Introduction to Differentiable Manifolds

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Contents

Main references

[Lee10] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2010.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second edition. Springer, 2013.

Both books have online versions available in the library.

0.1 Practical remarks about the course

0.1.1 Planning 2021: 14 classes

This information is preliminary and may be changed as we go.

- 1. (21.09) Topological and Differentiable manifolds
- 2. (28.09) Differentiable maps. Partitions of unity.
- 3. (05.10) Tangent bundle. Differential of a map.
- 4. (12.10) Vector bundles
- 5. (19.10) Vector fields and flows
- 6. (26.10) Submanifolds
- 7. (02.11) Whitney embedding. Regular levels sets are submanifolds.
- 8. (09.11) Differential forms
- 9. (16.11) Differential forms
- 10.(23.11) Exterior derivative
- 11.(30.11) Integration, orientation
- 12.(07.12) Manifolds with boundary
- 13.(14.12) Stokes' theorem
- 14.(21.12) Not assigned

0.1.2 Content for the exam

Studying the content under a heading marked by an asterisk is not mandatory by itself. You should read it only if you find it interesting or helpful for understanding the rest.

1 Manifolds

The goal of this course is to extend differential and integral calculus from Euclidean space $\mathbb{R}^n$ to all *differentiable manifolds* such as the n-sphere, the n-torus, etc. Roughly speaking, a differentiable manifold is a space that

- is endowed with a certain topology,
- has, in addition, a *differentiable structure* that allows us to distinguish whether a map is differentiable or not, rather than just continuous, and
- locally looks like Euclidean space $\mathbb{R}^n$.

1.1 Topological manifolds

[Lee13], Chapter 1 and [Lee10], Chapter 2

Let us pospone the question of differentiability and focus on topology. As said, we want to study spaces that "locally look like" Euclidean space $\mathbb{R}^n$.

Definition 1.1.1 (Locally Euclidean space)**.** Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is **locally Euclidean** of dimension n at a point $p \in M$ if the point p has an open neighborhood that is homeomorphic[1] to an open subset of $\mathbb{R}^n$. If this holds for all points $p \in M$, we say that M is locally Euclidean of dimension n.

A typical example is the circle: it is locally Euclidean of dimension 1 but not globally homeomorphic to any subset of $\mathbb{R}$.

Example 1.1.2. The **circle** $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (with the subspace topology) is locally Euclidean of dimension 1: let $(x_0, y_0) \in \mathbb{S}^1$, wlog $y_0 > 0$, then $U := (\mathbb{R} \times \mathbb{R}^+) \cap \mathbb{S}^1$ is an open subset of $\mathbb{S}^1$ containing (x_0, y_0) and homeomorphic to $(-1, 1)$ via the map $U \to (-1, 1)$ that sends $(x, y) \mapsto x$.

We will see more examples later on (see e.g. Examples 1.1.11 below). Let us make some general comments.

Remark 1.1.3. If a space M is locally Euclidean of dimension 0, then every point has a neighborhood homeomorphic to $\mathbb{R}^0 = \{0\}$, i.e. a point. In other words, M is a discrete topological space.

Remark 1.1.4. In the definition of locally Euclidean space, we could have replaced "...homeomorphic to an open subset of $\mathbb{R}^n$" by "...homeomorphic to $\mathbb{R}^n$". (Exercise.)

Remark 1.1.5. Brouwer's theorem of *invariance of domain*, given here without proof, says that if two nonempty open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$. It follows that the dimension of a locally Euclidean space at each point can be defined unambiguously. Furthermore, it is easy to prove that the dimension is constant throughtout each connected component. Thus the only way to get a locally Euclidean space of mixed dimensions is to make a disjoint union of components of different dimensions. Anyway, in the definition of topological manifold (see below) we will not admit this kind of spaces.

[1] Recall that two topological spaces X, Y are **homeomorphic** if there exists a bijection $\varphi : X \to Y$ such that both φ and its inverse are continuous.

For the definition of *topological manifold* we demand some further topological properties that ensure that the space is topologically "well-behaved". (For instance, we want the limit of every sequence to be unique.)

Definition 1.1.6. A **topological manifold of dimension** n, or **topological** n**-manifold**, is a topological space M that is locally Euclidean of dimension n, Hausdorff[2] and second countable.[3]

A **topological manifold** is a topological space that is a topological n-manifold for some n.

Side note: Make sure you are familiar with some basic definitions from topology such as Hausdorff, second countable, connected and compact spaces, and the construction of subspace, product, coproduct and quotient topologies. Chapters 2 and 3 in [Lee10] provides a succinct overview of everything we need.

Remark 1.1.7. The conditions of Hausdorff resp. second countable in Definition 1.1.6 are not redundant. For example, the *line with two origins* (see Exercises) is a locally Euclidean, second countable space that is not Hausdorff. The *long line* and the *Prüfer surface* (see Wikipedia if interested) are locally Euclidean of dimension 1 and 2 respectively, Hausdorff, and connected, but not second countable.

The homeomorphisms that locally identify a topological manifold with Euclidean space are called *charts*:

Definition 1.1.8 (Coordinate charts)**.** Let M be a topological n-manifold. A **chart** (or **coordinate chart**) for M is a homeomorphism $\varphi : U \to V$, where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets. Its inverse φ^{-1} is a **local parametrization** of M. An **atlas** for M is a collection of charts whose domains cover M.

For the moment, we can see an atlas simply as a way of showing that a space is locally Euclidean.

