Introduction to Differentiable Manifolds

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Contents

Main references

[Lee10] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2010.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second edition. Springer, 2013.

Both books have online versions available in the library.

0.1 Practical remarks about the course

0.1.1 Content for the exam

Studying the content under a heading marked by an asterisk is not mandatory by itself. You should read it only if you find it interesting or helpful for understanding the rest.

1 Manifolds

The goal of this course is to extend differential and integral calculus from Euclidean space $\mathbb{R}^n$ to all *differentiable manifolds* such as the n-sphere, the n-torus, etc. Roughly speaking, a differentiable manifold is a space that

- is endowed with a certain topology,
- has, in addition, a *differentiable structure* that allows us to distinguish whether a map is differentiable or not, rather than just continuous, and
- locally looks like Euclidean space $\mathbb{R}^n$.

1.1 Topological manifolds

[Lee13], Chapter 1 and [Lee10], Chapter 2

Let us pospone the question of differentiability and focus on topology. As said, we want to study spaces that "locally look like" Euclidean space $\mathbb{R}^n$.

Definition 1.1.1 (Locally Euclidean space)**.** Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is **locally Euclidean** of dimension n at a point $p \in M$ if the point p has an open neighborhood that is homeomorphic[1] to an open subset of $\mathbb{R}^n$. If this holds for all points $p \in M$, we say that M is locally Euclidean of dimension n.

A typical example is the circle: it is locally Euclidean of dimension 1 but not globally homeomorphic to any subset of $\mathbb{R}$.

Example 1.1.2. The **circle** $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (with the subspace topology) is locally Euclidean of dimension 1: let $(x_0, y_0) \in \mathbb{S}^1$, wlog $y_0 > 0$, then $U := (\mathbb{R} \times \mathbb{R}^+) \cap \mathbb{S}^1$ is an open subset of $\mathbb{S}^1$ containing (x_0, y_0) and homeomorphic to $(-1, 1)$ via the map $U \to (-1, 1)$ that sends $(x, y) \mapsto x$.

We will see more examples later on (see e.g. Examples 1.1.11 below). Let us make some general comments.

Remark 1.1.3. If a space M is locally Euclidean of dimension 0, then every point has a neighborhood homeomorphic to $\mathbb{R}^0 = \{0\}$, i.e. a point. In other words, M is a discrete topological space.

Remark 1.1.4. In the definition of locally Euclidean space, we could have replaced "...homeomorphic to an open subset of $\mathbb{R}^n$" by "...homeomorphic to $\mathbb{R}^n$". (Exercise.)

Remark 1.1.5. Brouwer's theorem of *invariance of domain*, given here without proof, says that if two nonempty open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$. It follows that the dimension of a locally Euclidean space at each point can be defined unambiguously. Furthermore, it is easy to prove that the dimension is constant throughtout each connected component. Thus the only way to get a locally Euclidean space of mixed dimensions is to make a disjoint union of components of different dimensions. Anyway, in the definition of topological manifold (see below) we will not admit this kind of spaces.

[1] Recall that two topological spaces X, Y are **homeomorphic** if there exists a bijection $\varphi : X \to Y$ such that both φ and its inverse are continuous.

For the definition of *topological manifold* we demand some further topological properties that ensure that the space is topologically "well-behaved". (For instance, we want the limit of every sequence to be unique.)

Definition 1.1.6. A **topological manifold of dimension** n, or **topological** n**-manifold**, is a topological space M that is locally Euclidean of dimension n, Hausdorff[2] and second countable.[3]

A **topological manifold** is a topological space that is a topological n-manifold for some n.

Side note: Make sure you are familiar with some basic definitions from topology such as Hausdorff, second countable, connected and compact spaces, and the construction of subspace, product, coproduct and quotient topologies. Chapters 2 and 3 in [Lee10] provides a succinct overview of everything we need.

Remark 1.1.7. The conditions of Hausdorff resp. second countable in Definition 1.1.6 are not redundant. For example, the *line with two origins* (see Exercises) is a locally Euclidean, second countable space that is not Hausdorff. The *long line* and the *Prüfer surface* (see Wikipedia if interested) are locally Euclidean of dimension 1 and 2 respectively, Hausdorff, and connected, but not second countable.

The homeomorphisms that locally identify a topological manifold with Euclidean space are called *charts*:

Definition 1.1.8 (Coordinate charts)**.** Let M be a topological n-manifold. A **chart** (or **coordinate chart**) for M is a homeomorphism $\varphi : U \to V$, where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets. Its inverse φ^{-1} is a **local parametrization** of M. An **atlas** for M is a collection of charts whose domains cover M.

For the moment, we can see an atlas simply as a way of showing that a space is locally Euclidean.

Remark 1.1.9. Some authors define a chart for M as a *pair* (φ, U) or even a *triple* (φ, U, V) where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets and $\varphi : U \to V$ is a homeomorphism. Here, instead, we consider the sets U and V as part of the function φ, namely, its domain $\mathrm{Dom}(\varphi)$ and codomain $\mathrm{Cod}(\varphi)$, thus there's no need not specify them separately.[4] From this point of view, the letters "U", "V" are just shorter names for the sets $\mathrm{Dom}(\varphi)$, $\mathrm{Cod}(\varphi)$.

Convention: When talking about subsets (resp. quotients, products, disjoints unions) of topological spaces we'll assume that they are endowed with the subspace (resp. quotient, product, coproduct) topology unless otherwise stated.

Using this convention, let us mention some easy ways to construct new topological manifolds from old ones.

Proposition 1.1.10 (New manifolds from old)**.** *The properties of being Hausdorff or second countable are preserved by taking subspaces, finite products and countable coproducts. In consequence:*

- *An open subset of a topological n-manifold is a topological n-manifold.*

[2] Recall that a topological space X is **Hausdorff** if every two different points $x, y \in X$ have disjoint neighborhoods.

[3] Recall that a topological space X is **second countable** if its topology admits a countable base. A **base** for a topology is a family $\mathcal{B}$ of open sets such that every open set is a union of some sets of $\mathcal{B}$.

[4] Formally, a function f is a triple $f = (X, Y, \Gamma)$ where X, Y are sets (called the **domain** and **codomain** of f, and denoted $\mathrm{Dom}(f)$ and $\mathrm{Cod}(f)$), and Γ is a subset of $X \times Y$ (called the **graph** of f, denoted $\mathrm{Gra}(f)$), such that for each $x \in X$ there is a unique $y \in Y$ (called the **image** of x by f, denoted $f(x)$) such that $(x, y) \in \Gamma$.

- *A disjoint union $M = \coprod_i M_i$ of countably many topological n-manifolds M_i is a topological n-manifold.*
- *A product $M = \prod_i M_i$ of finitely many topological manifolds M_i is a topological manifold of dimension* $\dim(M) = \sum_i \dim(M_i)$.

Proof. Exercise. $\square$

Example 1.1.11 (Examples of topological manifolds)**.**

(a) Of course any open subset of $\mathbb{R}^n$ is a topological manifold.

(b) An example of topological n-manifold is the **graph**

$$\Gamma_f := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^m$$

of a continuous function $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set. Indeed, it is homeomorphic to U via the **graph parametrization**

$$\begin{array}{ccc} U & \to & \Gamma_f \\ x & \mapsto & (x, f(x)), \end{array} \quad \text{whose inverse is the projection} \quad \begin{array}{ccc} \Gamma_f & \to & U \\ (x,y) & \mapsto & x. \end{array}$$

(c) The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a topological manifold. Being a subset of $\mathbb{R}^{n+1}$ it is Hausdorff and second countable. A possible choice of atlas is given by the so-called **graph coordinates**: Cover $\mathbb{S}^n$ by the $2(n+1)$ open sets $U_i^\pm := \{x \in \mathbb{R}^n \mid \pm x_i > 0\}$, then $\mathbb{S}^n \cap U_i^\pm$ is homemorphic to the open unit n-ball $\mathbb{B}^n$ via the projection[5]

$$\begin{array}{cccc} \varphi_i^\pm : & \mathbb{S}^n \cap U_i^\pm & \to & \mathbb{B}^n \\ & (x_0, \dots, x_n) & \to & (x_0, \dots, \widehat{x_i}, \dots, x_n). \end{array}$$

The maps $\varphi_i^\pm$ are coordinate charts for $\mathbb{S}^n$; we call them *graph coordinates*. Locally this is a special case of the previous item (b): each set $\mathbb{S}^n \cap U_i^\pm$ is (up to permutation of coordinates) the graph of the continuous function on the unit n-ball $\mathbb{B}^n(0)$:

$$\mathbb{B}^n \to \mathbb{R} : y \mapsto \pm\sqrt{1 - \textstyle\sum_i y_i^2}.$$

(d) Real projective space $\mathbb{P}^n$ is a topological n-manifold (exercise).

(e) The torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is a topological n-manifold (exercise).

(f) More generally, If M is a topological n-manifold and G is a group of homeomorphisms of M that acts properly discontinuously and without fixed points, then the quotient space M/G is a topological n-manifold.

(g) If M is a topological n-manifold and $\pi : N \to M$ is a covering map (with N connected), then N is a topological n-manifold. In particular, the universal covering space of any connected topological manifold is a topological manifold.

We will not prove the following result (although it can be done elementarily).

Theorem 1.1.12 (Classification of topological 1-manifolds)**.** *Every connected topological 1-manifold is homeomorphic to either $\mathbb{S}^1$ (if it is compact) or to $\mathbb{R}$ (if it is not compact).*

1.2 Differentiable manifolds

Our next goal is to define a kind of spaces and maps called *differentiable manifolds* and *differentiable maps* (or, more precisely $\mathcal{C}^k$ manifolds and maps) with which we can actually do differential calculus. Topological manifolds do not have enough structure because a topology does not allow us to determine whether a function is differentiable or not; it only distinguishes continuous functions. Differentiable manifolds should be

[5]The hat on $\widehat{x_i}$ means that we omit the respective coordinate x_i.

locally equivalent to Euclidean open sets (where we already have a well defined notion of $\mathcal{C}^k$ maps; see below), but at the global level they should be allowed to have a more interesting topology. In particular, the sphere, torus, projective space, etc. should become differentiable manifolds.

Before defining $\mathcal{C}^k$ manifolds, let us set up some terminology for $\mathcal{C}^k$ maps in $\mathbb{R}^n$.

