Introduction to Differentiable Manifolds
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Exercise series 2, with solutions **2021–10–12**

Exercise 2.1 (Stereographic projection.)**.** Let $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ be the *north pole* and $S = -N$ the *south pole* of the sphere $\mathbb{S}^n$. Define stereographic projection $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_0, \dots, x_n) = \frac{1}{1 - x_n}(x_0, \dots, x_{n-1}).$$

Let $\widetilde{\sigma}(x) = \sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) Show that σ is bijective, and

$$\sigma^{-1}(u_0, \dots, u_{n-1}) = \frac{1}{|u|^2 + 1}(2u_0, \dots, 2u_{n-1}, |u|^2 - 1).$$

Solution. To show that σ is bijective and σ^{-1} is its inverse, it is sufficient to verify that

$$\sigma^{-1} \circ \sigma = \mathrm{id},$$
$$\sigma \circ \sigma^{-1} = \mathrm{id}.$$

Let us show first how one could find the formulas for σ and σ^{-1}. We use the following notation: if $u \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we denote (u, a) the point of $\mathbb{R}^{n+1}$ whose first n coordinates are the u_i's and whose last coordinate is a. Thus the hyperplane $\Pi = \mathbb{R}^n \times \{0\} \equiv \mathbb{R}^n$ contains the points of the form $(u, 0)$.

Every non-horizontal line r containing the point N intersects the sphere $\mathbb{S}^n$ at one point x (other than N) and intersects the plane Π at a point $(u, 0)$. We want the formulas for the maps $\sigma : x \mapsto u$ and $\sigma^{-1} : u \mapsto x$. If we know x, then r is the image of the map

$$t \in \mathbb{R} \mapsto N + t(x - N) = (tx_0, \dots, tx_{n-1}, 1 + t(x_n - 1))$$

The intersection with the plane Π occurs when the last coordinate is 0, that is, when $t = \frac{1}{1-x_n}$. The point of intersection is $(u, 0)$, where

$$u = (tx_0, \dots, tx_{n-1}) = \frac{1}{1 - x_n}(x_0, \dots, x_{n-1}).$$

This gives the formula for σ.

To compute σ^{-1} suppose we know the point u. Then r is the image of the map

$$t \in \mathbb{R} \mapsto N + t((u, 0) - N)) = (tu_0, \dots, tu_n, 1 - t)$$

This point is contained in $\mathbb{S}^n$ if and only if $t^2|u|^2 + (1-t)^2 = 1$. We rewrite the equation as $(|u|^2 + 1)t^2 - 2t = 0$ and find the solutions $t = 0$ (corresponding to the north pole) and $t = \frac{2}{|u|^2+1}$, corresponding to the point

$$x = \frac{1}{|u|^2 + 1}(2u_0, \dots, 2u_{n-1}, |u|^2 - 1).$$

This gives the formula for σ^{-1}. □

(b) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that $\{\sigma, \widetilde{\sigma}\}$ is a smooth atlas for $\mathbb{S}^n$.

Solution. The domains of σ and $\widetilde{\sigma}$ cover $\mathbb{S}^n$. The transition map $\widetilde{\sigma} \circ \sigma^{-1}$, defined on $\mathbb{R}^n \setminus \{0\}$ by the formula

$$\widetilde{\sigma} \circ \sigma^{-1}(u) = \sigma\left(-\frac{2u_0, \dots, 2u_{n-1}, |u|^2 - 1}{|u|^2 + 1}\right) = -\frac{(u_0, \dots, u_{n-1})}{|u|^2},$$

is smooth. □

(c) Show that the smooth structure defined by the atlas $\{\sigma, \widetilde{\sigma}\}$ is the same as the one defined via graph coordinates in the lecture.

Solution. It suffices to show that each chart in $\{\sigma, \widetilde{\sigma}\}$ is compatible with all the graph charts $\phi_i^{\pm}$. The transition function is given by the formula

$$\begin{aligned}\phi_i^+ \circ \sigma^{-1}(y) &= \phi_i^+ \left(\frac{2y_0, \dots, 2y_{n-1}, |y|^2 - 1}{|y|^2 + 1}\right) \\ &= \left(\frac{2y_0, \dots, 2y_{i-1}, 2y_{i+1}, \dots, 2y_{n-1}, |y|^2 - 1}{|y|^2 + 1}\right)\end{aligned}$$

and is therefore a smooth map. The inverse map $\sigma \circ \phi_i^+(-1)$ is defined on $\sigma^{-1}(\mathbb{S}^n \cap U_i^+)$ and is [...] □

Exercise 2.2. Show that $\mathbb{P}^n$ is a smooth manifold. (Use exercise from series 1.)

