

Introduction to Differentiable Manifolds

modified by Marcos Cossarini
original notes by Anna Kiesenhofer

EPFL, autumn 2021
date of this version: 2021-09-28

Contents

0.1	Practical remarks about the course	3
0.1.1	Planning 2021: 14 classes	3
0.1.2	Content for the exam	3
1	Manifolds	4
1.1	Topological manifolds	4
1.2	Differentiable manifolds	6
1.3	Differentiable maps	9

Main references

- [Lee10] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2010.
- [Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second edition. Springer, 2013.

Both books have online versions available in the library.

0.1 Practical remarks about the course

0.1.1 Planning 2021: 14 classes

This information is preliminary and may be changed as we go.

- 1. (21.09) Topological and Differentiable manifolds
- 2. (28.09) Differentiable maps. Partitions of unity.
- 3. (05.10) Tangent bundle. Differential of a map.
- 4. (12.10) Vector bundles
- 5. (19.10) Vector fields and flows
- 6. (26.10) Submanifolds
- 7. (02.11) Whitney embedding. Regular levels sets are submanifolds.
- 8. (09.11) Differential forms
- 9. (16.11) Differential forms
- 10.(23.11) Exterior derivative
- 11.(30.11) Integration, orientation
- 12.(07.12) Manifolds with boundary
- 13.(14.12) Stokes' theorem
- 14.(21.12) Not assigned

0.1.2 Content for the exam

Studying the content under a heading marked by an asterisk is not mandatory by itself. You should read it only if you find it interesting or helpful for understanding the rest.

1 Manifolds

The goal of this course is to extend differential and integral calculus from Euclidean space \mathbb{R}^n to all *differentiable manifolds* such as the n -sphere, the n -torus, etc. Roughly speaking, a differentiable manifold is a space that

- is endowed with a certain topology,
- has, in addition, a *differentiable structure* that allows us to distinguish whether a map is differentiable or not, rather than just continuous, and
- locally looks like Euclidean space \mathbb{R}^n .

1.1 Topological manifolds

[Lee13], Chapter 1 and [Lee10], Chapter 2

Let us postpone the question of differentiability and focus on topology. As said, we want to study spaces that “locally look like” Euclidean space \mathbb{R}^n .

Definition 1.1.1 (Locally Euclidean space). Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is **locally Euclidean** of dimension n at a point $p \in M$ if the point p has an open neighborhood that is homeomorphic¹ to an open subset of \mathbb{R}^n . If this holds for all points $p \in M$, we say that M is locally Euclidean of dimension n .

A typical example is the circle: it is locally Euclidean of dimension 1 but not globally homeomorphic to any subset of \mathbb{R} .

Example 1.1.2. The **circle** $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (with the subspace topology) is locally Euclidean of dimension 1: let $(x_0, y_0) \in \mathbb{S}^1$, wlog $y_0 > 0$, then $U := (\mathbb{R} \times \mathbb{R}^+) \cap \mathbb{S}^1$ is an open subset of \mathbb{S}^1 containing (x_0, y_0) and homeomorphic to $(-1, 1)$ via the map $U \rightarrow (-1, 1)$ that sends $(x, y) \mapsto x$.

We will see more examples later on (see e.g. Examples 1.1.11 below). Let us make some general comments.

Remark 1.1.3. If a space M is locally Euclidean of dimension 0, then every point has a neighborhood homeomorphic to $\mathbb{R}^0 = \{0\}$, i.e. a point. In other words, M is a discrete topological space.

Remark 1.1.4. In the definition of locally Euclidean space, we could have replaced “...homeomorphic to an open subset of \mathbb{R}^n ” by “...homeomorphic to \mathbb{R}^n ”. (Exercise.)

Remark 1.1.5. Brouwer’s theorem of *invariance of domain*, given here without proof, says that if two nonempty open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$. It follows that the dimension of a locally Euclidean space at each point can be defined unambiguously. Furthermore, it is easy to prove that the dimension is constant throughout each connected component. Thus the only way to get a locally Euclidean space of mixed dimensions is to make a disjoint union of components of different dimensions. Anyway, in the definition of topological manifold (see below) we will not admit this kind of spaces.

