A computational analysis of the multidimensional knapsack problem: a backbone

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Abstract

This article contains a backbone for an in proceeding article about the hardness of the Multidimensional Knapsack Problem and the performance of the algorithms existing on literature for its solution.

1 Introduction

The 0-1 Multidimensional Knapsack Problem (MKP) is a generalization of the Knapsack Problem where an item expends more than a single resource type. A MKP having n items and m dimensions can be defined as following:

$$\begin{aligned} \text{maximize} & \sum_{j=1}^{n} p_j x_j \\ \text{subject to} & \sum_{j=1}^{n} c_{ij} x_j \leq b_i \quad i \in \{1, \ldots, m\} \\ & x_j \in \{0, 1\}, \quad j \in \{1, \ldots, n\}. \end{aligned}$$

The problem can be applied on budget planning scenarios, subset project selections, cutting stock problems, task scheduling and allocation of processors and databases in distributed computer programs. The problem is a generalization of the well-known Knapsack Problem (KP) in which $m = 1$. Several algorithms like FPTAS and dynamic programming have been developed for KP with considerable success. However the MKP have shown to be an intractable problem for some relative small instances, especially when m grows. Indeed little is known about what exactly makes a hard instance.

Several contributions have been made proposing exact, heuristic, approximation and probabilistic approaches for the MKP. The purpose of the work is to evaluate the performance the main approaches presented on literature over different instances of the problem identifying what makes an instance hard to solve for the proposed algorithms.

2 The main approaches for solving MKP

Due its simple definition and challenging difficulty, the MKP turned a quite addressed problem for experiments with metaheuristics in recent years, although few of them have exhibit competitive performance compared to comercial MIP solvers (at least for the instances addressed by literature). Among the heuristics reporting competitive performance compared to MIP solvers the newest are those proposed by Fleszar and Hindi [3] and Hanafi and Wilbaut [5].

According to literature MKP does not allow a FPTAS (unless $P = NP$) but a PTAS was proposed by Frieze and Clarke [4] based on the used of the dual simplex algorithm for linear programming. Dyer and Frieze [2] proposed a probabilistic algorithm which, given a $\epsilon > 0$ answers computes the optimal solution for the problem with probability at least $1 - \epsilon$ with polynomial time.

At the present moment the most powerful exact method for solving the MKP seems to be the multi-level search strategy proposed by Boussier [1] which, until now, is the only approach which have found the optimal solution for some popular instances with 500 items and 30 contraints.

3 The instances

The MKP instances from the OR-LIBRARY are the most popular ones and are used for performance measurement of state-of-art algorithms. The generation of thoses instances are generaly guided by two parameters: the profit-weight *correlation* of the items and the *tightness ratio* δ of the knapsack.

The profit-weight correlation $\frac{p_j}{\sum_{i=1}^{m} w_{ij}}$ states how much an item contributes for the objective function relative to its weight. A high variation on the items correlation tends to procude easier instances, once it easier to decide which items are more profitable.

The tightness ratio of a knapsack rules its relative capacity: given a tightness ratio $\delta \in [0, 1]$ the capacity b_i is setted to $b_i = \delta \sum_{i=1}^{n} x_i$. An extremaly high tightness ratio tends to produce easy instances, once we are able to fill the knapsack with most of the items. However extremaly low tightness ratios reduces the possibilities of filling the knapsack. Those facts lead us to expect a *transition phase* around $\delta = 0.5$.

4 Research questions

The main objective of the work is to understand what makes a MKP instance hard to solve. Some questions that may be interesting to answer:

- How the hardness of the problem varies against parameters like tightness ratio and item correlation?
- How the algorithms for solving MKP performes over instances from others NP-Complete problems?

- A MKP instance is hard even with sparce weight matrices?
- For what amount of computational effort the heuristics outperforms exact algorithms?
- Does the PTAS and probabilistic methods (in practice) offer better warranty than the exact method or the later indeed offers a better warranty on quality which is merely theoretically hard to prove?

References

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