Remark 1.1.9. Some authors define a chart for M as a *pair* (φ, U) or even a *triple* (φ, U, V) where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets and $\varphi : U \to V$ is a homeomorphism. Here, instead, we consider the sets U and V as part of the function φ, namely, its domain $\mathrm{Dom}(\varphi)$ and codomain $\mathrm{Cod}(\varphi)$, thus there's no need not specify them separately.[4] From this point of view, the letters "U", "V" are just shorter names for the sets $\mathrm{Dom}(\varphi)$, $\mathrm{Cod}(\varphi)$.

Convention: When talking about subsets (resp. quotients, products, disjoints unions) of topological spaces we'll assume that they are endowed with the subspace (resp. quotient, product, coproduct) topology unless otherwise stated.

Using this convention, let us mention some easy ways to construct new topological manifolds from old ones.

Proposition 1.1.10 (New manifolds from old)**.** *The properties of being Hausdorff or second countable are preserved by taking subspaces, finite products and countable coproducts. In consequence:*

- *An open subset of a topological* n*-manifold is a topological* n*-manifold.*

[2] Recall that a topological space X is **Hausdorff** if every two different points $x, y \in X$ have disjoint neighborhoods.

[3] Recall that a topological space X is **second countable** if its topology admits a countable basis. A **basis** for a topology is a family $\mathcal{B}$ of open sets such that every open set is a union of some sets of $\mathcal{B}$.

[4] Formally, a function f is a triple $f = (X, Y, \Gamma)$ where X, Y are sets (called the **domain** and **codomain** of f, and denoted $\mathrm{Dom}(f)$ and $\mathrm{Cod}(f)$), and Γ is a subset of $X \times Y$ (called the **graph** of f, denoted $\mathrm{Gra}(f)$), such that for each $x \in X$ there is a unique $y \in Y$ (called the **image** of x by f, denoted $f(x)$) such that $(x, y) \in \Gamma$.

- *A disjoint union $M = \coprod_i M_i$ of countably many topological n-manifolds M_i is a topological n-manifold.*
- *A product $M = \prod_i M_i$ of finitely many topological manifolds M_i is a topological manifold of dimension* $\dim(M) = \sum_i \dim(M_i)$.

Proof. Exercise. □

Example 1.1.11 (Examples of topological manifolds)**.**

(a) Of course any open subset of $\mathbb{R}^n$ is a topological manifold.

(b) An example of topological n-manifold is the **graph**

$$\Gamma_f := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^m$$

of a continuous function $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set. Indeed, it is homeomorphic to U via the **graph parametrization**

$$\begin{array}{ccc} U & \to & \Gamma_f \\ x & \mapsto & (x, f(x)), \end{array} \quad \text{whose inverse is the projection} \quad \begin{array}{ccc} \Gamma_f & \to & U \\ (x,y) & \mapsto & x. \end{array}$$

(c) The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a topological manifold. Being a subset of $\mathbb{R}^{n+1}$ it is Hausdorff and second countable. A possible choice of atlas is given by the so-called **graph coordinates**: Cover $\mathbb{S}^n$ by the $2(n+1)$ open sets $U_i^\pm := \{x \in \mathbb{R}^n \mid \pm x_i > 0\}$, then $\mathbb{S}^n \cap U_i^\pm$ is homemorphic to the open unit n-ball $\mathbb{B}^n$ via the projection[5]

$$\begin{array}{cccc} \varphi_i^\pm : & \mathbb{S}^n \cap U_i^\pm & \to & \mathbb{B}^n \\ & (x_0, \ldots, x_n) & \to & (x_0, \ldots, \widehat{x_i}, \ldots, x_n). \end{array}$$

The maps $\varphi_i^\pm$ are coordinate charts for $\mathbb{S}^n$; we call them *graph coordinates*. Locally this is a special case of the previous item (b): each set $\mathbb{S}^n \cap U_i^\pm$ is (up to permutation of coordinates) the graph of the continuous function on the unit n-ball $\mathbb{B}^n(0)$:

$$\mathbb{B}^n \to \mathbb{R} : y \mapsto \pm\sqrt{1 - \textstyle\sum_i y_i^2}.$$

(d) Real projective space $\mathbb{P}^n$ is a topological n-manifold (exercise).

(e) The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a topological n-manifold (exercise).

(f) More generally, If M is a topological n-manifold and G is a group of homeomomorphisms of M that acts properly discontinuously and without fixed points, then the quotient space M/G is a topological n-manifold.

(g) If M is a topological n-manifold and $\pi : N \to M$ is a covering map (with N connected), then N is a topological n-manifold. In particular, the universal covering space of any connected topological manifold is a topological manifold.

We will not prove the following result (although it can be done elementarily).

Theorem 1.1.12 (Classification of topological 1-manifolds)**.** *Every connected topological 1-manifold is homeomorphic to either $\mathbb{S}^1$ (if it is compact) or to $\mathbb{R}$ (if it is not compact).*

1.2 Differentiable manifolds

Our next goal is to define a kind of spaces and maps called *differentiable manifolds* and *differentiable maps* (or, more precisely $\mathcal{C}^k$ manifolds and maps) with which we can actually do differential calculus. Topological manifolds do not have enough structure because a topology does not allow us to determine whether a function is differentiable or not; it only distinguishes continuous functions. Differentiable manifolds should be

[5]The hat on $\widehat{x_i}$ means that we omit the respective coordinate x_i.

locally equivalent to Euclidean open sets (where we already have a well defined notion of $\mathcal{C}^k$ maps; see below), but at the global level they should be allowed to have a more interesting topology. In particular, the sphere, torus, projective space, etc. should become differentiable manifolds.

Before defining $\mathcal{C}^k$ manifolds, let us set up some terminology for $\mathcal{C}^k$ maps in $\mathbb{R}^n$.