For the definition of *topological manifold* we demand some further topological properties that ensure that the space is topologically “well-behaved”. (For instance, we want the limit of every sequence to be unique.)

¹Recall that two topological spaces X , Y are **homeomorphic** if there exists a bijection $\varphi : X \rightarrow Y$ such that both φ and its inverse are continuous.

Definition 1.1.6. A **topological manifold of dimension n** , or **topological n -manifold**, is a topological space M that is locally Euclidean of dimension n , Hausdorff² and second countable.³

A **topological manifold** is a topological space that is a topological n -manifold for some n .

Side note: Make sure you are familiar with some basic definitions from topology such as Hausdorff, second countable, connected and compact spaces, and the construction of subspace, product, coproduct and quotient topologies. Chapters 2 and 3 in [Lee10] provides a succinct overview of everything we need.

Remark 1.1.7. The conditions of Hausdorff resp. second countable in Definition 1.1.6 are not redundant. For example, the *line with two origins* (see Exercises) is a locally Euclidean, second countable space that is not Hausdorff. The *long line* and the *Prüfer surface* (see Wikipedia if interested) are locally Euclidean of dimension 1 and 2 respectively, Hausdorff, and connected, but not second countable.

The homeomorphisms that locally identify a topological manifold with Euclidean space are called *charts*:

Definition 1.1.8 (Coordinate charts). Let M be a topological n -manifold. A **chart** (or **coordinate chart**) for M is a homeomorphism $\varphi : U \rightarrow V$, where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets. Its inverse φ^{-1} is a **local parametrization** of M . An **atlas** for M is a collection of charts whose domains cover M .

For the moment, we can see an atlas simply as a way of showing that a space is locally Euclidean.

Remark 1.1.9. Some authors define a chart for M as a *pair* (φ, U) or even a *triple* (φ, U, V) where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets and $\varphi : U \rightarrow V$ is a homeomorphism. Here, instead, we consider the sets U and V as part of the function φ , namely, its domain $\text{Dom}(\varphi)$ and codomain $\text{Cod}(\varphi)$, thus there's no need not specify them separately.⁴ From this point of view, the letters “ U ”, “ V ” are just shorter names for the sets $\text{Dom}(\varphi)$, $\text{Cod}(\varphi)$.

Convention: When talking about subsets (resp. quotients, products, disjoint unions) of topological spaces we'll assume that they are endowed with the subspace (resp. quotient, product, coproduct) topology unless otherwise stated.

Using this convention, let us mention some easy ways to construct new topological manifolds from old ones.

Proposition 1.1.10 (New manifolds from old). *The properties of being Hausdorff or second countable are preserved by taking subspaces, finite products and countable coproducts. In consequence:*

- An open subset of a topological n -manifold is a topological n -manifold.
- A disjoint union $M = \coprod_i M_i$ of countably many topological n -manifolds M_i is a topological n -manifold.
- A product $M = \prod_i M_i$ of finitely many topological manifolds M_i is a topological manifold of dimension $\dim(M) = \sum_i \dim(M_i)$.

Proof. Exercise. □

²Recall that a topological space X is **Hausdorff** if every two different points $x, y \in X$ have disjoint neighborhoods.

³Recall that a topological space X is **second countable** if its topology admits a countable basis. A **basis** for a topology is a family \mathcal{B} of open sets such that every open set is a union of some sets of \mathcal{B} .

⁴Formally, a function f is a triple $f = (X, Y, \Gamma)$ where X, Y are sets (called the **domain** and **codomain** of f , and denoted $\text{Dom}(f)$ and $\text{Cod}(f)$), and Γ is a subset of $X \times Y$ (called the **graph** of f , denoted $\text{Gra}(f)$), such that for each $x \in X$ there is a unique $y \in Y$ (called the **image** of x by f , denoted $f(x)$) such that $(x, y) \in \Gamma$.

