

Exercise 13.1 (Properties of the integral). Let M be an oriented differentiable n -manifold and let ω, η be two continuous, compactly supported n -forms on M . Prove the following:

(a) Linearity: If $a, b \in \mathbb{R}$, then

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

Solution. Consider two continuous, compactly supported n -forms ω, η . Linearity: We have to show that

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta$$

for any two numbers $a, b \in \mathbb{R}$.

First case: The manifold M is an open subset U of \mathbb{R}^n , with the standard orientation. Then we may write

$$\omega = h \, dx^0 \wedge \cdots \wedge dx^{n-1}$$

$$\eta = g \, dx^0 \wedge \cdots \wedge dx^{n-1}$$

and we have $a\omega + b\eta = (ah + bg) \, dx^0 \wedge \cdots \wedge dx^{n-1}$, therefore

$$\int_M (a\omega + b\eta) = \int_U (ah + bg) = a \int_U h + b \int_U g = a \int_M \omega + b \int_M \eta$$

where the integrals over U are Riemann integrals.

Second case: M is a general manifold, but the supports of ω and η are contained in the image $\varphi(U)$ of a single parametrization $\varphi : U \rightarrow M$ that has constant sign $\text{sgn } \varphi = \pm 1$. Then $\text{supp}(a\omega + b\eta) \subseteq \text{supp } \omega \cup \text{supp } \eta \subseteq \varphi(U)$, and we have

$$\begin{aligned} \int_M (a\omega + b\eta) &= \text{sgn } \varphi \cdot \int_U \varphi^*(a\omega + b\eta) \\ &= \text{sgn } \varphi \cdot \int_U (a\varphi^*\omega + b\varphi^*\eta) \\ &= a \text{sgn } \varphi \cdot \int_U \varphi^*\omega + b \text{sgn } \varphi \cdot \int_U \varphi^*\eta = a \int_M \omega + b \int_M \eta \end{aligned}$$

General case: We use a family of local parametrizations (U_i, φ_i) whose images $\varphi_i(U_i)$ cover $\text{supp } \omega$ and $\text{supp } \eta$, and a partition of unity $(\chi_i)_i$ subordinate to the open cover $(U_i)_i$ of the set $\bigcup_i \varphi_i(U_i)$. Then by definition of \int_M we have

$$\begin{aligned} \int_M (a\omega + b\eta) &= \sum_i \int_M \chi_i (a\omega + b\eta) \\ &= \sum_i \int_M (a\chi_i \omega + b\chi_i \eta). \end{aligned}$$

Since the forms $\chi_i \omega$ and $\chi_i \eta$ have their support contained in $\varphi_i(U_i)$, by the previous case the last integral is equal to

$$\begin{aligned} &= \sum_i \left(a \int_M \chi_i \omega + b \int_M \chi_i \eta \right) \\ &= a \sum_i \int_M \chi_i \omega + b \sum_i \int_M \chi_i \eta \\ &= a \int_M \omega + b \int_M \eta. \end{aligned}$$

□

- (b) Positivity: If $\text{sgn}(\omega|_p)$ coincides with the orientation of M at every point $p \in M$ where $\omega|_p \neq 0$, then $\int_M \omega \geq 0$, and the inequality is strict unless ω is identically zero.

Solution. First case: Let M be an open set $U \subseteq \mathbb{R}^n$, with the standard orientation. In this case we can write $\omega = h dx^0 \wedge \cdots \wedge dx^{n-1}$ for some continuous function $h : U \rightarrow \mathbb{R}$, and we have $\text{sgn}(\omega_p) = \text{sgn } h(p)$ for any point $p \in U$, because $dx^0 \wedge \cdots \wedge dx^{n-1}$ is a positive n -form. Therefore the condition that $\text{sgn}(\omega_p)$ coincides with the orientation of M for all points $p \in U$ where $\omega_p \neq 0$ is equivalent to the condition that $h(p) \geq 0$ for all $p \in U$.

Thus we have $\int_M \omega = \int_U h \geq 0$, and the inequality is strict unless $h \equiv 0$, which means $\omega \equiv 0$.

