Introduction to Differentiable Manifolds

modified by Marcos Cossarini
original notes by Anna Kiesenhofer

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Contents

Main references

[Lee10] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2010.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second edition. Springer, 2013.

Both books have online versions available in the library.

0.1 Practical remarks about the course

0.1.1 Planning 2021: 14 classes

- 1. (21.09) Topological and Differentiable manifolds
- 2. (28.09) Differentiable maps. Partitions of unity.
- 3. (05.10) Tangent bundle. Differential of a map.
- 4. (12.10) Vector bundles
- 5. (19.10) Vector fields and flows
- 6. (26.10) Submanifolds
- 7. (02.11) Whitney embedding. Regular levels sets are submanifolds.
- 8. (09.11) Differential forms
- 9. (16.11) Differential forms
- 10.(23.11) Exterior derivative
- 11.(30.11) Integration, orientation
- 12.(07.12) Manifolds with boundary
- 13.(14.12) Stokes' theorem
- 14.(21.12) Not assigned

0.1.2 Content for the exam

For the exam, studying the content labelled as a comment, side-note, or footnote, or whose title is marked by an asterisk, is not mandatory by itself. You should read it if you find it interesting or helpful for undertanding the rest.

1 Manifolds

1.1 Topological manifolds

[Lee13], Chapter 1 and [Lee10], Chapter 2

We want to study spaces that "locally look like" Euclidean space $\mathbb{R}^n$.

Definition 1.1.1. *Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is* ***locally Euclidean*** *of dimension n if every point $p \in M$ has an open neighborhood that is homeomorphic*[1] *to an open subset of $\mathbb{R}^n$.*

Remark 1.1.2.
- In the case $n = 0$ this definition means that every point of M has a neighborhood homeomorphic to $\mathbb{R}^0 = \{0\}$, i.e. a point. In other words, M is a discrete topological space.
- We could have replaced "...homeomorphic to an open subset of $\mathbb{R}^n$" in the definition above by "...homeomorphic to $\mathbb{R}^n$". (Exercise.)
- If M is locally Euclidean of dimension m, then it cannot be locally Euclidean of another dimension $n \neq m$. This is a consequence of Brouwer's *invariance of domain* theorem (given without proof): If two open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Of course any open subset of $\mathbb{R}^n$ is locally Euclidean of dimension n. A more interesting example is the following.

Example 1.1.3. The circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (with the subspace topology) is locally Euclidean of dimension 1: let $(x_0, y_0) \in \mathbb{S}^1$, wlog $y_0 > 0$, then $U := (\mathbb{R} \times \mathbb{R}^+) \cap \mathbb{S}^1$ is an open subset of $\mathbb{S}^1$ containing (x_0, y_0) and homeomorphic to $(-1, 1)$ via the map $U \to (-1, 1)$ that sends $(x, y) \mapsto x$. We will see more examples later on (see e.g. Examples 1.1.8 below).

For the definition of *topological manifold* we demand some further topological properties that ensure that the space is topologically "well-behaved". (For instance, we want the limit of every sequence to be unique.)

Definition 1.1.4. *A* ***topological manifold of dimension*** *n, or* ***topological n-manifold****, is a topological space M that is locally Euclidean of dimension n, Hausdorff*[2] *and second countable.*[3]

A ***topological manifold*** *is a topological space that is a topological n-manifold for some n.*

Side note: Make sure you are familiar with some basic definitions from topology such as Hausdorff, second countable, connected and compact spaces, and the construction of subspace, product, coproduct and quotient topologies. Chapters 2 and 3 in [Lee10] provides a succinct overview of everything we need.

Remark 1.1.5. The conditions of Hausdorff resp. second countable in Definition 1.1.4 are not redundant. For example, the *line with two origins* (see Exercises) is a locally Euclidean, second countable space that is not Hausdorff. The *long line* and the *Prüfer surface* (see Wikipedia if interested) are locally Euclidean of dimension 1 and 2 respectively, Hausdorff and connected, but not second countable.

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[1] Recall that two topological spaces X, Y are **homeomorphic** if there exists a bijection $\varphi : X \to Y$ such that both φ and its inverse are continuous.

[2] Recall that a topological space X is **Hausdorff** if every two different points $x, y \in X$ have disjoint neighborhoods.

[3] Recall that a topological space X is **second countable** if its topology admits a countable basis. A **basis** for a topology is a family $\mathcal{B}$ of open sets such that every open set is a union of some sets of $\mathcal{B}$.