Example 1.1.11 (Examples of topological manifolds).

- (a) Of course any open subset of \mathbb{R}^n is a topological manifold.
- (b) An example of topological n -manifold is the **graph**

$$\Gamma_f := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^m$$

of a continuous function $f : U \rightarrow \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set. Indeed, it is homeomorphic to U via the **graph parametrization**

$$\begin{array}{ccc} U & \xrightarrow{\Gamma_f} & \Gamma_f \\ x & \mapsto & (x, f(x)), \end{array} \quad \text{whose inverse is the projection} \quad \begin{array}{ccc} \Gamma_f & \xrightarrow{\Gamma_f} & U \\ (x, y) & \mapsto & x. \end{array}$$

- (c) The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a topological manifold. Being a subset of \mathbb{R}^{n+1} it is Hausdorff and second countable. A possible choice of atlas is given by the so-called **graph coordinates**: Cover \mathbb{S}^n by the $2(n+1)$ open sets $U_i^\pm := \{x \in \mathbb{R}^n \mid \pm x_i > 0\}$, then $\mathbb{S}^n \cap U_i^\pm$ is homeomorphic to the open unit n -ball \mathbb{B}^n via the projection⁵

$$\begin{array}{ccc} \varphi_i^\pm : & \mathbb{S}^n \cap U_i^\pm & \rightarrow \mathbb{B}^n \\ & (x_0, \dots, x_n) & \rightarrow (x_0, \dots, \hat{x}_i, \dots, x_n). \end{array}$$

The maps φ_i^\pm are coordinate charts for \mathbb{S}^n ; we call them *graph coordinates*. Locally this is a special case of the previous item (b): each set $\mathbb{S}^n \cap U_i^\pm$ is (up to permutation of coordinates) the graph of the continuous function on the unit n -ball $\mathbb{B}^n(0)$:

$$\mathbb{B}^n \rightarrow \mathbb{R} : y \mapsto \pm \sqrt{1 - \sum_i y_i^2}.$$

- (d) Real projective space \mathbb{P}^n is a topological n -manifold (exercise).
- (e) The torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is a topological n -manifold (exercise).
- (f) More generally, If M is a topological n -manifold and G is a group of homeomorphisms of M that acts properly discontinuously and without fixed points, then the quotient space M/G is a topological n -manifold.
- (g) If M is a topological n -manifold and $\pi : N \rightarrow M$ is a covering map (with N connected), then N is a topological n -manifold. In particular, the universal covering space of any connected topological manifold is a topological manifold.

We will not prove the following result (although it can be done elementarily).

Theorem 1.1.12 (Classification of topological 1-manifolds). *Every connected topological 1-manifold is homeomorphic to either \mathbb{S}^1 (if it is compact) or to \mathbb{R} (if it is not compact).*

1.2 Differentiable manifolds

Our next goal is to define a kind of spaces and maps called *differentiable manifolds* and *differentiable maps* (or, more precisely \mathcal{C}^k manifolds and maps) with which we can actually do differential calculus. Topological manifolds do not have enough structure because a topology does not allow us to determine whether a function is differentiable or not; it only distinguishes continuous functions. Differentiable manifolds should be locally equivalent to Euclidean open sets (where we already have a well defined notion of \mathcal{C}^k maps; see below), but at the global level they should be allowed to have a more interesting topology. In particular, the sphere, torus, projective space, etc. should become differentiable manifolds.

Before defining \mathcal{C}^k manifolds, let us set up some terminology for \mathcal{C}^k maps in \mathbb{R}^n .

⁵The hat on \hat{x}_i means that we omit the respective coordinate x_i .

Definition 1.2.1 (Euclidean open sets and Euclidean \mathcal{C}^k maps). A **Euclidean open set** is an open subset of some Euclidean space \mathbb{R}^n .

