Introduction to Differentiable Manifolds

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Contents

Main references

[Lee10] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2010.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second edition. Springer, 2013.

Both books have online versions available in the library.

0.1 Practical remarks about the course

0.1.1 Content for the exam

Studying the content under a heading marked by an asterisk is not mandatory by itself. You should read it only if you find it interesting or helpful for understanding the rest.

1 Manifolds

The goal of this course is to extend differential and integral calculus from Euclidean space $\mathbb{R}^n$ to all *differentiable manifolds* such as the n-sphere, the n-torus, etc. Roughly speaking, a differentiable manifold is a space that

- is endowed with a certain topology,
- has, in addition, a *differentiable structure* that allows us to distinguish whether a map is differentiable or not, rather than just continuous, and
- locally looks like Euclidean space $\mathbb{R}^n$.

1.1 Topological manifolds

[Lee13], Chapter 1 and [Lee10], Chapter 2

Let us pospone the question of differentiability and focus on topology. As said, we want to study spaces that "locally look like" Euclidean space $\mathbb{R}^n$.

Definition 1.1.1 (Locally Euclidean space)**.** Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is **locally Euclidean** of dimension n at a point $p \in M$ if the point p has an open neighborhood that is homeomorphic[1] to an open subset of $\mathbb{R}^n$. If this holds for all points $p \in M$, we say that M is locally Euclidean of dimension n.

A typical example is the circle: it is locally Euclidean of dimension 1 but not globally homeomorphic to any subset of $\mathbb{R}$.

Example 1.1.2. The **circle** $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (with the subspace topology) is locally Euclidean of dimension 1: let $(x_0, y_0) \in \mathbb{S}^1$, wlog $y_0 > 0$, then $U := (\mathbb{R} \times \mathbb{R}^+) \cap \mathbb{S}^1$ is an open subset of $\mathbb{S}^1$ containing (x_0, y_0) and homeomorphic to $(-1, 1)$ via the map $U \to (-1, 1)$ that sends $(x, y) \mapsto x$.

We will see more examples later on (see e.g. Examples 1.1.11 below). Let us make some general comments.

Remark 1.1.3. If a space M is locally Euclidean of dimension 0, then every point has a neighborhood homeomorphic to $\mathbb{R}^0 = \{0\}$, i.e. a point. In other words, M is a discrete topological space.

Remark 1.1.4. In the definition of locally Euclidean space, we could have replaced "...homeomorphic to an open subset of $\mathbb{R}^n$" by "...homeomorphic to $\mathbb{R}^n$". (Exercise.)

Remark 1.1.5. Brouwer's theorem of *invariance of domain*, given here without proof, says that if two nonempty open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$. It follows that the dimension of a locally Euclidean space at each point can be defined unambiguously. Furthermore, it is easy to prove that the dimension is constant throughtout each connected component. Thus the only way to get a locally Euclidean space of mixed dimensions is to make a disjoint union of components of different dimensions. Anyway, in the definition of topological manifold (see below) we will not admit this kind of spaces.

[1] Recall that two topological spaces X, Y are **homeomorphic** if there exists a bijection $\varphi : X \to Y$ such that both φ and its inverse are continuous.

For the definition of *topological manifold* we demand some further topological properties that ensure that the space is topologically "well-behaved". (For instance, we want the limit of every sequence to be unique.)

Definition 1.1.6. A **topological manifold of dimension** n, or **topological** n**-manifold**, is a topological space M that is locally Euclidean of dimension n, Hausdorff[2] and second countable.[3]

A **topological manifold** is a topological space that is a topological n-manifold for some n.

Side note: Make sure you are familiar with some basic definitions from topology such as Hausdorff, second countable, connected and compact spaces, and the construction of subspace, product, coproduct and quotient topologies. Chapters 2 and 3 in [Lee10] provides a succinct overview of everything we need.

Remark 1.1.7. The conditions of Hausdorff resp. second countable in Definition 1.1.6 are not redundant. For example, the *line with two origins* (see Exercises) is a locally Euclidean, second countable space that is not Hausdorff. The *long line* and the *Prüfer surface* (see Wikipedia if interested) are locally Euclidean of dimension 1 and 2 respectively, Hausdorff, and connected, but not second countable.

The homeomorphisms that locally identify a topological manifold with Euclidean space are called *charts*:

Definition 1.1.8 (Coordinate charts)**.** Let M be a topological n-manifold. A **chart** (or **coordinate chart**) for M is a homeomorphism $\varphi : U \to V$, where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets. Its inverse φ^{-1} is a **local parametrization** of M. An **atlas** for M is a collection of charts whose domains cover M.

For the moment, we can see an atlas simply as a way of showing that a space is locally Euclidean.

Remark 1.1.9. Some authors define a chart for M as a *pair* (φ, U) or even a *triple* (φ, U, V) where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets and $\varphi : U \to V$ is a homeomorphism. Here, instead, we consider the sets U and V as part of the function φ, namely, its domain $\mathrm{Dom}(\varphi)$ and codomain $\mathrm{Cod}(\varphi)$, thus there's no need not specify them separately.[4] From this point of view, the letters "U", "V" are just shorter names for the sets $\mathrm{Dom}(\varphi)$, $\mathrm{Cod}(\varphi)$.

Convention: When talking about subsets (resp. quotients, products, disjoints unions) of topological spaces we'll assume that they are endowed with the subspace (resp. quotient, product, coproduct) topology unless otherwise stated.

Using this convention, let us mention some easy ways to construct new topological manifolds from old ones.

Proposition 1.1.10 (New manifolds from old)**.** *The properties of being Hausdorff or second countable are preserved by taking subspaces, finite products and countable coproducts. In consequence:*

[2] Recall that a topological space X is **Hausdorff** if every two different points $x, y \in X$ have disjoint neighborhoods.

[3] Recall that a topological space X is **second countable** if its topology admits a countable base. A **base** for a topology is a family $\mathcal{B}$ of open sets such that every open set is a union of some sets of $\mathcal{B}$.

[4] Formally, a function f is a triple $f = (X, Y, \Gamma)$ where X, Y are sets (called the **domain** and **codomain** of f, and denoted $\mathrm{Dom}(f)$ and $\mathrm{Cod}(f)$), and Γ is a subset of $X \times Y$ (called the **graph** of f, denoted $\mathrm{Gra}(f)$), such that for each $x \in X$ there is a unique $y \in Y$ (called the **image** of x by f, denoted $f(x)$) such that $(x, y) \in \Gamma$.

- *An open subset of a topological n-manifold is a topological n-manifold.*
- *A disjoint union $M = \coprod_i M_i$ of countably many topological n-manifolds M_i is a topological n-manifold.*
- *A product $M = \prod_i M_i$ of finitely many topological manifolds M_i is a topological manifold of dimension* $\dim(M) = \sum_i \dim(M_i)$.

Proof. Exercise. □

Example 1.1.11 (Examples of topological manifolds)**.**

(a) Of course any open subset of $\mathbb{R}^n$ is a topological manifold.

(b) An example of topological n-manifold is the **graph**

$$\Gamma_f := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^m$$

of a continuous function $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set. Indeed, it is homeomorphic to U via the **graph parametrization**

$$\begin{array}{ccc} U & \to & \Gamma_f \\ x & \mapsto & (x, f(x)), \end{array} \quad \text{whose inverse is the projection} \quad \begin{array}{ccc} \Gamma_f & \to & U \\ (x, y) & \mapsto & x. \end{array}$$

(c) The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a topological manifold. Being a subset of $\mathbb{R}^{n+1}$ it is Hausdorff and second countable. A possible choice of atlas is given by the so-called **graph coordinates**: Cover $\mathbb{S}^n$ by the $2(n+1)$ open sets $U_i^\pm := \{x \in \mathbb{R}^n \mid \pm x_i > 0\}$, then $\mathbb{S}^n \cap U_i^\pm$ is homemorphic to the open unit n-ball $\mathbb{B}^n$ via the projection[5]

$$\begin{array}{cccc} \varphi_i^\pm : & \mathbb{S}^n \cap U_i^\pm & \to & \mathbb{B}^n \\ & (x_0, \ldots, x_n) & \to & (x_0, \ldots, \widehat{x_i}, \ldots, x_n). \end{array}$$

The maps $\varphi_i^\pm$ are coordinate charts for $\mathbb{S}^n$; we call them *graph coordinates.* Locally this is a special case of the previous item (b): each set $\mathbb{S}^n \cap U_i^\pm$ is (up to permutation of coordinates) the graph of the continuous function on the unit n-ball $\mathbb{B}^n(0)$:

$$\mathbb{B}^n \to \mathbb{R} : y \mapsto \pm\sqrt{1 - \textstyle\sum_i y_i^2}.$$

(d) Real projective space $\mathbb{P}^n$ is a topological n-manifold (exercise).

(e) The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a topological n-manifold (exercise).

(f) More generally, If M is a topological n-manifold and G is a group of homeomorphisms of M that acts properly discontinuously and without fixed points, then the quotient space M/G is a topological n-manifold.

(g) If M is a topological n-manifold and $\pi : N \to M$ is a covering map (with N connected), then N is a topological n-manifold. In particular, the universal covering space of any connected topological manifold is a topological manifold.

We will not prove the following result (although it can be done elementarily).

Theorem 1.1.12 (Classification of topological 1-manifolds)**.** *Every connected topological 1-manifold is homeomorphic to either $\mathbb{S}^1$ (if it is compact) or to $\mathbb{R}$ (if it is not compact).*

[5]The hat on $\widehat{x_i}$ means that we omit the respective coordinate x_i.

1.2 Differentiable manifolds

Our next goal is to define a kind of spaces and maps called *differentiable manifolds* and *differentiable maps* (or, more precisely $\mathcal{C}^k$ manifolds and maps) with which we can actually do differential calculus. Topological manifolds do not have enough structure because a topology does not allow us to determine whether a function is differentiable or not; it only distinguishes continuous functions. Differentiable manifolds should be locally equivalent to Euclidean open sets (where we already have a well defined notion of $\mathcal{C}^k$ maps; see below), but at the global level they should be allowed to have a more interesting topology. In particular, the sphere, torus, projective space, etc. should become differentiable manifolds.

Before defining $\mathcal{C}^k$ manifolds, let us set up some terminology for $\mathcal{C}^k$ maps in $\mathbb{R}^n$.

Definition 1.2.1 (Euclidean open sets and Euclidean $\mathcal{C}^k$ maps)**.** A **Euclidean open set** is an open subset of some Euclidean space $\mathbb{R}^n$.
Let $k \in \{0, 1, \ldots, \infty\}$. A function $f : U \to V$ between Euclidean open sets is $\mathcal{C}^k$ at a point $p \in U$ if its partial derivatives of order $\leq k$ are defined in a neighborhood of p and continuous at p. We say that f is $\mathcal{C}^k$ (and we call it a **Euclidean $\mathcal{C}^k$ map**) if it is $\mathcal{C}^k$ at all points $p \in U$.
An **Euclidean $\mathcal{C}^k$ isomorphism** is an Euclidean $\mathcal{C}^k$ map that has an Euclidean $\mathcal{C}^k$ inverse.

Note that every Euclidean $\mathcal{C}^k$ map is continuous because the function f itself is a partial derivative (of order 0) of f. In fact, a $\mathcal{C}^0$ map is the same thing as a continuous map.

We are now ready to define $\mathcal{C}^k$ manifolds. The key to turn a topological manifold into a $\mathcal{C}^k$ manifold is to choose an appropriate atlas.

Definition 1.2.2 ($\mathcal{C}^k$ manifolds)**.** Let M be a topological n-manifold and $k = 0, \ldots, \infty$. Two charts φ, ψ for M, with respective domains $U, V \subseteq M$, are **$\mathcal{C}^k$-compatible** if the **transition map** from φ to ψ, that is, the homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

is a $\mathcal{C}^k$ isomorphism (i.e. itself and its inverse are both Euclidean $\mathcal{C}^k$ maps).

A **$\mathcal{C}^k$-consistent atlas** (or **$\mathcal{C}^k$ atlas**, for short) is an atlas for M whose charts are $\mathcal{C}^k$ compatible with each other. Two $\mathcal{C}^k$ atlases for M are **$\mathcal{C}^k$-equivalent** if their union is $\mathcal{C}^k$-consistent. A **$\mathcal{C}^k$ structure** on M is a maximal $\mathcal{C}^k$ atlas, i.e., a $\mathcal{C}^k$ atlas that is not contained in any other strictly larger $\mathcal{C}^k$ atlas. A **$\mathcal{C}^k$ manifold** is a topological manifold M endowed with a $\mathcal{C}^k$ structure $\mathcal{A}$. (More formally, the $\mathcal{C}^k$ manifold is the pair $(M, \mathcal{A})$.)

Note that a $\mathcal{C}^0$ manifold is the same thing as a topological manifold. A $\mathcal{C}^k$ manifold with $k \geq 1$ is called a **$\mathcal{C}^k$-differentiable manifold**. A **smooth manifold** is a $\mathcal{C}^\infty$ manifold.

Remark 1.2.3 (Domains and codomains of functions)**.** To be precise, the transition map that we wrote as $\psi \circ \varphi^{-1}$ should actually be defined as $\psi|_{U\cap V}^{\psi(U\cap V)} \circ \left(\varphi|_{U\cap V}^{\varphi(U\cap V)}\right)^{-1}$, using the *restricted* charts

$$\varphi|_{U\cap V}^{\varphi(U\cap V)} : U \cap V \to \varphi(U \cap V), \quad \psi|_{U\cap V}^{\psi(U\cap V)} : U \cap V \to \psi(U \cap V).$$

In general we will not write the restrictions explicitly because it is cumbersome. When we compose functions, it should be understood that the resulting composite function is defined in principle at all points where it is possible. (Maybe no points at all!)

We may further restrict a function by specifying a reduced domain or codomain. On the other hand, we shall never specify a domain containing points where the function is not defined, nor a codomain that does not contain the image of the specified domain. Thus a function "$f : A \to B$" always has domain A and codomain B.

The next proposition shows that it suffices to give any $\mathcal{C}^k$-consistent atlas (not necessarily a maximal one) to determine a $\mathcal{C}^k$ structure.

