Geometry

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1 Rigid Transformations

1.1 Reflections

1.1.1 Exercises

Let V a reflection, then $V(\mathbf{a}) \cdot V(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$

Proof. By definition of reflection:

$$V: \quad V(x, y) = x\mathbf{v} - y\mathbf{v}^{\perp}, \qquad \|\mathbf{v}\| = 1 \tag{1}$$

therefore, $V(\mathbf{a}) \cdot V(\mathbf{b})$ can be represented as:

$$V(\mathbf{a}) \cdot V(\mathbf{b}) = \left(a_1\mathbf{v} - a_2\mathbf{v}^{\perp}\right) \cdot \left(b_1\mathbf{v} - b_2\mathbf{v}^{\perp}\right) \cdot \tag{2}$$

$$= a_1b_1\mathbf{v} \cdot \mathbf{v} - a_1b_2\mathbf{v} \cdot \mathbf{v}^{\perp} - a_2b_1\mathbf{v}^{\perp} \cdot \mathbf{v} + a_2b_2\mathbf{v}^{\perp} \cdot \mathbf{v}^{\perp} \tag{3}$$

$$= a_1b_1\|\mathbf{v}\|^2 - a_1b_2\mathbf{v} \cdot \mathbf{v}^{\perp} - a_2b_1\mathbf{v} \cdot \mathbf{v}^{\perp} + a_2b_2\|\mathbf{v}\|^2 \tag{4}$$

applying the property $\mathbf{v} \cdot \mathbf{v}^{\perp} = 0$:

$$= a_1b_1\|\mathbf{v}\|^2 + a_2b_2\|\mathbf{v}\|^2 \tag{5}$$

$$= (a_1b_1 + a_2b_2)\,\|\mathbf{v}\|^2 \tag{6}$$

$$= (a_1b_1 + a_2b_2) \tag{7}$$

$$= \mathbf{a} \cdot \mathbf{b} \tag{8}$$

□

1.2 Rotations

1.2.1 Exercises

Let U a rotation, prove the identity: $U(\mathbf{a}) \cdot U(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$

Proof. Applying the definition of rotation:

$$U: \quad U(x, y) = x\mathbf{u} + y\mathbf{u}^{\perp}, \qquad \|\mathbf{u}\| = 1 \tag{1}$$

thus, the expression $U(\mathbf{a}) \cdot U(\mathbf{b})$ is equal to:

$$U(\mathbf{a}) \cdot U(\mathbf{b}) = \left(a_1\mathbf{u} + a_2\mathbf{u}^{\perp}\right) \cdot \left(b_1\mathbf{u} + b_2\mathbf{u}^{\perp}\right) \cdot \tag{2}$$

$$= a_1b_1\mathbf{u} \cdot \mathbf{u} + a_1b_2\mathbf{u} \cdot \mathbf{u}^{\perp} + a_2b_1\mathbf{u}^{\perp} \cdot \mathbf{u} + a_2b_2\mathbf{u}^{\perp} \cdot \mathbf{u}^{\perp} \tag{3}$$

$$= a_1b_1\|\mathbf{u}\|^2 + a_1b_2\mathbf{u} \cdot \mathbf{u}^{\perp} + a_2b_1\mathbf{u} \cdot \mathbf{u}^{\perp} + a_2b_2\|\mathbf{u}\|^2 \tag{4}$$

applying the property $\mathbf{u} \cdot \mathbf{u}^{\perp} = 0$:

$$= a_1b_1\|\mathbf{u}\|^2 + a_2b_2\|\mathbf{u}\|^2 \tag{5}$$

$$= (a_1b_1 + a_2b_2)\,\|\mathbf{u}\|^2 \tag{6}$$

$$= (a_1b_1 + a_2b_2) \tag{7}$$

$$= \mathbf{a} \cdot \mathbf{b} \tag{8}$$

□