Let $k \in \{0, 1, \dots, \infty\}$. A function $f : U \rightarrow V$ between Euclidean open sets is \mathcal{C}^k at a point $p \in U$ if its partial derivatives of order $\leq k$ are defined in a neighborhood of p and continuous at p . We say that f is \mathcal{C}^k (and we call it a **Euclidean \mathcal{C}^k map**) if it is \mathcal{C}^k at all points $p \in U$.

An **Euclidean \mathcal{C}^k isomorphism** is an Euclidean \mathcal{C}^k map that has an Euclidean \mathcal{C}^k inverse.

Note that every Euclidean \mathcal{C}^k map is continuous because the function f itself is a partial derivative (of order 0) of f . In fact, a \mathcal{C}^0 map is the same thing as a continuous map.

We are now ready to define \mathcal{C}^k manifolds. The key to turn a topological manifold into a \mathcal{C}^k manifold is to choose an appropriate atlas.

Definition 1.2.2 (\mathcal{C}^k manifolds). Let M be a topological n -manifold and $k = 0, \dots, \infty$. Two charts φ, ψ for M , with respective domains $U, V \subseteq M$, are **\mathcal{C}^k -compatible** if the **transition map** from φ to ψ , that is, the homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V),$$

is a \mathcal{C}^k isomorphism (i.e. itself and its inverse are both Euclidean \mathcal{C}^k maps).

A **\mathcal{C}^k -consistent atlas** (or **\mathcal{C}^k atlas**, for short) is an atlas for M whose charts are \mathcal{C}^k compatible with each other. Two \mathcal{C}^k atlases for M are **\mathcal{C}^k -equivalent** if their union is \mathcal{C}^k -consistent.

A **\mathcal{C}^k structure** on M is a maximal \mathcal{C}^k atlas, i.e., a \mathcal{C}^k atlas that is not contained in any other strictly larger \mathcal{C}^k atlas. A **\mathcal{C}^k manifold** is a topological manifold M endowed with a \mathcal{C}^k structure.

Note that a \mathcal{C}^0 manifold is the same thing as a topological manifold. A \mathcal{C}^k manifold with $k \geq 1$ is called a **differentiable manifold**, and a **smooth manifold** is a \mathcal{C}^∞ manifold.

Remark 1.2.3 (Domains and codomains of functions). To be precise, the transition map that we wrote as $\psi \circ \varphi^{-1}$ should actually be defined as $\psi|_{U \cap V}^{\psi(U \cap V)} \circ \left(\varphi|_{U \cap V}^{\varphi(U \cap V)} \right)^{-1}$, using the *restricted* charts

$$\varphi|_{U \cap V}^{\varphi(U \cap V)} : U \cap V \rightarrow \varphi(U \cap V), \quad \psi|_{U \cap V}^{\psi(U \cap V)} : U \cap V \rightarrow \psi(U \cap V).$$

In general we will not write the restrictions explicitly because it is cumbersome. When we compose functions, it should be understood that the resulting composite function is defined in principle at all points where it is possible. (Maybe no points at all!)

We may further restrict a function by specifying a reduced domain or codomain. On the other hand, we shall never specify a domain containing points where the function is not defined, nor a codomain that does not contain the image of the specified domain. Thus a function “ $f : A \rightarrow B$ ” always has domain A and codomain B .

The next proposition shows that it suffices to give any \mathcal{C}^k -consistent atlas (not necessarily a maximal one) to determine a \mathcal{C}^k structure.

Proposition 1.2.4 (\mathcal{C}^k atlas defines \mathcal{C}^k structure). *For a fixed topological manifold M , each \mathcal{C}^k atlas \mathcal{A} is contained in a unique maximal \mathcal{C}^k atlas $\overline{\mathcal{A}}$, which consists of all charts for M that are \mathcal{C}^k -compatible with those of \mathcal{A} . Any other \mathcal{C}^k atlas \mathcal{B} for M is equivalent to \mathcal{A} if and only if it is contained in $\overline{\mathcal{A}}$.*