Proposition 1.2.4 ($\mathcal{C}^k$ atlas defines $\mathcal{C}^k$ structure)**.** *For a fixed topological manifold M, each $\mathcal{C}^k$ atlas $\mathcal{A}$ is contained in a unique maximal $\mathcal{C}^k$ atlas $\overline{\mathcal{A}}$, which consists of all charts for M that are $\mathcal{C}^k$-compatible with those of $\mathcal{A}$. Any other $\mathcal{C}^k$ atlas $\mathcal{B}$ for M is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$.*

Proof. Let $\mathcal{A}$ be any $\mathcal{C}^k$ atlas for M. Define

$$\overline{\mathcal{A}} := \{\varphi \text{ chart for } M \text{ that is } \mathcal{C}^k \text{ compatible with all charts } \theta \in \mathcal{A}\}.$$

Clearly $\overline{\mathcal{A}}$ contains $\mathcal{A}$. We claim that $\overline{\mathcal{A}}$ is a $\mathcal{C}^k$ atlas. To prove this we have to show that if $\varphi, \psi \in \overline{\mathcal{A}}$ are charts with respective domains $U, V \subseteq M$, then the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is $\mathcal{C}^k$. Take any point $\psi(p) \in \psi(U \cap V)$ and let $\theta \in \mathcal{A}$ be a chart whose domain W contains the point $p \in U \cap V$. Then $\psi(U \cap V \cap W)$ is an open neighborhood of $\psi(p)$ and we can write the restriction[6]

$$\varphi \circ \psi^{-1} : \psi(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$

as the composition $(\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, which is $\mathcal{C}^k$ because $\varphi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are $\mathcal{C}^k$ by assumption. This proves that $\varphi \circ \psi^{-1}$ in a neighborhood of $\psi(p)$, but the same reasoning is valid at any point of $\psi(U \cap V)$, therefore $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$.

Finally, from the definition of $\overline{\mathcal{A}}$ it is clear that it is maximal, and that any atlas $\mathcal{B}$ is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$. In particular, any atlas $\mathcal{B}$ containing $\mathcal{A}$ is equivalent to $\mathcal{A}$ (by definition of equivalent atlases), and is therefore contained in $\overline{\mathcal{A}}$. Therefore a maximal atlas containing $\mathcal{A}$ is contained in $\overline{\mathcal{A}}$, but in fact it must be equal to $\overline{\mathcal{A}}$ (by maximality). We conclude that $\overline{\mathcal{A}}$ is the unique maximal $\mathcal{C}^k$ atlas containing $\mathcal{A}$. □

In consequence, given a topological manifold M and some $\mathcal{C}^k$ atlas $\mathcal{A}$ on M we can speak without ambiguity of *the* $\mathcal{C}^k$ structure $\overline{\mathcal{A}}$ determined by $\mathcal{A}$.

Remark 1.2.5. For practical purposes the concept of a maximal $\mathcal{C}^k$ atlas is not really important. We usually work with a smaller $\mathcal{C}^k$ atlas and this is all we need e.g. for checking that a function is $\mathcal{C}^k$ (see next section). (In fact, we could have defined a $\mathcal{C}^k$ structure on M as an equivalence class of $\mathcal{C}^k$ atlases, rather than as a maximal $\mathcal{C}^k$ atlas.) In general we won't give any name to the maximal atlas and we'll just speak about "a $\mathcal{C}^k$ manifold M" with the maximal $\mathcal{C}^k$ atlas $\mathcal{A}$ being implicit.

Also, when we say "a $\mathcal{C}^k$ chart" or simply "a chart" of M, we mean a chart $\varphi \in \mathcal{A}$. In the rare case that we may need a $\mathcal{C}^l$ chart φ with $l \leq k$ (which means φ is only $\mathcal{C}^l$ compatible with the charts of $\mathcal{A}$), we will say it explicitely. In particular, a "topological chart" is a $\mathcal{C}^0$ chart, i.e. a homeomorphism.

[6]see Remark 1.2.3

Remark 1.2.6. Every $\mathcal{C}^k$ manifold is automatically a $\mathcal{C}^l$ manifold for every $l \leq k$, because every $\mathcal{C}^k$ atlas is a $\mathcal{C}^l$ atlas. In the other direction, Whitney (*Differentiable Manifolds*, 1936) proved that every $\mathcal{C}^k$ structure contains a (non unique!) $\mathcal{C}^l$ structure for any $l > k$. The proof is reproduced in Munkres' *Elementary Differential Topology* and in Hirsch's *Differential Topology.*

Example 1.2.7 (Examples of smooth manifolds)**.**

1. $\mathbb{R}^n$ (with the atlas consisting of the single chart $\mathrm{id}_{\mathbb{R}^n}$) is a smooth manifold. In general, any topological manifold endowed with a single-chart atlas is automatically a smooth manifold. For example, the graph Gra_f of any *continuous* (sic) function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ as described in Example 1.1.11, endowed with the projection chart, is a smooth manifold.
2. Any open subset U of a $\mathcal{C}^k$ manifold M has a natural $\mathcal{C}^k$ structure consisting of the $\mathcal{C}^k$ charts of M whose domain is contained in U. (Exercise.) We will also see in the exercices that finite products of $\mathcal{C}^k$ manifolds have a natural $\mathcal{C}^k$ structure.
3. The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Indeed, the atlas given by the graph coordinates (Example 1.1.11) is smooth. To see this, we compute the transition functions (wlog $i < j$):

$$\begin{aligned}\varphi_i^+ \circ (\varphi_j^\pm)^{-1} :& \varphi_j^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \to \varphi_i^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n)\\ &(y_0, \dots, y_{n-1}) \mapsto (y_0, \dots, \hat{y}_i, \dots, y_{j-1}, \pm\sqrt{1 - \textstyle\sum_i (y_i)^2}, y_j, \dots, y_{n-1})\end{aligned}$$

   and a similar formula works if we replace φ_i^+ by φ_i^-. Hence all the transition maps are smooth.
4. More generally, any subset M of $\mathbb{R}^k$ given as the *regular level set* of a smooth map $F : \mathbb{R}^k \to \mathbb{R}^\ell$ is a $k - \ell$ dimensional smooth manifold "in a natural way[7]". (Being a level set means $M = F^{-1}(\{c\})$ for some $c \in \mathbb{R}^\ell$ and being a regular level set means that, moreover, the Jacobian $D|_p F$ is surjective for all $p \in M$.) You can prove this quite easily using the implicit function theorem and writing M locally as a graph of smooth functions (analogous to the graph coordinates for the sphere). We will show a more general statement later on when discussing submanifolds (Chapter 3).
5. Projective space $\mathbb{P}^n$ is naturally a smooth manifold; see exercises.
6. The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is naturally a smooth manifold. (Exercise.)
7. On the topological 1-manifold $M = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$, each of the two projections $\pi_0, \pi_1 : M \to \mathbb{R}$ (given by $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$) is a chart defined on the whole manifold M, but these two charts are not $\mathcal{C}^k$ compatible for any $k \geq 1$. (The transition functions are $\pi_1 \circ (\pi_0)^{-1} : x \mapsto x^3$ and its inverse $\pi_0 \circ (\pi_1)^{-1} : y \mapsto \sqrt[3]{y}$, which is not differentiable.) Thus these charts determine two different $\mathcal{C}^k$ structures on M.

1.2.1 One-step $\mathcal{C}^k$ manifolds*

Here it is convenient to work with local parametrizations rather than charts. Recall that an (n-dimensional) **local parametrization** of a topological space M is an homeomorphism from an open subset U of $\mathbb{R}^n$ to an open subset of M or, equivalently, an open injective map $U \to M$. A family $\mathcal{A}$ of such maps whose images cover M is called a **parametrization atlas** for M, and such an atlas is $\mathcal{C}^k$ if the transition map $\psi^{-1} \circ \psi$ is $\mathcal{C}^k$ for each $\phi, \psi \in \mathcal{A}$. Equivalently, a local parametrization is the inverse of a chart, and $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas iff $\mathcal{A}^{-1} := \{\varphi^{-1} : \varphi \in \mathcal{A}\}$ is a $\mathcal{C}^k$ atlas.

[7] In particular, the inclusion map $M \hookrightarrow \mathbb{R}^k$ is smooth.

Note that if M is a $\mathcal{C}^k$ manifold, then the topology of M is determined by the $\mathcal{C}^k$ structure. Indeed, if $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas, then

$$\text{a set } U \subseteq M \text{ is open iff } \phi^{-1}(U) \text{ is open in } \mathrm{Dom}(\phi) \subseteq \mathbb{R}^n \text{ for all } \phi \in \mathcal{A}. \qquad (1.1)$$

In other words, the topology of M is the *final topology* induced by the family $\mathcal{A}$.[8]

We can ask then, given a set M and a collection $\mathcal{A}$ of functions $\phi : U_\phi \subseteq \mathbb{R}^n \to M$, whether there exists a topology on M such that $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas of M. Such a topology is necessarily unique by (1.1). This provides a way to define a $\mathcal{C}^k$ manifold without constructing first a topological space.

Proposition 1.2.8 (One-step manifolds)**.** *Consider a set M endowed with a family $\mathcal{A}$ of injections $\phi : U_\phi \subseteq \mathbb{R}^n \to M$ satisfying the the following properties: (Denote $\widetilde{U}_\phi = \phi(U_\phi)$ the image of each $\phi \in \mathcal{A}$.)*

(1) For all $\phi, \psi \in \mathcal{A}$, the set $U_\phi^\psi := \phi^{-1}(\widetilde{U}_\psi)$ is open in $\mathbb{R}^n$,

(2) and the transition map $\psi^{-1} \circ \phi : U_\phi^\psi \to U_\psi^\phi$ is $\mathcal{C}^k$.

(3) Countably many of the sets $\widetilde{U}_\phi$ cover M.

(4) For every two distinct points $p, q \in M$ there exist maps $\phi, \psi \in \mathcal{A}$ containing p, q in their respective images, and open neighborhoods $V \subseteq U_\phi$, $W \subseteq U_\psi$ of $\phi^{-1}(p)$, $\psi^{-1}(q)$ respectively, such that $\phi(V) \cap \psi(W) = \emptyset$.

Then the set M endowed with the final topology w.r.t. $\mathcal{A}$ is a topological n-manifold admitting $\mathcal{A}$ as a $\mathcal{C}^k$ parametrization atlas.

Remarks:

- The hypothesis implies that each set $U_\phi \subseteq \mathbb{R}^n$ is open, because it is equal to U_ϕ^ϕ.
- Each transition map $\psi^{-1} \circ \phi : U_\phi^\psi \to U_\psi^\phi$ is a $\mathcal{C}^k$ isomorphism because it is $\mathcal{C}^k$ and bijective, with inverse $\phi^{-1} \circ \psi$, which is also $\mathcal{C}^k$.

Proof. We endow M with the final topology w.r.t $\mathcal{A}$, so that a set $U \subseteq M$ is open if and only if $\psi_\alpha^{-1}(U)$ is open in U_ϕ for all α. This is clearly a topology on M.

We claim that each map $\phi \in \mathcal{A}$ is open. Indeed, if $W \subseteq U_\phi$ is an open set, then $\phi(W)$ is open in M because for any $\psi \in \mathcal{A}$, the set $\psi^{-1}(\phi(W)) = (\psi^{-1} \circ \phi)(W \cap U_\phi^\psi)$ is open in U_ψ^ϕ since $\psi^{-1} \circ \phi$ is a homeomorphism $U_\phi^\psi \to U_\psi^\phi$.

It follows that each $\phi \in \mathcal{A}$ is a homeomorphism onto its image, because it is open an injective. Therefore the maps ϕ are n-dimensional local parametrizations of M ensuring that M is locally Euclidean of dimension n. Moreover, M is a topological manifold because it is second countable by (3) and Hausdorff by (4). Finally, the parametrization atlas $\mathcal{A}$ is $\mathcal{C}^k$-consistent by (2). □

An example of manifold that can be constructed in this way is the Grassman manifold $G_k(V)$ of k-subspaces of $V \simeq \mathbb{R}^n$; see [Lee13, p. 1.36].

1.3 Differentiable maps

We are about to define $\mathcal{C}^k$ maps between $\mathcal{C}^k$ manifolds. The plan is to reduce the question of differentiability to the case of a map between Euclidean open sets. We will do so by using charts.

In general, when studying a map $f : M \to N$ between manifolds, charts allow us to *locally* express f as a map between subsets of Euclidean space.

[8] The **final topology** on a set M induced by a family of maps $\phi : U_\phi \to M$, where U_ϕ are topological spaces, is the topology defined by the formula (1.1). Exercise: check that it is indeed a topology.

Definition 1.3.1 (Local expression of a map)**.** Let M, N be $\mathcal{C}^k$ manifolds and let $f : M \to N$ be any function (not necessarily continuous). A **local expression** (or **coordinate representation**) of f at some point $p \in M$ is a composite map

$$f|_\varphi^\psi := \psi \circ f \circ \varphi^{-1},$$

where φ and ψ are charts of M and N whose domains U, V contain the points p and $f(p)$ respectively. The composite is defined at all possible points; see Remark 1.2.3.

Local expressions allow us, for instance, to determine whether a function is continuous or not.

Remark 1.3.2. Exercise: Show that f is continuous at p if and only if the local expression $f|_\varphi^\psi$ at p is defined in a neighborhood of $\varphi(p)$ and is continuous at $\varphi(p)$. This fact holds however we choose the charts φ, ψ (provided their domains contain the points p, $f(p)$ respectively, of course).

The idea is to use local expressions to *define* whether a map is $\mathcal{C}^k$ or not.

Definition 1.3.3 ($\mathcal{C}^k$ maps between manifolds)**.** We say a function $f : M \to N$ between $\mathcal{C}^k$ manifolds is $\mathcal{C}^k$ at a point $p \in M$ if there exists a local expression $f|_\varphi^\psi$ of f at p such that

$$f|_\varphi^\psi \text{ is defined in a neighborhood of } \varphi(p) \text{ and is } \mathcal{C}^k \text{ at the point } \varphi(p). \tag{1.2}$$

If this holds for all points $p \in M$, we say that f is a $\mathcal{C}^k$ map. The set of $\mathcal{C}^k$ maps $M \to N$ is denoted $\mathcal{C}^k(M, N)$. A $\mathcal{C}^k$ **isomorphism** is a $\mathcal{C}^k$ map that has a $\mathcal{C}^k$ inverse.

If $k \geq 1$, a $\mathcal{C}^k$ map is also called $\mathcal{C}^k$**-differentiable**, and a $\mathcal{C}^k$ isomorphism is also called a $\mathcal{C}^k$ **diffeomorphism** (or a $\mathcal{C}^k$ **diffeo**, for short).

In fact the condition (1.2), that determines whether f is $\mathcal{C}^k$ or not, does not depend on how we choose the charts φ, ψ.

