2 \to 2$ process is given by:

$$d\sigma = \frac{d\Phi_2}{4F} |\overline{\mathcal{M}}|^2 \tag{1}$$

The flux factor in the massless limit is simply $F = p_1 \cdot p_2$, while the differential phase space is:

$$d\Phi_2 = \frac{d^3\mathbf{p_3}}{(2\pi)^3 2E_3} \frac{d^3\mathbf{p_4}}{(2\pi)^3 2E_4} (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \frac{d\cos\theta}{16\pi} \tag{2}$$

Define the Mandelstam variables:

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2 \tag{3}$$

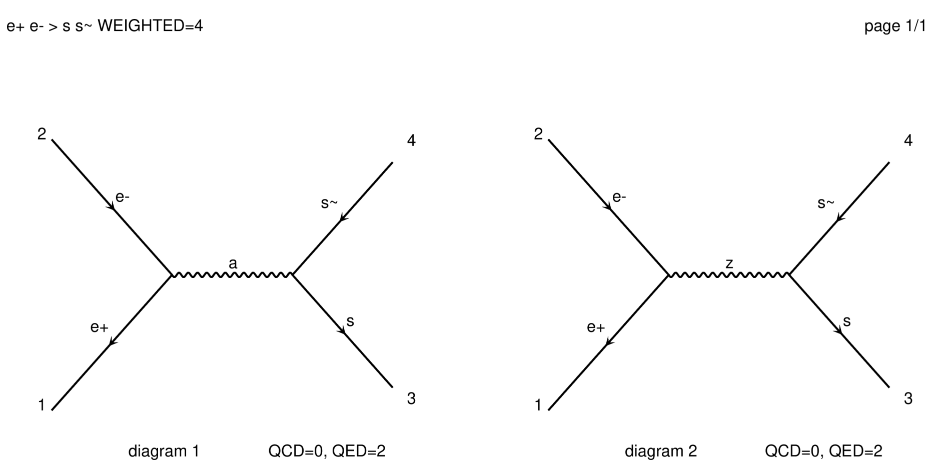

Figure 1: Leading order diagrams contributing to the $e^+e^- \to q\bar{q}$ process.

such as $s+t+u=0$ in the massless limit. The differential cross section can be written as:

$$d\sigma = \frac{d\,\cos\theta}{32\pi\, s}|\overline{\mathcal{M}}|^2 \tag{4}$$

where a factor 2π arises from the integration over the azimuthal angle φ, since matrix elements do not depend upon it.

1.1.1 $e^+e^- \to \gamma^* \to q\bar{q}$

The amplitude for the process e^+e^- into $q\bar{q}$, mediated by a photon, γ is:

$$\mathcal{M}_\gamma = 4\pi i\alpha Q_f \bar{v}(p_1)\gamma^\mu u(p_2)\,\frac{g_{\mu\nu}}{q^2}\,\bar{u}(p_3)\gamma^\nu v(p_4) \tag{5}$$

where $q = p_1 + p_2$, so that $s = q^2$. Then, after summing over the final state polarizations, averaging over the initial ones and adding the color factor (the photon is color blind), the matrix element squared is:

$$|\overline{\mathcal{M}_\gamma}|^2 = N_c \frac{(4\pi\alpha Q_f)^2}{4s^2}\mathrm{Tr}\big[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\big]\,\mathrm{Tr}\big[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu\big] \tag{6}$$

This is possible because for Dirac massless spinors, the sum over polarizations formulae hold:

$$\sum_r \bar{u}_r(p)u_r(p) = \not{p} \qquad \sum_r \bar{v}_r(p)v_r(p) = \not{p} \tag{7}$$

Now, compute the traces over Dirac matrices with the help of the following identity:

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \tag{8}$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor. The trace in the four dimensional space is:

$$\mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] = 4\,(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1\cdot p_2)\,\eta^{\mu\nu}) \tag{9}$$

Then the matrix element squared is:

$$|\overline{\mathcal{M}_\gamma}|^2 = N_c \frac{(4\pi\alpha Q_f)^2}{4s^2} 32\big[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\big] \tag{10}$$

Because of momentum conservation and masslessness of partons, finally:

$$|\overline{\mathcal{M}_\gamma}|^2 = N_c \frac{(4\pi\alpha Q_f)^2}{4s^2} 8(t^2+u^2) = 32\pi^2 N_c \alpha^2 Q_f^2\,\frac{t^2+u^2}{s^2} \tag{11}$$

Putting this result into eqn. 4 yields:

$$\frac{d\sigma}{d\cos\theta} = N_c\big(\sum_f Q_f^2\big)\,\frac{\pi\alpha^2}{s}\frac{t^2+u^2}{s^2} \tag{12}$$