Proof. Let \mathcal{A} be any \mathcal{C}^k atlas for M . Define

$$\overline{\mathcal{A}} := \{ \varphi \text{ chart for } M \text{ that is } \mathcal{C}^k \text{ compatible with all charts } \theta \in \mathcal{A} \}.$$

Clearly $\overline{\mathcal{A}}$ contains \mathcal{A} . We claim that $\overline{\mathcal{A}}$ is a \mathcal{C}^k atlas. To prove this we have to show that if $\varphi, \psi \in \overline{\mathcal{A}}$ are charts with respective domains $U, V \subseteq M$, then the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$ is \mathcal{C}^k . Take any point $\psi(p) \in \psi(U \cap V)$ and let $\theta \in \mathcal{A}$ be a chart whose domain W contains the point $p \in U \cap V$. Then $\psi(U \cap V \cap W)$ is an open neighborhood of $\psi(p)$ and we can write the restriction⁶

$$\varphi \circ \psi^{-1} : \psi(U \cap V \cap W) \rightarrow \varphi(U \cap V \cap W)$$

as the composition $(\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, which is \mathcal{C}^k because $\varphi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are \mathcal{C}^k by assumption. This proves that $\varphi \circ \psi^{-1}$ is \mathcal{C}^k in a neighborhood of $\psi(p)$, but the same reasoning is valid at any point of $\psi(U \cap V)$, therefore $\varphi \circ \psi^{-1}$ is \mathcal{C}^k .

Finally, from the definition of $\overline{\mathcal{A}}$ it is clear that it is maximal, and that any atlas \mathcal{B} is equivalent to \mathcal{A} if and only if it is contained in $\overline{\mathcal{A}}$. In particular, any atlas \mathcal{B} containing \mathcal{A} is equivalent to \mathcal{A} (by definition of equivalent atlases), and is therefore contained in $\overline{\mathcal{A}}$. Therefore a maximal atlas containing \mathcal{A} is contained in $\overline{\mathcal{A}}$, but in fact it must be equal to $\overline{\mathcal{A}}$ (by maximality). We conclude that $\overline{\mathcal{A}}$ is the unique maximal \mathcal{C}^k atlas containing \mathcal{A} . \square

In consequence, given a topological manifold M and some \mathcal{C}^k atlas \mathcal{A} on M we can speak without ambiguity of *the* \mathcal{C}^k structure $\overline{\mathcal{A}}$ determined by \mathcal{A} .

Remark 1.2.5. For practical purposes the concept of a maximal \mathcal{C}^k atlas is not really important. We usually work with a smaller \mathcal{C}^k atlas and this is all we need e.g. for checking that a function is \mathcal{C}^k (see next section). (In fact, we could have defined a \mathcal{C}^k structure on M as an equivalence class of \mathcal{C}^k atlases, rather than as a maximal \mathcal{C}^k atlas.) In general we won't give any name to the maximal atlas and we'll just speak about "a \mathcal{C}^k manifold M " with the maximal atlas being implicit.

Remark 1.2.6. Any open subset U of a \mathcal{C}^k manifold M has a natural \mathcal{C}^k structure consisting of the \mathcal{C}^k charts of M whose domain is contained in U . (Exercise.) We will also see in the exercises that finite products of \mathcal{C}^k manifolds have a natural \mathcal{C}^k structure.

Example 1.2.7 (Examples of smooth manifolds).

- \mathbb{R}^n (with the atlas consisting of the single chart $\text{id}_{\mathbb{R}^n}$) is a smooth manifold. In general, any topological manifold endowed with a single-chart atlas is automatically a smooth manifold. For example, the graph Γ_f of any *continuous* (sic) function $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ as described in Example 1.1.11, endowed with the projection chart, is a smooth manifold.
- The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Indeed, the atlas given by the graph coordinates (Example 1.1.11) is smooth. To see this, we compute the transition functions (wlog $i < j$):

$$\begin{aligned} \varphi_i^+ \circ (\varphi_j^\pm)^{-1} : \varphi_j^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) &\rightarrow \varphi_i^\pm(U_i^\pm \cap U_i^+ \cap \mathbb{S}^n) \\ (y_0, \dots, y_{n-1}) &\mapsto (y_0, \dots, \hat{y}_i, \dots, y_{j-1}, \pm \sqrt{1 - \sum_i (y_i)^2}, y_j, \dots, y_{n-1}) \end{aligned}$$

and a similar formula works if we replace φ_i^+ by φ_i^- . Hence all the transition maps are smooth.