Definition 1.2.1 (Euclidean open sets and Euclidean $\mathcal{C}^k$ maps)**.** A **Euclidean open set** is an open subset of some Euclidean space $\mathbb{R}^n$.
Let $k \in \{0, 1, \dots, \infty\}$. A function $f : U \to V$ between Euclidean open sets is $\mathcal{C}^k$ at a point $p \in U$ if its partial derivatives of order $\leq k$ are defined in a neighborhood of p and continuous at p. We say that f is $\mathcal{C}^k$ (and we call it a **Euclidean $\mathcal{C}^k$ map**) if it is $\mathcal{C}^k$ at all points $p \in U$.
An **Euclidean $\mathcal{C}^k$ isomorphism** is an Euclidean $\mathcal{C}^k$ map that has an Euclidean $\mathcal{C}^k$ inverse.

Note that every Euclidean $\mathcal{C}^k$ map is continuous because the function f itself is a partial derivative (of order 0) of f. In fact, a $\mathcal{C}^0$ map is the same thing as a continuous map.

We are now ready to define $\mathcal{C}^k$ manifolds. The key to turn a topological manifold into a $\mathcal{C}^k$ manifold is to choose an appropriate atlas.

Definition 1.2.2 ($\mathcal{C}^k$ manifolds)**.** Let M be a topological n-manifold and $k = 0, \dots, \infty$. Two charts φ, ψ for M, with respective domains $U, V \subseteq M$, are **$\mathcal{C}^k$-compatible** if the **transition map** from φ to ψ, that is, the homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

is a $\mathcal{C}^k$ isomorphism (i.e. itself and its inverse are both Euclidean $\mathcal{C}^k$ maps).

A **$\mathcal{C}^k$-consistent atlas** (or **$\mathcal{C}^k$ atlas**, for short) is an atlas for M whose charts are $\mathcal{C}^k$ compatible with each other. Two $\mathcal{C}^k$ atlases for M are **$\mathcal{C}^k$-equivalent** if their union is $\mathcal{C}^k$-consistent. A **$\mathcal{C}^k$ structure** on M is a maximal $\mathcal{C}^k$ atlas, i.e., a $\mathcal{C}^k$ atlas that is not contained in any other strictly larger $\mathcal{C}^k$ atlas. A **$\mathcal{C}^k$ manifold** is a topological manifold M endowed with a $\mathcal{C}^k$ structure $\mathcal{A}$. (More formally, the $\mathcal{C}^k$ manifold is the pair $(M, \mathcal{A})$.)

Note that a $\mathcal{C}^0$ manifold is the same thing as a topological manifold. A $\mathcal{C}^k$ manifold with $k \geq 1$ is called a **$\mathcal{C}^k$-differentiable manifold**. A **smooth manifold** is a $\mathcal{C}^\infty$ manifold.

Remark 1.2.3 (Domains and codomains of functions)**.** To be precise, the transition map that we wrote as $\psi \circ \varphi^{-1}$ should actually be defined as $\psi|_{U\cap V}^{\psi(U\cap V)} \circ \left(\varphi|_{U\cap V}^{\varphi(U\cap V)}\right)^{-1}$, using the *restricted* charts

$$\varphi|_{U\cap V}^{\varphi(U\cap V)} : U \cap V \to \varphi(U \cap V), \quad \psi|_{U\cap V}^{\psi(U\cap V)} : U \cap V \to \psi(U \cap V).$$

In general we will not write the restrictions explicitly because it is cumbersome. When we compose functions, it should be understood that the resulting composite function is defined in principle at all points where it is possible. (Maybe no points at all!)

We may further restrict a function by specifying a reduced domain or codomain. On the other hand, we shall never specify a domain containing points where the function is not defined, nor a codomain that does not contain the image of the specified domain. Thus a function "$f : A \to B$" always has domain A and codomain B.

The next proposition shows that it suffices to give any $\mathcal{C}^k$-consistent atlas (not necessarily a maximal one) to determine a $\mathcal{C}^k$ structure.

Proposition 1.2.4 ($\mathcal{C}^k$ atlas defines $\mathcal{C}^k$ structure)**.** *For a fixed topological manifold M, each $\mathcal{C}^k$ atlas $\mathcal{A}$ is contained in a unique maximal $\mathcal{C}^k$ atlas $\overline{\mathcal{A}}$, which consists of all charts for M that are $\mathcal{C}^k$-compatible with those of $\mathcal{A}$. Any other $\mathcal{C}^k$ atlas $\mathcal{B}$ for M is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$.*

Proof. Let $\mathcal{A}$ be any $\mathcal{C}^k$ atlas for M. Define

$$\overline{\mathcal{A}} := \{\varphi \text{ chart for } M \text{ that is } \mathcal{C}^k \text{ compatible with all charts } \theta \in \mathcal{A}\}.$$

Clearly $\overline{\mathcal{A}}$ contains $\mathcal{A}$. We claim that $\overline{\mathcal{A}}$ is a $\mathcal{C}^k$ atlas. To prove this we have to show that if $\varphi, \psi \in \overline{\mathcal{A}}$ are charts with respective domains $U, V \subseteq M$, then the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is $\mathcal{C}^k$. Take any point $\psi(p) \in \psi(U \cap V)$ and let $\theta \in \mathcal{A}$ be a chart whose domain W contains the point $p \in U \cap V$. Then $\psi(U \cap V \cap W)$ is an open neighborhood of $\psi(p)$ and we can write the restriction[6]

$$\varphi \circ \psi^{-1} : \psi(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$

as the composition $(\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, which is $\mathcal{C}^k$ because $\varphi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are $\mathcal{C}^k$ by assumption. This proves that $\varphi \circ \psi^{-1}$ in a neighborhood of $\psi(p)$, but the same reasoning is valid at any point of $\psi(U \cap V)$, therefore $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$.

Finally, from the definition of $\overline{\mathcal{A}}$ it is clear that it is maximal, and that any atlas $\mathcal{B}$ is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$. In particular, any atlas $\mathcal{B}$ containing $\mathcal{A}$ is equivalent to $\mathcal{A}$ (by definition of equivalent atlases), and is therefore contained in $\overline{\mathcal{A}}$. Therefore a maximal atlas containing $\mathcal{A}$ is contained in $\overline{\mathcal{A}}$, but in fact it must be equal to $\overline{\mathcal{A}}$ (by maximality). We conclude that $\overline{\mathcal{A}}$ is the unique maximal $\mathcal{C}^k$ atlas containing $\mathcal{A}$. □

In consequence, given a topological manifold M and some $\mathcal{C}^k$ atlas $\mathcal{A}$ on M we can speak without ambiguity of *the* $\mathcal{C}^k$ structure $\overline{\mathcal{A}}$ determined by $\mathcal{A}$.

Remark 1.2.5. For practical purposes the concept of a maximal $\mathcal{C}^k$ atlas is not really important. We usually work with a smaller $\mathcal{C}^k$ atlas and this is all we need e.g. for checking that a function is $\mathcal{C}^k$ (see next section). (In fact, we could have defined a $\mathcal{C}^k$ structure on M as an equivalence class of $\mathcal{C}^k$ atlases, rather than as a maximal $\mathcal{C}^k$ atlas.) In general we won't give any name to the maximal atlas and we'll just speak about "a $\mathcal{C}^k$ manifold M" with the maximal $\mathcal{C}^k$ atlas $\mathcal{A}$ being implicit.

Also, when we say "a $\mathcal{C}^k$ chart" or simply "a chart" of M, we mean a chart $\varphi \in \mathcal{A}$. In the rare case that we may need a $\mathcal{C}^l$ chart φ with $l \leq k$ (which means φ is only $\mathcal{C}^l$ compatible with the charts of $\mathcal{A}$), we will say it explicitely. In particular, a "topological chart" is a $\mathcal{C}^0$ chart, i.e. a homeomorphism.

Remark 1.2.6. Every $\mathcal{C}^k$ manifold is automatically a $\mathcal{C}^l$ manifold for every $l \leq k$, because every $\mathcal{C}^k$ atlas is a $\mathcal{C}^l$ atlas. In the other direction, Whitney (*Differentiable Manifolds*, 1936) proved that every $\mathcal{C}^k$ structure contains a (non unique!) $\mathcal{C}^l$ structure for any $l > k$. The proof is reproduced in Munkres' *Elementary Differential Topology* and in Hirsch's *Differential Topology*.

Example 1.2.7 (Examples of smooth manifolds)**.**

1. $\mathbb{R}^n$ (with the atlas consisting of the single chart $\mathrm{id}_{\mathbb{R}^n}$) is a smooth manifold. In general, any topological manifold endowed with a single-chart atlas is automatically a smooth manifold. For example, the graph Γ_f of any *continuous* (sic) function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ as described in Example 1.1.11, endowed with the projection chart, is a smooth manifold.
2. Any open subset U of a $\mathcal{C}^k$ manifold M has a natural $\mathcal{C}^k$ structure consisting of the $\mathcal{C}^k$ charts of M whose domain is contained in U. (Exercise.) We will also see in the exercices that finite products of $\mathcal{C}^k$ manifolds have a natural $\mathcal{C}^k$ structure.
3. The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Indeed, the atlas given by the graph coordinates (Example 1.1.11) is smooth. To see this, we compute the transition functions (wlog $i < j$):

$$\begin{aligned}\varphi_i^+ \circ (\varphi_j^\pm)^{-1} :& \varphi_j^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \to \varphi_i^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \\ & (y_0, \dots, y_{n-1}) \mapsto (y_0, \dots, \hat{y}_i, \dots, y_{j-1}, \pm\sqrt{1 - \textstyle\sum_i (y_i)^2}, y_j, \dots, y_{n-1})\end{aligned}$$

[6] see Remark 1.2.3

and a similar formula works if we replace φ_i^+ by φ_i^-. Hence all the transition maps are smooth.