Proposition 1.3.4. *If f is $\mathcal{C}^k$ at the point p, then every local expression $f|_\varphi^\psi$ of f at p satisfies* (1.2).

Proof. Assume that the local expression $f|_\varphi^\psi$ satisfies (1.2). This implies that f is continuous at p, by Remark 1.3.2. Consider a second local expression $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ of f at p. This new local expression is defined on some neighborhood of $\widetilde{\varphi}(p)$ (again by Remark 1.3.2), and is related to the old one by the **chart-change formula**

$$f|_{\widetilde{\varphi}}^{\widetilde{\psi}} \equiv (\psi \circ \widetilde{\psi}^{-1}) \circ f|_\varphi^\psi \circ (\varphi \circ \widetilde{\varphi}^{-1}), \tag{1.3}$$

which holds at all points where the two expressions are defined (in particular, in some neighborhood of $\widetilde{\varphi}(p)$). Since the transition maps $(\psi \circ \widetilde{\psi}^{-1})$ and $(\varphi^{-1} \circ \widetilde{\varphi})$ are $\mathcal{C}^k$, we conclude that $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ is $\mathcal{C}^k$ at the point $\widetilde{\varphi}(p)$. □

Example 1.3.5.
1. The identity map of any $\mathcal{C}^k$ manifold is a $\mathcal{C}^k$ map. (Exercise.)
2. A composite map $g \circ f$ is $\mathcal{C}^k$ at a point p if f is $\mathcal{C}^k$ at p and g is $\mathcal{C}^k$ at $f(p)$. (Exercise.)
3. If M is a $\mathcal{C}^k$ manifold, then every $\mathcal{C}^k$ chart of M, as well as its inverse, are $\mathcal{C}^k$ maps. (Exercise.)

A $\mathcal{C}^k$ structure on a topological manifold M allows us to determine which maps that go to M are $\mathcal{C}^k$. But the reciprocal property also holds: if we know which maps to M are $\mathcal{C}^k$, this information determines the $\mathcal{C}^k$ structure of M.

Proposition 1.3.6. *Let $\mathcal{A}_0$, $\mathcal{A}_1$ be two $\mathcal{C}^k$ atlases on a topological manifold M, defining two $\mathcal{C}^k$ manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases $\mathcal{A}_i$ are equivalent if and only if the following property holds:*
For every function $f : N \to M$ (where N is a $\mathcal{C}^k$ manifold), the function f is $\mathcal{C}^k$ as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Proof. Exercise. □

1.4 Partitions of unity

In this section we develop *partitions of unity*, a tool often used to turn a local construction, obtained by working in coordinates, into a global one. (Don't worry if this sounds vague; we will see examples later on.) Existence of partitions of unity on a manifold is easier to prove on a compact manifold; we will deal first with this case. The general proof relies on a topological property of manifolds called *paracompactness*, which is in turn a consequence of being Hausdorff, second countable and locally compact.

Recall that the **support** of a continuous function $\eta : M \to \mathbb{R}$ is the closed set

$$\operatorname{supp}(\eta) := \overline{\{p \in M \mid \eta(p) \neq 0\}}.$$

Definition 1.4.1. A $\mathcal{C}^k$ **partition of unity** (or **POU**, for short) on a $\mathcal{C}^k$ manifold M is a family $(\eta_i)_i$ of $\mathcal{C}^k$ functions $\eta_i : M \to [0, +\infty)$ satisfying

- Sum condition: $\sum_i \eta_i(x) = 1$ for every $x \in M$,
- Local finiteness: Every point of M has a neighborhood where all except finitely many of the functions η_i vanish.

A partition of unity $(\eta_i)_i$ is **subordinate** to an open cover $\mathcal{U}$ of M if every η_i has its support contained in some open set $U \in \mathcal{U}$.

Theorem 1.4.2 (Existence of Partitions of Unity.)**.** *For any open cover $\mathcal{U}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$ such that the functions η_i have compact support (in fact, their supports are closed coordinate balls; see definition below).*

Another version of the theorem is the following.

Corollary 1.4.3 (Existence of Partitions of Unity, alternate form)**.** *For any open cover $\mathcal{U} = \{U_j\}_{j\in J}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\xi_j)_{j\in J}$ such that* $\operatorname{supp}(\xi_j) \subseteq U_j$ *for all j.*

This version of the theorem can be deduced from the first one (exercise). Note that the functions g_j may not have compact support in this case.

For the proof of Theorem 1.4.2 we will use *bump functions.*

Lemma 1.4.4 (Bump functions on Euclidean space)**.** *For any numbers $0 < a < b$ there exists a smooth **bump function** $h : \mathbb{R}^n \to [0, 1]$ satisfying $h(x) = 1$ iff $\|x\| \leq a$ and $h(x) = 0$ iff $\|x\| > b$.*

Proof. It suffices to let $h(x) = g(\|x\|)$, where $g : \mathbb{R} \to [0,1]$ is a smooth **cutoff function** satisfying $g(t) = 1$ iff $t < a$ and $g(t) = 0$ iff $t > b$. This cutoff function, in turn, may be defined as

$$g(t) = \frac{f(b-t)}{f(b-t) + f(t-a)},$$

where $f : \mathbb{R} \to [0, +\infty)$ is a smooth function such that $f(t) > 0$ iff $t > 0$. Such a function f may be given e.g. by the formula

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

□

We may use charts to transport these bump functions from Euclidean space to any manifold. The resulting bump function will be supported on a closed *coordinate ball.*

Definition 1.4.5. A **closed coordinate ball** in a $\mathcal{C}^k$ manifold M is the preimage $\varphi^{-1}(\overline{B})$ of a closed Euclidean ball $\overline{B}$ by a $\mathcal{C}^k$ chart φ containing $\overline{B}$ in its codomain.

Proof of Thm. 1.4.2 for compact manifolds. Let M be a compact $\mathcal{C}^k$ manifold and let $\mathcal{U}$ be an open cover of M. In this case we'll obtain a *finite* $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$. In fact, it is sufficient to find finitely many $\mathcal{C}^k$ functions $\widetilde{\eta_i} : M \to [0, +\infty)$, each of which has its support $\operatorname{supp}(\widetilde{\eta_i})$ contained in some $U \in \mathcal{U}$, and such that $\sum_i \widetilde{\eta_i}(x) > 0$ for all x. The functions η_i can then be obtained by dividing each $\widetilde{\eta_i}$ by the strictly positive $\mathcal{C}^k$ function $\widetilde{\eta} = \sum_i \widetilde{\eta_i}$.

To construct the functions η_i we proceed as follows. Each point of a set $U \in \mathcal{U}$ is contained in the interior of some closed coordinate ball $D \subseteq U$. Thus there is a family of closed coordinates balls, each of them contained in some $U \in \mathcal{U}$, whose interiors cover M. By compactness, we may take a finite subfamily of balls D_i whose interiors still cover M. Write each ball D_i as $\varphi_i^{-1}(\overline{B_i})$, where B_i is an open Euclidean ball and φ_i is a $\mathcal{C}^k$ chart containing $\overline{B_i}$ in its codomain. Then let $h_i : \mathbb{R}^n \to [0, +\infty)$ be a $\mathcal{C}^k$ bump function that is supported on the closed ball $\overline{B_i}$ and strictly positive on the interior B_i. Finally, define a $\mathcal{C}^k$ function $\widetilde{\eta_i} : M \to [0,1]$ by the formula

$$\widetilde{\eta_i} = \begin{cases} h_i \circ \varphi_i & \text{on } \operatorname{Dom}(\varphi) \\ 0 & \text{on } M \setminus D_i \end{cases}$$

This function is supported on D_i, which is contained in some $U \in \mathcal{U}$, thus the function family $(\widetilde{\eta_i})$ is subordinate to $\mathcal{U}$. In addition, $\widetilde{\eta_i}$ is strictly positive on B_i, and since the balls B_i cover M, we conclude that $\sum_i \widetilde{\eta_i}(x) > 0$ for all $x \in M$, as required. □

1.4.1 Paracompactness*

Definition 1.4.6 (Paracompact space)**.** Let X be a topological space.

- An **open cover** of X is a family $\mathcal{U}$ of open sets whose union is X.
- Another open cover $\mathcal{V}$ **refines** $\mathcal{U}$ if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.
- The space X is **paracompact** if every open cover $\mathcal{U}$ admits as refinement some open cover $\mathcal{V}$ that is locally finite.
- (A family of subsets of X is **locally finite** if every point $x \in X$ has a neighborhood that intersects only finitely many sets of the family.)

Proposition 1.4.7. *If a topological space X is Hausdorff, second countable and locally compact, then it is paracompact.*

In fact, given any topological base $\mathcal{B}$ of X, every open cover $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ consisting of open sets $B \in \mathcal{B}$ with their closures $\overline{B}$ contained in some $U \in \mathcal{U}$.

Proof.

Lemma 1.4.8. *X admits an exhaustion by compact sets, i.e. a sequence $(K_i)_{i\in\mathbb{N}}$ of compact sets that cover X and satisfy $K_i \subseteq \operatorname{Int}(K_{i+1})$.*

Proof of lemma. Let $(V_j)_{j\in\mathbb{N}}$ be a countable open cover of M where each V_j has compact closure. Let $K_0 = \emptyset$, and define inductively for each $i \in \mathbb{N}$ an integer $j_i \geq i$ such that the sets $(V_j)_{j<j_i}$ cover K_i, and a compact set $K_{i+1} = \overline{V_i} \cup \bigcup_{j<j_i} \overline{V_j}$. These compact sets K_i cover M and satisfy $K_i \subseteq \operatorname{Int}(K_{i+1})$. □

Consider the compact sets $L_i = K_i \setminus \operatorname{Int}(K_{i-1})$ and their respective open neighborhoods $W_i = \operatorname{Int} K_{i+1} \setminus K_{i-2}$.

Each point $x \in L_i$ is contained in some open $U_x \in \mathcal{U}$ and has a basic neighborhood $B^i_x \in \mathcal{B}$ such that $\overline{B^i_x} \subseteq W_i \cap U$.

Take a finite subfamily $(B^i_{x_j})_j$ that covers L_i. Doing this for each i we obtain a family of basics $(B^i_{x_j})_{i,j}$ that cover M. Their closures satisfy $\overline{B^i_{x_j}} \subseteq U_{x_j}$, which gives the subordination condition, and $\overline{B^i_{x_j}} \subseteq W_i$, which ensures local finiteness since every point $x \in L_i$ is contained in at most three sets W_ℓ, namely, those with $|\ell - i| \leq 1$. □

Using paracompactness, we can prove existence of partitions of unity without the hypothesis of compactness.

Proof of Thm. 1.4.2. Let M be a manifold and $\mathcal{U}$ an open cover.

Let $\mathcal{B}$ be the topological base of M consisting of the interiors of closed coordinate balls. By Proposition 1.4.7, there exists a family of closed coordinate balls D_i, each contained in some open set $U \in \mathcal{U}$, whose interiors cover M.

The proof finishes as in the compact case. We take for each i a $\mathcal{C}^k$ function $\widetilde{\eta}_i : M \to [0,1]$ that is strictly positive on $\operatorname{Int}(D_i)$ and supported on D_i. Then the functions $\eta_i = \frac{\widetilde{\eta}_i}{\sum_j \widetilde{\eta}_j}$ form a $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ that is subordinate to the open cover $\mathcal{U}$. □

1.4.2 Applications

Corollary 1.4.9 (Bump functions)**.** *If M is a $\mathcal{C}^k$ manifold, $A \subset M$ a closed set and $U \subseteq M$ an open neighborhood of A, then there exists a $\mathcal{C}^k$ function $\eta : M \to [0,1]$ such that $\eta \equiv 1$ on A and $\operatorname{supp}(\eta) \subset U$.*

We call η a *bump function* for A supported in U.

Proof. Just take the open cover $\{V_0 = U, V_1 = M \setminus A\}$ of M and a partition of unity $(\eta_i)_{i=0,1}$ satisfying $\operatorname{supp} \eta_i \subseteq V_i$, then set $\eta = \eta_0$. □

We have already defined differentiability for function $M \to N$ between $\mathcal{C}^k$ manifolds. For functions on closed sets we make the following definition:

Definition 1.4.10. Let $f : A \to N$ be a function where $A \subseteq M$ is a closed set and M, N are $\mathcal{C}^k$ manifolds. We say that f is $\mathcal{C}^k$ if it can be extended to a $\mathcal{C}^k$ function defined on an open neighborhood of A.

As a corollary of the existence of bump functions we can extend a smooth function on a closed set to a smooth function on the whole manifold:

Corollary 1.4.11 (Extension lemma)**.** *Let $f : A \to \mathbb{R}$ be $\mathcal{C}^k$, where $A \subseteq M$ is a closed subset of a $\mathcal{C}^k$ manifold M, and let $U \subseteq M$ be an open set containing A. Then there exists a $\mathcal{C}^k$ function $\widetilde{f} : M \to \mathbb{R}$ such that $\widetilde{f}|_A = f$ and $\operatorname{supp} \widetilde{f} \subseteq U$.*

Proof. By definition f can be extended to a $\mathcal{C}^k$ function (say, also called f) on some open set $W \supseteq A$; wlog $W \subseteq U$. We take a $\mathcal{C}^k$ bump function η for A supported in W. Then ηf has support in W and therefore extending the function by 0 outside W we obtain a $\mathcal{C}^k$ function $\widetilde{f}$ on M with the desired properties. $\square$

2 Tangent vectors

Recall that if a map $f : U \to V$ between Euclidean open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ is $\mathcal{C}^1$ at a point $p \in U$, then there exists a unique linear transformation $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$, called the **differential transformation** of f at p, which gives a first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_p f(v) + r_p(v)$$

where $\frac{r_p(v)}{\|v\|} \to 0$ as $v \to 0$.

To define the differential of a map between $\mathcal{C}^k$ manifolds, we need a notion of tangent space.

Definition 2.0.1. Let M be an n-dimensional $\mathcal{C}^k$ manifold with $k \geq 1$. A **coordinatized tangent vector** on M is a triple (p, φ, v) where $p \in M$ is a point, φ is a $\mathcal{C}^k$ chart of M defined at p, and $v \in \mathbb{R}^n$ is a vector in Euclidean space. A **tangent vector** on M is the equivalence class $[p, \varphi, v]$ of a coordinatized tangent vector (p, φ, v) under the equivalence relation

$$(p, \varphi, v) \sim (\widetilde{p}, \widetilde{\varphi}, \widetilde{v}) \quad \Longleftrightarrow \quad \widetilde{p} = p \quad \text{and} \quad \widetilde{v} = \mathrm{D}_{\varphi(p)}(\widetilde{\varphi}\,\varphi^{-1})(v)$$

The set TM of tangent vectors is the **tangent bundle** of M, and there is a canonic projection map $\pi_{TM} : TM \to M$ sending $[p, \varphi, v] \mapsto p$.