- More generally, any subset M of \mathbb{R}^k given as the *regular level set* of a smooth map $F : \mathbb{R}^k \rightarrow \mathbb{R}^\ell$ is a $k - \ell$ dimensional smooth manifold "in a natural way"⁷. (Being a level set means $M = F^{-1}(\{c\})$ for some $c \in \mathbb{R}^\ell$ and being a regular level set means that, moreover, the Jacobian $D|_p F$ is surjective for all $p \in M$.) You can prove this quite easily using the implicit function theorem and writing M locally as a graph of smooth functions (analogous to the graph coordinates for the sphere). We will show a more general statement later on when discussing submanifolds (Chapter ??).

⁶see Remark 1.2.3

⁷In particular, the inclusion map $M \hookrightarrow \mathbb{R}^k$ is smooth.

- Projective space \mathbb{P}^n is naturally a smooth manifold (see exercises).
- On the topological 1-manifold $M = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$, each of the two projections $\pi_0, \pi_1 : M \rightarrow \mathbb{R}$ (given by $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$) is a chart defined on the whole manifold M , but these two charts are not \mathcal{C}^k compatible for any $k \geq 1$. (The transition functions are $\pi_1 \circ (\pi_0)^{-1} : x \mapsto x^3$ and its inverse $\pi_0 \circ (\pi_1)^{-1} : y \mapsto \sqrt[3]{y}$, which is not differentiable.) Thus these charts determine two different \mathcal{C}^k structures on M .

1.3 Differentiable maps

We are about to define \mathcal{C}^k maps between \mathcal{C}^k manifolds. The plan is to reduce the question of differentiability to the case of a map between Euclidean open sets. We will do so by using charts.

In general, when studying a map $f : M \rightarrow N$ between manifolds, charts allow us to *locally* express f as a map between subsets of Euclidean space.

Definition 1.3.1 (Local expression of a map). Let M, N be topological manifolds and let $f : M \rightarrow N$ be any function. A **local expression** (or **coordinate representation**) of f at some point $p \in M$ is a composite map

$$f|_{\varphi}^{\psi} := \psi \circ f \circ \varphi^{-1},$$

where φ and ψ are charts of M and N whose domains U, V contain the points p and $f(p)$ respectively.

In particular, we will use local expressions to decide whether a map is \mathcal{C}^k or not.

Definition 1.3.2 (\mathcal{C}^k maps between manifolds). Let $f : M \rightarrow N$ be a function between \mathcal{C}^k manifolds and let $p \in M$. Take a local expression $f|_{\varphi}^{\psi} = \psi \circ f \circ \varphi^{-1}$ of f at p , where φ, ψ are \mathcal{C}^k charts of M and N respectively. We say that f is \mathcal{C}^k at the point p if

the local expression $f|_{\varphi}^{\psi}$ is defined on a neighborhood of $\varphi(p)$ and is \mathcal{C}^k at the point $\varphi(p)$. (1.1)

The function $f : M \rightarrow N$ is \mathcal{C}^k if it is \mathcal{C}^k at all points $p \in M$.

A \mathcal{C}^k **isomorphism** is a \mathcal{C}^k map that has a \mathcal{C}^k inverse. If $k \geq 1$, a \mathcal{C}^k isomorphism is also called a \mathcal{C}^k **diffeomorphism** (or a \mathcal{C}^k **diffeo**, for short).