4. More generally, any subset M of $\mathbb{R}^k$ given as the *regular level set* of a smooth map $F:\mathbb{R}^k\to\mathbb{R}^\ell$ is a $k-\ell$ dimensional smooth manifold "in a natural way[7]". (Being a level set means $M=F^{-1}(\{c\})$ for some $c\in\mathbb{R}^\ell$ and being a regular level set means that, moreover, the Jacobian $D|_pF$ is surjective for all $p\in M$.) You can prove this quite easily using the implicit function theorem and writing M locally as a graph of smooth functions (analogous to the graph coordinates for the sphere). We will show a more general statement later on when discussing submanifolds (Chapter 4).
5. Projective space $\mathbb{P}^n$ is naturally a smooth manifold; see exercises.
6. The torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ is naturally a smooth manifold. (Exercise.)
7. On the topological 1-manifold $M=\{(x,y)\in\mathbb{R}^2 \mid y=x^3\}$, each of the two projections $\pi_0,\pi_1:M\to\mathbb{R}$ (given by $\pi_0(x,y)=x$ and $\pi_1(x,y)=y$) is a chart defined on the whole manifold M, but these two charts are not $\mathcal{C}^k$ compatible for any $k\geq 1$. (The transition functions are $\pi_1\circ(\pi_0)^{-1}:x\mapsto x^3$ and its inverse $\pi_0\circ(\pi_1)^{-1}:y\mapsto\sqrt[3]{y}$, which is not differentiable.) Thus these charts determine two different $\mathcal{C}^k$ structures on M.

1.2.1 One-step $\mathcal{C}^k$ manifolds*

Here it is convenient to work with local parametrizations rather than charts. Recall that an (n-dimensional) **local parametrization** of a topological space M is an homeomorphism from an open subset U of $\mathbb{R}^n$ to an open subset of M or, equivalently, an open injective map $U\to M$. A family $\mathcal{A}$ of such maps whose images cover M is called a **parametrization atlas** for M, and such an atlas is $\mathcal{C}^k$ if the transition map $\psi^{-1}\circ\psi$ is $\mathcal{C}^k$ for each $\phi,\psi\in\mathcal{A}$. Equivalently, a local parametrization is the inverse of a chart, and $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas iff $\mathcal{A}^{-1}:=\{\varphi^{-1}:\varphi\in\mathcal{A}\}$ is a $\mathcal{C}^k$ atlas.

Note that if M is a $\mathcal{C}^k$ manifold, then the topology of M is determined by the $\mathcal{C}^k$ structure. Indeed, if $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas, then

$$\text{a set } U\subseteq M \text{ is open iff } \phi^{-1}(U) \text{ is open in } \mathrm{Dom}(\phi)\subseteq\mathbb{R}^n \text{ for all } \phi\in\mathcal{A}. \tag{1.1}$$

In other words, the topology of M is the *final topology* induced by the family $\mathcal{A}$.[8]

We can ask then, given a set M and a collection $\mathcal{A}$ of functions $\phi:U_\phi\subseteq\mathbb{R}^n\to M$, whether there exists a topology on M such that $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas of M. Such a topology is necessarily unique by (1.1). This provides a way to define a $\mathcal{C}^k$ manifold without constructing first a topological space.

Proposition 1.2.8 (One-step manifolds)**.** *Consider a set M endowed with a family $\mathcal{A}$ of injections $\phi:U_\phi\subseteq\mathbb{R}^n\to M$ satisfying the the following properties:*
(Denote $\widetilde{U}_\phi=\phi(U_\phi)$ the image of each $\phi\in\mathcal{A}$.)

(1) For all $\phi,\psi\in\mathcal{A}$, the set $U_\phi^\psi:=\phi^{-1}(\widetilde{U}_\psi)$ is open in $\mathbb{R}^n$,

(2) and the transition map $\psi^{-1}\circ\phi:U_\phi^\psi\to U_\psi^\phi$ is $\mathcal{C}^k$.

(3) Countably many of the sets $\widetilde{U}_\phi$ cover M.

(4) For every two distinct points $p,q\in M$ there exist maps $\phi,\psi\in\mathcal{A}$ containing p,q in their respective images, and open neighborhoods $V\subseteq U_\phi$, $W\subseteq U_\psi$ of $\phi^{-1}(p)$, $\psi^{-1}(q)$ respectively, such that $\phi(V)\cap\psi(W)=\emptyset$.

Then the set M endowed with the final topology w.r.t. $\mathcal{A}$ is a topological n-manifold admitting $\mathcal{A}$ as a $\mathcal{C}^k$ parametrization atlas.

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[7] In particular, the inclusion map $M\hookrightarrow\mathbb{R}^k$ is smooth.

[8] The **final topology** on a set M induced by a family of maps $\phi:U_\phi\to M$, where U_ϕ are topological spaces, is the topology defined by the formula (1.1). Exercise: check that it is indeed a topology.

Remarks:

- The hypothesis implies that each set $U_\phi \subseteq \mathbb{R}^n$ is open, because it is equal to U_ϕ^ϕ.
- Each transition map $\psi^{-1} \circ \phi : U_\phi^\psi \to U_\psi^\phi$ is a $\mathcal{C}^k$ isomorphism because it is $\mathcal{C}^k$ and bijective, with inverse $\phi^{-1} \circ \psi$, which is also $\mathcal{C}^k$.

Proof. We endow M with the final topology w.r.t $\mathcal{A}$, so that a set $U \subseteq M$ is open if and only if $\psi_\alpha^{-1}(U)$ is open in U_ϕ for all α. This is clearly a topology on M.

We claim that each map $\phi \in \mathcal{A}$ is open. Indeed, if $W \subseteq U_\phi$ is an open set, then $\phi(W)$ is open in M because for any $\psi \in \mathcal{A}$, the set $\psi^{-1}(\phi(W)) = (\psi^{-1} \circ \phi)(W \cap U_\phi^\psi)$ is open in U_ψ^ϕ since $\psi^{-1} \circ \phi$ is a homeomorphism $U_\phi^\psi \to U_\psi^\phi$.

It follows that each $\phi \in \mathcal{A}$ is a homeomorphism onto its image, because it is open an injective. Therefore the maps ϕ are n-dimensional local parametrizations of M ensuring that M is locally Euclidean of dimension n. Moreover, M is a topological manifold because it is second countable by (3) and Hausdorff by (4). Finally, the parametrization atlas $\mathcal{A}$ is $\mathcal{C}^k$-consistent by (2). □

An example of manifold that can be constructed in this way is the Grassman manifold $G_k(V)$ of k-subspaces of $V \simeq \mathbb{R}^n$; see [Lee13, p. 1.36].

1.3 Differentiable maps

We are about to define $\mathcal{C}^k$ maps between $\mathcal{C}^k$ manifolds. The plan is to reduce the question of differentiability to the case of a map between Euclidean open sets. We will do so by using charts.

In general, when studying a map $f : M \to N$ between manifolds, charts allow us to *locally* express f as a map between subsets of Euclidean space.

Definition 1.3.1 (Local expression of a map)**.** Let M, N be $\mathcal{C}^k$ manifolds and let $f : M \to N$ be any function (not necessarily continuous). A **local expression** (or **coordinate representation**) of f at some point $p \in M$ is a composite map

$$f|_\varphi^\psi := \psi \circ f \circ \varphi^{-1},$$

where φ and ψ are charts of M and N whose domains U, V contain the points p and $f(p)$ respectively. The composite is defined at all possible points; see Remark 1.2.3.

Local expressions allow us, for instance, to determine whether a function is continuous or not.

Remark 1.3.2. Exercise: Show that f is continuous at p if and only if the local expression $f|_\varphi^\psi$ at p is defined in a neighborhood of $\varphi(p)$ and is continuous at $\varphi(p)$. This fact holds however we choose the charts φ, ψ (provided their domains contain the points p, $f(p)$ respectively, of course).

The idea is to use local expressions to *define* whether a map is $\mathcal{C}^k$ or not.

Definition 1.3.3 ($\mathcal{C}^k$ maps between manifolds)**.** We say a function $f : M \to N$ between $\mathcal{C}^k$ manifolds is $\mathcal{C}^k$ at a point $p \in M$ if there exists a local expression $f|_\varphi^\psi$ of f at p such that

$$f|_\varphi^\psi \text{ is defined in a neighborhood of } \varphi(p) \text{ and is } \mathcal{C}^k \text{ at the point } \varphi(p). \tag{1.2}$$

If this holds for all points $p \in M$, we say that f is a $\mathcal{C}^k$ map. The set of $\mathcal{C}^k$ maps $M \to N$ is denoted $\mathcal{C}^k(M, N)$. A $\mathcal{C}^k$ **isomorphism** is a $\mathcal{C}^k$ map that has a $\mathcal{C}^k$ inverse.

If $k \geq 1$, a $\mathcal{C}^k$ map is also called $\mathcal{C}^k$**-differentiable**, and a $\mathcal{C}^k$ isomorphism is also called a $\mathcal{C}^k$ **diffeomorphism** (or a $\mathcal{C}^k$ **diffeo**, for short).

In fact the condition (1.2), that determines whether f is $\mathcal{C}^k$ or not, does not depend on how we choose the charts φ, ψ.

Proposition 1.3.4. *If f is $\mathcal{C}^k$ at the point p, then every local expression $f|_\varphi^\psi$ of f at p satisfies* (1.2)*.*

Proof. Assume that the local expression $f|_\varphi^\psi$ satisfies (1.2). This implies that f is continuous at p, by Remark 1.3.2. Consider a second local expression $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ of f at p. This new local expression is defined on some neighborhood of $\widetilde{\varphi}(p)$ (again by Remark 1.3.2), and is related to the old one by the **chart-change formula**

$$f|_{\widetilde{\varphi}}^{\widetilde{\psi}} \equiv (\psi \circ \widetilde{\psi}^{-1}) \circ f|_\varphi^\psi \circ (\varphi \circ \widetilde{\varphi}^{-1}), \tag{1.3}$$

which holds at all points where the two expressions are defined (in particular, in some neighborhood of $\widetilde{\varphi}(p)$). Since the transition maps $(\psi \circ \widetilde{\psi}^{-1})$ and $(\varphi^{-1} \circ \widetilde{\varphi})$ are $\mathcal{C}^k$, we conclude that $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ is $\mathcal{C}^k$ at the point $\widetilde{\varphi}(p)$. □

Example 1.3.5.
1. The identity map of any $\mathcal{C}^k$ manifold is a $\mathcal{C}^k$ map. (Exercise.)
2. A composite map $g \circ f$ is $\mathcal{C}^k$ at a point p if f is $\mathcal{C}^k$ at p and g is $\mathcal{C}^k$ at $f(p)$. (Exercise.)
3. If M is a $\mathcal{C}^k$ manifold, then every $\mathcal{C}^k$ chart of M, as well as its inverse, are $\mathcal{C}^k$ maps. (Exercise.)