The **tangent space** at a point $p \in M$ is the set $T_pM := \pi^{-1}(p)$. It is a vector space with vector addition

$$[p, \varphi, v] + [p, \varphi, w] := [p, \varphi, v + w]$$

and vector scaling

$$\lambda\,[p, \varphi, v] := [p, \varphi, \lambda v] \quad \text{for } \lambda \in \mathbb{R}.$$

Remark 2.0.2. 1. $\sim$ is indeed an equivalence relation. (Exercise.)

2. Fixed a point $p \in M$ and a $\mathcal{C}^k$ chart φ defined on p, the function $\iota : \mathbb{R}^n \to T_pM$ sending $v \mapsto [p, \varphi, v]$ is a bijection. (Exercise.)
3. A tangent vector $X = [p, \varphi, v] \in T_pM$ can be considered as a function

$$\begin{array}{rcl} \{\text{charts of } M \text{ def. at } p\} & \to & \mathbb{R}^n \\ \psi & \mapsto & D_{\varphi(p)}(\psi\,\varphi^{-1})(v). \end{array}$$

   The vector $D_{\varphi(p)}(\psi\,\varphi^{-1})(v)$ is the only $w \in \mathbb{R}^n$ such that $[p, \psi, w] = X$.
4. Vector addition and scaling are well defined and make T_pM a vector space isomorphic to $\mathbb{R}^n$. (Exercise.)

Remark 2.0.3. If $U \subseteq \mathbb{R}^n$ is an open set (considered as a smooth manifold), we identify $TU \equiv U \times \mathbb{R}^n$ by the bijection

$$(p, v) \in U \times \mathbb{R}^n \mapsto [p, \mathrm{id}_U, v] \in TU.$$

Thus for each $p \in U$ we have $T_pU \equiv \{p\} \times \mathbb{R}^n \equiv \mathbb{R}^n$, and for $p \in U$ and $v \in \mathbb{R}^n$ we write

$$v|_p := [p, \mathrm{id}_U, v] \in T_pU. \tag{2.1}$$

Coordinate base for the tangent space We can construct a base of the tangent space T_pM as follows. Consider the canonic base $(e_i)_i$ of $\mathbb{R}^n$, and take a chart φ defined at p. Then the vectors

$$\frac{\partial}{\partial \varphi^i}|_p := [p, \varphi, e_i],$$

called the **coordinate vectors** at p associated to the chart φ, form a base of T_pM.

If ψ is another chart defined at p, then this chart determines a second base consising of vectors $\frac{\partial}{\partial \psi^j}|_p$. This second base is related to the first one by the formula

$$\frac{\partial}{\partial \varphi^i}|_p = \sum_j \frac{\partial \psi^j}{\partial \varphi^i}|_{\varphi(p)} \frac{\partial}{\partial \psi^j}|_p.$$

where $\frac{\partial \psi^j}{\partial \varphi^i}|_{\varphi(p)}$ is the partial derivative of $\psi \circ \varphi^{-1}$ that appears as the coefficient (j, i) of the matrix expression of the linear map $\mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$.

Remark 2.0.4. In concrete examples it is common to use more intuitive symbols for φ^i, e.g. the polar coordinates (r, φ) or the spherical coordinates (r, ϕ, θ). The standard coordinates $(x^0, \ldots, x^{n-1})$ on $\mathbb{R}^n$ are usually written (x, y) for $n = 2$ and (x, y, z) for $n = 3$. Bear in mind that an expression like (r, φ) can mean either the map (chart) or the coordinates of a particular point; see example below.

Example 2.0.5 (Polar coordinates)**.** Let $W := \mathbb{R}^+ \times (0, 2\pi)$. The map

$$\Psi : W \to \mathbb{R}^2 : (r, \varphi) \mapsto (r \cos \varphi, r \sin \varphi)$$

is a diffeomorphism onto its image $U := \Psi(W) = \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$. Its inverse $\Psi^{-1} : U \to W$ is therefore a smooth chart for $\mathbb{R}^2$. The components of Ψ^{-1} are usually written (r, φ) and called *polar coordinates*. On the other hand, we have the standard coordinates (x, y) (i.e. the identity map) on $\mathbb{R}^2$. Take a point $p = (x, y) \in U$ and let $(r, \varphi) = \Psi^{-1}(x, y)$ be its polar coordinates. The polar coordinate vectors $\frac{\partial}{\partial r}\Big|_p, \frac{\partial}{\partial \varphi}\Big|_p$ can be expressed as a linear combination of the standard coordinate vectors $\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p$ using the change of coordinates formula:

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -r\sin(\varphi)\frac{\partial}{\partial x} + r\cos(\varphi)\frac{\partial}{\partial y}.
\end{aligned}$$

Here we do certain standard abuses of notation. First, the letters r, φ, x, y represent functions when preceeded by a ∂, otherwise they are numbers (obtained by evaluating these functions at the point p). Second, we have omitted the "$|_p$" on the vectors and the evaluation at (r, φ) for the partial derivatives.

2.0.1 Differential of a $\mathcal{C}^k$ map between manifolds

Now that we have tangent spaces, we can define the differential of a $\mathcal{C}^k$ map.

Definition 2.0.6. The **differential transformation** (or **differential**, for short) of a map $f : M \to N$ that is $\mathcal{C}^k$ at a point $p \in M$ is the linear operator

$$\begin{array}{rccc}
D_p f : & T_pM & \to & T_{f(p)}N \\
 & [p, \varphi, v] & \mapsto & [f(p), \psi, D_{\varphi(p)} f|_\varphi^\psi(v)].
\end{array}$$

where φ, ψ are charts of M, N defined at the points p, $f(p)$ respectively, and $f|_\varphi^\psi = \psi \circ f \circ \varphi^{-1}$ is the local expression of f with respect to the charts φ, ψ.

If f is $\mathcal{C}^k$ everywhere, the union of the maps $D_p f$ for all $p \in M$ is a map $Df : TM \to TN$, also denoted f_* and called the **pushforward** by f.

Note that $D_p f$ is well defined (independent of φ, ψ) and linear. To see that it is well defined, we compute twice

$$\begin{array}{lll} D_p f: \; X = [p, \varphi, v] & \mapsto & Y = [f(p), \psi, w = D_{\varphi(p)} f|_{\varphi}^{\psi}(v)]. \\ D_p f: \; \widetilde{X} = [p, \widetilde{\varphi}, \widetilde{v}] & \mapsto & \widetilde{Y} = [f(p), \widetilde{\psi}, \widetilde{w} = D_{\widetilde{\varphi}(p)} f|_{\widetilde{\varphi}}^{\widetilde{\psi}}(v)]. \end{array}$$

and verify that $X = \widetilde{X}$ implies $Y = \widetilde{Y}$. That is, we must check that $\widetilde{v} = D_{\varphi(p)}(\widetilde{\varphi}\varphi^{-1})(v)$ (or, equivalently, $v = D_{\widetilde{\varphi}(p)}(\varphi\widetilde{\varphi}^{-1})(\widetilde{v})$) implies $\widetilde{w} = D_{\psi(p)}(\widetilde{\psi}\psi^{-1})(w)$. This follows from the equation $f|_{\widetilde{\varphi}}^{\widetilde{\psi}} = (\widetilde{\psi}\psi^{-1}) f|_{\varphi}^{\psi}(\varphi\widetilde{\varphi}^{-1})$ by the chain rule (in its Euclidean version).

The linearity of $D_p f$ follows from the linearity of $D_{\varphi(p)} f|_{\varphi}^{\psi}$.

The chain rule has the following version for maps between manifolds.

Proposition 2.0.7 (Chain rule)**.** *If $f : M \to N$ is $\mathcal{C}^k$ at some point p and $g : N \to L$ is $\mathcal{C}^k$ at $f(p)$, then $g \circ f$ is $\mathcal{C}^k$ at p and has differential*

$$D_p(g \circ f) = D_{f(p)} g \circ D_p f.$$

In particular, for a diffeo f, the differential $D_p f$ is a linear isomorphism whose inverse is $D_{f(p)}(f^{-1})$. (Exercise.)

Example 2.0.8 (Derivative of a chart)**.** Let M be an n-dimensional $\mathcal{C}^k$-manifold, let $p \in M$, and let $\varphi : U \to V$ be a chart of M defined at p. Then we have

$$\mathrm{D}\,\varphi(\frac{\partial}{\partial \varphi^i}|_p) = e_i|_{\phi(p)} \in \mathrm{T}_{\phi(p)}\,\mathbb{R}^n.$$

To see this, using the local expression $\varphi|_{\varphi}^{\mathrm{id}_V} = \mathrm{id}_V \circ \varphi \circ \varphi^{-1} = \mathrm{id}_V$ we compute

$$\begin{aligned} \mathrm{D}\,\varphi(\frac{\partial}{\partial \varphi^i}|_p) &= \mathrm{D}\,\varphi[p, \varphi, e_i] \\ &= [\varphi(p), \mathrm{id}_V, D_{\varphi(p)}(\varphi|_{\varphi}^{\mathrm{id}_V})(e_i))] = [\varphi(p), \mathrm{id}_V, e_i] = e_i|_p \in \mathrm{T}_p\,\mathbb{R}^n. \end{aligned}$$

Example 2.0.9 (Velocity of a curve)**.** For a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ on a manifold M, we define its **velocity vector** at an instant $t \in I$ as the vector $\gamma'(t) := D_t\gamma(1|_t) \in T_{\gamma(t)}M$ where $1|_t$ represents the element $[t, \mathrm{id}_I, 1]$ of $T_t I$ according to the identification $TI \cong I \times \mathbb{R}$ given in Remark 2.0.3.
Exercise: Show that for any vector $X \in TM$ there is a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(0) = X$.

Tangent vectors as derivations

Definition 2.0.10. A **derivation** on a $\mathcal{C}^k$ differentiable manifold M at a point $p \in M$ is a linear function $D : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ satisfying the Leibniz identity

$$D(f\,g) = D(f)\,g(p) + f(p)\,D(g).$$

The set $Der_p M$ of derivations on M at p is a vector space with the operations

$$\begin{aligned} (D + E)(f) &:= D(f) + E(f), \\ (\lambda D)(f) &:= \lambda(D(f)) \end{aligned}$$

defined for $D, E \in Der_p M$ and $\lambda \in \mathbb{R}$.

Each vector $X \in \mathrm{T}_p\,M$ induces a derivation $D_X \in Der_p M$ defined by the formula

$$D_X(f) := \mathrm{D}_p\,f(X) \in \mathrm{T}_{f(p)}\,\mathbb{R} \equiv \mathbb{R}.$$

(Here we use the identification of Remark 2.0.3).

Proposition 2.0.11. *The map $\nu_p : X \in \mathrm{T}_p M \mapsto D_X \in Der_p M$ is a linear injection. (Exercise.)*

Therefore we may identify a tangent vector X with the derivation D_X and write $X(f) := D_X(f)$. In fact, the map ν is bijective if M is smooth (see e.g. [Lee13, Prop. 3.2]). Therefore in some books (e.g. [Lee13]), the tangent space $\mathrm{T}_p M$ of a smooth manifold M is *defined* as the vector space of derivations at p. Here we do not use that definition because it does not work for non-smooth manifolds (i.e. C^k manifolds with $k < \infty$).

Tangent vectors as derivations

In this chapter we give an alternative definition of the "tangent space" using derivations. This second definition is equivalent to the one given in the previous chapter, but is more similar to the one used in most books (e.g. Lee's book [Lee13]).

In more detail, for a point p of a differentiable manifold M, we will consider first a vector space $\operatorname{Der}_p M$ of *derivation operators* (or *derivations*, for short) and then we will define the tangent space $\mathrm{T}_p M$ as a certain subspace of $\operatorname{Der}_p M$. Tangent vectors are the derivations that correspond (via a chart) to a derivation defined by a vector of $\mathbb{R}^n$. Equivalently, tangent vectors are those derivations that can be expressed as the velocity vector of a curve.

The objects that we will define are analogous to those defined in the previous chapter (except that the differential of a function will be called "tangent map"). The plan is the following:

(1) For each $\mathcal{C}^k$-differentiable n-manifold M and each point $p \in M$, define a vector space $\mathrm{T}_p M \simeq \mathbb{R}^n$, called the *tangent space* of M at p.

(2) For each $\mathcal{C}^k$ map $f : M \to N$ and each point $p \in M$, define a linear map $\mathrm{T}_p f : \mathrm{T}_p M \to \mathrm{T}_p N$ called the *tangent map* of f at p.

If the manifolds M, N are open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively, then the map $\mathrm{T}_p f$ will be essentially equivalent to the **differential transformation** studied in calculus courses, that is, the unique linear map $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$ which gives the first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_p f(v) + r(v)$$

where $\frac{r(v)}{\|v\|} \to 0$ as $v \to 0$.

Motivation for definition of tangent vectors Let M be a $\mathcal{C}^k$ manifold M with $k \geq 1$ and let $p \in M$. We want to have a vector space $\mathrm{T}_p M$ of vectors tangent to M at p. How can we define the notion of a "tangent vector"?

For example, let $\gamma : I \subseteq \mathbb{R} \to M$ be a differentiable curve such that $\gamma(0) = p$. This curve should have a "velocity vector" X at the instant $t = 0$, which should be vector tangent to M at p. This object X should tells us the *direction and speed* of movement of $\gamma(t)$ at time $t = 0$. How can we measure this "direction and speed" ?

Idea: take a function $h \in \mathcal{C}^k(M, \mathbb{R})$. The composite $h \circ \gamma$ is a function $I \to \mathbb{R}$ that is differentiable at $t = 0$, and the number number $(h \circ \gamma)'(0)$ tells us something about how γ is moving at $t = 0$. If we know this number for all functions h, let us agree that we know the velocity vector X completely. Therefore we propose to *define* the velocity vector X simply as the function $h \mapsto (h \circ \gamma)'(0)$. This function has certain algebraic properties that make it a *derivation*.