Lemma 1.3.3. *In Definition 1.3.2, the proposition (1.1) does not depend on which pair of \mathcal{C}^k charts φ, ψ are chosen (provided, of course, that their domains U, V contain p and $f(p)$ respectively). The proposition (1.1) also implies that f is continuous at p , in fact, it is equivalent to the continuity of f at p in the case $k = 0$.*

Proof. Suppose first that f is continuous at p . Then there is some open neighborhood U' of p such that $f(U') \subseteq V$, and we may assume $U' \subseteq U$. Furthermore, the restriction $f|_{U'}^V$ is continuous at p . It follows that the local expression $f|_{\varphi}^{\psi}$ is defined on $\varphi(U')$ (which is a neighborhood of $\varphi(p)$) and continuous at $\varphi(p)$. Thus the proposition (1.1) holds for $k = 0$.

Now suppose that the proposition (1.1) holds for some $k \geq 0$. Let us show that f is continuous at p . By hypothesis there is some open neighborhood of $\varphi(p)$ where $f|_{\varphi}^{\psi}$ is defined. We can write this open neighborhood as $\varphi(U')$, where U' is an open neighborhood of p . The fact that $f|_{\varphi}^{\psi}$ is defined on $\varphi(U')$ is equivalent to saying that $f(U') \subseteq V$. Thus we can define the restriction $f|_{U'}^V$, which can be written as

$$f|_{U'}^V = \psi^{-1} \circ f|_{\varphi}^{\psi} \circ \varphi : U' \rightarrow V,$$

and is therefore continuous at p because $f|_{\varphi}^{\psi}$ is continuous at $\varphi(p)$. This shows that f is continuous at p .

Finally, let us show that (1.1) does not depend on the choice of charts. Thus we assume again that the proposition (1.1) holds, and we construct a second local expression $f|_{\tilde{\varphi}}^{\tilde{\psi}}$ using any pair of charts $\tilde{\varphi}$ and $\tilde{\psi}$ with domains $\tilde{U} \subseteq M$, $\tilde{V} \subseteq N$ containing p and $f(p)$ respectively. This new local expression is defined on some neighborhood of $\tilde{\varphi}(p)$ (since f is continuous at p), and is related to the old one by the equation

$$f|_{\tilde{\varphi}}^{\tilde{\psi}} = (\psi \circ \tilde{\psi}^{-1}) \circ f|_{\varphi}^{\psi} \circ (\varphi \circ \tilde{\varphi}^{-1})$$

Since the transition maps $(\psi \circ \tilde{\psi}^{-1})$ and $(\varphi^{-1} \circ \tilde{\varphi})$ are \mathcal{C}^k , we conclude that $f|_{\tilde{\varphi}}^{\tilde{\psi}}$ is \mathcal{C}^k at the point $\tilde{\varphi}(p)$. This shows that the proposition (1.1) is independent of the choice of φ and ψ . \square

Example 1.3.4. 1. The identity map of any \mathcal{C}^k manifold is a \mathcal{C}^k map. (Exercise.)
 2. The composite of two \mathcal{C}^k maps is a \mathcal{C}^k map. (Exercise.)
 3. If M is a \mathcal{C}^k manifold, then every \mathcal{C}^k chart of M , as well as its inverse, are \mathcal{C}^k maps. (Exercise.)

A \mathcal{C}^k structure on a topological manifold M allows us to determine which maps that go to M are \mathcal{C}^k . But the reciprocal property also holds: if we know which maps to M are \mathcal{C}^k , this information determines the \mathcal{C}^k structure of M .

Proposition 1.3.5. *Let $\mathcal{A}_0, \mathcal{A}_1$ be two \mathcal{C}^k atlases on a topological manifold M , defining two \mathcal{C}^k manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases \mathcal{A}_i are equivalent if and only if the following property holds:*

For every function $f : N \rightarrow M$ (where N is a \mathcal{C}^k manifold), the function f is \mathcal{C}^k as a map $N \rightarrow M_0$ if and only if it is \mathcal{C}^k as a map $N \rightarrow M_1$.

Proof. Exercise. \square