A $\mathcal{C}^k$ structure on a topological manifold M allows us to determine which maps that go to M are $\mathcal{C}^k$. But the reciprocal property also holds: if we know which maps to M are $\mathcal{C}^k$, this information determines the $\mathcal{C}^k$ structure of M.

Proposition 1.3.6. *Let $\mathcal{A}_0$, $\mathcal{A}_1$ be two $\mathcal{C}^k$ atlases on a topological manifold M, defining two $\mathcal{C}^k$ manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases $\mathcal{A}_i$ are equivalent if and only if the following property holds:*
For every function $f : N \to M$ (where N is a $\mathcal{C}^k$ manifold), the function f is $\mathcal{C}^k$ as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Proof. Exercise. □

1.4 Partitions of unity and paracompactness

In this section we develop *partitions of unity*, a tool often used to turn a local construction, obtained by working in coordinates, into a global one. (Don't worry if this sounds vague; we will see examples later on.) Existence of partitions of unity on a manifold is easier to prove on a compact manifold; we will deal first with this case. The general proof relies on a topological property of manifolds called *paracompactness*, which is in turn a consequence of being Hausdorff, second countable and locally compact.

Recall that the **support** of a continuous function $\eta : M \to \mathbb{R}$ is the closed set

$$\mathrm{supp}(\eta) := \overline{\{p \in M \mid \eta(p) \neq 0\}}.$$

Definition 1.4.1. A $\mathcal{C}^k$ **partition of unity** (or **POU**, for short) on a $\mathcal{C}^k$ manifold M is a family $(\eta_i)_i$ of $\mathcal{C}^k$ functions $\eta_i : M \to [0, +\infty)$ satisfying
- Sum condition: $\sum_i \eta_i(x) = 1$ for every $x \in M$,
- Local finiteness: Every point of M has a neighborhood where all except finitely many of the functions η_i vanish.

A partition of unity $(\eta_i)_i$ is **subordinate** to an open cover $\mathcal{U}$ of M if every η_i has its support contained in some open set $U \in \mathcal{U}$.

Theorem 1.4.2 (Existence of Partitions of Unity.)**.** *For any open cover $\mathcal{U}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$ such that the functions η_i have compact support (in fact, their supports are closed coordinate balls; see definition below).*

Another version of the theorem is the following.

Corollary 1.4.3 (Existence of Partitions of Unity, alternate form)**.** *For any open cover $\mathcal{U} = \{U_j\}_{j\in J}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\xi_j)_{j\in J}$ such that* $\operatorname{supp}(\xi_j) \subseteq U_j$ *for all j.*

This version of the theorem can be deduced from the first one (exercise).

Proof. By Theorem 1.4.2, there exists a partition of unity $(\eta_i)_{i\in I}$ where each η_i has its support $\operatorname{supp}(\eta_i)$ contained in U_j for some $j = j_i \in J$. (We also know that $\operatorname{supp}(\eta_i)$ compact, but that is not useful for the proof.) Note that the functions η_i may be many more than the sets U_j. For example, if M is noncompact, since the compact supports $\operatorname{supp}(\eta_i)$ cover M, we see that I is infinite even if J is finite. However, we can "group" the functions η_i as follows.

For each $j \in J$ let $I_j = \{i \in I \mid j_i = j\}$. The sets $(I_j)_{j\in J}$ form a partition of I, i.e., each $i \in I$ is contained in exactly one of the sets I_j (namely, when $j = j_i$).

Define for each $j \in J$ the function $\xi_j = \sum_{i\in I_j} \eta_i$. This function ξ_j is $\mathcal{C}^k$ because locally it is a finite sum of $\mathcal{C}^k$ functions. The functions η_j are $\mathcal{C}^k$ and form a partition of unity because

$$\sum_{j\in J} \xi_j = \sum_{j\in J} \sum_{i\in I_j} \eta_i = \sum_{i\in I} \eta_i = 1$$

To finish, we must check that each ξ_j is supported on U_j.

$$\operatorname{supp}(\xi_j) = \overline{\bigcup_{i\in I_j} \{x \in \mathbb{R} : \eta_i(x) > 0\}} \subseteq \bigcup_{i\in I_j} \overline{\{x \in \mathbb{R} : \eta_i(x) > 0\}} = \bigcup_{i\in I_j} \operatorname{supp}(\eta_i)$$

(Here we are using again the local finiteness of the family $(\operatorname{supp}(\eta_i))_{i\in I}$ in general we have just an inclusion $\overline{\bigcup A_i} \supseteq \bigcup_i \overline{A_i}$, and the equality holds, for instance, if the sets A_i are sets, but in general the closure of a union of sets A_i may be greater than the union of the closures. The equality holds for example if the family

[to be continued] ∎

Note that the functions g_j may not have compact support in this case.

For the proof of Theorem 1.4.2 we will use *bump functions.*

Lemma 1.4.4 (Bump functions on Euclidean space)**.** *For any numbers $0 < a < b$ there exists a smooth **bump function** $h : \mathbb{R}^n \to [0, 1]$ satisfying $h(x) = 1$ iff $\|x\| \le a$ and $h(x) = 0$ iff $\|x\| > b$.*

Proof. It suffices to let $h(x) = g(\|x\|)$, where $g : \mathbb{R} \to [0, 1]$ is a smooth **cutoff function** satisfying $g(t) = 1$ iff $t < a$ and $g(t) = 0$ iff $t > b$. This cutoff function, in turn, may be defined as

$$g(t) = \frac{f(b-t)}{f(b-t) + f(t-a)},$$

where $f : \mathbb{R} \to [0, +\infty)$ is a smooth function such that $f(t) > 0$ iff $t > 0$. Such a function f may be given e.g. by the formula

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \le 0 \end{cases}.$$

∎

We may use charts to transport these bump functions from Euclidean space to any manifold. The resulting bump function will be supported on a closed *coordinate ball.*

Definition 1.4.5. A **closed coordinate ball** in a $\mathcal{C}^k$ manifold M is the preimage $\varphi^{-1}(\overline{B})$ of a closed Euclidean ball $\overline{B}$ by a $\mathcal{C}^k$ chart φ containing $\overline{B}$ in its codomain.

Proof of Thm. 1.4.2 for compact manifolds. Let M be a compact $\mathcal{C}^k$ manifold and let $\mathcal{U}$ be an open cover of M. In this case we'll obtain a *finite* $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$. In fact, it is sufficient to find finitely many $\mathcal{C}^k$ functions $\widetilde{\eta}_i : M \to [0, +\infty)$, each of which has its support $\mathrm{supp}(\widetilde{\eta}_i)$ contained in some $U \in \mathcal{U}$, and such that $\sum_i \widetilde{\eta}_i(x) > 0$ for all x. The functions η_i can then be obtained by dividing each $\widetilde{\eta}_i$ by the strictly positive $\mathcal{C}^k$ function $\widetilde{\eta} = \sum_i \widetilde{\eta}_i$.

To construct the functions η_i we proceed as follows. Each point of a set $U \in \mathcal{U}$ is contained in the interior of some closed coordinate ball $D \subseteq U$. Thus there is a family of closed coordinates balls, each of them contained in some $U \in \mathcal{U}$, whose interiors cover M. By compactness, we may take a finite subfamily of balls D_i whose interiors still cover M. Write each ball D_i as $\varphi_i^{-1}(\overline{B_i})$, where B_i is an open Euclidean ball and φ_i is a $\mathcal{C}^k$ chart containing $\overline{B_i}$ in its codomain. Then let $h_i : \mathbb{R}^n \to [0, +\infty)$ be a $\mathcal{C}^k$ bump function that is supported on the closed ball $\overline{B_i}$ and strictly positive on the interior B_i. Finally, define a $\mathcal{C}^k$ function $\widetilde{\eta}_i : M \to [0, 1]$ by the formula

$$\widetilde{\eta}_i = \begin{cases} h_i \circ \varphi_i & \text{on } \mathrm{Dom}(\varphi) \\ 0 & \text{on } M \setminus D_i \end{cases}$$

This function is supported on D_i, which is contained in some $U \in \mathcal{U}$, thus the function family $(\widetilde{\eta}_i)$ is subordinate to $\mathcal{U}$. In addition, $\widetilde{\eta}_i$ is strictly positive on B_i, and since the balls B_i cover M, we conclude that $\sum_i \widetilde{\eta}_i(x) > 0$ for all $x \in M$, as required. □

1.4.1 Paracompactness*

Definition 1.4.6 (Paracompact space)**.** Let X be a topological space.

- An **open cover** of X is a family $\mathcal{U}$ of open sets whose union is X.
- Another open cover $\mathcal{V}$ **refines** $\mathcal{U}$ if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.
- The space X is **paracompact** if every open cover $\mathcal{U}$ admits as refinement some open cover $\mathcal{V}$ that is locally finite.
- (A family of subsets of X is **locally finite** if every point $x \in X$ has a neighborhood that intersects only finitely many sets of the family.)

Proposition 1.4.7. *If a topological space X is Hausdorff, second countable and locally compact, then it is paracompact.*

In fact, given any topological base $\mathcal{B}$ of X, every open cover $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ consisting of open sets $B \in \mathcal{B}$ with their closures $\overline{B}$ contained in some $U \in \mathcal{U}$.

Proof.

Lemma 1.4.8. *X admits an exhaustion by compact sets, i.e. a sequence $(K_i)_{i\in\mathbb{N}}$ of compact sets that cover X and satisfy $K_i \subseteq \mathrm{Int}(K_{i+1})$.*

Proof of lemma. Let $(V_j)_{j\in\mathbb{N}}$ be a countable open cover of M where each V_j has compact closure. Let $K_0 = \emptyset$, and define inductively for each $i \in \mathbb{N}$ an integer $j_i \geq i$ such that the sets $(V_j)_{j<j_i}$ cover K_i, and a compact set $K_{i+1} = \overline{V_i} \cup \bigcup_{j<j_i} \overline{V_j}$. These compact sets K_i cover M and satisfy $K_i \subseteq \mathrm{Int}(K_{i+1})$. □

Consider the compact sets $L_i = K_i \setminus \mathrm{Int}(K_{i-1})$ and their respective open neighborhoods $W_i = \mathrm{Int}\, K_{i+1} \setminus K_{i-2}$.