Definition 1.2.1 (Euclidean open sets and Euclidean $\mathcal{C}^k$ maps)**.** A **Euclidean open set** is an open subset of some Euclidean space $\mathbb{R}^n$.
Let $k \in \{0, 1, \dots, \infty\}$. A function $f : U \to V$ between Euclidean open sets is $\mathcal{C}^k$ at a point $p \in U$ if its partial derivatives of order $\leq k$ are defined in a neighborhood of p and continuous at p. We say that f is $\mathcal{C}^k$ (and we call it a **Euclidean $\mathcal{C}^k$ map**) if it is $\mathcal{C}^k$ at all points $p \in U$.
An **Euclidean $\mathcal{C}^k$ isomorphism** is an Euclidean $\mathcal{C}^k$ map that has an Euclidean $\mathcal{C}^k$ inverse.

Note that every Euclidean $\mathcal{C}^k$ map is continuous because the function f itself is a partial derivative (of order 0) of f. In fact, a $\mathcal{C}^0$ map is the same thing as a continuous map.

We are now ready to define $\mathcal{C}^k$ manifolds. The key to turn a topological manifold into a $\mathcal{C}^k$ manifold is to choose an appropriate atlas.

Definition 1.2.2 ($\mathcal{C}^k$ manifolds)**.** Let M be a topological n-manifold and $k = 0, \dots, \infty$. Two charts φ, ψ for M, with respective domains $U, V \subseteq M$, are **$\mathcal{C}^k$-compatible** if the **transition map** from φ to ψ, that is, the homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

is a $\mathcal{C}^k$ isomorphism (i.e. itself and its inverse are both Euclidean $\mathcal{C}^k$ maps).

A **$\mathcal{C}^k$-consistent atlas** (or **$\mathcal{C}^k$ atlas**, for short) is an atlas for M whose charts are $\mathcal{C}^k$ compatible with each other. Two $\mathcal{C}^k$ atlases for M are **$\mathcal{C}^k$-equivalent** if their union is $\mathcal{C}^k$-consistent.

A **$\mathcal{C}^k$ structure** on M is a maximal $\mathcal{C}^k$ atlas, i.e., a $\mathcal{C}^k$ atlas that is not contained in any other strictly larger $\mathcal{C}^k$ atlas. A **$\mathcal{C}^k$ manifold** is a topological manifold M endowed with a $\mathcal{C}^k$ structure.

Note that a $\mathcal{C}^0$ manifold is the same thing as a topological manifold. A $\mathcal{C}^k$ manifold with $k \geq 1$ is called a **differentiable manifold**, and a **smooth manifold** is a $\mathcal{C}^\infty$ manifold.

Remark 1.2.3 (Domains and codomains of functions)**.** To be precise, the transition map that we wrote as $\psi \circ \varphi^{-1}$ should actually be defined as $\psi|_{U \cap V}^{\psi(U \cap V)} \circ \left(\varphi|_{U \cap V}^{\varphi(U \cap V)}\right)^{-1}$, using the *restricted* charts

$$\varphi|_{U \cap V}^{\varphi(U \cap V)} : U \cap V \to \varphi(U \cap V), \quad \psi|_{U \cap V}^{\psi(U \cap V)} : U \cap V \to \psi(U \cap V).$$

In general we will not write the restrictions explicitly because it is cumbersome. When we compose functions, it should be understood that the resulting composite function is defined in principle at all points where it is possible. (Maybe no points at all!)

We may further restrict a function by specifying a reduced domain or codomain. On the other hand, we shall never specify a domain containing points where the function is not defined, nor a codomain that does not contain the image of the specified domain. Thus a function "$f : A \to B$" always has domain A and codomain B.

The next proposition shows that it suffices to give any $\mathcal{C}^k$-consistent atlas (not necessarily a maximal one) to determine a $\mathcal{C}^k$ structure.

Proposition 1.2.4 ($\mathcal{C}^k$ atlas defines $\mathcal{C}^k$ structure)**.** *For a fixed topological manifold M, each $\mathcal{C}^k$ atlas $\mathcal{A}$ is contained in a unique maximal $\mathcal{C}^k$ atlas $\overline{\mathcal{A}}$, which consists of all charts for M that are $\mathcal{C}^k$-compatible with those of $\mathcal{A}$. Any other $\mathcal{C}^k$ atlas $\mathcal{B}$ for M is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$.*

Proof. Let $\mathcal{A}$ be any $\mathcal{C}^k$ atlas for M. Define

$$\overline{\mathcal{A}} := \{\varphi \text{ chart for } M \text{ that is } \mathcal{C}^k \text{ compatible with all charts } \theta \in \mathcal{A}\}.$$

Clearly $\overline{\mathcal{A}}$ contains $\mathcal{A}$. We claim that $\overline{\mathcal{A}}$ is a $\mathcal{C}^k$ atlas. To prove this we have to show that if $\varphi, \psi \in \overline{\mathcal{A}}$ are charts with respective domains $U, V \subseteq M$, then the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is $\mathcal{C}^k$. Take any point $\psi(p) \in \psi(U \cap V)$ and let $\theta \in \mathcal{A}$ be a chart whose domain W contains the point $p \in U \cap V$. Then $\psi(U \cap V \cap W)$ is an open neighborhood of $\psi(p)$ and we can write the restriction[6]

$$\varphi \circ \psi^{-1} : \psi(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$

as the composition $(\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, which is $\mathcal{C}^k$ because $\varphi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are $\mathcal{C}^k$ by assumption. This proves that $\varphi \circ \psi^{-1}$ in a neighborhood of $\psi(p)$, but the same reasoning is valid at any point of $\psi(U \cap V)$, therefore $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$.