Now consider the center of mass frame kinematics:

$$\begin{aligned} p_1 &= (E,0,0,E) & p_2 &= (E,0,0,-E) \\ p_3 &= (E,0,E\sin\theta,E\cos\theta) & p_4 &= (E,0,-E\sin\theta,-E\cos\theta) \end{aligned}$$

That gives:

$$\begin{aligned} s &= 2p_1\cdot p_2 = 4E^2 \\ t &= -2p_1\cdot p_3 = -2E^2\,(1-\cos\theta) \\ u &= -2p_1\cdot p_4 = -2E^2\,(1+\cos\theta) \end{aligned}$$

which means that t and u can be expressed in terms of s and $\cos\theta$, leading to the final results:

$$|\overline{\mathcal{M}_\gamma}|^2 = 16N_c\pi^2\alpha^2Q_f^2\,(1+\cos^2\theta) \tag{13}$$

$$\frac{d\sigma}{d\cos\theta} = N_c\Big(\sum_f Q_f^2\Big)\,\frac{\pi\alpha^2}{2s}\,(1+\cos^2\theta) \tag{14}$$

This is the differential cross section of the process $e^+e^-\to\gamma^*\to q\bar{q}$ in the center of mass frame in the high energy limit: $s\gg m_f$. At fixed center of mass energy $\sqrt{s}$, only flavor quarks that have production energy threshold lower than s contribute to the sum of the squared charges $\sum_f Q_f^2$. Therefore, the cross section as a function of s takes the well-known step-like slope. Note that in this frame and for massless particles, the squared matrix element is finite, the $s=0$ divergence has been canceled. At the cross section level, the divergence is still present but comes from the two-body phase space.

1.1.2 $e^+e^-\to Z^*\to q\bar{q}$

The amplitude for the process e^+e^- into $q\bar{q}$ mediated by the Z boson is:

$$\mathcal{M}_Z = -\frac{\pi i\alpha}{2s_w^2}\bar{v}(p_1)\gamma^\mu\big(P_L-2s_w^2\big)u(p_2)\,\Pi_{\mu\nu}(q)\,\bar{u}(p_3)\gamma^\nu\big(P_L+2Q_f^2s_w^2\big)v(p_4) \tag{15}$$

with $s_w^2\equiv\sin^2_W$ and the propagator $\Pi_{\mu\nu}(q)$ and the chiral projectors $P_{R/L}$ being:

$$\begin{aligned} \Pi_{\mu\nu}(q) &= \frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{m_Z^2}}{q^2-m_Z^2} \\ P_{\frac{R}{L}} &= \frac{1\pm\gamma_5}{2} \end{aligned}$$

The propagator's factor proportional to q_μ in the amplitude does not contribute, since Dirac equation for massless particles $\not{p}u(p)=0$ gives a vanishing contribution.

The squared amplitude is proportional to the following trace:

$$|\overline{\mathcal{M}_Z}|^2 \sim \mathrm{Tr}[\not{p}_1\gamma^\mu\big(P_L-2s_w^2\big)\not{p}_2\big(P_R-2s_w^2\big)\gamma^\nu] \tag{16}$$

Use the following identities:

$$\mathrm{Tr}[\gamma_5] = \mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0$$
$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\epsilon^{\mu\nu\rho\sigma}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita totally antisymmetric tensor. And the anti-commutator $\{\gamma^\mu, \gamma_5\} = 0$. In view of these relations, the trace can be reduced to:

$$|\overline{\mathcal{M}_Z}|^2 \sim \mathrm{Tr}\Big[\not{p}_1\gamma^\mu\Big(\frac{1-4s_w^2}{2} + \frac{1+4s_w^2}{2}\gamma_5\Big)\not{p}_2\gamma^\nu\Big] \tag{17}$$

Recalling the previous result, eqn 11, it can be shown that:

$$|\overline{\mathcal{M}_Z}|^2 \sim \frac{1-4s_w^2}{2}\frac{1+4Q_f^2s_w^2}{2}8(t^2+u^2) - \frac{1+4s_w^2}{2}\frac{1-4Q_f^2s_w^2}{2}16\mathrm{A} \tag{18}$$

where

$$\begin{aligned}
A &= \epsilon^{\rho\mu\sigma\nu}\epsilon_{\alpha\mu\beta\nu}p_{1\rho}p_{2\sigma}p_3^\alpha p_4^\beta = \\
&= 2(\delta^\rho_\alpha\,\delta^\sigma_\beta - \delta^\rho_\beta\,\delta^\sigma_\alpha)p_{1\rho}p_{2\sigma}p_3^\alpha p_4^\beta = \\
&= 2\big[(p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3)\big] = \\
&= \frac{1}{2}(t^2-u^2)
\end{aligned}$$

The Z boson diagram has the following amplitude squared:

$$|\overline{\mathcal{M}_Z}|^2 = N_c\frac{\pi^2\alpha^2}{8s_w^4\Big(1-\frac{m_Z^2}{s}\Big)^2}\Big[Z_1(Q_f^2)(1+\cos^2\theta) + Z_2(Q_f^2)\,2\cos\theta\Big] \tag{19}$$

where:

$$Z_1(Q_f^2) = \big(1-4s_w^2\big)\big(1+4Q_f^2s_w^2\big) \qquad Z_2(Q_f^2) = \big(1+4s_w^2\big)\big(1-4Q_f^2s_w^2\big) \tag{20}$$

Since probabilities for different processes sum incoherently, a sum over quark flavors must be performed, in order to obtain the differential cross section.