Second case: Suppose $\text{supp } \omega \subseteq U$ for some local parametrization $\varphi : \tilde{U} \rightarrow U$ that has constant sign $\text{sgn } \varphi = \pm 1$. For a point $p = \varphi(\tilde{p})$ where $\omega_p \neq 0$, we have

$$\text{sgn}((\varphi^* \omega)|_p) = \text{sgn } \varphi \cdot \underbrace{\text{sgn}(\omega|_p)}_{=+1},$$

which implies that $\text{sgn}(\varphi^* \omega)|_p = \text{sgn } \varphi$. Therefore

$$\int_M \omega = \text{sgn } \varphi \int_{\tilde{U}} \varphi^* \omega \geq 0$$

and the equality holds if and only if $\varphi^* \omega \equiv 0$, iff $\omega \equiv 0$.

General case: We use a family of local parametrizations (U_i, φ_i) whose images $\varphi_i(U_i)$ cover $\text{supp } \omega$ and $\text{supp } \eta$, and a partition of unity $(\chi_i)_i$ subordinate to the open cover $(U_i)_i$ of the set $\bigcup_i \varphi_i(U_i)$. Since $\chi_i \geq 0$, we have $\text{sgn}(\chi_i \omega) \geq 0$, and therefore $\int_M \omega = \sum_i \int_M \chi_i \omega \geq 0$ with equality iff $\chi_i \omega = 0$ for all i , iff $\omega \equiv 0$. □

- (c) Diffeomorphism invariance: If $f : N \rightarrow M$ is an diffeomorphism of constant sign $\text{sgn}(f) = \pm 1$ (i.e. f is either orientation preserving or orientation reversing), then

$$\int_N f^* \omega = \text{sgn } f \cdot \int_M \omega.$$

Solution. Suppose $\text{supp}(f^* \omega) \subseteq \varphi(\tilde{U})$ for some local parametrization $\varphi : \tilde{U} \rightarrow N$. Then $f \circ \varphi$ is a local parametrization of M such that $\text{supp } \omega \subseteq f(\varphi(U))$, and we have

$$\int_N f^* \omega = \int_{\tilde{U}} \varphi^*(f^* \omega) = \int_{\tilde{U}} (f \circ \varphi)^* \omega = \int_M \omega.$$

In the general case, the result is deduced easily using partitions unity. □

- (d) Orientation reversal: If $-M$ denotes M with the reversed orientation, then

$$\int_{-M} \omega = - \int_M \omega.$$

Solution. Let \mathcal{O} be the orientation of M , so that $-\mathcal{O}$ is the orientation of $-M$. Here the most important case is when $\text{supp}(\omega) \subseteq \varphi(\tilde{U})$ for some local parametrization $\varphi : \tilde{U} \rightarrow M$. In the formula

$$\int_M \omega := \text{sgn}_{\mathcal{O}} \varphi \cdot \int_{\tilde{U}} \varphi^* \omega,$$

if we reverse the orientation of M , then the sign of φ is also reversed:

$$\int_{-M} \omega = \text{sgn}_{-\mathcal{O}} \varphi \cdot \int_{\tilde{U}} \varphi^* \omega = - \text{sgn}_{\mathcal{O}} \varphi \cdot \int_{\tilde{U}} \varphi^* \omega = - \int_M \omega.$$

In the general case, the result is deduced easily using partitions unity. □

Exercise 13.2. Prove that a continuous k -form is determined by the value of its integrals (Proposition 7.3.12). **Hint:** Use a chart to move the problem to \mathbb{R}^n , then integrate on small pieces of coordinate planes.

Solution. Suppose that our manifold is an open set $U \subseteq \mathbb{R}^n$ (considered as a \mathcal{C}^1 manifold) and $\omega \in \Omega^k(U)$ is a continuous k -form. We write $\omega = \sum_{I \in \underline{n}^k} \omega_I dx^I$. Let $p \in U$ and $I = (i_0, \dots, i_{k-1}) \in \underline{n}^k$ an increasing k -index. We want to show that $\omega_I(p)$ is determined by values of k -dimensional integrals of ω .

Let $\iota = \iota_{p,I} : \mathbb{R}^k \rightarrow \mathbb{R}^n : y \mapsto x$ where

$$x^i = \begin{cases} y^s & \text{if } i = i_s \text{ for some } s \in \underline{k} \\ p^i & \text{otherwise.} \end{cases}$$

We restrict ι to the open set $V = \iota^{-1}U$ and note that the point $\tilde{p} = \iota^{-1}(p) = (p^{i_s})_{s \in \underline{k}} \subseteq \mathbb{R}^k$ is contained in V .

The pullback by ι of ω is the continuous k -form $\iota^*\omega = h dy^0 \wedge \dots \wedge dy^{k-1} \in \Omega^k(V)$, where $h = \omega_I \circ \iota \in \mathcal{C}(V, \mathbb{R})$.