Finally, from the definition of $\overline{\mathcal{A}}$ it is clear that it is maximal, and that any atlas $\mathcal{B}$ is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$. In particular, any atlas $\mathcal{B}$ containing $\mathcal{A}$ is equivalent to $\mathcal{A}$ (by definition of equivalent atlases), and is therefore contained in $\overline{\mathcal{A}}$. Therefore a maximal atlas containing $\mathcal{A}$ is contained in $\overline{\mathcal{A}}$, but in fact it must be equal to $\overline{\mathcal{A}}$ (by maximality). We conclude that $\overline{\mathcal{A}}$ is the unique maximal $\mathcal{C}^k$ atlas containing $\mathcal{A}$. □

In consequence, given a topological manifold M and some $\mathcal{C}^k$ atlas $\mathcal{A}$ on M we can speak without ambiguity of *the* $\mathcal{C}^k$ structure $\overline{\mathcal{A}}$ determined by $\mathcal{A}$.

Remark 1.2.5. For practical purposes the concept of a maximal $\mathcal{C}^k$ atlas is not really important. We usually work with a smaller $\mathcal{C}^k$ atlas and this is all we need e.g. for checking that a function is $\mathcal{C}^k$ (see next section). (In fact, we could have defined a $\mathcal{C}^k$ structure on M as an equivalence class of $\mathcal{C}^k$ atlases, rather than as a maximal $\mathcal{C}^k$ atlas.) In general we won't give any name to the maximal atlas and we'll just speak about "a $\mathcal{C}^k$ manifold M" with the maximal atlas being implicit.

Remark 1.2.6. Any open subset U of a $\mathcal{C}^k$ manifold M has a natural $\mathcal{C}^k$ structure consisting of the $\mathcal{C}^k$ charts of M whose domain is contained in U. (Exercise.) We will also see in the exercices that finite products of $\mathcal{C}^k$ manifolds have a natural $\mathcal{C}^k$ structure.

Example 1.2.7 (Examples of smooth manifolds)**.**

- $\mathbb{R}^n$ (with the atlas consisting of the single chart $\mathrm{id}_{\mathbb{R}^n}$) is a smooth manifold. In general, any topological manifold endowed with a single-chart atlas is automatically a smooth manifold. For example, the graph Γ_f of any *continuous* (sic) function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ as described in Example 1.1.11, endowed with the projection chart, is a smooth manifold.
- The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Indeed, the atlas given by the graph coordinates (Example 1.1.11) is smooth. To see this, we compute the transition functions (wlog $i < j$):

$$\begin{aligned}\varphi_i^+ \circ (\varphi_j^\pm)^{-1} :&\varphi_j^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \to \varphi_i^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n)\\ &(y_0, \dots, y_{n-1}) \mapsto (y_0, \dots, \hat{y}_i, \dots, y_{j-1}, \pm\sqrt{1 - \textstyle\sum_i (y_i)^2}, y_j, \dots, y_{n-1})\end{aligned}$$

  and a similar formula works if we replace φ_i^+ by φ_i^-. Hence all the transition maps are smooth.

[6] see Remark 1.2.3

- More generally, any subset M of $\mathbb{R}^k$ given as the *regular level set* of a smooth map $F : \mathbb{R}^k \to \mathbb{R}^\ell$ is a $k-\ell$ dimensional smooth manifold "in a natural way[7]". (Being a level set means $M = F^{-1}(\{c\})$ for some $c \in \mathbb{R}^\ell$ and being a regular level set means that, moreover, the Jacobian $D|_p F$ is surjective for all $p \in M$.) You can prove this quite easily using the implicit function theorem and writing M locally as a graph of smooth functions (analogous to the graph coordinates for the sphere). We will show a more general statement later on when discussing submanifolds (Chapter **??**).
- Projective space $\mathbb{P}^n$ is naturally a smooth manifold (see exercises).
- On the topological 1-manifold $M = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$, each of the two projections $\pi_0, \pi_1 : M \to \mathbb{R}$ (given by $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$) is a chart defined on the whole manifold M, but these two charts are not $\mathcal{C}^k$ compatible for any $k \geq 1$. (The transition functions are $\pi_1 \circ (\pi_0)^{-1} : x \mapsto x^3$ and its inverse $\pi_0 \circ (\pi_1)^{-1} : y \mapsto \sqrt[3]{y}$, which is not differentiable.) Thus these charts determine two different $\mathcal{C}^k$ structures on M.

1.3 Differentiable maps

We are about to define $\mathcal{C}^k$ maps between $\mathcal{C}^k$ manifolds. The plan is to reduce the question of differentiability to the case of a map between Euclidean open sets. We will do so by using charts.

In general, when studying a map $f : M \to N$ between manifolds, charts allow us to *locally* express f as a map between subsets of Euclidean space.

Definition 1.3.1 (Local expression of a map)**.** Let M, N be topological manifolds and let $f : M \to N$ be any function. A **local expression** (or **coordinate representation**) of f at some point $p \in M$ is a composite map

$$f|_\varphi^\psi := \psi \circ f \circ \varphi^{-1},$$

where φ and ψ are charts of M and N whose domains U, V contain the points p and $f(p)$ respectively.

In particular, we will use local expressions to decide whether a map is $\mathcal{C}^k$ or not.

