Introduction to Differentiable Manifolds

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Exercise sheet 1 **2019–09–21**

Convention: We understand a subset of a topological space to be automatically endowed with the subspace topology and a product of topological spaces to be endowed with the product topology (unless stated otherwise).

Exercise 1.1 (Locally Euclidean spaces)**.** Show that the following definition of locally Euclidean spaces is equivalent to the one given in the lecture notes:

Definition. Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is **locally Euclidean** of dimension n if every point $p \in M$ has a neighbourhood that is homeomorphic to $\mathbb{R}^n$.

Exercise 1.2 (Examples of locally Euclidean spaces)**.** Which of the following spaces are locally Euclidean? Which are (globally) homeomorphic to Euclidean space?

- an open ball in $\mathbb{R}^n, n \in \mathbb{N}$
- the closed interval $[0, 1] \subset \mathbb{R}$
- the circle $S^1 \subset \mathbb{R}^2$
- the zero set of the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = xy$
- the "corner" $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, xy = 0\}$.

Exercise 1.3 ("The line with two origins")**.** Let $X := \mathbb{R} \times \{1\} \cup \mathbb{R} \times \{-1\}$ and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ iff $x \in \mathbb{R} \backslash \{0\}$. We endow M with the quotient topology. Show that M is locally Euclidean and second countable, but not Hausdorff.

Exercise 1.4 (New manifolds from old)**.** Convince yourself that[1]:

(a) A subset of a Hausdorff (resp. second countable) topological space is Hausdorff (resp. second countable).
(b) An open subset of a topological n-manifold is a topological n-manifold.
(c) The product of two Hausdorff (resp. second countable) spaces is Hausdorff (resp. second countable).
(d) The product of two topological manifolds is a topological manifold. What is its dimension?

Exercise 1.5 (Projective space)**.** We define $\mathbb{P}^n$ as the quotient space of $\mathbb{R}^{n+1} \backslash \{0\}$ under the equivalence relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R}$. For $x \in \mathbb{R}^{n+1} \backslash \{0\}$ let $[x]$ denote its equivalence class in $\mathbb{P}^n$. We endow $\mathbb{P}^n$ with the quotient topology under the map $\pi : x \mapsto [x]$. Show that $\mathbb{P}^n$ is a topological manifold. *Hint: To show that it is locally Euclidean consider for $i = 0, 1, \dots, n$ the sets*

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\}$$

and the coordinate chart on U_i:

$$\varphi_i : U_i \to \mathbb{R}^n : [(x_0, x_1, \dots, x_n)] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

Exercise 1.6. Show that the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, defined as the quotient of $\mathbb{R}^n$ by the equivalence relation

$$x \sim y \quad \Longleftrightarrow \quad y - x \in \mathbb{Z}^n,$$

is a topological n-manifold.

[1]This means that if you find the exercise trivial you don't have to write down a detailed proof.