Denote $D_{\tilde{p}, \varepsilon} \subseteq \mathbb{R}^k$ be the closed ball of center \tilde{p} and radius $\varepsilon > 0$ in \mathbb{R}^k , and let $|D_{\tilde{p}, \varepsilon}|$ be the volume of this ball. We take ε small enough so that $D_{\tilde{p}, \varepsilon} \subseteq V$. Then

$$\frac{1}{|D_{\tilde{p}, \varepsilon}|} \int_{D_{\tilde{p}, \varepsilon}} \iota^*\omega = \frac{1}{|D_{\tilde{p}, \varepsilon}|} \int_{D_{\tilde{p}, \varepsilon}} h = (\text{average value of } h \text{ on } D_{\tilde{p}, \varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} h(\tilde{p}) = \omega_I(p).$$

This means that we can find out the value of $\omega_I(p)$ if we know the value of the integral $\int_{D_{\tilde{p}, \varepsilon}} \iota_{p,I}^* \omega$ for every $\varepsilon > 0$. Doing this for each increasing k -index $I \in \underline{n}^k$, we find out the value of $\omega|_p$ at the point $p \in U$. Thus any continuous k -form $\omega \in \Omega^k(U)$ is determined by the value of its integrals of the form $\int_{D_{\tilde{p}, \varepsilon}} \iota_{p,I}^* \omega$.

If M is a general \mathcal{C}^1 manifold and $\omega \in \Omega^k(M)$ is a continuous k -form, we use a local parametrization $\varphi : U \rightarrow M$ to get a k -form $\varphi^*\omega \in \Omega^k(U)$, and then as explained above we can determine this k -form if we know the value of the integrals of the kind

$$\int_{D_{\tilde{p}, \varepsilon}} \iota_{p,I}^*(\varphi^*\omega) = \int_{D_{\tilde{p}, \varepsilon}} (\varphi \circ \iota_{p,I})^*\omega$$

But knowing $\varphi^*\omega$ is equivalent to knowing $\omega|_{\varphi(U)}$, thus using different parametrizations (U, φ) we can know ω at all points of M . \square

Exercise 13.3.* Let $f : M \rightarrow N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have

$$f^*(d\omega) = d(f^*\omega).$$

Exercise 13.4.* Let (x, y, z) be the standard coordinates on \mathbb{R}^3 and let (v, w) be the standard coordinates on \mathbb{R}^2 . Let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined as $\phi(x, y, z) = (x + z, xy)$. Let $\alpha = e^w dv + v dw$ and $\beta = v dv \wedge dw$ be 2-forms on \mathbb{R}^2 . Compute the following differential forms:

$$\alpha \wedge \beta, \quad \phi^*(\alpha), \quad \phi^*(\beta), \quad \phi^*(\alpha) \wedge \phi^*(\beta).$$

Solution. (a) Since $\Omega^3(\mathbb{R}^2)$ is just the zero form, whatever α and β are, $\alpha \wedge \beta = 0$

(b) We can either use the definition of $\phi^*(\alpha)$ to compute the components of this 1-form, or the property that, in local coordinates,

$$\phi^*(\alpha_I dy^I) = (\alpha_I \circ \phi) d(y^I \circ \phi)$$

If we want to use the definition, consider that a basis for $\Omega^1(\mathbb{R}^3)$ is given by dx, dy, dz . Following a standard convention, we denote for convenience

$\frac{\partial}{\partial x} = \partial_x$. Thus we have

$$\begin{aligned}\phi^*(\alpha)(\partial_x) &= \alpha(d\phi(\partial_x)) = \alpha(\partial_v + y\partial_w) = e^w + yv = e^{xy} + (x+z)y \\ \phi^*(\alpha)(\partial_y) &= \alpha(d\phi(\partial_y)) = \alpha(x\partial_w) = xv = (x+z)x \\ \phi^*(\alpha)(\partial_z) &= \alpha(d\phi(\partial_z)) = \alpha(\partial_v) = e^w = e^{xy}\end{aligned}$$

Thus we obtain

$$\phi^*\alpha = e^{xy} + (x+z)y dx + (x+z)x dy + e^{xy} dz$$

- (c) To compute $\phi^*\beta$ we use the other (perhaps more direct) method. It uses the following important property of the pull-back:

$$\phi^*(f(x) dx^1 \wedge \cdots \wedge dx^k) = (f \circ \phi) d(x^1 \circ \phi) \wedge \cdots \wedge d(x^k \circ \phi)$$

