Vector Projections

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1 Vector Projections

1.1 Exercises

1.1.1 Exercise 1

Show that if $\mathbf{b}$ and $\mathbf{b}' \in \mathbb{R}^2$ are parallel and not null vectors, then $\mathbf{Proy_b a} = \mathbf{Proy_{b'} a}$ for every vector $\mathbf{a} \in \mathbb{R}^2$.

Proof. Since $\mathbf{b}$ is parallel to $\mathbf{b}'$, then

$$\mathbf{b}' = r\mathbf{b} \text{ for some } r \in \mathbb{R} \tag{1}$$

Remember that $\mathbf{a} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{b}'$ and $\mathbf{b}'^{\perp}$:

$$\mathbf{a} = s\mathbf{b}' + t\mathbf{b}'^{\perp} \text{ for some } s \text{ and } t \in \mathbb{R} \tag{2}$$

Applying scalar product with $\mathbf{b}'$ and $\mathbf{b}'^{\perp}$ the following expressions are obtained:

$$\mathbf{a} \cdot \mathbf{b}' = s\|\mathbf{b}'\|^2 \tag{3}$$

$$\mathbf{a} \cdot \mathbf{b}'^{\perp} = t\|\mathbf{b}'\|^2 \tag{4}$$

From Equation (3) and Equation (4) the values for s and t can be obtained to replace in the Equation (2):

$$\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}'}{\|\mathbf{b}'\|^2} [\mathbf{b}'] + \frac{\mathbf{a} \cdot \mathbf{b}'^{\perp}}{\|\mathbf{b}'\|^2} \left[\mathbf{b}'^{\perp}\right] \tag{5}$$

Equation (5) is equivalent to:

$$\mathbf{a} = \mathbf{Proy}_{\mathbf{b}'}\mathbf{a} + \mathbf{Proy}_{\mathbf{b}'^{\perp}}\mathbf{a} \tag{6}$$

Reviewing Equation (5) and Equation (6), it can be concluded that:

$$\mathbf{Proy}_{\mathbf{b}'}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}'}{\|\mathbf{b}'\|^2}\,[\mathbf{b}'] \tag{7}$$

Substituting Equation (1) into Equation (7) gives the following:

$$\mathbf{Proy}_{\mathbf{b}'}\mathbf{a} = \frac{\mathbf{a} \cdot r\mathbf{b}}{\|r\mathbf{b}\|^2}\,[r\mathbf{b}] \tag{8}$$

Equation (8) can be simplified as follows:

$$\mathbf{Proy}_{\mathbf{b}'}\mathbf{a} = \frac{r^2}{r^2}\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\,[\mathbf{b}] \tag{9}$$

Since $\mathbf{b}$ and $\mathbf{b}'$ are not null vectors, then $r \neq 0$ and it can be concluded that Equation (9) is equal to:

$$\mathbf{Proy}_{\mathbf{b}'}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\,[\mathbf{b}] = \mathbf{Proy}_{\mathbf{b}}\mathbf{a} \tag{10}$$

$\square$