Definition 1.3.2 ($\mathcal{C}^k$ maps between manifolds)**.** Let $f : M \to N$ be a function between $\mathcal{C}^k$ manifolds and let $p \in M$. Take a local expression $f|_\varphi^\psi = \psi \circ f \circ \varphi^{-1}$ of f at p, where φ, ψ are $\mathcal{C}^k$ charts of M and N respectively. We say that f is $\mathcal{C}^k$ at the point p if

$$\text{the local expression } f|_\varphi^\psi \text{ is defined on a neighbrhd of } \varphi(p) \text{ and is } \mathcal{C}^k \text{ at the point } \varphi(p). \tag{1.1}$$

The function $f : M \to N$ is $\mathcal{C}^k$ if it is $\mathcal{C}^k$ at all points $p \in M$.

A $\mathcal{C}^k$ **isomorphism** is a $\mathcal{C}^k$ map that has a $\mathcal{C}^k$ inverse. If $k \geq 1$, a $\mathcal{C}^k$ isomorphism is also called a $\mathcal{C}^k$ **diffeomorphism** (or a $\mathcal{C}^k$ **diffeo**, for short).

Lemma 1.3.3. *In Definition 1.3.2, the proposition* (1.1) *does not depend on which pair of $\mathcal{C}^k$ charts φ, ψ are chosen (provided, of course, that their domains U, V contain p and $f(p)$ respectively). The proposition* (1.1) *also implies that f is continuous at p, in fact, it is equivalent to the continuity of f at p in the case $k = 0$.*

Proof. Suppose first that f is continuous at p. Then there is some open neighborhood U' of p such that $f(U') \subseteq V$, and we may assume $U' \subseteq U$. Furthermore, the restriction $f|_{U'}^V$ is continuous at p. It follows that the local expression $f|_\varphi^\psi$ is defined on $\varphi(U')$ (which is a neighborhood of $\varphi(p)$) and continuous at $\varphi(p)$. Thus the proposition (1.1) holds for $k = 0$.

[7] In particular, the inclusion map $M \hookrightarrow \mathbb{R}^k$ is smooth.

Now suppose that the proposition (1.1) holds for some $k \geq 0$. Let us show that f is continuous at p. By hypothesis there is some open neighborhood of $\varphi(p)$ where $f|_{\varphi}^{\psi}$ is defined. We can write this open neighborhood as $\varphi(U')$, where U' is an open neighborhood of p. The fact that $f|_{\varphi}^{\psi}$ is defined on $\varphi(U')$ is equivalent to saying that $f(U') \subseteq V$. Thus we can define the restriction $f|_{U'}^{V}$, which can be written as

$$f|_{U'}^{V} = \psi^{-1} \circ f|_{\varphi}^{\psi} \circ \varphi : U' \to V,$$

and is therefore continuous at p because $f|_{\varphi}^{\psi}$ is continuous at $\varphi(p)$. This shows that f is continuous at p.

Finally, let us show that (1.1) does not depend on the choice of charts. Thus we assume again that the proposition (1.1) holds, and we construct a second local expression $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ using any pair of charts $\widetilde{\varphi}$ and $\widetilde{\psi}$ with domains $\widetilde{U} \subseteq M$, $\widetilde{V} \subseteq N$ containing p and $f(p)$ respectively. This new local expression is defined on some neighborhood of $\widetilde{\varphi}(p)$ (since f is continuous at p), and is related to the old one by the equation

$$f|_{\widetilde{\varphi}}^{\widetilde{\psi}} = (\psi \circ \widetilde{\psi}^{-1}) \circ f|_{\varphi}^{\psi} \circ (\varphi \circ \widetilde{\varphi}^{-1})$$

Since the transition maps $(\psi \circ \widetilde{\psi}^{-1})$ and $(\varphi^{-1} \circ \widetilde{\varphi})$ are $\mathcal{C}^k$, we conclude that $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ is $\mathcal{C}^k$ at the point $\widetilde{\varphi}(p)$. This shows that the proposition (1.1) is independent of the choice of φ and ψ. ☐

Example 1.3.4. 1. The identity map of any $\mathcal{C}^k$ manifold is a $\mathcal{C}^k$ map. (Exercise.)

2. The composite of two $\mathcal{C}^k$ maps is a $\mathcal{C}^k$ map. (Exercise.)
3. If M is a $\mathcal{C}^k$ manifold, then every $\mathcal{C}^k$ chart of M, as well as its inverse, are $\mathcal{C}^k$ maps. (Exercise.)

A $\mathcal{C}^k$ structure on a topological manifold M allows us to determine which maps that go to M are $\mathcal{C}^k$. But the reciprocal property also holds: if we know which maps to M are $\mathcal{C}^k$, this information determines the $\mathcal{C}^k$ structure of M.

Proposition 1.3.5. *Let $\mathcal{A}_0$, $\mathcal{A}_1$ be two $\mathcal{C}^k$ atlases on a topological manifold M, defining two $\mathcal{C}^k$ manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases $\mathcal{A}_i$ are equivalent if and only if the following property holds:*
For every function $f : N \to M$ (where N is a $\mathcal{C}^k$ manifold), the function f is $\mathcal{C}^k$ as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Proof. Exercise. ☐

1.4 Partitions of unity and paracompactness

In this section we develop *partitions of unity*, a tool often used to turn a local construction, obtained by working in coordinates, into a global one. (Don't worry if this sounds vague; we will see examples later on.) Existence of partitions of unity on a manifold is easier to prove on a compact manifold; we will deal first with this case. The general proof relies on a topological property of manifolds called *paracompactness*, which is in turn a consequence of being Hausdorff, second countable and locally compact.