The homeomorphisms that locally identify a topological manifold with Euclidean space are called *charts*:

Definition 1.1.6 (Coordinate charts). *Let M be a topological n-manifold. A* ***coordinate chart*** *(or* ***chart*** *for short) for M is a homeomorphism $\varphi : U \to V$, where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets. Its inverse φ^{-1} is a* ***local parametrization*** *of M. An* ***atlas*** *for M is a collection of charts whose domains cover M.*

Thus a way of showing that a space is locally Euclidean is by exhibiting an atlas.

Convention: When talking about subsets (resp. quotients, products, disjoints unions) of topological spaces we'll assume that they are endowed with the subspace (resp. quotient, product, coproduct) topology unless otherwise stated.

Using this convention, let us mention some easy ways to construct new topological manifolds from old ones.

Proposition 1.1.7 (New manifolds from old). *The properties of being Hausdorff or second countable are preserved by taking subspaces, finite products and countable coproducts. In consequence:*

- *An open subset of a topological n-manifold is a topological n-manifold.*
- *A disjoint union $M = \coprod_i M_i$ of countably many topological n-manifolds M_i is a topological n-manifold.*
- *A product $M = \prod_i M_i$ of finitely many topological manifolds M_i is a topological manifold of dimension $\dim(M) = \sum_i \dim(M_i)$.*

Proof. Exercise. □

Example 1.1.8 (Examples of topological manifolds).

(a) Of course any open subset of $\mathbb{R}^n$ is a topological manifold.

(b) The graph Γ_f of a continuous function $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set,

$$\Gamma_f := \{(x, f(x)) \mid x \in U\}$$

is a topological n-manifold. Being a subset of $\mathbb{R}^m \times \mathbb{R}^n$, by Proposition 1.1.7 it is Hausdorff and second countable. It is globally homeomorphic to U via the projection

$$\Gamma_f \to U : (x, f(x)) \mapsto x.$$

(c) The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a topological manifold: being a subset of $\mathbb{R}^{n+1}$ it is Hausdorff and second countable; a possible choice of atlas is given by the so-called **graph coordinates**: Cover $\mathbb{S}^n$ by the $2(n+1)$ open sets $U_i^{\pm} := \{x \in \mathbb{R}^n \mid \pm x_i > 0\}$, then $\mathbb{S}^n \cap U_i^{\pm}$ is homemorphic to the open unit n-ball $\mathbb{B}^n$ via the projection[4]

$$\begin{array}{cccc} \varphi_i^{\pm} : & \mathbb{S}^n \cap U_i^{\pm} & \to & \mathbb{B}^n \\ & (x_0, \ldots, x_n) & \to & (x_0, \ldots, \hat{x}_i, \ldots, x_n). \end{array}$$

The maps $\varphi_i^{\pm}$ are coordinate charts for $\mathbb{S}^n$; we call them *graph coordinates*. Locally this is a special case of the previous item (b): each set $\mathbb{S}^n \cap U_i^{\pm}$ is (up to permutation of coordinates) the graph of the continuous function on the unit n-ball $\mathbb{B}^n(0)$:

$$\mathbb{B}^n \to \mathbb{R} : y \mapsto \pm\sqrt{1 - \textstyle\sum_i y_i^2}.$$

(d) Real projective space $\mathbb{P}^n$ is a topological n-manifold (exercise).

(e) The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a topological n-manifold (exercise).

[4]The hat on $\hat{x}_i$ means that we omit the respective coordinate x_i.

(f) More generally, if M is a topological manifold and G is a group of homeomomorphisms of M that acts properly discontinuously and without fixed points, then the quotient M/Γ is a topological n-manifold.

(g) If M is a topological n-manifold and $\pi : N \to M$ is a covering map (where N is connected), then N is a topological n-manifold. In particular, the universal covering space of any connected topological manifold is a topological manifold.

We will not prove the following result (although it can be done elementarily).

Theorem 1.1.9 (Classification of topological 1-manifolds)**.** *Every connected topological 1-manifold is homeomorphic to either $\mathbb{S}^1$ (if it is compact) or to $\mathbb{R}$ (if it is not compact).*

1.2 Differentiable manifolds

Our next goal is to define a kind of spaces called *differentiable manifolds* and a kind of maps called *differentiable maps* (or, more precisely $\mathcal{C}^k$ manifolds and maps) so that we can actually do differential calculus with them. These spaces should be locally equivalent to Euclidean open sets (where we already have a notion of differentiable or $\mathcal{C}^k$ maps; see next definition), but at the global scale they should be allowed to have a more interesting topology, as topological manifolds often do. In particular, the topological manifolds discussed above, such as the sphere, torus, projective space, etc. should become differentiable manifolds.

Before continuing, let us set up some terminology for $\mathcal{C}^k$ maps in $\mathbb{R}^n$.