Each point $x \in L_i$ is contained in some open $U_x \in \mathcal{U}$ and has a basic neighborhood $B^i_x \in \mathcal{B}$ such that $\overline{B^i_x} \subseteq W_i \cap U$.

Take a finite subfamily $(B^i_{x_j})_j$ that covers L_i. Doing this for each i we obtain a family of basics $(B^i_{x_j})_{i,j}$ that cover M. Their closures satisfy $\overline{B^i_{x_j}} \subseteq U_{x_j}$, which gives the subordination condition, and $\overline{B^i_{x_j}} \subseteq W_i$, which ensures local finiteness since every point $x \in L_i$ is contained in at most three sets W_ℓ, namely, those with $|\ell - i| \leq 1$. □

Using paracompactness, we can prove existence of partitions of unity without the hypothesis of compactness.

Proof of Thm. 1.4.2. Let M be a manifold and $\mathcal{U}$ an open cover.

Let $\mathcal{B}$ be the topological base of M consisting of the interiors of closed coordinate balls. By Proposition 1.4.7, there exists a family of closed coordinate balls D_i, each contained in some open set $U \in \mathcal{U}$, whose interiors cover M.

The proof finishes as in the compact case. We take for each i a $\mathcal{C}^k$ function $\widetilde{\eta}_i : M \to [0,1]$ that is strictly positive on $\mathrm{Int}(D_i)$ and supported on D_i. Then the functions $\eta_i = \frac{\widetilde{\eta}_i}{\sum_j \widetilde{\eta}_j}$ form a $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ that is subordinate to the open cover $\mathcal{U}$. □

1.4.2 Applications

Corollary 1.4.9 (Bump functions). *If M is a $\mathcal{C}^k$ manifold, $A \subset M$ a closed set and $U \subseteq M$ an open neighborhood of A, then there exists a $\mathcal{C}^k$ function $\eta : M \to [0,1]$ such that $\eta \equiv 1$ on A and* $\mathrm{supp}(\eta) \subset U$.

We call η a *bump function* for A supported in U.

Proof. Just take the open cover $\{V_0 = U, V_1 = M \ \setminus A\}$ of M and a partition of unity $(\eta_i)_{i=0,1}$ satisfying $\mathrm{supp}\, \eta_i \subseteq V_i$, then set $\eta = \eta_0$. □

We have already defined differentiability for function $M \to N$ between $\mathcal{C}^k$ manifolds. For functions on closed sets we make the following definition:

Definition 1.4.10. Let $f : A \to N$ be a function where $A \subseteq M$ is a closed set and M, N are $\mathcal{C}^k$ manifolds. We say that f is $\mathcal{C}^k$ if it can be extened to a $\mathcal{C}^k$ function defined on an open neighborhood of A.

As a corollary of the existence of bump functions we can extend a smooth function on a closed set to a smooth function on the whole manifold:

Corollary 1.4.11 (Extension lemma). *Let $f : A \to \mathbb{R}$ be $\mathcal{C}^k$, where $A \subseteq M$ is a closed subset of a $\mathcal{C}^k$ manifold M, and let $U \subseteq M$ be an open set containing A. Then there exists a $\mathcal{C}^k$ function $\widetilde{f} : M \to \mathbb{R}$ such that $\widetilde{f}|_A = f$ and* $\mathrm{supp}\, \widetilde{f} \subseteq U$.

Proof. By definition f can be extended to a $\mathcal{C}^k$ function (say, also called f) on some open set $W \supseteq A$; wlog $W \subseteq U$. We take a $\mathcal{C}^k$ bump function η for A supported in W. Then ηf has support in W and therefore extending the function by 0 outside W we obtain a $\mathcal{C}^k$ function $\widetilde{f}$ on M with the desired properties. □

2 Tangent vectors

Recall that if a map $f : U \to V$ between Euclidean open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ is $\mathcal{C}^1$ at a point $p \in U$, then there exists a unique linear transformation $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$, called the **differential transformation** of f at p, which gives a first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_p f(v) + r_p(v)$$

where $\frac{r_p(v)}{\|v\|} \to 0$ as $v \to 0$.

To define the differential of a map between $\mathcal{C}^k$ manifolds, we need a notion of tangent space.

Definition 2.0.1. Let M be an n-dimensional $\mathcal{C}^k$ manifold with $k \geq 1$. A **coordinatized tangent vector** on M is a triple (p, φ, v) where $p \in M$ is a point, φ is a $\mathcal{C}^k$ chart of M defined at p, and $v \in \mathbb{R}^n$ is a vector in Euclidean space. A **tangent vector** on M is the equivalence class $[p, \varphi, v]$ of a coordinatized tangent vector (p, φ, v) under the equivalence relation

$$(p, \varphi, v) \sim (\widetilde{p}, \widetilde{\varphi}, \widetilde{v}) \quad \Longleftrightarrow \quad \widetilde{p} = p \quad \text{and} \quad \widetilde{v} = \mathrm{D}_{\varphi(p)}(\widetilde{\varphi}\,\varphi^{-1})(v)$$

The set TM of tangent vectors is the **tangent bundle** of M, and there is a canonic projection map $\pi_{TM} : TM \to M$ sending $[p, \varphi, v] \mapsto p$.

The **tangent space** at a point $p \in M$ is the set $T_pM := \pi^{-1}(p)$. It is a vector space with vector addition

$$[p, \varphi, v] + [p, \varphi, w] := [p, \varphi, v + w]$$

and vector scaling

$$\lambda\,[p, \varphi, v] := [p, \varphi, \lambda v] \quad \text{for } \lambda \in \mathbb{R}.$$

Remark 2.0.2.

1. $\sim$ is indeed an equivalence relation. (Exercise.)
2. Fixed a point $p \in M$ and a $\mathcal{C}^k$ chart φ defined on p, the function $\iota : \mathbb{R}^n \to T_pM$ sending $v \mapsto [p, \varphi, v]$ is a bijection. (Exercise.)
3. A tangent vector $X = [p, \varphi, v] \in T_pM$ can be considered as a function

$$\begin{array}{rcl} \{\text{charts of } M \text{ def. at } p\} & \to & \mathbb{R}^n \\ \psi & \mapsto & D_{\varphi(p)}(\psi\,\varphi^{-1})(v). \end{array}$$

   The vector $D_{\varphi(p)}(\psi\,\varphi^{-1})(v)$ is the only $w \in \mathbb{R}^n$ such that $[p, \psi, w] = X$.
4. Vector addition and scaling are well defined and make T_pM a vector space isomorphic to $\mathbb{R}^n$. (Exercise.)

Remark 2.0.3. If $U \subseteq \mathbb{R}^n$ is an open set (considered as a smooth manifold), we identify $TU \equiv U \times \mathbb{R}^n$ by the bijection

$$(p, v) \in U \times \mathbb{R}^n \mapsto [p, \mathrm{id}_U, v] \in TU.$$

Thus for each $p \in U$ we have $T_pU \equiv \{p\} \times \mathbb{R}^n \equiv \mathbb{R}^n$, and for $p \in U$ and $v \in \mathbb{R}^n$ we write

$$v|_p := [p, \mathrm{id}_U, v] \in T_pU. \tag{2.1}$$

Coordinate base for the tangent space We can construct a base of the tangent space T_pM as follows. Consider the canonic base $(e_i)_i$ of $\mathbb{R}^n$, and take a chart φ defined at p. Then the vectors

$$\frac{\partial}{\partial \varphi^i}|_p := [p, \varphi, e_i],$$

called the **coordinate vectors** at p associated to the chart φ, form a base of T_pM.

If ψ is another chart defined at p, then this chart determines a second base consising of vectors $\frac{\partial}{\partial \psi^j}|_p$. This second base is related to the first one by the formula

$$\frac{\partial}{\partial \varphi^i}|_p = \sum_j \frac{\partial \psi^j}{\partial \varphi^i}|_{\varphi(p)} \frac{\partial}{\partial \psi^j}|_p.$$

where $\frac{\partial \psi^j}{\partial \varphi^i}|_{\varphi(p)}$ is the partial derivative of $\psi \circ \varphi^{-1}$ that appears as the coefficient (j, i) of the matrix expression of the linear map $\mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$.

Remark 2.0.4. In concrete examples it is common to use more intuitive symbols for φ^i, e.g. the polar coordinates (r, φ) or the spherical coordinates (r, ϕ, θ). The standard coordinates $(x^0, \dots, x^{n-1})$ on $\mathbb{R}^n$ are usually written (x, y) for $n = 2$ and (x, y, z) for $n = 3$. Bear in mind that an expression like (r, φ) can mean either the map (chart) or the coordinates of a particular point; see example below.

Example 2.0.5 (Polar coordinates)**.** Let $W := \mathbb{R}^+ \times (0, 2\pi)$. The map

$$\Psi : W \to \mathbb{R}^2 : (r, \varphi) \mapsto (r \cos \varphi, r \sin \varphi)$$

is a diffeomorphism onto its image $U := \Psi(W) = \mathbb{R}^2 \backslash (\mathbb{R}_{\geq 0} \times \{0\})$. Its inverse $\Psi^{-1} : U \to W$ is therefore a smooth chart for $\mathbb{R}^2$. The components of Ψ^{-1} are usually written (r, φ) and called *polar coordinates*. On the other hand, we have the standard coordinates (x, y) (i.e. the identity map) on $\mathbb{R}^2$. Take a point $p = (x, y) \in U$ and let $(r, \varphi) = \Psi^{-1}(x, y)$ be its polar coordinates. The polar coordinate vectors $\left.\frac{\partial}{\partial r}\right|_p, \left.\frac{\partial}{\partial \varphi}\right|_p$ can be expressed as a linear combination of the standard coordinate vectors $\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p$ using the change of coordinates formula:

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -r\sin(\varphi)\frac{\partial}{\partial x} + r\cos(\varphi)\frac{\partial}{\partial y}.
\end{aligned}$$

Here we do certain standard abuses of notation. First, the letters r, φ, x, y represent functions when preceeded by a ∂, otherwise they are numbers (obtained by evaluating these functions at the point p). Second, we have omitted the "$|_p$" on the vectors and the evaluation at (r, φ) for the partial derivatives.