Recall that the **support** of a function $\eta : M \to \mathbb{R}$ is the closed set

$$\operatorname{supp}(\eta) := \overline{\{p \in M \mid \eta(p) \neq 0\}}.$$

Definition 1.4.1. A $\mathcal{C}^k$ **partition of unity** (or **POU**, for short) on a $\mathcal{C}^k$ manifold M is a family $(\eta_i)_i$ of $\mathcal{C}^k$ functions $\eta_i : M \to [0, +\infty)$ satisfying

- Sum condition: $\sum_i \eta_i(x) = 1$ for every $x \in M$,

- Local finiteness: Every point of M has a neighborhood where all except finitely many of the functions η_i vanish.

A partition of unity $(\eta_i)_i$ is **subordinate** to an open cover $\mathcal{U}$ of M if every η_i has its support contained in some open set $U \in \mathcal{U}$.

Theorem 1.4.2 (Existence of Partitions of Unity.)**.** *For any open cover $\mathcal{U}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$ such that the functions η_i have compact support (in fact, their supports are closed coordinate balls; see definition below).*

Another version of the theorem is the following.

Corollary 1.4.3 (Existence of Partitions of Unity, alternate form)**.** *For any open cover $\mathcal{U} = \{U_j\}_{j\in J}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(g_j)_{j\in J}$ such that* $\operatorname{supp}(\eta_j) \subseteq U_j$ *for all j.*

This version of the theorem can be deduced from the first one (exercise). Note that the functions g_j may not have compact support in this case.

For the proof of Theorem 1.4.2 we will use *bump functions.*

Lemma 1.4.4 (Bump functions on Euclidean space)**.** *For any numbers $0 < a < b$ there exists a smooth* ***bump function*** *$h : \mathbb{R}^n \to [0,1]$ satisfying $h(x) = 1$ iff $\|x\| \le a$ and $h(x) = 0$ iff $\|x\| > b$.*

Proof. It suffices to let $h(x) = g(\|x\|)$, where $g : \mathbb{R} \to [0,1]$ is a smooth **cutoff function** satisfying $g(t) = 1$ iff $t < a$ and $g(t) = 0$ iff $t > b$. This cutoff function, in turn, may be defined as

$$g(t) = \frac{f(b-t)}{f(b-t) + f(t-a)},$$

where $f : \mathbb{R} \to [0, +\infty)$ is a smooth function such that $f(t) > 0$ iff $t > 0$. Such a function f may be given e.g. by the formula

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \le 0 \end{cases}.$$

□

We may use charts to transport these bump functions from Euclidean space to any manifold. The resulting bump function will be supported on a closed *coordinate ball.*

Definition 1.4.5. A **closed coordinate ball** in a $\mathcal{C}^k$ manifold M is the preimage $\varphi^{-1}(\overline{B})$ of a closed Euclidean ball $\overline{B}$ by a $\mathcal{C}^k$ chart φ containing $\overline{B}$ in its codomain.

Proof of Thm. 1.4.2 for compact manifolds. Let M be a compact $\mathcal{C}^k$ manifold and let $\mathcal{U}$ be an open cover of M. In this case we'll obtain a *finite* $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$. In fact, it is sufficient to find finitely many $\mathcal{C}^k$ functions $\widetilde{\eta}_i : M \to [0, +\infty)$, each of which has its support $\operatorname{supp}(\widetilde{\eta}_i)$ contained in some $U \in \mathcal{U}$, and such that $\sum_i \widetilde{\eta}_i(x) > 0$ for all x. The functions η_i can then be obtained by dividing each $\widetilde{\eta}_i$ by the strictly positive $\mathcal{C}^k$ function $\widetilde{\eta} = \sum_i \widetilde{\eta}_i$.

To construct the functions η_i we proceed as follows. Each point of a set $U \in \mathcal{U}$ is contained in the interior of some closed coordinate ball $D \subseteq U$. Thus there is a family of closed coordinates balls, each of them contained in some $U \in \mathcal{U}$, whose interiors cover M. By compactness, we may take a finite subfamily of balls D_i whose interiors still cover M. Write each ball D_i as $\varphi_i^{-1}(\overline{B_i})$, where B_i is an open Euclidean ball and φ_i is a $\mathcal{C}^k$ chart containing $\overline{B_i}$ in its codomain. Then let $h_i : \mathbb{R}^n \to [0, +\infty)$ be a $\mathcal{C}^k$ bump function that is supported on the closed ball $\overline{B_i}$ and strictly positive on the interior B_i. Finally, define a $\mathcal{C}^k$ function $\widetilde{\eta_i} : M \to [0,1]$ by the formula

$$\widetilde{\eta_i} = \begin{cases} h_i \circ \varphi_i & \text{on } \mathrm{Dom}(\varphi) \\ 0 & \text{on } M \setminus D_i \end{cases}$$

This function is supported on D_i, which is contained in some $U \in \mathcal{U}$, thus the function family $(\widetilde{\eta_i})$ is subordinate to $\mathcal{U}$. In addition, $\widetilde{\eta_i}$ is strictly positive on B_i, and since the balls B_i cover M, we conclude that $\sum_i \widetilde{\eta_i}(x) > 0$ for all $x \in M$, as required. □

1.4.1 Paracompactness*

Definition 1.4.6 (Paracompact space)**.** Let X be a topological space.

- An **open cover** of X is a family $\mathcal{U}$ of open sets whose union is X.
- Another open cover $\mathcal{V}$ **refines** $\mathcal{U}$ if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.
- The space X is **paracompact** if every open cover $\mathcal{U}$ admits as refinement some open cover $\mathcal{V}$ that is locally finite.
- (A family of subsets of X is **locally finite** if every point $x \in X$ has an open neighborhood W that intersects only finitely many sets of the family.)

Proposition 1.4.7. *If a topological space X is Hausdorff, second countable and locally compact, then it is paracompact. In fact, given any topological basis $\mathcal{B}$ of X, every open cover $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ consisting of open sets $B \in \mathcal{B}$ with their closures $\overline{B}$ contained in some $U \in \mathcal{U}$.*

Proof.