Definition 1.2.1 (Euclidean open sets and Euclidean $\mathcal{C}^k$ maps)**.** *A **Euclidean open set** is an open subset of some Euclidean space $\mathbb{R}^n$.*
*Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be Euclidean open sets and $k \in \{0, 1, \ldots, \infty\}$. We say that a function $f : U \to V$ is $\mathcal{C}^k$ at a point $p \in U$ if its partial derivatives of order $\leq k$ are defined in a neighborhood of p and continuous at p. We say that f is $\mathcal{C}^k$ (and we call it an **Euclidean $\mathcal{C}^k$ map**) if it is $\mathcal{C}^k$ at all points $p \in U$.*
*A **smooth map** is a $\mathcal{C}^\infty$ map.*

Remark 1.2.2. Every Euclidean $\mathcal{C}^k$ map is continuous because the function f itself is a partial derivative (of order 0) of f. In fact, a $\mathcal{C}^0$ map is the same thing as a continuous map.

Remark 1.2.3. The composite $g \circ f$ of two Euclidean $\mathcal{C}^k$ maps $f : U \to V$, $g : V \to W$ is a $\mathcal{C}^k$ map. Also, the identity map of any Euclidean open set is $\mathcal{C}^k$ for all k.

The key to make a topological manifold into a differentiable manifold is to choose an appropriate atlas.

Definition 1.2.4 (Differentiable manifolds)**.** *Let M be a topological n-manifold and $k = 0, \ldots, \infty$. Two charts φ, ψ for M, with respective domains $U, V \subseteq M$, are $\mathcal{C}^k$ **compatible** if the homeomorphism*

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

*called the **transition map** from φ to ψ, is a $\mathcal{C}^k$ isomorphism (i.e. itself and its inverse are both Euclidean $\mathcal{C}^k$ maps).*

*A $\mathcal{C}^k$ **atlas** is an atlas for M whose charts are $\mathcal{C}^k$ compatible with each other. Two $\mathcal{C}^k$ atlases for M are $\mathcal{C}^k$ **equivalent** if their union is a $\mathcal{C}^k$ atlas. A $\mathcal{C}^k$ **structure** on M is a $\mathcal{C}^k$ atlas that is maximal, i.e., not contained in any other strictly larger $\mathcal{C}^k$ atlas for M. A $\mathcal{C}^k$ **manifold** is a topological manifold M endowed with a $\mathcal{C}^k$ structure. A **differentiable manifold** is a $\mathcal{C}^k$ manifold with $k \geq 1$, and a **smooth manifold** is a $\mathcal{C}^\infty$ manifold.*

Remark 1.2.5. To be precise, the transition map that we wrote as $\psi \circ \varphi^{-1}$ should actually be defined as $\psi|_{U\cap V}^{\psi(U\cap V)} \circ \left(\varphi|_{U\cap V}^{\varphi(U\cap V)}\right)^{-1}$, using the *restricted* charts

$$\varphi|_{U\cap V}^{\varphi(U\cap V)} : U \cap V \to \varphi(U \cap V), \quad \psi|_{U\cap V}^{\psi(U\cap V)} : U \cap V \to \psi(U \cap V)$$

We will often not write this more precise expression because it is cumbersome. In general, when we produce a map by composition, it should be understood that the map is defined in principle at all points where it is possible (maybe no points at all!), and we may further restrict the map by specifying a reduced domain and/or codomain.

The next proposition shows that it suffices to specify any $\mathcal{C}^k$ atlas (not necessarily a maximal one) to determine a smooth structure.

Proposition 1.2.6. *For a fixed topological manifold M, each $\mathcal{C}^k$ atlas $\mathcal{A}$ is contained in a unique maximal $\mathcal{C}^k$ atlas $\overline{\mathcal{A}}$. Any other $\mathcal{C}^k$ atlas $\mathcal{B}$ for M is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$.*

Proof. Let $\mathcal{A}$ be any $\mathcal{C}^k$ atlas for M. Define

$$\overline{\mathcal{A}} := \{\varphi \text{ chart for } M \text{ that is } \mathcal{C}^k \text{ compatible with all charts in } \mathcal{A}\}.$$

Clearly $\overline{\mathcal{A}}$ contains $\mathcal{A}$. Moreover, $\overline{\mathcal{A}}$ is a $\mathcal{C}^k$ atlas: if $\varphi, \psi \in \overline{\mathcal{A}}$, denoting $U, V \subseteq M$ their respective domains, we have to show that $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is $\mathcal{C}^k$. Take any point $\psi(p) \in \psi(U \cap V)$ and let $\theta \in \mathcal{A}$ be a chart whose domain W contains the point $p \in U \cap V$. Then $\psi(U \cap V \cap W)$ is an open neighborhood of p and we can write the restriction

$$\varphi \circ \psi^{-1} : \psi(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$

as the composition $(\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, which is $\mathcal{C}^k$ because $\varphi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are $\mathcal{C}^k$ by assumption.