2.0.1 Differential of a $\mathcal{C}^k$ map between manifolds

Now that we have tangent spaces, we can define the differential of a $\mathcal{C}^k$ map.

Definition 2.0.6. The **differential transformation** (or **differential**, for short) of a map $f : M \to N$ that is $\mathcal{C}^k$ at a point $p \in M$ is defined as

$$\begin{array}{rccc}
D_p f : & T_pM & \to & T_{f(p)}N \\
 & [p, \varphi, v] & \mapsto & [f(p), \psi, D_{\varphi(p)} f|_\varphi^\psi(v)].
\end{array}$$

where φ, ψ are charts of M, N defined at the points p, $f(p)$ respectively, and $f|_\varphi^\psi = \psi \circ f \circ \varphi^{-1}$ is the local expression of f with respect to the charts φ, ψ.

If f is $\mathcal{C}^k$ everywhere, the union of the maps $D_p f$ for all $p \in M$ is a map $Df : TM \to TN$, also denoted f_* and called the **pushforward** by f.

Note that $D_p f$ is well defined (independent of φ, ψ) and linear. To see that it is well defined, we compute twice

$$\begin{aligned} D_p f: \quad X = [p, \varphi, v] &\mapsto Y = [f(p), \psi, w = D_{\varphi(p)} f|_{\varphi}^{\psi}(v)]. \\ D_p f: \quad \widetilde{X} = [p, \widetilde{\varphi}, \widetilde{v}] &\mapsto \widetilde{Y} = [f(p), \widetilde{\psi}, \widetilde{w} = D_{\widetilde{\varphi}(p)} f|_{\widetilde{\varphi}}^{\widetilde{\psi}}(v)]. \end{aligned}$$

and verify that $X = \widetilde{X}$ implies $Y = \widetilde{Y}$. That is, we must check that $\widetilde{v} = D_{\varphi(p)}(\widetilde{\varphi}\varphi^{-1})(v)$ (or, equivalently, $v = D_{\widetilde{\varphi}(p)}(\varphi\widetilde{\varphi}^{-1})(\widetilde{v})$) implies $\widetilde{w} = D_{\psi(p)}(\widetilde{\psi}\psi^{-1})(w)$. This follows from the equation $f|_{\widetilde{\varphi}}^{\widetilde{\psi}} = (\widetilde{\psi}\psi^{-1}) f|_{\varphi}^{\psi}(\varphi\widetilde{\varphi}^{-1})$ by the chain rule (in its Euclidean version).

The linearity of $D_p f$ follows from the linearity of $D_{\varphi(p)} f|_{\varphi}^{\psi}$.

The chain rule has the following version for maps between manifolds.

Proposition 2.0.7 (Chain rule)**.** *If $f : M \to N$ is $\mathcal{C}^k$ at some point p and $g : N \to L$ is $\mathcal{C}^k$ at $f(p)$, then $g \circ f$ is $\mathcal{C}^k$ at p and has differential*

$$D_p(g \circ f) = D_{f(p)} g \circ D_p f.$$

In particular, for a diffeo f, the differential $D_p f$ is a linear isomorphism whose inverse is $D_{f(p)}(f^{-1})$. (Exercise.)

Example 2.0.8 (Derivative of a chart)**.** Let M be an n-dimensional $\mathcal{C}^k$-manifold, let $p \in M$, and let $\varphi : U \to V$ be a chart of M defined at p. Then we have

$$\mathrm{D}\,\varphi\left(\frac{\partial}{\partial \varphi^i}|_p\right) = e_i|_p \in T_p\mathbb{R}^n.$$

To see this, using the local expression $\varphi|_{\varphi}^{\mathrm{id}_V} = \mathrm{id}_V \circ \varphi \circ \varphi^{-1} = \mathrm{id}_V$ we compute

$$\begin{aligned} \mathrm{D}\,\varphi\left(\frac{\partial}{\partial \varphi^i}|_p\right) &= \mathrm{D}\,\varphi[p, \varphi, e_i] \\ &= [\varphi(p), \mathrm{id}_V, D_{\varphi(p)}(\varphi|_{\varphi}^{\mathrm{id}_V})(e_i))] = [\varphi(p), \mathrm{id}_V, e_i] = e_i|_p \in T_p\mathbb{R}^n. \end{aligned}$$

Example 2.0.9 (Velocity of a curve)**.** For a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ on a manifold M, we define its **velocity vector** at an instant $t \in I$ as the vector $\gamma'(t) := D_t\gamma(1|_t) \in T_{\gamma(t)}M$ where $1|_t$ represents the element $[t, \mathrm{id}_I, 1]$ of T_tI according to the identification $TI \cong I \times \mathbb{R}$ given in Remark 2.0.3.
Exercise: Show that for any vector $X \in TM$ there is a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(0) = X$.

Tangent vectors as derivations

Definition 2.0.10. A **derivation** on a $\mathcal{C}^k$ differentiable manifold M at a point $p \in M$ is a linear function $D : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ satisfying the Leibniz identity

$$D(f\,g) = D(f)\,g(p) + f(p)\,D(g).$$

The set Der_pM of derivations on M at p is a vector space with the operations

$$\begin{aligned} (D + E)(f) &:= D(f) + E(f), \\ (\lambda D)(f) &:= \lambda(D(f)) \end{aligned}$$

defined for $D, E \in Der_pM$ and $\lambda \in \mathbb{R}$.

Each vector $X \in T_pM$ induces a derivation $D_X \in Der_pM$ defined by the formula

$$D_X(f) := \mathrm{D}_p\, f(X) \in T_{f(p)}\mathbb{R} \equiv \mathbb{R}.$$

(Here we use the identification of Remark 2.0.3).

Proposition 2.0.11. *The map $\nu_p : X \in T_pM \mapsto D_X \in Der_pM$ is a linear injection. (Exercise.)*

Therefore we may identify a tangent vector X with the derivation D_X and write $X(f) := D_X(f)$. In fact, the map ν is bijective if M is smooth (see e.g. [Lee13, Prop. 3.2]). Therefore in some books (e.g. [Lee13]), the tangent space T_pM of a smooth manifold M is *defined* as the vector space of derivations at p. Here we do not use that definition because it does not work for non-smooth manifolds (i.e. C^k manifolds with $k < \infty$).

3 Tangent vectors as derivations

In this chapter we give an alternative definition of tangent vectors using derivations. This second definition is equivalent to the one given in the previous chapter, but is more similar to the one used in Lee's book [Lee13].

The objects that we will define are analogous to those defined in the previous chapter (except that the differential of a function will be called *tangent map*). The plan is the following:

(1) For each $\mathcal{C}^k$-differentiable n-manifold M and each point $p \in M$, define a vector space $\mathrm{T}_p\, M \simeq \mathbb{R}^n$, called the **tangent space** of M at p.

(2) For each $\mathcal{C}^k$ map $f : M \to N$ and each point $p \in M$, define a linear map $\mathrm{T}_p\, f : \mathrm{T}_p\, M \to \mathrm{T}_p\, N$ called the **tangent map** of f at p.

(3) Prove the chain rule: $\mathrm{T}_p(g \circ f) = \mathrm{T}_{f(p)}\, g \circ \mathrm{T}_p\, f$ (and $\mathrm{T}_p\, \mathrm{id}_M = \mathrm{id}_{\mathrm{T}_p\, M}$).

If M, N are open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively, then the map $\mathrm{T}_p\, f$ should be essentially equivalent to the **differential transformation** that we already know, that is, the unique unique linear map $\mathrm{D}_p\, f : \mathbb{R}^m \to \mathbb{R}^n$ which gives the first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_p\, f(v) + r(v)$$

where $\frac{r(v)}{\|v\|} \to 0$ as $v \to 0$.

Motivation for definition of tangent vectors Let M be a $\mathcal{C}^k$ manifold M with $k \geq 1$. How should we define the space $\mathrm{T}_p\, M$ of vectors tangent to M at p?

For example, let $\gamma : I \subseteq \mathbb{R} \to M$ be a differentiable curve such that $\gamma(0) = p$. This curve should have a velocity vector $X \in \mathrm{T}_p\, M$ at the instant $t = 0$, which should be a tangent vector at p. This object X should contain information that tells us the direction and speed of movement of $\gamma(t)$ at time $t = 0$.

Idea: take a function $h \in \mathcal{C}^k(M, \mathbb{R})$. The composite $h \circ \gamma$ is a function $I \to \mathbb{R}$ that is differentiable at $t = 0$, and the number number $(h \circ \gamma)'(0)$ tells us something about how γ is moving at $t = 0$. If we know this number for all functions h, let us agree that we know the velocity vector X completely. Therefore we can *define* the velocity vector X simply as the function $h \mapsto (h \circ \gamma)'(0)$. This function has certain algebraic properties that make it a *derivation*.

Definition 3.0.1. A **derivation** of a $\mathcal{C}^k$-differentiable manifold M at a point $p \in M$ is a function $X : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ that is $\mathbb{R}$-linear and satisfies the **Leibniz identity**

$$X(gh) = X(g)\, h(p) + g(p)\, X(h). \tag{3.1}$$

The set $\mathrm{Der}_p(M)$ of derivations of M at p is a real vector space by defining

$$X + \lambda Y : h \mapsto X(h) + \lambda Y(h) \quad \text{for } X, Y \in \mathrm{Der}_p\, M \text{ and } \lambda \in \mathbb{R}.$$

We define the set $\mathrm{Der}_M := \coprod_{p \in M} \mathrm{Der}_p\, M$ and the natural projection $\pi_{\mathrm{Der}\, M} : \mathrm{Der}\, M \to M$ that maps each $x \in \mathrm{Der}_p\, M$ to the point p.

The first example of derivation is the one we already mentioned:

Example 3.0.2. The **velocity vector** of a curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ (where γ is differentiable) is the derivation

$$\begin{array}{rccc} X : & \mathcal{C}^k(M, \mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & (h \circ \gamma)'(t) \end{array}$$

This function is indeed a derivation of M at the point $\gamma(t)$ (check it!). It may be denoted $(\gamma, \gamma')(t)$. Sometimes we also denote it $\gamma'(t)$, but it is different from the usual $\gamma'(t)$ defined for a curve in $\mathbb{R}^n$, because it also contains the information of the point $\gamma(t)$ where the derivation is located.