Solution. From exercise series 1, we already know that $\mathbb{P}^n$ is a topological manifold. Convention: All indices i, j, k are in the set $n' = n + 1 = \{0, \dots, n\}$. For each i we have a chart $\phi_i : U_i \to \mathbb{R}^{n' \setminus \{k\}} \equiv \mathbb{R}^n$, given by

$$U_i = \{[x] \in \mathbb{P}^n : x^i \neq 0\} \subseteq \mathbb{P}^n,$$

$$\phi_i : [x] \mapsto \left(\frac{x_j}{x_i}\right)_{j \neq i}.$$

Its inverse is $\phi_i^{-1} : (y^j)_{j \neq i} \mapsto [x^j]_j$ where $x^j := y^j$ if $j \neq i$ and $x^i := 1$.
The nontrivial transition functions are $\phi_k \circ \phi_i^{-1}$, with $k \neq i$, defined on

$$\phi_i(U_k) = \{x \in \mathbb{R}^{n' \setminus \{i\})} : x_k \neq 0\}$$

by the formula

$$\phi_k \circ \phi_i^{-1} : y \mapsto (\frac{x^j}{y_k})_{j \neq k},$$

where the x^j is defined as above: $x^j = y^j$ if $j \neq i$, $x^i = 1$. The transition maps are smooth, therefore the atlas is smooth. □

Exercise 2.3 (Open submanifolds)**.** Let N be an open subset of a $\mathcal{C}^k$ n-manifold $(M, \mathcal{A})$, and let $\mathcal{B}$ be the set of all charts $\varphi \in \mathcal{A}$ whose domain is contained in N. Prove that:

(1) $\mathcal{B}$ is a $\mathcal{C}^k$ structure for N, making N into a $\mathcal{C}^k$ n-manifold. We call the $\mathcal{C}^k$ manifold $(N, \mathcal{B})$ an *open submanifold* of M.

Solution. N is a topological n-manifold because it is an open subset of M. Each element of $\mathcal{B}$ is a topological chart of N, because it is an homeomorphism $\varphi : U \to V$, where $U \subseteq N$ is an open subset of M (hence of N) and V is an open subset of $\mathbb{R}^n$. These charts are smoothly compatible because they are taken from the $\mathcal{C}^k$ structure of M. Finally, let us see that the atlas $\mathcal{B}$ covers N, because for each point $p \in N$ there is a chart $\varphi \in \mathcal{A}$ of M with domain $U \ni p$, and then the restriction $\psi = \varphi|_{U \cap N} : U \cap N \to \varphi(U \cap N)$ is a chart in $\mathcal{B}$ that is defined at p. □

(2) The inclusion map $\iota : N \hookrightarrow M$ is $\mathcal{C}^k$.

Solution. We just need to check that the local expressions of ι are $\mathcal{C}^k$. These local expressions are of the form

$$\iota_\psi^\varphi = \varphi \circ \iota \circ \psi^{-1} = \varphi \circ \psi^{-1}$$

with $\varphi \in \mathcal{B} \subseteq \mathcal{A}$ and $\psi \in \mathcal{A}$. They are $\mathcal{C}^k$ because they are transition maps of the atlas $\mathcal{A}$. □

(3) A function $f : L \to N$ (where L is a $\mathcal{C}^k$ manifold) is $\mathcal{C}^k$ if and only if the composite $\iota \circ f$ is $\mathcal{C}^k$.

Solution. If f is $\mathcal{C}^k$, the composite $\iota \circ f$ is $\mathcal{C}^k$ because ι is $\mathcal{C}^k$. Reciprocally, suppose ι is $\mathcal{C}^k$. Then f is $\mathcal{C}^k$, because any local expression f_ξ^ψ (with ξ a chart of L and $\psi \in \mathcal{B}$) is also a local expression of $\iota \circ f$. Indeed,

$$f_\xi^\psi = \psi \circ f \circ \xi^{-1} = \psi \circ \iota \circ f \circ \xi^{-1}.$$

□

Use this to show that the general linear group $GL(n, \mathbb{R})$, i.e. the set consisting of invertible $n \times n$ matrices, is naturally a smooth manifold.