Definition 2.0.12. A **derivation operator** (or **derivation**, for short) on a $\mathcal{C}^k$-differentiable manifold M at a point $p \in M$ is a function $X : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ that is $\mathbb{R}$-linear, that is, such that

$$X(a \cdot g + b \cdot h) = a \cdot X(g) + b \cdot X(h) \quad \text{for } g, h \in \mathcal{C}^k(M, \mathbb{R}) \text{ and } a, b \in \mathbb{R},$$

and satisfies the **Leibniz identity**

$$X(g \cdot h) = X(g) \cdot h(p) + g(p) \cdot X(h). \tag{2.2}$$

The set $\mathrm{Der}_p(M)$ of derivations on M at p is a real vector space: for $X, Y \in \mathrm{Der}_p M$ and $a, b \in \mathbb{R}$ we define the derivation $a\,X + b\,Y \in \mathrm{Der}_p M$ by the formula

$$(a \cdot X + b \cdot Y)(h) := a \cdot X(h) + b \cdot Y(h).$$

We also define the set $\mathrm{Der}\, M := \coprod_{p \in M} \mathrm{Der}_p M$.

Before getting to the derivations that we are interested in (namely, those we'll call "tangent vectors"), let us prove some properties of general derivations.

Proposition 2.0.13. *Consider any derivation $X \in \mathrm{Der}_p M$ at a point p of a $\mathcal{C}^k$-diferentiable manifold M. Then for all functions $g, h \in \mathcal{C}^k(M, \mathbb{R})$ we have:*

(a) If h is constant, then $X(h) = 0$.

(b) If both g and h vanish at p (i.e. $g(p) = h(p) = 0$), then $X(g \cdot h) = 0$.

(c) Locality: If $g \equiv h$ on a neighborhood of p, then $X(g) = X(h)$.

Proof. To prove (a), since X is linear, it suffices to show that $X(h) = 0$ if $h \equiv 1$. And indeed, in this case we have

$$X(h) = X(h^2) = X(h) \cdot h(p) + h(p) \cdot X(h) = 2X(h)$$

which implies $X(h) = 0$.

Fact (b) follows from the Leibniz identity.

To prove (c), since X is linear, it suffices to show that $X(f) = 0$ if $f \equiv 0$ in some open neighborhood U of p. Let $\eta : \mathcal{C}^k(M, \mathbb{R})$ be a bump function that is constantly 1 on the closed set $M \setminus U$ and whose support is contained in the open set $M \setminus \{p\}$. Note that $f \cdot \eta = f$, therefore $X(f) = X(f \cdot \eta) = 0$ since $f(p) = \eta(p) = 0$. $\square$

One first example of derivation is the one we already mentioned: velocity vectors.

Definition 2.0.14. The **velocity vector** of a curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ (where γ is differentiable) is the derivation $\mathrm{Vel}_\gamma(t) \in \mathrm{Der}_{\gamma(t)} M$ defined by

$$\begin{array}{rccc} \mathrm{Vel}_\gamma(t) : & \mathcal{C}^k(M, \mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & (h \circ \gamma)'(t). \end{array}$$

Let us check that the function $X = \mathrm{Vel}_\gamma(t_0)$ is indeed a derivation on M at the point $p = \gamma(t_0)$. For a function $h \in \mathcal{C}^k(M, \mathbb{R})$ we denote $h_\gamma := h \circ \gamma \in \mathcal{C}^k(I, \mathbb{R})$. The fact that X satisfies the Leibniz identity for functions $g, h \in \mathcal{C}^k(M, \mathbb{R})$ follows from the known Leibniz identity for the functions $g_\gamma, h_\gamma : I \to \mathbb{R}$, since

$$\begin{aligned} X(g \cdot h) = (g \cdot h)'_\gamma(t_0) = (g_\gamma \cdot h_\gamma)'(t_0) &= g'_\gamma(t_0) \cdot h_\gamma(t_0) + g_\gamma(t_0) \cdot h'_\gamma(t_0) \\ &= X(g) \cdot h(p) + g(p) \cdot X(h) \end{aligned}$$

The fact that X is $\mathbb{R}$-linear can be proven in a similar way.

Remark 2.0.15. The velocity vector $\mathrm{Vel}_\gamma(t)$ is sometimes denoted $\gamma'(t)$, but it is different from the usual $\gamma'(t)$ defined for a curve in $\mathbb{R}^n$, because it also contains the information of the point $\gamma(t)$ where the derivation is located.
Exercise: If γ, β are differentiable curves in $\mathbb{R}^n$, show that $\mathrm{Vel}_\gamma(t) = \mathrm{Vel}_\beta(t)$ if and only if $\gamma(t) = \beta(t)$ and $\gamma'(t) = \beta'(t)$.

Another kind of derivations are *vectorial derivations* (also known as *directional derivations*[1]). Before considering general manifolds, let us recall how these derivation operators are defined on an open subset of $\mathbb{R}^n$.

[1] I don't like this name because it suggests that the derivation depends only on the direction of the vector, and not on its size.

Definition 2.0.16. A **vectorial derivation** at a point p of an open set $U \subseteq \mathbb{R}^n$ is a derivation $\delta_{p,v} \in \operatorname{Der}_p U$ determined by a vector $v \in \mathbb{R}^n$ by the formula

$$\begin{array}{rccc} \delta_{p,v}: & \mathcal{C}^k(U,\mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \mathrm{D}_p\, h(v). \end{array} \tag{2.3}$$

Note that $\delta_{p,v}$ is a derivation; in fact, it is the velocity vector at time $t = 0$ of any curve $\gamma : (-\varepsilon, \varepsilon) \to U$ satisfying $\gamma'(0) = p$ and $\gamma'(0) = v$. (For example, the curve $\gamma(t) = p + tv$.) Indeed, if γ is such a curve, then for any function $h \in \mathcal{C}^k(U,\mathbb{R})$ we have

$$\begin{aligned} \delta_{p,v}(h) &= \mathrm{D}_p\, h(v) \\ &= \mathrm{D}_p\, h(\gamma'(0)) \\ &= (h \circ \gamma)'(0) = \mathrm{Vel}_\gamma(0)(h) \end{aligned}$$

The definition of vectorial derivation can be adapted to a general manifold by using a chart.

Definition 2.0.17. A **vectorial derivation** on a $\mathcal{C}^k$ differentiable n-manifold M at a point $p \in M$ is a derivation of the form

$$\begin{array}{rccc} \delta_{p,\phi,v}: & \mathcal{C}^k(M,\mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \mathrm{D}_{\phi(p)}(h \circ \phi^{-1})(v) \end{array} \tag{2.4}$$

where ϕ is a chart defined at p and $v \in \mathbb{R}^n$. The triple (p, ϕ, v) is called a **coordinate expression** of $\delta_{p,\phi,v}$.

A **tangent vector** to M at p is the same thing as a vectorial derivation on M at p.

The **tangent space** of M at p is the set $\mathrm{T}_p\, M \subseteq \operatorname{Der}_p M$ of vectors tangent to M at p. (We will show that $\mathrm{T}_p\, M$ is in fact an n-dimensional subspace of $\operatorname{Der}_p M$.)

The **tangent bundle** of M is the set $TM := \coprod_{p \in M} \mathrm{T}_p\, M$.

Exercise 2.0.18. Prove that tangent vectors are derivations indeed. In fact, tangent vectors are the same thing as velocity vectors. That is, every tangent vector is the velocity vector of some curve, and every velocity vector can also be expressed as a vectorial derivation.

Remark 2.0.19 (Domain of tangent vectors)**.** Although the "official" domain of a tangent vector $X \in \mathrm{T}_p\, M$ is the set $\mathcal{C}^k(M,\mathbb{R})$, in fact we can apply X to any real-valued h that is defined on a neighborhood U of p and is differentiable at p. The number $X(h)$ is defined using the same formula (2.4) that we would use for a function $h \in \mathcal{C}^k(M,\mathbb{R})$.

Remark 2.0.20 (Locality of tangent vectors)**.** It is also clear from the definition that the number $X(h)$ depends only on the values of h near p. Thus if two functions g, h coincide on a neighborhood of p, then $X(g) = X(h)$. (We have already shown in Proposition 2.0.13 that this is a property of general derivations, but in the case of vectorial derivations it is especially evident.)

Let us show that the tangent space $\mathrm{T}_p\, M$ of an n-dimensional differentiable manifold M at any point $p \in M$ is an n-dimensional vector space.

Proposition 2.0.21. *Let M be a $\mathcal{C}^k$-differentiable manifold and let $p \in M$. Then for any chart ϕ of M defined at p, the map*

$$\begin{array}{rccc} \iota_{p,\phi}: & \mathbb{R}^n & \to & \operatorname{Der}_p M \\ & v & \mapsto & \delta_{p,\phi,v} \end{array}$$

is a linear isomorphism onto the tangent space $\mathrm{T}_p\, M$.

In other words, *any single* chart ϕ defined at p allows us to express *all* tangent vectors $X \in \mathrm{T}_p M$ in the form $X = \delta_{p,\phi,v}$, with the vector $v \in \mathbb{R}^n$ being unique for each X.

Proof. To show that $\iota_{p,\phi}$ is an isomorphism we develop some formulas that will be useful elsewhere.

Lemma 2.0.22. *Any tangent vector $X = \delta_{p,\phi,v} \in \mathrm{T}_p M$ applied to a component function ϕ^i of the chart ϕ gives*

$$X(\phi^i) = v^i. \tag{2.5}$$

In consequence, $\iota_{p,\phi}$ is injective.

Proof of lemma. Note first that $\phi^i = \pi^i \circ \phi$, where $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is the projection on the i-th coordinate axis. Knowing this, we can compute

$$X(\phi^i) = \mathrm{D}_p(\phi^i \circ \phi^{-1})(v) = \mathrm{D}_p(\pi^i)(v) = \pi^i(v) = v^i.$$

Here we used the fact that $\mathrm{D}_p(\pi^i) = \pi^i$ since π^i is linear.

Note that the functions ϕ^i are not in the "official" domain $\mathcal{C}^k(M, \mathbb{R})$ of the operator $\delta_{p,\phi,v}$, because they are defined on $U \subseteq M$ (the domain of ϕ) rather than on the whole manifold M. However, we can construct functions $\widetilde{\phi^i} \in \mathcal{C}^k(M, \mathbb{R})$ that coincide with ϕ^i on a neighborhood of p. (For example, using bump functions; see Lemma 1.4.11.) Using the functions $\widetilde{\phi^i}$ instead of ϕ^i we obtain an analogous formula $X(\widetilde{\phi^i}) = v^i$. This formula allows us to recover the components of the vector $v \in \mathbb{R}^n$ from the derivation $\delta_{p,\phi,v}$. It follows that the map $\iota_{p,\phi} : v \mapsto \delta_{p,\phi,v}$ is injective. ☐

Lemma 2.0.23. *For any two charts ϕ, ψ defined at p, and for any two vectors $v, w \in \mathbb{R}^n$, we have*

$$\begin{aligned} \delta_{p,\phi,v} = \delta_{p,\psi,w} \quad &\iff \quad w = \mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})(v) \\ &\iff \quad v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w) \end{aligned} \tag{2.6}$$

In consequence, the map $\iota_{p,\phi}$ is surjective onto $\mathrm{T}_p M$.

Proof of lemma. Note first that the two equations on the right are equivalent since the linear transformations $\mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})$ and $\mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})$ are inverse of each other.

Suppose first that these equations hold. Then for any function $h \in \mathcal{C}^k(M, \mathbb{R})$ we have

$$\begin{aligned} \delta_{p,\psi,w}(h) &= \mathrm{D}_{\psi(p)}(h \circ \psi^{-1})(w) \\ &= \mathrm{D}_{\psi(p)}(h \circ \psi^{-1})(\mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})(v)) \\ &= \mathrm{D}_{\phi(p)}(h \circ \phi^{-1})(v) \\ &= \delta_{p,\phi,v}(h). \end{aligned}$$

It follows that $\iota_{p,\phi}$ is surjective onto $\mathrm{T}_p M$, since any vector $X \in \mathrm{T}_p M$ is of the form $X = \delta_{p,\psi,w}$ for some chart ψ and some vector $w \in \mathbb{R}^n$, and we can obtain $\iota_{p,\phi}(v) = X$ by putting $v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)$.

Finally, suppose that $\delta_{p,\phi,v} = \delta_{p,\psi,w}$. We have already shown that this equation holds when $v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)$, and it cannot hold for any other value of v because the map $\iota_{p,\phi} : v \mapsto \delta_{p,\phi,v}$ is injective. We conclude that $v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)$. ☐

This finishes the proof that $\iota_{p,\phi}$ is an isomorphism, and hence $\mathrm{T}_p M \simeq \mathbb{R}^n$. $\square$

Remark 2.0.24. Lemma 2.0.23 shows that a tangent vector $X \in \mathrm{T}_p M$ can be thought of as an equivalence class of triples (p, ϕ, v) (where ϕ is a chart of M and $v \in \mathbb{R}^n$) under the equivalence relation

$$\begin{aligned}(p, \phi, v) \sim (p, \psi, w) \quad &\iff \quad w = \mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})(v) \\ &\iff \quad v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)\end{aligned}$$

This is another way to define tangent vectors, equivalent to ours.

Example 2.0.25 (Tangent bundle of an open subset of $\mathbb{R}^n$)**.** Recall that any open set $U \subseteq \mathbb{R}^n$ is naturally a $\mathcal{C}^k$ differentiable manifold with an atlas consisting of the single chart id_U. This single chart is sufficient to express all tangent vectors, by Proposition 2.0.21. That is, all tangent vectors $X \in \mathrm{T}\, U$ are of the form $X = \delta_{p,\mathrm{id}_U,v}$, with $p \in U$ and $v \in \mathbb{R}^n$. Thus there is a natural identification[2]

$$\begin{array}{ccc}\mathrm{T}\, U & \equiv & U \times \mathbb{R}^n \\ \delta_{p,\mathrm{id}_U,v} & \leftrightarrow & (p, v).\end{array}$$

Note also that the operator $\delta_{p,\mathrm{id}_U,v}$ coincides with the operator $\delta_{p,v}$ of Definition 2.0.17, therefore the two definitions of "vectorial derivation" (2.0.17 and 2.0.16) are equivalent for an open set $U \subseteq \mathbb{R}^n$.

2.0.2 Pushforward and tangent operators

Any differentiable map between manifolds induces a map on derivations called the **pushforward operator**.