Example 3.0.3. Let $U \subseteq \mathbb{R}^n$ be an open set. A **vectorial derivation**[1] of U at a point $p \in U$ is a function

$$\begin{array}{rccc} \delta_{p,v}: & \mathcal{C}^k(U, \mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \mathrm{D}_p\, h(v), \end{array}$$

determined by some vector $v \in \mathbb{R}^n$. Note that $\delta_{p,v}$ is indeed a derivation of U at p; in fact it is the velocity vector at $t = 0$ of any curve γ in U satisfying $\gamma(0) = p$ and $\gamma'(0) = v$, for example, the curve $\gamma_{p,v}(t) = p + tv$. The vectorial derivations form an n-dimensional subspace of $\mathrm{Der}_p\, U$ because $v \in \mathbb{R}^n \mapsto \mathrm{D}_{p,v} \in \mathrm{Der}_p\, U$ is an injective linear map. Injectivity follows from the equation $D_{p,v}(\pi^i) = v^i$, where $\pi^i : x \in U \to x^i \in \mathbb{R}$ is the natural projection on the i-th coordinate axis.

Remark 3.0.4. If M is a smooth manifold, then one one can define the tangent space $\mathrm{T}_p\, M$ simply as the space $\mathrm{Der}_p\, M$ of derivations, because one can prove that this space is n-dimensional (see exercises). However, if M is a $\mathcal{C}^k$ manifold with $k < \infty$, then the space of $\mathrm{Der}_p\, M$ is infinite dimensional (see exercises).

Therefore, we will define the tangent space $\mathrm{T}_p\, M$ as a subspace of $\mathrm{Der}_p\, M$, namely, that consisting on the derivations that correspond (via some coordinate system) to some vectorial derivation of an open subset of $\mathbb{R}^n$.

To map derivations from one manifold to another one we will use the following operator.

Definition 3.0.5 (Pushforward of derivations)**.** The **pushforward operator** of a $\mathcal{C}^k$-differentiable map $f : M \to N$ at some point $p \in M$ is the linear map

$$\begin{array}{rccc} f_*|_p: & \mathrm{Der}_p\, M & \to & \mathrm{Der}_{f(p)}\, N \\ & X & \mapsto & (h \mapsto X(f \circ h)). \end{array}$$

The union of the operators $f_*|_p$ over all $p \in M$ is denoted $f_* : \mathrm{Der}\, M \to \mathrm{Der}\, N$.

Example 3.0.6. Let $f : M \to M$ be a $\mathcal{C}^k$ map. Let $\gamma : I \subseteq \mathbb{R}$ be a curve that is differentiable at some instant $t \in I$, and let X be its velocity vector at that instant. Then f_* is the velocity vector of the curve $f \circ \gamma$ at the same instant.

In the next result we study an important example of pushforward map.

Proposition 3.0.7. *Let M be a $\mathcal{C}^k$-diferentiable manifold and let $p \in M$. Then for any derivation $X \in \mathrm{Der}_p\, M$ we have:*

(a) If $h \in \mathcal{C}^k(M, \mathbb{R})$ is constant, then $X(h) = 0$.

(b) If $g, h \in \mathcal{C}^k(M, \mathbb{R})$ both vanish at p, then $X(gh) = 0$.

(c) Locality: If two functions $g, h \in \mathcal{C}^k(M, \mathbb{R})$ coincide on a neighborhood of p, then $X(g) = X(h)$.

In consequence, if U is an open neighborhood of p and $\iota : U \to M$ is the inclusion map, then the pushforward

$$\begin{array}{rccc} \iota_*|_p: & \mathrm{Der}_p\, U & \to & \mathrm{Der}_p\, M \\ & X & \mapsto & (h \mapsto X(h|_U)), \end{array}$$

is a linear isomorphism. Therefore we may identify $\mathrm{Der}_p\, M \equiv \mathrm{Der}_p\, U$.

[1] These derivations are sometimes called “directional derivations”, but I don’t like this name because it suggests that it depends only on the direction of the vector v.

Proof. To prove (a), since X is linear, it suffices to show that $X(h) = 0$ if $h \equiv 1$. And indeed, in this case we have

$$X(h) = X(h^2) = X(h)\,h(p) + h(p)\,X(h) = 2X(h)$$

which implies $X(h) = 0$.

Fact (b) follows immediately from the Leibniz identity.

To prove (c), since X is linear, it suffices to show that $X(f) = 0$ if $f \equiv 0$ in some open neighborhood U of p. Let $\eta : \mathcal{C}^k(M, \mathbb{R})$ be a bump function that is constantly 1 on the closed set $M \setminus U$ and whose support is contained in the open set $M \setminus \{p\}$. Note that $f\eta = f$, therefore $X(f) = X(f\eta) = 0$ since $f(p) = \eta(p) = 0$.

Finally, to show that $\iota_*|_p$ we construct an inverse

$$\begin{array}{rccc} r: & \mathrm{Der}_p\, M & \to & \mathrm{Der}_p\, U \\ & Y & \mapsto & (g \mapsto Y(\widetilde{g}))\,, \end{array}$$

where $\widetilde{g} \in \mathcal{C}^k(M, \mathbb{R})$ is any function that coincides with $g \in \mathcal{C}^k(U, \mathbb{R})$ on a neighborhood of p. For example, we may put $\widetilde{g} := g\,\eta$, where $\eta \in \mathcal{C}^k(M, \mathbb{R})$ is a bump function supported on U that is $\equiv 1$ on a neighborhood of p.

It is easy to verify that r is inverse of $\iota_*|_p$ (check it!). □

Now we are ready to define tangent vectors.

Definition 3.0.8 (Tangent vectors and tangent maps)**.** A **tangent vector** to a $\mathcal{C}^k$ differentiable n-manifold M at a point $p \in M$ is a derivation of the form

$$\begin{array}{rccc} \delta_{p,\phi,v}: & \mathcal{C}^k(M, \mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \mathrm{D}_{\phi(p)}(h \circ \phi^{-1})(v) \end{array} \tag{3.2}$$

where ϕ is a chart defined at p and $v \in \mathbb{R}^n$. Equivalently, $\delta_{p,\phi,v}$ is the pushforward of the vectorial derivation $\delta_{\phi(p),v}$ by the map $\phi^{-1} : U \to M$. The triple (p, ϕ, v) is called a **coordinate expression** of the tangent vector $\delta_{p,\phi,v} \in \mathrm{T}_p\, M$.

The **tangent space** of M at point p is the set $\mathrm{T}_p\, M \subseteq \mathrm{Der}_p\, M$ of tangent vectors to M at p. The **tangent bundle** of M is the set $TM := \coprod_{p \in M} \mathrm{T}_p\, M$. It has a natural projection $\pi_{\mathrm{T}\, M} : \mathrm{T}\, M \to M$ that maps each $X \in \mathrm{T}_p\, M$ to the point p.

The **tangent transformation** of a $\mathcal{C}^k$ map $f : M \to N$ at a point $p \in M$ is the map $\mathrm{T}_p\, f : \mathrm{T}_p\, M \to \mathrm{T}_{f(p)}\, N$ obtained by restricting the pushforward map $f_*|_p : \mathrm{Der}_p\, M \to \mathrm{Der}_{f(p)}\, N$. The **tangent map** of f is the map $\mathrm{T}\, f : TM \to TN$ obtained as union of the transformations $\mathrm{T}_p\, f$ over all points $p \in M$.

We must check that f_* indeed maps tangent vectors to tangent vectors. More precisely, let us show for any tangent vector $\mathrm{D}_{p,\phi,v} \in \mathrm{T}_p\, M$ and any chart ψ of N at $f(p)$ we have

$$f_*|_p : \mathrm{D}_{p,\phi,v} \mapsto \mathrm{D}_{\phi(p),\psi,w} \quad \text{where} \quad w = \mathrm{D}_{\phi(p)}(f|_\phi^\psi)(v). \tag{3.3}$$

Indeed, for any $h \in \mathcal{C}^k(N, \mathbb{R})$ we have

$$\begin{aligned} (f_*|_p\, \delta_{p,\phi,v})(h) &= \delta_{p,\phi,v}(h\, f) \\ &= \mathrm{D}_{\phi(p)}(h\, f\, \phi^{-1})(v) \\ &= \mathrm{D}_{\psi(f(p))}(h\, \psi^{-1})\left(\mathrm{D}_{\phi(p)}(\psi\, f\, \phi^{-1})(v)\right) \\ &= \delta_{f(p),\psi,w}(h). \end{aligned}$$

Remark 3.0.9. A tangent vector $X \in \mathrm{T}_p\, M$ can be applied to any real-valued function h that is defined on a neighborhood of p and is differentiable at the point p. In this case, we define $X(h)$ in the same way as for a function $h \in \mathcal{C}^k(M, \mathbb{R})$, using formula (3.2). Similarly, the tangent transformation $\mathrm{T}_p\, f : \mathrm{T}_p\, M \to \mathrm{T}_{f(p)}\, N$ can be defined for any function $f : M \to N$ that is differentiable at p.

Example 3.0.10. Let p be a point of a $\mathcal{C}^k$ manifold M. Any tangent vector $X \in \mathrm{T}_p M$ can be written as $X = \delta_{p,\phi,v}$ where $\phi : U \to V$ is a chart defined at p. We claim that the derivation $X \in \mathrm{Der}_p M \equiv \mathrm{Der}_p U$ applied to one of the chart components $\phi^i : U \to \mathbb{R}$ gives

$$X(\phi^i) = v^i. \tag{3.4}$$

To see this, we note first that $\phi^i = \pi^i \circ \phi$, where $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is the projection on the i-th coordinate. Then we compute

$$X(\phi^i) = \mathrm{D}_p(\phi^i \circ \phi^{-1})(v) = \mathrm{D}_p(\pi^i)(v) = \pi^i(v) = v^i.$$

Here we used the fact that $\mathrm{D}_p(\pi^i) = \pi^i$ since π^i is linear.

We may now show that the tangent space of a differentiable n-manifold at any point is an n-dimensional vector space.