Lemma 1.4.8. *X admits an exhaustion by compact sets, i.e. a sequence $(K_i)_{i\in\mathbb{N}}$ of compact sets that cover X and satisfy $K_i \subseteq \mathrm{Int}(K_{i+1})$.*

Proof of lemma. Let $(V_j)_{j\in\mathbb{N}}$ be a countable open cover of M where each V_j has compact closure. Let $K_0 = \emptyset$, and define inductively for each $i \in \mathbb{N}$ an integer $j_i \geq i$ such that the sets $(V_j)_{j<j_i}$ cover K_i, and a compact set $K_{i+1} = \overline{V_i} \cup \bigcup_{j<j_i} \overline{V_j}$. These compact sets K_i cover M and satisfy $K_i \subseteq \mathrm{Int}(K_{i+1})$. □

Consider the compact sets $L_i = K_i \setminus \mathrm{Int}(K_{i-1})$ and their respective open neighborhoods $W_i = \mathrm{Int}\, K_{i+1} \setminus K_{i-2}$.

Each point $x \in L_i$ is contained in some open $U_x \in \mathcal{U}$ and has a basic neighborhood $B^i_x \in \mathcal{B}$ such that $\overline{B^i_x} \subseteq W_i \cap U$.

Take a finite subfamily $(B^i_{x_j})_j$ that covers L_i. Doing this for each i we obtain a family of basics $(B^i_{x_j})_{i,j}$ that cover M. Their closures satisfy $\overline{B^i_{x_j}} \subseteq U_{x_j}$, which gives the subordination condition, and $\overline{B^i_{x_j}} \subseteq W_i$, which ensures local finiteness since every point $x \in L_i$ is contained in at most three sets W_ℓ, namely, those with $|\ell - i| \leq 1$. □

Using paracompactness, we can prove existence of partitions of unity without the hypothesis of compactness.

Proof of Thm. 1.4.2. Let M be a manifold and $\mathcal{U}$ an open cover.

Let $\mathcal{B}$ be the topological basis of M consisting of the interiors of closed coordinate balls. By Proposition 1.4.7, there exists a family of closed coordinate balls D_i, each contained in some open set $U \in \mathcal{U}$, whose interiors cover M.

The proof finishes as in the compact case. We take for each i a $\mathcal{C}^k$ function $\widetilde{\eta_i}$: $M \to [0,1]$ that is strictly positive on $\mathrm{Int}(D_i)$ and supported on D_i. Then the functions $\eta_i = \frac{\widetilde{\eta_i}}{\sum_j \widetilde{\eta_j}}$ form a $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ that is subordinate to the open cover $\mathcal{U}$. □

1.4.2 Applications

Corollary 1.4.9 (Bump functions)**.** *If M is a $\mathcal{C}^k$ manifold, $A \subset M$ a closed set and $U \subseteq M$ an open neighborhood of A, then there exists a $\mathcal{C}^k$ function $\eta : M \to [0,1]$ such that $\eta \equiv 1$ on A and* $\operatorname{supp}(\eta) \subset U$.

We call η a *bump function* for A supported in U.

Proof. Just take the open cover $\{V_0 = U, V_1 = M \ \setminus A\}$ of M and a partition of unity $(\eta_i)_{i=0,1}$ satisfying $\operatorname{supp} \eta_i \subseteq V_i$, then set $\eta = \eta_0$. □

We have already defined differentiability for function $M \to N$ between $\mathcal{C}^k$ manifolds. For functions on closed sets we make the following definition:

Definition 1.4.10. Let $f : A \to N$ be a function where $A \subseteq M$ is a closed set and M, N are $\mathcal{C}^k$ manifolds. We say that f is $\mathcal{C}^k$ if it can be extened to a $\mathcal{C}^k$ function defined on an open neighborhood of A.

As a corollary of the existence of bump functions we can extend a smooth function on a closed set to a smooth function on the whole manifold:

Corollary 1.4.11 (Extension lemma)**.** *Let $f : A \to \mathbb{R}$ be $\mathcal{C}^k$, where $A \subseteq M$ is a closed subset of a $\mathcal{C}^k$ manifold M, and let $U \subseteq M$ be an open set containing A. Then there exists a $\mathcal{C}^k$ function $\widetilde{f} : M \to \mathbb{R}$ such that $\widetilde{f}|_A = f$ and* $\operatorname{supp} \widetilde{f} \subseteq U$.

Proof. By definition f can be extended to a $\mathcal{C}^k$ function (say, also called f) on some open set $W \supseteq A$; wlog $W \subseteq U$. We take a $\mathcal{C}^k$ bump function η for A supported in W. Then ηf has support in W and therefore extending the function by 0 outside W we obtain a $\mathcal{C}^k$ function $\widetilde{f}$ on M with the desired properties. □

2 Tangent vectors

Recall that if a map $f : U \to V$ between Euclidean open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ is $\mathcal{C}^1$ at a point $p \in U$, then there is a linear transformation $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$, called the **differential** of f at p, which gives a first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_p f(v) + r_p(v)$$

where $\frac{r_p(v)}{\|v\|} \to 0$ as $v \to 0$.

To define the differential of a map between $\mathcal{C}^k$ manifolds, we need a notion of tangent space.