Solution. The set $GL(n, \mathbb{R})$ is an open subset of the smooth manifold $M(n, \mathbb{R}) \equiv \mathbb{R}^{n \times n}$ therefore it is naturally a smooth manifold. □

Exercise 2.4 (Some basic properties of $\mathcal{C}^k$ manifolds)**.** Let M be a topological manifold and let $\mathcal{A}$ be a $\mathcal{C}^k$ atlas, with $k \geq 1$. Show that:

(1) Let $\mathcal{A}'$ be another smooth atlas on M. Then $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure on M if and only if their union is a smooth atlas.

Solution. Let $\bar{\mathcal{A}}$ denote the maximum atlas determined by $\mathcal{A}$: the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. We have to show that $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$ if and only if $\mathcal{A} \cup \mathcal{A}'$ is an atlas. We can define the equivalence relation between charts $\phi \sim \psi$ iff ϕ is smoothly compatible with ψ. Then the argument follow from the transitivity property of the smoothly compatible relationship. If $\mathcal{A} \cup \mathcal{A}'$ is an atlas then, $\phi \sim \psi$ for every $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}'$, This implies $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$. Conversely, $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$ implies that for $\xi \in \bar{\mathcal{A}}$, $\xi \sim \phi$ for every $(U, \phi) \in \mathcal{A}$ then $\xi \sim \psi$ for every $(V, \psi) \in \mathcal{A}'$. Therefore, $\phi \sim \psi$ for every $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}'$. So $\mathcal{A} \cup \mathcal{A}'$ is an atlas. □

(2) Every $\mathcal{C}^k$ chart $\varphi : U \to V$ of M is a $\mathcal{C}^k$ diffeomorphism.

Solution. We know that $\phi : U \to \phi(U)$ is an homeomorphism. Moreover ϕ is a smooth map iff the composition $id_{\mathbb{R}^n} \circ \phi \circ \phi^{-1}$ is smooth. Since $id_{\mathbb{R}^n} \circ \phi \circ \phi^{-1} = id_{\mathbb{R}^n}$ is a smooth map so ϕ is a smooth map. One can show that ϕ^{-1} is smooth as well by a similar argument. □

(3) The composite $g \circ f$ of two $\mathcal{C}^k$ maps $f : M \to N$, $g : N \to P$ is a $\mathcal{C}^k$ map.

Solution. Let $f : M \to N$ and $g : N \to P$ be smooth maps. Then by definitions the maps $\phi \circ f \circ \psi^{-1}$ and $\psi \circ g \circ \xi^{-1}$ are smooth for every smooth local chart ϕ, ψ, and ξ on M, N, and P respectively. Then the composition $h = f \circ g$ is smooth since the function

$$\phi \circ h \circ \xi^{-1} = \phi \circ f \circ g \circ \xi^{-1} = \phi \circ f \circ \psi^{-1} \circ \psi \circ g \circ \xi^{-1}$$

is smooth for every local chart. □

(4) Let $\mathcal{A}_0$, $\mathcal{A}_1$ be two $\mathcal{C}^k$ atlases on M, defining two $\mathcal{C}^k$ manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases $\mathcal{A}_i$ are equivalent iff the following holds:
For every function $f : N \to M$ (where N is a $\mathcal{C}^k$ manifold), the function f is $\mathcal{C}^k$ as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Solution. ($\Rightarrow$) is clear. Let us prove prove ($\Leftarrow$). Thuss assuming the last property holds, let us show that the atlases $\mathcal{A}_0$, $\mathcal{A}_1$ are equivalent. For this, it suffices to prove the following:

Claim: *For every $\varphi \in \mathcal{A}_0$, $\psi \in \mathcal{A}_1$ the transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are $\mathcal{C}^k$.*

Proof of claim: The function φ is a $\mathcal{C}^k$ isomorphism $U \to V$, where $U \subseteq M_0$ and $V \subseteq \mathbb{R}^n$ are open sets. Therefore its inverse $\varphi^{-1} : V \to M_0$ is $\mathcal{C}^k$. Then, by the hypothesis, $\varphi^{-1} : V \to M_1$ is $\mathcal{C}^k$. Therefore $\psi \circ \varphi^{-1}$ is $\mathcal{C}^k$ for each $\psi \in \mathcal{A}_1$, as claimed. An analogous argument shows that $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$. □

Exercise 2.5 (Smooth structures on $\mathbb{R}$)**.** On the real line $\mathbb{R}$ (with the standard topology) we define two atlases $\mathcal{A} = \{\mathrm{id}_{\mathbb{R}}\}$, $\mathcal{B} = \{\varphi\}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^3$.