Proposition 3.0.11. *Let p be a point of a $\mathcal{C}^k$-differentiable manifold M, and let ϕ, ψ be two charts of M defined at p.. Then for any two vectors $v, w \in \mathbb{R}^n$ we have*

$$\delta_{p,\phi,v} = \delta_{p,\psi,w} \quad\iff\quad w = \mathrm{D}_{\phi(p)}(\psi\phi^{-1})(v) \quad\iff\quad v = \mathrm{D}_{\psi(p)}(\phi\psi^{-1})(w) \tag{3.5}$$

In consequence, the linear map $\nu : v \in \mathbb{R}^n \mapsto \delta_{p,\phi,v} \in \mathrm{Der}_p M$ is an isomorphism onto the tangent space $\mathrm{T}_p M$, and in particular, $\mathrm{T}_p M \simeq \mathbb{R}^n$.

Proof. Note first that the two equations $w = \mathrm{D}_{\phi(p)}(\psi\phi^{-1})(v)$ and $v = \mathrm{D}_{\psi(p)}(\phi\psi^{-1})(w)$ are equivalent since the linear transformations $\mathrm{D}_{\phi(p)}(\psi\phi^{-1})$ and $\mathrm{D}_{\psi(p)}(\phi\psi^{-1})$ are inverse of each other.

Suppose first that these equations hold. Then for any function $h \in \mathcal{C}^k(M, \mathbb{R})$ we have

$$\begin{aligned}
\delta_{p,\psi,w}(h) &= \mathrm{D}_{\psi(p)}(h\,\psi^{-1})(w) \\
&= \mathrm{D}_{\psi(p)}(h\,\psi^{-1})(\mathrm{D}_{\phi(p)}(\psi\,\phi^{-1})(v)) \\
&= \mathrm{D}_{\phi(p)}(h\,\phi^{-1})(v) \\
&= \delta_{p,\phi,v}(h).
\end{aligned}$$

It follows that ν is surjective onto $\mathrm{T}_p M$, since any vector $X \in \mathrm{T}_p M$ is of the form $X = \delta_{p,\psi,w}$ for some chart ψ and some vector $w \in \mathbb{R}^n$, and then we obtain $\nu(v) = X$ by putting $v = \mathrm{D}_{\psi(p)}(\phi\psi^{-1})(w)$.

The injectivity of ν follows from the formula (3.4), which gives a way to recover the components of v from the derivation $\nu(v) = \delta_{p,\phi,v}$ by applying this derivation to the functions ϕ^i.

Finally, suppose that $\delta_{p,\phi,v} = \delta_{p,\psi,w}$. We have already shown that this equation holds when $v = \mathrm{D}_{\psi(p)}(\phi\psi^{-1})(w)$, and it cannot hold for any other value of v, because the map $\nu : v \mapsto \delta_{p,\phi,v}$ is injective. Therefore we may conclude that $v = \mathrm{D}_{\psi(p)}(\phi\psi^{-1})(w)$. ☐

More to do:

- Vectorial derivations are the same thing as velocity vectors (but there are fewer velocity vectors on a manifold with boundary/corners).
- General derivations on $\mathcal{C}^k$ manifolds: smooth and nonsmooth case.

4 Submanifolds

Let us start with a short motivation. The "corner" in $\mathbb{R}^2$ given by $C = \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$ is intuitively not "smoothly contained" in $\mathbb{R}^2$. It is homeomorphic to $\mathbb{R}$ and can be given the structure of a smooth manifold (e.g. by taking just the single chart given by this homeomorphism), but the smooth structure does not "agree" with the ambient one. [1] To make this precise we need a few definitions:

Definition 4.0.1. Let $f : M \to N$ be a $\mathcal{C}^k$ map between C^k manifolds, with $k \geq 1$.
We say that f is a **submersion** if D_f is surjective for each $p \in M$.
We say that f is an **immersion** if $D_p f$ is injective for each $p \in M$.
We say that f is a **smooth embedding** if it is an immersion that is also a topological embedding (i.e. a homeomorphism onto its image, the latter being endowed with the subspace topology).
The **rank** of f at a point $p \in M$ is the rank of the linear map $D_p f$. If f has the same rank k at all points ("constant rank") then we write $\operatorname{rank} f = k$.

So to sum up, if $f : M \to N$ is $\mathcal{C}^k$, $\dim M = m$, $\dim N = n$, then f is a submersion iff $\operatorname{rank} f = n$ and it is an immersion iff $\operatorname{rank} f = m$. In both cases this means that the rank of f is maximal at any point of M.[2]

Example 4.0.2. The **standard immersion** $\iota|_k^n : \mathbb{R}^k \to \mathbb{R}^n$, for $k \leq n$, is the map

$$\iota_k^n : (x^0 \ldots, x^{k-1}) \mapsto (x^0, \ldots, x^{k-1}, 0, \ldots, 0).$$

This is also a smooth embedding.
The **standard submersion** $\pi_m^k = \mathbb{R}^m \to \mathbb{R}^k$, for $m \geq k$, is the map

$$\pi_m^k : (x^0, \ldots, x^{m-1}) \mapsto (x^0, \ldots, x^{k-1}).$$

Example 4.0.3.

- A smooth curve $\gamma : J \to M$, where $J \subset \mathbb{R}$ is an interval and M is a smooth manifold, is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$. E.g. the curve on $\mathbb{R}^2$ given by $t \mapsto (\cos t, \sin t)$, $t \in \mathbb{R}$, is an immersion, but the curve on $\mathbb{R}^2$ given by $t \mapsto (t^2, t^3)$, $t \in \mathbb{R}$, is not.
- The inclusion $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding (where we endow $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$ with the standard smooth structures).
- Let $M = M_0 \times \ldots \times M_{k-1}$ be a product of smooth manifolds. Then the standard projections $\pi_i : M \to M_i$ are submersions.
- *"The figure 8"*: Let $J = (-\frac{\pi}{2}, \frac{3\pi}{2})$ and consider the curve

  $$\gamma : J \to \mathbb{R}^2, \quad \gamma(t) = (\sin(2t), \cos t).$$

  It is easy to verify that γ is an injective immersion. However, it is not a homeomorphism onto its image $N := \gamma(J) \subset \mathbb{R}^2$ (endowed with the subspace topology): removing e.g. the point $(0, 1) = \gamma(0)$ the set $N \backslash \{(0, 1)\}$ is connected, but its preimage under γ is $(-\frac{\pi}{2}, 0) \cup (0, \frac{3\pi}{2})$ hence disconnected. Therefore γ is not an embedding.

[1] In the terminology introduced below, the inclusion is not a smooth embedding whatever smooth structure we choose on C.

[2] For a linear map $\mathbb{R}^m \to \mathbb{R}^n$ we say that its rank is maximal if it is equal to $\min\{m, n\}$, i.e. the maximum rank a linear map $\mathbb{R}^m \to \mathbb{R}^n$ can have.

4.1 Local diffeomorphisms

If $\dim M = \dim N$, then a $\mathcal{C}^k$ map $f : M \to N$ is immersion iff it is a submersion iff $D_p f$ is an isomorphism for all $p \in M$.

Definition 4.1.1 (Local diffeomorphism)**.** We say f is a **local $\mathcal{C}^k$ isomorphism** at some point $p \in M$ if there are open nbhds U, V of p, $f(p)$ (endowed with their natural $\mathcal{C}^k$ structures) such that the restriction $f|_U^V : U \to V$ is a $\mathcal{C}^k$ isomorphism. If this holds for all $p \in M$, we say that simply that f is a **local $\mathcal{C}^k$ isomorphism**.

Proposition 4.1.2. *Let $f : M \to N$ a $\mathcal{C}^k$ map, $k \geq 1$, and let $p \in M$ be such that $D_p f$ is a linear isomorphism. Then f is a **local diffeomorphism** at p, i.e, there exist respective open neighborhoods U, V of x, $f(x)$ such that $f|_U^V$ is a $\mathcal{C}^k$ diffeomorphism.*

This follows from the Inverse Function Theorem.

Theorem 4.1.3 (Inverse Function Theorem, or IFT)**.** *Let $f : U \to V$ be a $\mathcal{C}^k$ map, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ are open sets, and let $x \in U$ be a point such that $D_x f(x)$ is invertible. Then f is a local diffeomorphism at x, i.e. there exists respective open neighborhoods U', V' of x, $f(x)$ such that $f|_{U'}^{V'}$ is a $\mathcal{C}^k$ diffeomorphism.*

A very useful consequence of the IFT is the following:

Theorem 4.1.4 (Constant Rank Theorem)**.** *Let M and N be smooth manifolds of dimension m and n respectively. Let $F : M \to N$ be of constant rank in a neighborhood of a point $p \in M$. Then there exist charts (U, φ) centred at p and (V, ψ) centred at $F(p)$ such that the coordinate representation of F in a neighborhood of $\varphi(p)$ is*

$$\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

This theorem directly implies that any immersion (resp. submersion) locally has the "standard" form given in Example 4.0.2 above, since an immersion (resp. submersion) does have constant rank.

4.2 Embedded submanifolds

Roughly speaking, an embedded submanifold is a subset of a manifold that locally looks like $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^m$.

Definition 4.2.1. A subset of a $\mathcal{C}^k$ manifold M is a **$\mathcal{C}^k$-embedded n-submanifold** if it is the image of a $\mathcal{C}^k$ embedding $f : N \to M$, where $\dim N = n$.

Proposition 4.2.2. *Let N be a subset of a $\mathcal{C}^k$ m-manifold M. The following are equivalent:*

(a) N is a $\mathcal{C}^k$-embedded n-submanifold.

(b) Each point $p_0 \in N$ has an open neighborhood $U \subseteq M$ such that the set $N \cap U$ is the image of a $\mathcal{C}^k$ embedding $\varphi : V \to U$, where $V \subseteq \mathbb{R}^n$ is an open set.

*(c) N is locally an "n-slice": for each point $p_0 \in N$ there exists a chart φ of M defined at p_0 that is an n-**slice chart** for N, i.e, such that*

$$p \in N \iff \varphi(p) \in \mathbb{R}^n \times \{0\} \quad \text{for all } p \in \operatorname{Dom}\varphi.$$

Definition 4.2.3. Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map.

A point $q \in N$ is called a **regular value** of f if $\mathrm{D}_p f$ is surjective for every $p \in f^{-1}(c)$.

Theorem 4.2.4 (Regular preimage theorem)**.** *Let c be a regular value of a $\mathcal{C}^k$ map $f : M \to N$. Then the set $L = f^{-1}(c)$ is an embedded submanifold of M. Its tangent space at any point $p \in L$ is $\mathrm{T}_p L = \operatorname{Ker}(\mathrm{D}_p f)$.*