Thus we have

$$\begin{aligned}\phi^*(\beta) &= \phi^*(v dv \wedge dw) \\ &= v(\phi) d(v(\phi)) \wedge d(w(\phi)) \\ &= (x+z)(dx + dz) \wedge (y dx + x dy) \\ &= (x+z)(x dx \wedge dy - y dx \wedge dz - x dy \wedge dz)\end{aligned}$$

- (d) The property $\phi^*(\alpha) \wedge \phi^*(\beta) = \phi^*(\alpha \wedge \beta)$ implies that this is a null form. □

Exercise 13.5.* Compute the exterior derivative of the following forms:

- (a) on $\mathbb{R}^2 \setminus \{0\}$ $\theta = \frac{x dy - y dx}{x^2 + y^2}$.

Solution. Applying the definition of exterior derivative yields

$$d\theta = d\left(\frac{x}{x^2 + y^2}\right) \wedge dy + d\left(\frac{-y}{x^2 + y^2}\right) \wedge dx = \frac{-x^2 + y^2}{(x^2 + y^2)^2} dx \wedge dy - \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \wedge dx = 0$$

□

- (b) on \mathbb{R}^3 , $\varphi = \cos(x) dy \wedge dz$.

Solution. $d\varphi = d(\cos(x)) \wedge dy \wedge dz = -\sin(x) dx \wedge dy \wedge dz$. □

- (c) on \mathbb{R}^3 $\omega = A dx + B dy + C dz$.

Solution. Here consider A, B, C as smooth functions on \mathbb{R}^3 , by definition we have

$$\begin{aligned}d\omega &= dA \wedge dx + dB \wedge dy + dC \wedge dz \\ &= \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz\right) \wedge dx + \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy + \frac{\partial B}{\partial z} dz\right) \wedge dy \\ &\quad + \left(\frac{\partial C}{\partial x} dx + \frac{\partial C}{\partial y} dy + \frac{\partial C}{\partial z} dz\right) \wedge dz \\ &= -\frac{\partial A}{\partial y} dx \wedge dy + \frac{\partial A}{\partial z} dz \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy - \frac{\partial B}{\partial z} dy \wedge dz - \frac{\partial C}{\partial x} dz \wedge dx + \frac{\partial C}{\partial y} dy \wedge dz \\ &= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy - \left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx\end{aligned}$$

Remark: Notice the resemblance of $d\omega$ with the curl of a vector field in \mathbb{R}^3 . Recall that if $F = (A, B, C)$ is a vector field in \mathbb{R}^3 then

$$\text{curl} F = (C_y - B_z, A_z - C_x, B_x - A_y)$$

One can identify 2-forms with vector fields, by sending $a dx \wedge dy + b dy \wedge dz + c dz \wedge dx$ to the vector field (b, c, a) . □

Exercise 13.6.* Deduce the following classical theorems from Stokes' theorem.

- (a) **Green's theorem.** Let $D \subseteq \mathbb{R}^2$ be a smooth 2-dimensional compact embedded submanifold with boundary in \mathbb{R}^2 . Then for any differentiable 1-form $\omega = P dx + Q dy$ defined on an open neighborhood of D we have

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} P dx + Q dy.$$

Solution. The differential of ω is

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy \\ &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy. \end{aligned}$$

Hence this is a particular case of the Stokes' theorem. \square

- (b) **Divergence theorem.** Let $A \subset \mathbb{R}^3$ be a 3-dimensional compact embedded submanifold with boundary in \mathbb{R}^3 . Then for any smooth vector field $F : A \rightarrow \mathbb{R}^3$ we have

$$\int_A \operatorname{div} F dV = \int_{\partial A} F \cdot dS$$

where $dV := dx \wedge dy \wedge dz$ is the standard 3-form on \mathbb{R}^3 and on the right hand side we have the formal inner product with $dS = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$.

Solution. The expression $F \cdot dS$ is to be understood as in the lecture notes as the 2-form $\omega \in \Omega^2(\mathbb{R}^3)$ given as the formal inner product between F and $dS = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$, i.e.

$$\omega = F \cdot dS = F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy.$$

A direct computation then shows that

$$d\omega = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz = \operatorname{div}(F) dV.$$

Hence by Stokes Theorem:

$$\int_A \operatorname{div}(F) dV = \int_A d\omega = \int_{\partial A} \omega = \int_{\partial A} F \cdot dS$$

where of course all the forms in the integrals are understood to be restricted to A resp. ∂A . \square