

A fast dynamic programming multi-objective knapsack problem

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Abstract

This work addresses... The Multid Objective knapsack programming.
The dynamic programming method... The data structure...

1 Introduction

2 The Multiobjective Knapsack Problem

A general multiobjective optimization problem can be described as a vector function f that maps a tuple of n parameters (decision variables) to a tuple of k objectives. Formally:

$$\begin{aligned} \min/\max \mathbf{y} &= f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_k(\mathbf{x})) \\ \text{subject to } \mathbf{x} &= (x_1, x_2, \dots, x_n) \in X \end{aligned}$$

where \mathbf{x} is the *decision vector*, X denotes the set of feasible solutions, and \mathbf{y} is the *objective vector* where each objective has to be minimized (or maximized).

Considering two decision vectors $\mathbf{a}, \mathbf{b} \in X$, \mathbf{a} is said to *dominate* \mathbf{b} if, and only if \mathbf{a} is at least as good as \mathbf{b} in all objectives and better than \mathbf{b} in at least one objective. For shortening we will say that \mathbf{a} dominates \mathbf{b} by saying $dom(\mathbf{a}, \mathbf{b})$. Formally:

$$dom(\mathbf{a}, \mathbf{b}) = \begin{cases} \forall i \in \{1, 2, \dots, k\} : f_i(\mathbf{a}) \geq f_i(\mathbf{b}) \text{ and} \\ \exists j \in \{1, 2, \dots, k\} : f_j(\mathbf{a}) > f_j(\mathbf{b}) \end{cases}$$

This relation is based on the dominance relation proposed by Weingartner and Ness for the multidimensional knapsack problem [10].

A feasible solution $\mathbf{a} \in X$ is called *efficient* if its not dominated by any other feasible solution. The set of all efficient solutions of a multiobjective optimization problem is known as *Pareto optimal*. Solving a multiobjective problem consists in giving its Pareto optimal set.

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An instance of a multiobjective knapsack problem (MOKP) with k objectives consists of an integer capacity $W > 0$ and n items. Each item i has a positive weight w_i and non negative integer profits $p_i^1, p_i^2, \dots, p_i^k$. Each profit p_i^k represents the contribution of the i -th item for k -th objective. A solution is represented by a set $\mathbf{s} \subseteq \{1, \dots, n\}$ containing the indexes of the itens included in the solution. A solution is feasible if the total weight included in the knapsack does not exceed its capacity. Formally the definition of the problem is:

$$\begin{aligned} \max f(\mathbf{s}) &= (f_1(\mathbf{s}), f_2(\mathbf{s}), \dots, f_k(\mathbf{s})) \\ \text{subject to } w(\mathbf{s}) &< W \\ \mathbf{s} &\subseteq I_n \end{aligned}$$

where

$$\begin{aligned} I_n &= \{1, \dots, n\} \\ f_j(\mathbf{s}) &= \sum_{i \in \mathbf{s}} p_i^j \\ w(\mathbf{s}) &= \sum_{i \in \mathbf{s}} w_i \end{aligned}$$

The MOKP is considered a \mathcal{NP} -Hard problem since it is a generalization of the well-known 0–1 knapsack problem, in which $k = 1$. It is quite difficult to determine the Pareto optimal set for the MOKP, especially for high dimension instances, in which the solution it self (Pareto optimal set) tends to grow exponentially. Even for the bi-objective case, small problems may prove intractable. For this reason we are interested in developing efficient methods for handling large solution sets, which may bring tractability to previously intractable instances.

3 The Dynamic Programing Algorithm

Paragrafo de introducao da secao, justificando toda a explicacao que segue..

We will introduce the dynamic programming (DP) algorithm for the MOKP proposed in [1]. This algorithm is based on the classical dynamic programming algorithm proposed in [6], although with some optimizations.

A solution \mathbf{x} is called *deficient* if it has available space to fit more item, i.e., $W - w(\mathbf{x}) \leq \min\{w_i : i \notin \mathbf{x}\}$.

partial dominance Formally:

$$dom_p(\mathbf{x}, \mathbf{y}) = \begin{cases} dom(\mathbf{x}, \mathbf{y}) & \text{and} \\ w(\mathbf{x}) \leq w(\mathbf{y}) \end{cases}$$

3.1 The basic DP algorithm and the dominance filter

The sequential process used in the algorithm consists of n stages. The Pareto optimal set is obtained in this constructive way and finally returned after n -th stage.

It's pseudocode is presented in Algorithm 1. At the k -th stage the al-

Algorithm 1 Basic dynamic programming algorithm for MOKP

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1: function DP( $\mathbf{p}, \mathbf{w}, W$ )
2:    $S^0 = \{\emptyset\}$ 
3:   for  $k \leftarrow 1, n$  do
4:      $S_*^k = S^{k-1} \cup \{\mathbf{x} \cup k \mid \mathbf{x} \in S^{k-1}\}$ 
5:      $S^k = \{\mathbf{x} \mid \nexists \mathbf{a} \in S_*^k : \text{dom}_p(\mathbf{a}, \mathbf{x})\}$ 
6:   end for
7:    $P = \{\mathbf{x} \mid \nexists \mathbf{a} \in S_*^k : \text{dom}(\mathbf{a}, \mathbf{x})\}$ 
8:   return  $P$ 
9: end function

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gorithm receives a set S^{k-1} of solutions and generates the set S^k of solutions that correspond to subsets containing exclusively the first k items, i.e., $\forall \mathbf{x} \in S^k, \mathbf{x} \subseteq \{1, \dots, k\}$.

This is done by expanding S^{k-1} by adding a copy of each solution with the inclusion of k item (line 4). We will refer as *partial solutions* all the solutions handled by stages prior to n -th stage.

The clever part of the algorithm is that it uses the concept of dominance on partial solutions to filter solutions that will not lead to efficient solutions (line 5). Considering two partial solutions $\mathbf{x}, \mathbf{y} \in S^k$, if \mathbf{x} is dominated by \mathbf{y} then we may discard \mathbf{x} since all solutions generated from \mathbf{x} will not be

3.2 Avoiding deficient solutions

The first optimization that can be made on algorithm is avoiding deficient solution.

3.3 Removing unpromising solutions

3.4 Item order

4 The use of data structure

The k -d tree is a type of binary search tree for indexing multidimensional data with simple construction and low space usage. Despite its simplicity it efficiently supports operations like nearest neighbour search and range search [2]. For those reasons k -d tree is widely used on spacial geometry algorithms [8, 3], clustering [5, 4] and graphic rendering algorithms [7].

Like a standard binary search tree, the k -d tree subdivides data at each recursive level of the tree. Unlike a standard binary tree, that users only one key for all levels of the tree, the k -d tree uses k keys and cycles through these keys for successive levels of the tree.

Concerning it's efficiency, it is important to consider the number of dimensions k -d tree is indexing. As a general rule, a k -d tree is suitable for efficiently

indexing of n elements if n is much greater than 2^k . Otherwise, when k -d tree are used with high-dimensional data, most of the elements in the tree will be evaluated and the efficiency is no better than exhaustive search [9].

Indexing the solutions and range operations.

Tends to increase the feasibility on problems with higher dimensions.

5 Computational experiments

- Base de dados utilizaca
- Parametros dos algoritmos
- Anlise dos resultados (comparao)

6 Conclusions and future remarks

- Concluses dos resultados
- Trabalhos futuros

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