Definition 2.0.26 (Pushforward of derivations)**.** The **pushforward operator** of a $\mathcal{C}^k$-differentiable map $f : M \to N$ at some point $p \in M$ is the linear map

$$\begin{array}{rccc} f_*|_p : & \mathrm{Der}_p M & \to & \mathrm{Der}_{f(p)} N \\ & X & \mapsto & (h \in \mathcal{C}^k(N, \mathbb{R}) \mapsto X(h \circ f) \in \mathbb{R}).\end{array}$$

The union of the operators $f_*|_p$ over all points $p \in M$ is the **pushforward map**

$$f_* := \coprod_{p \in M} f_*|_p : \mathrm{Der}\, M \to \mathrm{Der}\, N.$$

Example 2.0.27. Each derivation $\delta_{p,\phi,v}$ of Definition 2.0.17 is the image of the derivation $\delta_{\phi(p),v}$ of Definition 2.0.16 by the pushforward operator of of the map ϕ^{-1} at the point $\phi(p)$. (Check it!)

The **tangent operator** of a differentiable map is obtained by restricting the pushforward operator to tangent vectors.

Definition 2.0.28. The **tangent operator** of a $\mathcal{C}^k$ map $f : M \to N$ at a point $p \in M$ is the map

$$\mathrm{T}_p f : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$$

obtained by restricting the pushforward map $f_*|_p : \mathrm{Der}_p M \to \mathrm{Der}_{f(p)} N$.

The **tangent map** of f is the union of the tangent operators $\mathrm{T}_p f$ over all points $p \in M$,

$$\mathrm{T} f := \coprod_{p \in M} \mathrm{T}_p f : \mathrm{T} M \to \mathrm{T} N.$$

[2] We write $A \equiv B$ when there is a natural isomorphism between two objects A, B.

To show that the tangent operator $\mathrm{T}_p f$ is well defined, we have to verify that the pushforward operator $f_*|_p$ indeed maps tangent vectors to tangent vectors (or, equivalently, it maps velocity vectors to velocity vectors).

Proposition 2.0.29 (Local expression of the tangent operator)**.** *If $f : M \to N$ is differentiable at a point $p \in M$, then for any tangent vector $\delta_{p,\phi,v} \in \mathrm{T}_p M$ and any chart ψ of N defined at $f(p)$ we have*

$$f_*|_p(\delta_{p,\phi,v}) = \delta_{\phi(p),\psi,w} \quad \textit{where} \quad w = \mathrm{D}_{\phi(p)}(f|_\phi^\psi)(v). \tag{2.7}$$

Proof. For any $h \in \mathcal{C}^k(N, \mathbb{R})$ we have

$$\begin{aligned}(f_*|_p\, \delta_{p,\phi,v})(h) &= \delta_{p,\phi,v}(h\, f) \\ &= \mathrm{D}_{\phi(p)}(h\, f\, \phi^{-1})(v) \\ &= \mathrm{D}_{\psi(f(p))}(h\, \psi^{-1})\left(\mathrm{D}_{\phi(p)}(\psi\, f\, \phi^{-1})(v)\right) \\ &= \delta_{f(p),\psi,w}(h).\end{aligned}$$

□

Proposition 2.0.30 (Pushforward of velocity vectors; exercise)**.** *Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map, and let $\gamma : I \subseteq \mathbb{R} \to M$ be a curve that is differentiable at some instant $t_0 \in I$. Then $f_*(\mathrm{Vel}_\gamma(t_0)) = \mathrm{Vel}_{f\circ\gamma}(t_0)$.*

Remark 2.0.31. The tangent transformation $\mathrm{T}_p f : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$ can in fact be defined for any function $f : U \subseteq M \to N$ that is defined on a neighborhood U of p and is differentiable at p. This follows from Remarks 2.0.19 and 2.0.20 (check it!).

Example 2.0.32. For a $\mathcal{C}^k$ differentiable map $f : U \to V$, where $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ are open sets, the tangent operator $\mathrm{T}_p f : \mathrm{T}_p U \to \mathrm{T}_p V$ is essentially equivalent to the differential $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$ because for all vectors $X \in \mathrm{T}_p U$ (which can be written as $X = \delta_{p,v}$ with $v \in \mathbb{R}^m$) we have

$$\mathrm{T}_p f(X) = \mathrm{T}_p f(\delta_{p,v}) = \delta_{p,w} \quad \text{where} \quad w = \mathrm{D}_p f(v).$$

The chain rule that you know from calculus also holds for maps between manifolds.

Proposition 2.0.33 (Chain rule; exercise)**.** *Let $f : M \to N$ and $g : N \to L$ be maps between differentiable manifolds.*

1. *If f and g are differentiable at points $p \in M$, $f(p) \in N$ respectively, then the composite map $g \circ f$ is differentiable at p, and*

$$\mathrm{T}_p(g \circ f) = \mathrm{T}_{f(p)}\, g \circ \mathrm{T}_p f. \tag{2.8}$$

2. *For any point $p \in M$ we have $T_p\, \mathrm{id}_M = \mathrm{id}_{\mathrm{T}_p M}$.*
3. *If f is a diffeomorphism, then $\mathrm{T}_p f$ is a linear isomorphism with inverse $\mathrm{T}_{f(p)}(f^{-1})$.*

2.0.3 Classical notation for tangent vectors

Here we could do the same as in the last chapter.

Given a chart ϕ of M at p, define a basis of $\mathrm{T}_p M$ consisting of vectors $\left.\frac{\partial}{\partial \phi^i}\right|_p := \delta_{p,\phi,e_i}$.

Note that for any function $f \in \mathcal{C}^k(M, \mathbb{R})$ $\left.\frac{\partial}{\partial \phi^i}\right|_p f = \frac{\partial}{\partial x^i}\big|_{\phi(p)} f|_\phi(x)$.

The vector $\left.\frac{\partial}{\partial \phi^i}\right|_p$ is the velocity vector at $t = 0$ of a curve $\gamma_i(t)$ that satisfies $\phi^j(\gamma_i(t)) = \phi^j(p) + \delta_i^j t$. That is, along this curve $\gamma_i(t)$ the i-th coordinate increases at unit speed while the other coordinates remain constant.

One can prove that $\left.\frac{\partial}{\partial \phi^i}\right|_p (\phi^j) = \delta_i^j$ and also give the coordinate change formula.

2.0.4 General derivations*

If M is a smooth manifold, then the space $\operatorname{Der}_p M$ contains no other derivations than the tangent vectors (see exercies). Thus one can define the tangent space $\mathrm{T}_p M$ of a smooth manifold M simply as the space $\operatorname{Der}_p M$ of derivations. However, if M is a $\mathcal{C}^k$ manifold with $k < \infty$, then the space $\operatorname{Der}_p M$ is infinite dimensional (see exercises).

3 Submanifolds

Let us start with a short motivation. The "bent line" in $\mathbb{R}^2$, defined as $C = \{(x,y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$, is intuitively not "smoothly contained" in $\mathbb{R}^2$. It is topological submanifold of $\mathbb{R}^2$ because, when endowed with the subspace topology, it is homeomorphic to $\mathbb{R}$. Moreover, it follows that C can be given the structure of a smooth manifold (using a single-chart atlas), but the smooth structure does not "agree" with the ambient one; the bent line is not a *smooth submanifold* of $\mathbb{R}^2$. To make this precise we need a few definitions:

Definition 3.0.1. Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map between manifolds of respective dimensions m, n.

We say that f is **submersive** at a point $p \in M$ if $\mathrm{T}_p f$ is surjective. If this holds for all points $p \in M$, then f is a **submersion**.

We say that f is **immersive** at a point $p \in M$ if $\mathrm{T}_p f$ is injective. If this holds for all points $p \in M$, then f is an **immersion**.

We say that f is an **embedding** if it is an immersion and also a topological embedding (i.e. a homeomorphism onto its image, the latter being endowed with the subspace topology). Its image is called a $\mathcal{C}^k$ **embedded submanifold** of N (or a $\mathcal{C}^k$ **embedded** m**-submanifold**, to highlight its dimension).

The **rank** of f at a point $p \in M$, denoted $\operatorname{rank}_p f$, is the rank of the tangent operator $\mathrm{T}_p f$. If f has the same rank k at all points ("constant rank") then we write $\operatorname{rank} f = k$.

So to sum up, a $\mathcal{C}^k$ map $f : M \to N$ is a submersion iff $\operatorname{rank} f = n = \dim N$ and is an immersion iff $\operatorname{rank} f = m = \dim M$. In both cases the tangent operator $\mathrm{T}_p f$ has maximal rank for all points $p \in M$.[1] Note that the set of points where f has maximal rank is open because the rank is an upper semicontinuous function: if $k = \operatorname{rank}_p f$, then there is a neighborhood of p where f has rank $\geq k$.

Example 3.0.2. The **standard immersion** $\iota|_k^n : \mathbb{R}^k \to \mathbb{R}^n$, defined for $k \leq n$, is the map

$$\iota_k^n : (x^0 \ldots, x^{k-1}) \mapsto (x^0, \ldots, x^{k-1}, 0, \ldots, 0).$$

This is also a smooth embedding.
The **standard submersion** $\pi_m^k = \mathbb{R}^m \to \mathbb{R}^k$, defined for $m \geq k$, is the map

$$\pi_m^k : (x^0, \ldots, x^{m-1}) \mapsto (x^0, \ldots, x^{k-1}).$$

Composing these we obtain the **standard map of rank** k

$$\iota_k^n \circ \pi_m^k d : \mathbb{R}^m \to \mathbb{R}^n : (x^0, \ldots, x^{m-1}) \mapsto (x^0, \ldots, x^{k-1}, 0, \ldots, 0).$$

Example 3.0.3.

- A smooth curve $\gamma : J \to M$, where $J \subset \mathbb{R}$ is an open interval and M is a smooth manifold, is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$. For example, the curve on $\mathbb{R}^2$ given by $t \mapsto (\cos t, \sin t)$, $t \in \mathbb{R}$, is an immersion, but the curve on $\mathbb{R}^2$ given by $t \mapsto (t^2, t^3)$, $t \in \mathbb{R}$, is not.

[1] A linear map $V \simeq \mathbb{R}^m \to W \simeq \mathbb{R}^n$ has **maximal rank** if its rank is equal to $\min\{m, n\}$, which is the maximum rank that a linear map $V \to W$ can have.

- The *bent line* $C = \{(x,y) \in \mathbb{R}^2 | xy = 0,\ x,y \geq 0\}$ is not an embedded submanifold. In fact, it is not the image of an immersion. (Exercise.)
- The inclusion $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding (where we endow $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$ with the standard smooth structures).
- Let $M = M_0 \times M_1$ be a product of smooth manifolds. Then the canonical projections $\pi^i : M \to M_i$ are submersions.
- *"The figure 8"*: Let $J = (-\frac{\pi}{2}, \frac{3\pi}{2})$ and consider the curve

  $$\gamma : J \to \mathbb{R}^2, \quad \gamma(t) = (\sin(2t), \cos t).$$

  It is easy to verify that γ is an injective immersion. However, it is not a homeomorphism onto its image $N := \gamma(J) \subset \mathbb{R}^2$ (endowed with the subspace topology): removing e.g. the point $(0,1) = \gamma(0)$ the set $N \backslash \{(0,1)\}$ is connected, but its preimage under γ is $(-\frac{\pi}{2}, 0) \cup (0, \frac{3\pi}{2})$ hence disconnected. Therefore γ is not an embedding.
- *Irrational line on the torus*: Consider the curve in the torus $f : \mathbb{R} \to \mathbb{T}^2$ defined by $f(t) = [t, \alpha\, t]$, where $\alpha \in \mathbb{R}$ is an irrational number. This map is an injective immersion, but is not an embedding since one can find a divergent sequence of numbers $t_i \in \mathbb{R}$ such that $f(t_i)$ converges to some point in the image of f. (See [Lee13, Example 4.20] for more details.)

Example 3.0.4 (Graphs as submanifolds)**.** The **graph** of a $\mathcal{C}^k$ map $f : M \to N$, defined as the set $\mathrm{Gra}_f = \{(x, f(x)) : x \in M\}$, is a submanifold of $M \times N$. Indeed, it is the image of the **graphing map** of f, i.e. the map $g : M \to M \times N$ defined by $g(x) = (x, f(x))$, which is an embedding.

To see that g is an immersion we note that for any point $p \in M$, the tangent operator $\mathrm{T}_p\, g : T_pM \to T_{p,f(p)}(M \times N) \equiv T_pM \times T_{f(p)}N$ maps every nonzero vector $v \in \mathrm{T}_p\, M$ to the vector $\mathrm{T}_p\, g(v) = (v, \mathrm{T}_p\, f(v))$, which is nonzero because its first component is nonzero.

To see that the map g is a topological embedding we note that it admits a continuous retraction that is the projection map $\pi : M \times N \to M : (x,y) \mapsto x$. (A **retraction** of g is a left inverse, i.e. a map ρ satisfying $\rho \circ g = \mathrm{id}$.) Equivalently, the map $g|^{\mathrm{Gra}_f} : M \to \mathrm{Gra}_f$ is a homeomorphism because its inverse is the continuous map $\pi|_{\mathrm{Gra}_f} : \mathrm{Gra}_f \to M$.

3.1 How to recognize an embedding

Suppose we are given a differentiable map $f : M \to N$ between manifolds, and we have already checked that it is an injective immersion. How can we prove that it is an embedding ? This is essentially a topological question: how can we show that a given injective map[2] is actually a topological embedding ? Here are some ways to do so:

Proposition 3.1.1. *An injective $\mathcal{C}^r$ immersion $f : M \to N$ is an embedding in any of the following cases:*

(a) f is an open map.

(b) $\dim M = \dim N$.

(c) f is a closed map.

(d) The domain M is compact.

(e) f is a proper map.

[2] A **map** between topological spaces is a continuous function.

(f) *The image $f(M)$ has an open neighborhood W such that the map $f|^W : M \to W$ is closed or proper.*

Proof. (a) In this case f sends open subsets of M to open subsets of N, which are clearly also open in the image $f(M)$. Thus f is an open (and bijective) map onto its image, hence an homeomeomorphism onto its image.

(b) In this case f is a local diffeomorphism by the IFT, hence an open map.

(c) Similar to (a), replacing "open" by "closed".

(d) A map from a compact space to a Hausdorff space is closed.

(e) A proper map is closed if its codomain is Hausdorff and locally compact; see e.g. [Lee13, A.57] or [Lee10, p. 4.95].

(f) In this case f is the composite of two embeddings: the closed embedding $f|^W$, followed by the inclusion map $\iota_W : W \to M$, which is an open embedding. □

Remark 3.1.2. The last case (f) is interesting because it is general: any embedded submanifold S has an open neighborhood that contains S as a closed subset. (Exercise.)

