Introduction to Differentiable Manifolds	
EPFL – Fall 2021	M. Cossarini, B. Santos Correia
Exercise Series 12 - Orientations, manifolds w/boundary	**2021–12–21**

Exercise 12.1. Which of the following manifolds are orientable?

(a) The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
(b) A product $M_0 \times \cdots \times M_{k-1}$ of several orientable manifolds.
(c) The projective plane $\mathbb{P}^2$.
(d) The Möbius band.

Exercise 12.2. Let M be a differentiable n-manifold. Show that the sign $\operatorname{sgn}\omega$ of a nonvanishing n-form on M, defined by

$$(\operatorname{sgn}\omega)|_p(X^0,\dots,X^{n-1}) := \operatorname{sgn}(\omega|_p(X^0,\dots,X^{n-1}))$$

for $p \in M$ and $X^0,\dots,X^{n-1} \in \mathrm{T}_pM$, is an orientation on M.

Exercise 12.3. Prove Proposition 7.2.6. (properties of the integral: linearity, etc).

Exercise 12.4. Let M be a differentiable manifold with boundary. Show that its interior $\operatorname{Int} M$ and its boundary ∂M are complementary subsets (i.e. they are disjoint and their union is M). Also show that ∂M is closed and $\operatorname{Int} M$ is open.

Exercise 12.5. Prove that a continuous k-form is determined by the value of its integrals (Proposition 7.3.12). ***Hint:*** Use a chart to move the problem to $\mathbb{R}^n$, then integrate on small pieces of coordinate planes.

Exercise 12.6. Let $f : M \to N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega).$$

Exercise 12.7. Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and let (v, w) be the standard coordinates on $\mathbb{R}^2$. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $\phi(x, y, z) = (x + z, xy)$. Let $\alpha = e^w\,\mathrm{d}v + v\,\mathrm{d}w$ and $\beta = v\ \mathrm{d}v \wedge \mathrm{d}w$ be 2-forms on $\mathbb{R}^2$. Compute the following differential forms:

$$\alpha\wedge\beta,\quad \phi^*(\alpha),\quad \phi^*(\beta),\quad \phi^*(\alpha)\wedge\phi^*(\beta).$$

Exercise 12.8. Compute the exterior derivative of the following forms:

(a) on $\mathbb{R}^2 \setminus \{0\}$ $\theta = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$.
(b) on $\mathbb{R}^3$, $\varphi = \cos(x)\,\mathrm{d}y \wedge \mathrm{d}z$.
(c) on $\mathbb{R}^3$ $\omega = A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z$.

Exercise 12.9. Deduce the following classical theorems from Stokes' theorem.

(a) **Green's theorem.** Let $D \subseteq \mathbb{R}^2$ be a smooth 2-dimensional compact embedded submanifold with boundary in $\mathbb{R}^2$. Then for any differentiable 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ defined on an open neighborhood of D we have

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\ \mathrm{d}y = \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

(b) **Divergence theorem.** Let $A \subset \mathbb{R}^3$ be a 3-dimensional compact embedded submanifold with boundary in $\mathbb{R}^3$. Then for any smooth vector field $F : A \to \mathbb{R}^3$ we have

$$\int_A \operatorname{div}F\ \mathrm{d}V = \int_{\partial A} F\cdot\mathrm{d}$$

wher $\mathrm{d}V := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the standard 3-form on $\mathbb{R}^3$ and on the right hand side we have the formal inner product with $\mathrm{d}S = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$.