Definition 2.0.1. Let M be an n-dimensional $\mathcal{C}^k$ manifold with $k \geq 1$. A **coordinatized tangent vector** on M is a triple (p, φ, v) where $p \in M$ is a point, φ is a $\mathcal{C}^k$ chart of M defined at p, and $v \in \mathbb{R}^n$ is a vector in Euclidean space. A **tangent vector** on M is the equivalence class $[p, \varphi, v]$ of a coordinatized tangent vector (p, φ, v) under the equivalence relation

$$(p, \varphi, v) \sim (\widetilde{p}, \widetilde{\varphi}, \widetilde{v}) \quad \iff \quad \widetilde{p} = p \quad \text{and} \quad \widetilde{v} = \mathrm{D}_{\varphi(p)}(\widetilde{\varphi}\,\varphi^{-1})(v)$$

The set TM of tangent vectors is the **tangent bundle** of M, and there is a canonic projection map $\pi_{TM} : TM \to M$ sending $[p, \varphi, v] \mapsto p$.

The **tangent space** at a point $p \in M$ is the set $T_pM := \pi^{-1}(p)$. It is a vector space with vector addition

$$[p, \varphi, v] + [p, \varphi, w] := [p, \varphi, v + w]$$

and vector scaling

$$\lambda\,[p, \varphi, v] := [p, \varphi, \lambda v] \quad \text{for } \lambda \in \mathbb{R}.$$

Remark 2.0.2.

1. $\sim$ is indeed an equivalence relation. (Exercise.)
2. Fixed a point $p \in M$ and a $\mathcal{C}^k$ chart φ defined on p, the function $\iota : \mathbb{R}^n \to T_pM$ sending $v \mapsto [p, \varphi, v]$ is a bijection. (Exercise.)
3. Vector addition and scalar multiplication are well defined and make T_pM a vector space isomorphic to $\mathbb{R}^n$. (Exercise.)

Remark 2.0.3. If $U \subseteq \mathbb{R}^n$ is an open set (considered as a smooth manifold), we identify $TU \equiv U \times \mathbb{R}^n$ by the bijection

$$(p, v) \in U \times \mathbb{R}^n \mapsto [p, \mathrm{id}_U, v] \in TU.$$

Thus for each $p \in U$ we have $T_pU \equiv \{p\} \times \mathbb{R}^n \equiv \mathbb{R}^n$.

Coordinate basis for the tangent space We can construct a basis of the tangent space T_pM as follows. Consider the canonic basis $(e_i)_i$ of $\mathbb{R}^n$, and take a chart φ defined at p. Then the vectors

$$\frac{\partial}{\partial \varphi^i}|_p := [p, \varphi, e_i],$$

called the **coordinate vectors** at p associated to the chart φ, form a basis of T_pM.

Tangent vectors as derivations

Definition 2.0.4. A **derivation** on a $\mathcal{C}^k$ differentiable manifold M at a point $p \in M$ is a linear function $D : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ satisfying the Leibniz identity

$$D(f\,g) = D(f)\,g(p) + f(p)\,D(g).$$

The set Der_pM of derivations on M at p is a vector space with the operations

$$(D + E)(f) := D(f) + E(f),$$
$$(\lambda D)(f) := \lambda(D(f))$$

defined for $D, E \in Der_pM$ and $\lambda \in \mathbb{R}$.

Each vector $X \in T_pM$ induces a derivation $D_X \in Der_pM$ defined by the formula

$$D_X(f) := \mathrm{D}_p\,f(X) \in T_{f(p)}\mathbb{R} \equiv \mathbb{R}.$$

(Here we use the identification of Remark 2.0.3). The map $x \in T_pM \mapsto D_x \in Der_pM$ is a linear injection, and is bijective if M is smooth (see e.g. [Lee13, Prop. 3.2]). Therefore in some books (e.g. [Lee13]), the tangent space T_pM of a smooth manifold M is *defined* as the vector space of derivations at p. Here we do not use this definition because it is not good for $\mathcal{C}^k$ manifolds. However, we may identify a tangent vector X with the derivation D_X and write $X(f)$ instead of $D_X(f)$.

Manifold structure on the tangent bundle The set TM is a $\mathcal{C}^{k-1}$ manifold (to be explained later...).

2.0.1 Differential of a $\mathcal{C}^k$ map between manifolds

Now that we have tangent spaces, we can define the differential of a $\mathcal{C}^k$ map.

Definition 2.0.5. The **differential transformation** (or **differential**, for short) of a map $f : M \to N$ that is $\mathcal{C}^k$ at a point $p \in M$ is defined as

$$\begin{array}{rccc} D_pf: & T_pM & \to & T_{f(p)}N \\ & [p, \varphi, v] & \mapsto & [f(p), \psi, D|_{\varphi(p)} f|_\varphi^\psi(v)]. \end{array}$$

where φ, ψ are charts of M, N defined at the points p, $f(p)$ respectively, and $f|_\varphi^\psi = \psi \circ f \circ \varphi^{-1}$ is, the local expression of f with respect to the charts φ, ψ.

If f is $\mathcal{C}^k$ everywhere, we define its **derivative** $Df : TM \to TN$ by the same formula. The map Df is also called **pushforward** by f and denoted f_*.

Note:

- Well defined and linear (Exercise.)
- Obeys the chain rule $D|_p(g \circ f) = D|_{f(p)}g \circ D|_pf$. In particular, for a diffeo f, the differential D_pf is a linear isomorphism with inverse $D_{f(p)}(f^{-1}) = (D_pf)^{-1}$. (Exercise.)
- The derivative $Df : TM \to TN$ is a $\mathcal{C}^{k-1}$ map. (Not yet proved...)

Velocity of a curve For a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ on a manifold M, we define its **velocity vector** at an instant $t \in I$ as the vector $g'(t) := D_tg(1) \in T_{\gamma(t)}M$. Exercise: Show that for any vector $X \in TM$ there is a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(0) = X$.