A different, arguably more direct way to check that a $\mathcal{C}^k$ map $f : M \to N$ is an embedding is by constructing a $\mathcal{C}^k$ **retraction** (a.k.a left inverse) of f, that is, a $\mathcal{C}^k$ map $g : N \to M$ satisfying $g \circ f = \mathrm{id}_M$. In fact, the retraction need not be defined everywhere; an open neighborhood of the image suffices.

Proposition 3.1.3. *Let $f : M \to N$ be a $\mathcal{C}^k$ map. Then f is an embedding if it admits a $\mathcal{C}^k$ **neighborhood retraction**, that is, a $\mathcal{C}^k$ map $g : U \to M$, where $U \subseteq N$ is an open neighborhood of the image $f(M)$, such that $g \circ f = \mathrm{id}_M$.*

Proof. Suppose $f : M \to N$ has a neighborhood retraction $g : U \to M$. Then f is clearly injective. Its tangent operator $\mathrm{T}_p f$ at any point $p \in M$ is injective as well because

$$\mathrm{T}_p \,\mathrm{id}\, M = \mathrm{T}_p(g \circ f) = \mathrm{T}_{f(p)}\, g \circ \mathrm{T}_p\, f$$

is an isomorphism. Therefore f is an immersion. Finally, f is a homeomorphism onto its image $f(M)$ because it admits as inverse the continuous map $g|_{f(M)}$. □

Note that this is the argument we used in Example 3.0.4 to prove that the graph of a $\mathcal{C}^r$ map $f : M \to N$ is a $\mathcal{C}^r$ submanifold of $M \times N$. The graphing map $g : M \to M \times N : x \mapsto (x, f(x))$ admits as a retraction the projection $\pi : M \times N \to M : (x, y) \mapsto x$.

Remark 3.1.4. This method for showing that a map is a submanifold is also general in the sense that every embedding $f : M \to N$ admits a neighborhood retraction (defined on a so-called *tubular neighborhood* of the submanifold $f(M)$). We won't prove that, but we'll show how to produce local neighborhood retractions, which are sufficient to prove that a subset is a submanifold (see Propositions 3.3.1 and 3.4.1 below).

3.2 Constant rank theorem

Many theorems about submanifolds are based on the inverse function theorem.

Theorem 3.2.1 (Inverse Function Theorem, or IFT)**.** *Let $f : M \to N$ be a $\mathcal{C}^k$ map, where M, N are n-manifolds, and let $p \in M$ be a point where the tangent operator $\mathrm{T}_p f$ is invertible. Then f is a **local diffeomorphism** at p, i.e. there exists respective open neighborhoods U, V of p, $f(p)$ such that $f|_U^V$ is a $\mathcal{C}^k$ diffeomorphism.*

In case that M, N are open subsets of $\mathbb{R}^n$, this is just the inverse function theorem that you know from calculus. The general case can be deduced easily, by using charts.

A very useful generalization (and corollary) of the IFT is the following:

Theorem 3.2.2 (Constant Rank Theorem, or CRT). *Let $f : M \to N$ be $\mathcal{C}^r$ map that has rank k at some point $p \in M$, and let $m = \dim M$ and $n = \dim N$. Suppose, further that f has constant rank k in a neighborhood of p. Then f admits at the point p a local expression $\widetilde{f} = f|_\phi^\psi$ of the form*

$$\begin{array}{cccc} \widetilde{f}: & \widetilde{U} \subseteq \mathbb{R}^m & \to & \widetilde{V} \subseteq \mathbb{R}^n \\ & x & \mapsto & (x^0, \ldots, x^{k-1}, 0, \ldots, 0). \end{array}$$

where $\phi : U \to \widetilde{U}$, $\psi : V \to \widetilde{V}$ are charts of M, N defined at p, $f(p)$ respectively. Moreover, one can assume that the chart images are product sets $\widetilde{U} = \widetilde{W} \times \widetilde{U}'$, $\widetilde{V} = \widetilde{W} \times \widetilde{V}'$, for some open sets $\widetilde{W} \subseteq \mathbb{R}^k$, $\widetilde{U}' \subseteq \mathbb{R}^{m-k}$ and $\widetilde{V}' \subseteq \mathbb{R}^{n-k}$.

The hypothesis of constant rank near p is satisfied automatically in the following cases:

- *If f is submersive at p (i.e. if $n = k$): In this case the chart ψ may be constructed by restricting an arbitrary chart. Equivalently, if N is an open submanifold of $\mathbb{R}^k$, one can take $\psi = \mathrm{id}_V$, where $V \subseteq N$ is an open neighborhood of $f(p)$.*
- *If f is immersive at p (i.e. if $m = k$): In this case the chart ϕ may be constructed by restricting an arbitrary chart. Equivalently, if M is an open submanifold of $\mathbb{R}^k$ one can take $\phi = \mathrm{id}_U$, where $U \subseteq M$ is an open neighborhood of p.*

This theorem says that any map of constant rank (in particular, any immersion or submersion) is locally equivalent to the standard map given in Example 3.0.2 above. If $\mathrm{T}_p f$ does not have maximum rank (i.e. if $\mathrm{rank}_p f =: k < m, n$), then the hypothesis of having constant rank k near p is hard to satisfy because the rank is lower semicontinuous but not upper semicontinuous in general. We will prove the constant rank theorem only in the cases $k = m$ and $k = n$, which are the most useful.

Example 3.2.3. If $f : V \to W$ is an *linear* map of rank k between topological spaces $V \simeq \mathbb{R}^m$ and $W \simeq \mathbb{R}^n$, then there are *linear* isomorphisms $\phi : V \to \mathbb{R}^m$, $\psi : W \to \mathbb{R}^n$ such that $\psi \circ f \circ \phi^{-1}$ is a standard map of rank k. (Exercise.)

Before proving the constant rank theorem, let us discuss the situation. We use the following terminology and notation for a point $x = (x^i)_{0 \leq i < m} \in \mathbb{R}^m$. Its **horizontal part** is the tuple of its first k-coordinates,

$$x^{[0,k)} := (x^0, \ldots, x^{k-1}) = (x^i)_{0 \leq i < k} \in \mathbb{R}^{[0,k)} = \mathbb{R}^k,$$

and its **vertical part** is the tuple of the remaining coordinates

$$x^{[k,m)} := (x^k, \ldots, x^{m-1}) = (x^i)_{k \leq i < m} \in \mathbb{R}^{[k,m)} \equiv \mathbb{R}^{m-k}.$$

Similarly, for a point $y \in \mathbb{R}^n$, its horizontal and vertical parts are $y^{[0,k)}$ and $y^{[k,n)}$.

From the constant rank theorem we conclude that the map $f|_U^V$ has the following features: its image is a k-submanifold of V, and each fiber (i.e. the preimage of a point of the image) is an $(m-k)$-submanifold of U. These features are evident in the local expression $f|_\phi^\psi$, but they cannot be created by the charts, therefore they must be already present in the function $f|_U^V$.

The main job of the chart ϕ is to straighten the fibers of $f|_U^V$ so that they appear in the chart image $\widetilde{U} \subseteq \mathbb{R}^m$ as the vertical slices $\{x^{[0,k)}\} \times \widetilde{U}'$. For this task, the first k coordinates must be chosen carefully, so that they are constant on the fibers of f. This straightening is not necessary if $k = m$ (because the fibers are points), hence we may choose ϕ arbitrarily in this case.

Similarly, the main job of the chart ψ is to straighten the image of $f|_U^V$ so that it appears in the chart image $\widetilde{V} \subseteq \mathbb{R}^m$ as the horizontal null plane $\widetilde{W} \times \{0_{\mathbb{R}^{[k,n)}}\}$. For this task, the last $n-k$ coordinates are the most important; they must vanish on the image of $f|_U^V$. This straightening is not necessary if $k = n$ (because the image is an open set, not a submanifold of smaller dimension), hence we may choose ψ arbitrarily in this case.

Finally, we must ensure that the local expression $f|_\phi^\psi$ maps each fiber $\{x^{[0,k)}\} \times \widetilde{U}'$ to the right point $(x^{[0,k)}, 0, \ldots, 0)$ of the image. (To see that something is needed consider a map $f : \mathbb{R}^m \to \mathbb{R}^n$ of the form $f(x) = (\theta(x^{[0,k)}), 0, \ldots, 0) \in \mathbb{R}^n$, where $\theta : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism. This map f has constant rank k but it is not in the standard form, even though its fibers are the vertical slices and its image is the horizontal null plane.) This third job can be done either by ϕ or by ψ, but it must be done by one of the two charts. This is why in the case $k = m = n$ we cannot choose both ϕ and ψ arbitrarily.

Proof of the constant rank theorem (for submersions and immersions). Since the theorem is local, we may assume that M, N are open submanifolds of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively and that $p = 0_{\mathbb{R}^m}$ and $f(p) = 0_{\mathbb{R}^n}$. At the beginning, we have the standard coordinates $(x^i)_{i\in[0,m)}$, $(y^j)_{j\in[0,n)}$. By permutting the indices, we may assume that the $k \times k$ matrix

$$\left.\frac{\partial f^{[0,k)}(x)}{\partial x^{[0,k)}}\right|_{x=p} := \left.\left(\frac{\partial f^j(x)}{\partial x^i}\right)_{\substack{i\in[0,k)\\ j\in[0,k)}}\right|_{x=p} \quad \text{is invertible.} \tag{3.1}$$

Note that no permutation of the j indices is needed if $k = n$, and no permutation of the i indices is needed if $k = m$.

Case f submersive at p (i.e. $k = n$): We define on M near p the coordinate system $\phi : x \mapsto u$ by the formulas

$$\begin{cases} u^{[0,k)} = f(x) \\ u^{[k,m)} = x^{[k,m)}. \end{cases}$$

The first k coordinates are defined in the only possible way that ensures that the map $f|_\phi^{\mathrm{id}_V} = f \circ \phi^{-1}$ has the right form, sending $u \mapsto u^{[0,k)}$. The last coordinates are just meant to complete the coordinate system. To verify that this correspondence $x \mapsto u$ really defines a chart ϕ on a neighborhood of p, we check that its matrix of partial derivatives at the point p is invertible by our hypothesis. And indeed, the matrix

$$\left.\frac{\partial u^{[k,m)}}{\partial x^{[k,m)}}\right|_{x=p} = \left(\begin{array}{c|c} \left.\frac{\partial f(x)}{\partial x^{[0,k)}}\right|_{x=p} & * \\ \hline 0 & I_{m-k} \end{array}\right)$$

is invertible because the block $\left.\frac{\partial f(x)}{\partial x^{[0,k)}}\right|_{x=p}$ is invertible. Thus by the IFT, there are respective open neighborhoods $U \subseteq M$, $\widetilde{U} \subseteq \mathbb{R}^m$ of p, $\phi(p)$, such that the restriction $\phi|_U^{\widetilde{U}}$ is a diffeomorphism. Moreover, by shrinking the sets U, $\widetilde{U}$ we can ensure that $\widetilde{U} = \widetilde{W} \times \widetilde{U}'$, where $\widetilde{W} \subseteq \mathbb{R}^k$ and $\widetilde{U}' \subseteq \mathbb{R}^{m-k}$ are open sets. On the manifold N we choose the chart $\psi = \mathrm{id}_{\widetilde{W}}$.

Case f immersive at p (i.e. $k = m$): Note that the horizontal part of f, i.e. the map $\theta(x) = f^{[0,k)}(x)$ is a diffeomorphism when given a suitable domain and codomain (open neighborhoods of p and $f(p)^{[0,k)}$, respectively). This follows from condition (3.1) by the IFT.

We define on N near $f(p)$ a coordinate system $\psi : y \mapsto v$ by the formulas

$$\begin{cases} v^{[0,k)} = \theta^{-1}(y^{[0,k)}) \\ v^{[k,n)} = y^{[k,n)} - f^{[k,n)}(\theta^{-1}(y^{[0,k)})). \end{cases}$$

On a point $y = f(x)$, the horizontal coordinates $v^{[0,k)}$ are

$$v^{[0,k)} = \theta^{-1}(f^{[0,k)}(x)) = x$$

while the vertical coordinates vanish:

$$v^{[k,n)} = y^{[k,n)} - f^{[k,n)}(\theta(y^{[0,k)})) = y^{[k,n)} - f^{[k,n)}(x) = 0.$$

Therefore the map $f|_{\mathrm{id}}^{\psi} = \psi \circ f$ sends $x \mapsto (x, 0)$, as required.

To ensure that this correspondence $y \mapsto v$ really defines a chart ψ on a neighborhood of p, we verify that the matrix

$$\left.\frac{\partial v^{[k,n)}}{\partial y^{[k,n)}}\right|_{y=f(p)} = \left(\begin{array}{c|c} \left.\frac{\partial \theta^{-1}(y)}{\partial y^{[0,k)}}\right|_{y=f(p)} & 0 \\ \hline * & I_{n-k} \end{array}\right)$$

is invertible. This follows because the matrix $\left.\frac{\partial \theta^{-1}(y)}{\partial y^{[0,k)}}\right|_{y=f(p)}$, which represents the linear transformation $\mathrm{D}_{f(p)}\,\theta^{-1}$, is invertible. Thus by the IFT, there are respective open neighborhoods $V \subseteq N$, $\widetilde{V} \subseteq \mathbb{R}^n$ of $f(p)$, $\psi(f(p))$, such that the restriction $\phi|_V^{\widetilde{V}}$ is a diffeomorphism. Moreover, by shrinking the sets $\widetilde{V}$, V we can ensure that $\widetilde{V} = \widetilde{W} \times \widetilde{V}'$, where $\widetilde{W} \subseteq \mathbb{R}^k$ and $\widetilde{V}' \subseteq \mathbb{R}^{n-k}$ are open sets. On the manifold M we choose the chart $\psi = \mathrm{id}_W$.

General case* (i.e. $k \leq m, n$) Here one can combine the two constructions. If interested, you can take it as an exercise or read the proof in p. 42 of Spivak's "Comprehensive introduction to Differential Geometry", volume 1. □

3.3 Some properties of immersions and embeddings

3.3.1 Local properties: slices and retractions

The following corollary of the constant rank theorem says that for any immersion $f : M \to N$, if we restrict f to a sufficiently small neighborhood U of any point $p \in M$, its image $f(U)$ looks like a "slice" of some open set $V \subseteq N$. This implies that f has a local retraction and is locally an embedding.

