

Exercise 13.1. Prove Proposition 7.2.6. (properties of the integral: linearity, etc).

Exercise 13.2. Prove that a continuous k -form is determined by the value of its integrals (Proposition 7.3.12). *Hint:* Use a chart to move the problem to \mathbb{R}^n , then integrate on small pieces of coordinate planes.

Exercise 13.3. Let $f : M \rightarrow N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have

$$f^*(d\omega) = d(f^*\omega).$$

Exercise 13.4. Let (x, y, z) be the standard coordinates on \mathbb{R}^3 and let (v, w) be the standard coordinates on \mathbb{R}^2 . Let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined as $\phi(x, y, z) = (x + z, xy)$. Let $\alpha = e^w dv + v dw$ and $\beta = v \, dv \wedge dw$ be 2-forms on \mathbb{R}^2 . Compute the following differential forms:

$$\alpha \wedge \beta, \quad \phi^*(\alpha), \quad \phi^*(\beta), \quad \phi^*(\alpha) \wedge \phi^*(\beta).$$

Exercise 13.5. Compute the exterior derivative of the following forms:

- (a) on $\mathbb{R}^2 \setminus \{0\}$ $\theta = \frac{x \, dy - y \, dx}{x^2 + y^2}$.
- (b) on \mathbb{R}^3 , $\varphi = \cos(x) \, dy \wedge dz$.
- (c) on \mathbb{R}^3 $\omega = A \, dx + B \, dy + C \, dz$.

Exercise 13.6. Deduce the following classical theorems from Stokes' theorem.

- (a) **Green's theorem.** Let $D \subseteq \mathbb{R}^2$ be a smooth 2-dimensional compact embedded submanifold with boundary in \mathbb{R}^2 . Then for any differentiable 1-form $\omega = P \, dx + Q \, dy$ defined on an open neighborhood of D we have

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \int_{\partial D} P \, dx + Q \, dy.$$

- (b) **Divergence theorem.** Let $A \subset \mathbb{R}^3$ be a 3-dimensional compact embedded submanifold with boundary in \mathbb{R}^3 . Then for any smooth vector field $F : A \rightarrow \mathbb{R}^3$ we have

$$\int_A \operatorname{div} F \, dV = \int_{\partial A} F \cdot d$$

wher $dV := dx \wedge dy \wedge dz$ is the standard 3-form on \mathbb{R}^3 and on the right hand side we have the formal inner product with $dS = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$.