(a) Convince yourself that $\mathcal{B}$ defines a smooth structure on $\mathbb{R}$.

Solution. The atlas $\mathcal{B}$ contains a single chart, therefore it is a smooth atlas, and it is contained in a unique maximal smooth atlas. □

(b) Show that $\mathcal{A}$ and $\mathcal{B}$ define distinct smooth structures.

Solution. The charts ϕ and $\mathrm{id}_{\mathbb{R}}$ are not smoothly compatible since the $\mathrm{id}_{\mathbb{R}} \circ \phi^{-1} : y \mapsto y^{1/3}$ is not smooth. Therefore, the atlases $\mathcal{A}$ and $\mathcal{B}$ define distinct smooth structures. □

(c) Find a smooth diffeomorphism $(\mathbb{R}, \overline{\mathcal{A}}) \to (\mathbb{R}, \overline{\mathcal{B}})$.

Solution. The homeomorphism $f : (\mathbb{R}, \overline{\mathcal{A}}) \to (\mathbb{R}, \overline{\mathcal{B}})$ defined by $f(x) = x^3$ is a diffeomorphism since the local expressions

$$f^{\varphi}_{\mathrm{id}_{\mathbb{R}}} = \mathrm{id}_{\mathbb{R}} \circ f \circ \varphi^{-1} = \mathrm{id}_{\mathbb{R}}$$

and

$$f^{-1}|^{\mathrm{id}_{\mathbb{R}}}_{\varphi} = \varphi \circ f^{-1} \circ \mathrm{id}_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$$

are smooth. □

Exercise 2.6 (Product manifolds)**.** Let M_0 and M_1 be $\mathcal{C}^k$ manifolds of dimension m_0 and m_1 respectively. Recall that $M = M_0 \times M_1$ (with the product topology) is a topological manifold of dimension $m_0 + m_1$.

(a) Find a natural $\mathcal{C}^k$ structure on $M_0 \times M_1$ such that two projections $\pi_i : M_0 \times M_1 \to M_i$ are $\mathcal{C}^k$ maps.

Solution. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be atlases that define the $\mathcal{C}^k$ structures of M_0 and M_1 respectively. We define for M the atlas $\mathcal{A}$ consisting of charts

$$\varphi = \varphi_0 \times \varphi_1 : U_0 \times U_1 \to V_0 \times V_1,$$

where $\mathcal{A}_0 \ni \varphi_0 : U_0 \to V_0$ and $\mathcal{A}_1 \ni \varphi_1 : U_1 \to V_1$. The transition functions are of the form

$$\varphi \circ \psi^{-1} = (\varphi_0 \circ \psi_0^{-1}) \times (\varphi_1 \circ \psi_1^{-1}),$$

thus they are $\mathcal{C}^k$ maps. Therefore $\mathcal{A}$ is a $\mathcal{C}^k$ atlas for M.

Let us show that the projection maps $\pi_i : M_0 \times M_1 \to M_i$ are $\mathcal{C}^k$. Indeed, for every point $p = (p_0, p_1) \in M$ we can find a chart $\varphi = \varphi_0 \times \varphi_1$ defined at p, then note that φ_i is a chart defined at $p_i = \pi_i(p)$. Thus we can write the local expression

$$\pi_i|^{\varphi_i}_{\varphi} = \varphi_i \circ \pi_i \circ (\varphi_0^{-1} \times \varphi_1{-1}) = \pi_i : V_0 \times V_1 \to V_i$$

which is $\mathcal{C}^k$. □

(b) Show that a map $f : N \to M_0 \times M_1$ (where N is another $\mathcal{C}^k$ manifold) is $\mathcal{C}^k$ if and only if the two composite maps $\pi_i \circ f$ are $\mathcal{C}^k$.

Solution. If f is $\mathcal{C}^k$, then the composite maps $f^i = \pi_i \circ f$ are $\mathcal{C}^k$ because the π_i are $\mathcal{C}^k$. Conversely, suppose that the composite maps f^i are $\mathcal{C}^k$. Then f is $\mathcal{C}^k$ because for every chart ψ of N and every chart $\varphi = \varphi_0 \times \varphi_1 \in \mathcal{A}$ of $M_0 \times M_1$, the local expression

$$\varphi \circ f \circ \psi^{-1} = (\varphi_0 \circ f_0 \circ \psi^{-1}, \varphi_1 \circ f_1 \circ \psi^{-1})$$

is $\mathcal{C}^k$. □