Lemma 3.3.1 (Slice property (and local retractions) for immersions)**.** *Let N be an n-manifold and let $f : M \to N$ be an immersion with $\dim M = k$. Then for each point p of M there exists a chart $\psi : V \to \widetilde{V}$ of N and an open neighborhood U of p such that*

$$f(U) = \{q \in V : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\},$$

and such that the map $f|_U^V$ is an embedding and has a $\mathcal{C}^k$ retraction $\rho : V \to U$.

Proof. By the constant rank theorem, the map f admits at the point p a local expression $\widetilde{f} = f|_\phi^\psi$ of the form $\widetilde{f}(x) = (x^0, \dots, x^{k-1}, 0, \dots 0)$, where $\phi : U \to \widetilde{U}$, $\psi : V \to \widetilde{V}$ are charts of M, N defined at the points p, $f(p)$ respectively. Moreover, we can assume that $\widetilde{V} = \widetilde{U} \times \widetilde{V}'$, where $\widetilde{V}' \subseteq \mathbb{R}^{n-k}$ is an open neighborhood of $0_{\mathbb{R}^{n-k}}$. It follows that $\widetilde{f}(\widetilde{U}) = \widetilde{U} \times \{0_{\mathbb{R}^{n-k}}\}$ and hence

$$f(U) = \psi^{-1}(\widetilde{U} \times \{0_{\mathbb{R}^{n-k}}\}) = \{q \in V : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\}.$$

To construct the retraction ρ, consider the projection map

$$\begin{array}{rrcl} \pi : & \widetilde{V} = \widetilde{U} \times \widetilde{V}' & \to & \widetilde{U} \\ & (x, y) & \mapsto & x. \end{array}$$

There is a unique map $\rho : V \to U$ that has local expression $\rho|_\psi^\phi = \pi$, namely, the map $\rho = \phi^{-1} \circ \pi \circ \psi : V \to U$. This map ρ is a $\mathcal{C}^r$ retraction of $f|_U^V$. The existence of this retraction implies that $f|_U^V$ is an embedding by Proposition 3.1.3. $\square$

Note that the image of f may intrude in the set V at points not located on the slice $f(U) = \{\psi^k = \cdots = \psi^{n-1} = 0\}$. (Exercise: Find an example (i.e. an immersion f and a point p) such that this happens for any V.) However, this situation can be avoided if f is an embedding.

Proposition 3.3.2 (Slice property for embedded submanifolds)**.** *Let S be a the image of a $\mathcal{C}^r$ embedding $f : M \to N$, and let $k = \dim M$ and $n = \dim N$. Then each point of S is contained in the domain V of a chart ψ that is k-**sliced** by S, meaning that*

$$S \cap V = \{q \in V : \phi^k(q) = \cdots = \phi^{n-1}(q) = 0\}. \tag{3.2}$$

Proof. Let $f : M \to N$ be an embedding with image S, and take a point $q_0 = f(p_0) \in S$. By Lemma 3.3.1, there exists a chart ψ of M with domain V' and a neighborhood U of p_0 such that $f(U) = \{q \in V' : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\}$. Since $f(U)$ is open in S, we can write $f(U) = S \cap V''$ using some open set $V'' \subseteq M$. But since $f(U) \subseteq V'$, we have $f(U) = S \cap V$, and therefore

$$S \cap V = f(U) = \{q \in V : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\}.$$

$\square$

3.3.2 The initial property

Recall that topological embeddings have the following *initial property*.

Proposition 3.3.3 (Initial property of topological embeddings)**.** *If a continuous map $f : X \to Y$ between topological spaces is an embedding, then any function $h : Z \to X$ (where Z is a topological space) is continuous if the composite $f \circ h$ is continuous.*

A $\mathcal{C}^r$ immersion has a similar property that allows us to show that a continuous function is $\mathcal{C}^r$.

Proposition 3.3.4 (Initial property of immersions and embeddings)**.** *Let $f : M \to N$ be a $\mathcal{C}^r$-differentiable map, and let L be another $\mathcal{C}^r$ manifold.*

(a) If f is an immersion, then any continuous map $h : L \to M$ is $\mathcal{C}^r$ if the composite $f \circ h$ is $\mathcal{C}^r$.

(b) *If f is an embedding, then any function $h : L \to M$ is $\mathcal{C}^r$ if the composite $f \circ h$ is $\mathcal{C}^r$.*

The proof is based on the fact that an immersion admits local retractions.

Proof. (a) Suppose that f is an immersion and that $h : L \to M$ is continuous at some point $z \in L$, and the composite $f \circ h$ is $\mathcal{C}^r$ at some point $z \in L$. Let us show that h is $\mathcal{C}^r$ at z.

Let $U \subseteq M$ and $V \subseteq N$ be open neighborhoods of the points $f(z)$ and $h(f(z))$ such that $f(U) \subseteq V$ and the map $f|_U^V$ has a $\mathcal{C}^r$ retraction $\rho : V \to U$. Thus we have $\rho \circ f|_U^V = \mathrm{id}_U$. Note that the set $W = h^{-1}(U)$ is a neighborhood of x since h is continuous at x. Thus we can write

$$h|_W^U = \mathrm{id}_U \circ h|_W^U = (\rho \circ f|_U^V) \circ h|_W^U = \rho \circ (f \circ h)|_W^V.$$

This factorization shows that h is $\mathcal{C}^r$ at the point z, since $f \circ h$ is $\mathcal{C}^r$ at z and ρ is $\mathcal{C}^r$ everywhere.

(b) Now suppose that f is a $\mathcal{C}^r$ embedding and that $h : L \to M$ is any function such that the composite $f \circ h$ is $\mathcal{C}^r$. Since f is a topological embedding, the function h is continuous by Proposition 3.3.3. Since f is an immersion, the map h is $\mathcal{C}^r$ by part (a). □

3.4 How to recognize an embedded submanifold

Sometimes we are just given a subset S of a manifold M, and we have to determine whether it is an embedded submanifold or not. There are several ways to do so.

Proposition 3.4.1. *Let S be a subset of a $\mathcal{C}^r$ manifold M, and let $k \le n = \dim M$. The following are equivalent:*

(a) *S is a $\mathcal{C}^r$-embedded k-submanifold of M.*

(b) *Local embedded submanifold: Each point of S has an open neighborhood W in M such that the set $S \cap W$ is a $\mathcal{C}^r$-embedded k-submanifold of W.*

(c) *Local slice: Each point of S is contained in the domain W of a chart ψ of M that is k-sliced by S.*

(d) *Local retract: Each point of S has an open neighborhood W in M such that the set $S \cap W$ is the image of a $\mathcal{C}^r$ map $\phi : U \to W$ (where $U \subseteq \mathbb{R}^k$ is an open set), which in turn admits a $\mathcal{C}^r$ retraction $\rho : W \to U$.*

Proof. (a) $\Rightarrow$ (b) Trivial: take $W = M$.
(a) $\Rightarrow$ (c) This is Proposition 3.3.2.
(b) $\Rightarrow$ (c) Same proof as (a) $\Rightarrow$ (c).
(c) $\Rightarrow$ (d) Let p be a point of S, and let $\psi : W \to \widetilde{W}$ be a chart of M, defined at p, that is k-sliced by S. By shrinking the sets W and $\widetilde{W}$, we may assume that $\widetilde{W} = \widetilde{U} \times \widetilde{V}$, for some open sets $\widetilde{U} \subseteq \mathbb{R}^k$ and $\widetilde{V} \subseteq \mathbb{R}^{n-k}$. The slice condition implies that

$$S \cap W = \psi^{-1}(\widetilde{U} \times \{0\}).$$

Consider the projection map $\pi : \widetilde{U} \times \widetilde{V} \to \widetilde{U} : (x, y) \mapsto x$ and its section $\sigma : \widetilde{U} \to \widetilde{U} \times \widetilde{V} : x \mapsto (x, 0)$. Note that $\pi \circ \sigma = \mathrm{id}_{\widetilde{U}}$.

Now we use the diffeomorphism $\psi : W \to \widetilde{W}$ to transport the maps π and ρ from the Euclidean set $\widetilde{W} \subseteq \mathbb{R}^n$ to the manifold subset $W \subseteq M$. That is, we define the $\mathcal{C}^r$ maps $\phi = \psi^{-1} \circ \sigma : \widetilde{U} \to W$ and $\rho = \pi \circ \psi : W \to \widetilde{U}$. Clearly ρ is a retraction of ϕ, that is, we have $\rho \circ \phi = \mathrm{id}_U$. The image of ϕ is the set

$$\mathrm{Img}(\phi) = \phi^{-1}(\mathrm{Img}(\sigma)) = \phi^{-1}(\widetilde{U} \times \{0\}) = S \cap W.$$

(d) $\Rightarrow$ (a) Suppose S is covered by open sets W_i such that each set $V_i := S \cap W_i$ is the image of a $\mathcal{C}^r$ maps $\phi_i : U_i \to W_i$ admitting a retraction $\rho_i : W_i \to U_i$.

Let us show that S admits a structure of k-dimensional $\mathcal{C}^r$ manifold such that the inclusion map $\iota_S : S \to M$ is a $\mathcal{C}^r$ embedding. The only topology we may put on S (so that ι_S a topological embedding) is the subspace topology. This topology is Hausdorff and second countable.

Let us construct a $\mathcal{C}^r$ atlas on S. The maps ϕ_i and ρ_i of course satisfy $\rho_i \circ \phi_i = \mathrm{id}_{U_i}$. Composing in the other order we get a map $\phi_i \circ \rho_i$ that fixes the image of ϕ_i, which is the set V_i. Therefore the continuous maps

$$\overline{\phi}_i := \phi_i|^{V_i} : U_i \to V_i, \quad \overline{\rho}_i := \rho_i|_{V_i} : V_i \to U_i$$

are inverse of each other. The maps $\overline{\rho}_i$ form a $\mathcal{C}^r$ atlas of S since the transition maps

$$\overline{\rho}_i \circ \overline{\rho}_j^{-1} = \overline{\rho}_i \circ \overline{\phi}_j = \rho_i \circ \phi_j$$

are $\mathcal{C}^r$.

The inclusion $\iota_S : S \to M$ is an immersion because composing with each local parametrization $\overline{\phi}_i$ we obtain a map $\iota_S \circ \overline{\phi}_i = \phi_i$ that is a $\mathcal{C}^k$ immersion, since it admits a $\mathcal{C}^k$ retraction. $\square$

Example 3.4.2. A subset $S \subseteq \mathbb{R}^n$ that can be locally expressed as a graph of a $\mathcal{C}^r$ function (with some $n-k$ coordinates expressed as a function of the other k coordinates) is a k-submanifold of $\mathbb{R}^n$ because it satisfies condition (d), since graphing functions admit projections as retractions. This is how we constructed the manifold structure on the sphere $\mathbb{S}^{n-1}$.

3.5 How to produce a submanifold as a level set

Definition 3.5.1. Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map.

A point $q \in N$ is called a **regular value** of f if $\mathrm{T}_p f$ is surjective for all points $p \in f^{-1}(q)$.

Theorem 3.5.2 (Regular preimage theorem)**.** *Let c be a regular value of a $\mathcal{C}^r$-differentiable map $f : M \to N$, and let $m = \dim M$, $n = \dim N$. Then the set $S = f^{-1}(c)$ is a $\mathcal{C}^r$-embedded submanifold of M of dimension $k = m - n$. Its tangent space at any point $p \in S$ is $\mathrm{T}_p S = \mathrm{Ker}(\mathrm{T}_p f)$.*

Proof. We will use the constant rank theorem to show that S fulfills the "local slice" condition of Proposition 3.4.1, therefore S is a submanifold.

At a point $p_0 \in S$ the map f is submersive, therefore by the CRT it admits at the point p_0 a local expression $\widetilde{f} = f|_{\phi}^{\psi}$ of the form $\widetilde{f}(x) = (x^k, \ldots, x^{m-1})$, where $\phi : U \to \widetilde{U}$, $\psi : V \to \widetilde{V}$ are charts of M, N defined at the points p_0, $f(p_0) = c$ respectively. In addition, we may assume that $\widetilde{U} = \widetilde{V} \times \widetilde{U}'$ where $U' \subseteq \mathbb{R}^{m-n}$ is an open set.

Note that $\psi(c) = 0$. Thus for a point $p \in U$, denoting $x = \phi(p) \in \mathbb{R}^m$, we have $p \in S$ iff $f(p) = c$ iff $\widetilde{f}(x) = 0$ iff the last n coordinates $x^k, \ldots, x^{m-1}$ vanish. In other words, the chart ϕ is k-sliced by $S = f^{-1}(c)$. Since we can find such a chart ϕ for any point $p_0 \in S$, we conclude that S is a $\mathcal{C}^r$-embedded k-submanifold of M.

Tangent space: exercise. $\square$

3.6 Whitney's theorem

Theorem 3.6.1 (Whitney)**.** *Every compact $\mathcal{C}^k$ manifold M can be $\mathcal{C}^k$ embedded in $\mathbb{R}^m$ for some m.*

Proof. Let $n = \dim M$, and denote $B(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ for $r > 0$. Since M is compact, there is a finite family $(\phi_i)_{i \in N}$ of charts $\phi_i : U_i \to B(2)$ such that the open sets $B_i := \phi_i^{-1} B(1)$ cover M.

For each i, take a $\mathcal{C}^k$ bump function $\eta_i : M \to [0,1]$ with support $\operatorname{supp} \eta_i \subseteq U_i$ such that $\eta_i \equiv 1$ on B_i, and define a function $f_i : M \to \mathbb{R}^n$ by

$$f_i(x) = \begin{cases} \eta_i(x)\, \phi_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \notin \operatorname{supp}(\eta_i) \end{cases}$$

Finally, define a $\mathcal{C}^k$ map $f : M \to \mathbb{R}^{Nn+N}$ by

$$f(x) = ((f_i(x))_{i \in N}, (\eta_i(x))_{i \in N}).$$

We claim that f is an embedding.

f is an immersion because each f_i is immersive on B_i, since it coincides with the chart ϕ_i.

f is injective: suppose that $f(x) = f(y)$. Take some i such that $x \in B_i$. Then $\eta_i(x) = 1$, and therefore $\eta_i(y) = 1$. But then $x, y \in U_i$ and we have $\phi_i(x) = f_i(x) = f_i(y) = \phi_i(y)$, which implies that $x = y$.

Since f is an injective map with compact domain (and Hausdorff codomain), it is a topological embedding. $\square$