Finally, from the definition of $\overline{\mathcal{A}}$ it is clear that it is maximal, and that any atlas $\mathcal{B}$ is equivalent to $\mathcal{A}$ if and only if it is contained in $\mathcal{A}$. In particular, any atlas $\mathcal{B}$ containing $\mathcal{A}$ is equivalent to $\mathcal{A}$ (by definition of equivalent atlases), and therefore is contained in $\overline{\mathcal{A}}$. Therefore a maximal atlas containing $\mathcal{A}$ is contained in $\overline{\mathcal{A}}$, but in fact it must be equal to $\overline{\mathcal{A}}$ (by maximality). We conclude that $\overline{\mathcal{A}}$ is the unique maximal $\mathcal{C}^k$ atlas containing $\mathcal{A}$. □

In consequence, given a topological manifold M and some smooth atlas $\mathcal{A}$ on M we can speak without ambiguity of *the* smooth structure determined by $\mathcal{A}$ and denote it $\overline{\mathcal{A}}$.

Remark 1.2.7. For practical purposes the concept of a maximal $\mathcal{C}^k$ atlas is not really important. We usually work with a smaller atlas and this is all we need e.g. for checking that a function is $\mathcal{C}^k$ (see next section). In fact, we could have defined a $\mathcal{C}^k$ structure on M as an equivalence class of $\mathcal{C}^k$ atlases, rather than as a maximal $\mathcal{C}^k$ atlas.

Remark 1.2.8. Any open subset of a $\mathcal{C}^k$ manifold is a $\mathcal{C}^k$ manifold in a natural way: if M is a manifold with a $\mathcal{C}^k$ atlas $\mathcal{A}$, and $U \subset M$ is open, then the set of charts

$$\mathcal{A}|_U := \{\varphi|_{U\cap \mathrm{Dom}(\varphi)} : \varphi \in \mathcal{A}\}$$

is a $\mathcal{C}^k$ atlas for U. (Exercise.) We will also see in the exercises that finite products of $\mathcal{C}^k$ manifolds have a natural $\mathcal{C}^k$ structure.

Example 1.2.9 (Examples of smooth manifolds)**.**

- $\mathbb{R}^n$ (with the atlas consisting of the single chart $\mathrm{id}_{\mathbb{R}^n}$) is a smooth manifold. In general, any topological manifold endowed with a single-chart atlas is automatically a smooth manifold. For example, the graph Γ_f of any *continuous* (sic) function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ as described in Example 1.1.8, endowed with the projection chart, is a smooth manifold.

- The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Indeed, the atlas given by the graph coordinates (Example 1.1.8) is smooth. To see this, we compute the transition functions (wlog $i < j$):

$$\begin{aligned} \varphi_i^+ \circ (\varphi_j^\pm)^{-1} :& \varphi_j^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \to \varphi_i^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \\ & (y_0, \dots, y_{n-1}) \mapsto (y_0, \dots, \hat{y}_i, \dots, y_{j-1}, \pm\sqrt{1 - \textstyle\sum_i (y_i)^2}, y_j, \dots, y_{n-1}) \end{aligned}$$

  and a similar formula works if we replace φ_i^+ by φ_i^-. Hence all the transition maps are smooth.
- More generally, any subset M of $\mathbb{R}^k$ given as the *regular level set* of a smooth map $F : \mathbb{R}^k \to \mathbb{R}^\ell$ is a $k - \ell$ dimensional smooth manifold "in a natural way[5]". (Being a level set means $M = F^{-1}(\{c\})$ for some $c \in \mathbb{R}^\ell$ and being a regular level set means that, moreover, the Jacobian $DF(p)$ is surjective for all $p \in M$.) You can prove this quite easily using the implicit function theorem and writing M locally as a graph of smooth functions (analogous to the graph coordinates for the sphere). We will show a more general statement later on when discussing embedded submanifolds (Chapter **??**).
- Projective space $\mathbb{P}^n$ is naturally a smooth manifold (see exercises).
- On the topological 1-manifold $M = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$, each of the two projections $\pi_0, \pi_1 : M \to \mathbb{R}$, given by $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$, is a chart defined on the whole manifold M, but these two charts are not $\mathcal{C}^k$ compatible for any $k \geq 1$. Thus they determine two different $\mathcal{C}^k$ structures on M.

[5] In particular, the inclusion map $M \hookrightarrow \mathbb{R}^k$ is smooth.