The last contribution to $e^+e^- \to \gamma^*, Z^* \to q\bar{q}$ is the interference term between the photon and the Z boson:

$$2\Re(\overline{\mathcal{M}_Z\mathcal{M}_\gamma^*}) = -N_c\frac{\pi^2\alpha^2Q_f}{4ss_w^2}\Re\Big(T_1\cdot T_2\Big) \tag{21}$$

with the two traces $T_{1,2}$:

$$T_1 = \mathrm{Tr}\Big[\not{p}_1\gamma^\mu\big(1-4s_w^2-\gamma_5\big)\not{p}_2\gamma^\nu\Big]$$
$$T_2 = \mathrm{Tr}\Big[\not{p}_3\gamma_\mu\big(1+4Q_f^2s_w^2-\gamma_5\big)\not{p}_4\gamma_\nu\Big]$$

The product of the two traces is similar to the one computed above, where there is no cross product between terms with and without $\gamma 5$. Hence:

$$T_1 \cdot T_2 = (1 - 4s_w^2)(1 + 4Q_f^2 s_w^2)8(t^2 + u^2) - 8(t^2 - u^2) \tag{22}$$

That yields the final result:

$$2\Re(\overline{\mathcal{M}_Z \mathcal{M}_\gamma^*}) = -N_c \frac{\pi^2 \alpha^2 Q_f}{s_w^2 \left(1 - \frac{m_Z^2}{s}\right)} \left[Z_1(Q_f^2)\,(1 + \cos^2\theta) - 2\cos\theta\right] \tag{23}$$

and again Z_1 is given by eqn. 20.

It is possible to put everything together (eqns. 13-19-23) and write the amplitude squared for the process $e^+e- \to \gamma^*, Z^* \to q\bar{q}$:

$$|\overline{\mathcal{M}_{\gamma,Z}}|^2 = 16 N_c \pi^2 \alpha^2 \sum_f \left[A_1(Q_f)\,(1 + \cos^2\theta) + A_2(Q_f)\,2\cos\theta\right] \tag{24}$$

where the coefficients A_1, A_2 are defined as:

$$A_1(Q_f) = Q_f^2 + Z_1(Q_f^2)\left(\frac{1}{128 s_w^4 \left(1 - \frac{m_Z^2}{s}\right)^2} - \frac{1}{16 s_w^2 \left(1 - \frac{m_Z^2}{s}\right)}\right)$$
$$A_2(Q_f) = \frac{Z_2(Q_f^2)}{128 s_w^4 \left(1 - \frac{m_Z^2}{s}\right)^2} + \frac{Q_f}{16 s_w^2 \left(1 - \frac{m_Z^2}{s}\right)}$$

Finally the differential cross section for the whole process is:

$$\frac{d\sigma}{d\cos\theta} = N_c \frac{\pi\alpha^2}{2s} \sum_f \left[A_1(Q_f)\,(1 + \cos^2\theta) + A_2(Q_f)\,2\cos\theta\right] \tag{25}$$

The $\cos\theta$ distribution was initially an even function of this variable. This is due to the fact that the photon, when interacting with a charged particle, couples equally with the two particle helicities. The axial contributions (proportional to γ_5) from the left and right-handed particles exactly cancel. This is not true anymore when the Z boson is taken into account: indeed, it has a non-zero axial contribution given by $A_f = T_f^3$, where T_f^3 is the particle's isospin. This axial factor leads to the term proportional to $\cos\theta$ in the cross section, coming either from the Z Feynman diagram and the interference between Z and γ channels.

Hence, the $\cos\theta$ distribution does not present anymore a definite parity, yields a non-zero forward-backward asymmetry parameter A_{FB}:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} \tag{26}$$

where $\sigma_{F/B}$ are:

$$\sigma_F = \int_0^1 d\cos\theta \, \frac{d\sigma}{d\cos\theta} \qquad \sigma_F = \int_{-1}^0 d\cos\theta \, \frac{d\sigma}{d\cos\theta}$$

A positive forward-backward asymmetry parameter tells that there is more probability to emit a quark in the forward beam direction $\hat{z}$; a negative A_{FB} parameter, instead, will produce the opposite case of more events with quark momentum pointing backward.