Logistic map

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The dynamics of nuclear decay can be described by the first-order iterative map $N(t + \Delta t) = N(t)(1 - \alpha\Delta t)$. This allows for the calculation of the future value $N(t + 1)$ based on the current value $N(t)$ of the dynamic variable of interest. This map is a map of the form $x(t + 1) = \lambda x(t)$, where we have $\lambda = (1 - \alpha\Delta t)$.

In general, the first-order iterative map is:

$$x(t + 1) = \lambda x(t) (1 - x(t)) . \tag{1}$$

Depending on the value of λ, we have two possible scenarios. For $\lambda < 1$ we have exponential decay, where the future value $x(t+1) < x(t)$. On the other hand, for $\lambda > 1$, we have exponential growth, where the future value $x(t+1) > x(t)$. In the case of $\lambda > 1$, we could represent the growth of a colony of bacteria reproducing on a microscope slide. If we count the number $N(t)$ of bacteria at a given time t, and then count $N(t + \Delta t)$ after a time interval Δt, we will see that the value has increased. In this case, $\Delta N = N(t + \Delta t) - N(t)$ is the number of bacteria that have duplicated during that time interval. Mathematically, we can write:

$$\Delta N = \alpha\Delta t N(t).$$

Where α would be a constant value that depends on the reproductive capacity of bacteria. Therefore, the iterative map for exponential growth is:

$$N(t + \Delta t) = N(t)(1 + \alpha\Delta t)$$

We can also add a term $-\lambda x^2(t)$, which will limit the incessant growth of the map. For example, we know that in the case of bacterial growth, there is a growth limit dictated by the environment. In addition, we will consider $\Delta t = 1$ in the simulations, and therefore $\lambda = 1 - \alpha$. The logistic map will be:

$$x(t + 1) = \lambda x(t)(1 - x(t))$$

For long times ($t \to \infty$), I will call $x(\infty) = x^*$, which is known in the literature as a fixed point, being the value to which the map converges. Substituting $x(\infty) = x^*$ in equation (7), we have two possible solutions for the steady state:

$$x^* = \begin{cases} 0 \\ \frac{\lambda-1}{\lambda} \end{cases} \tag{2}$$

To verify this, I will adopt $\lambda = 1.01$ and $x(0) = 0.0001$, obtaining the graph .

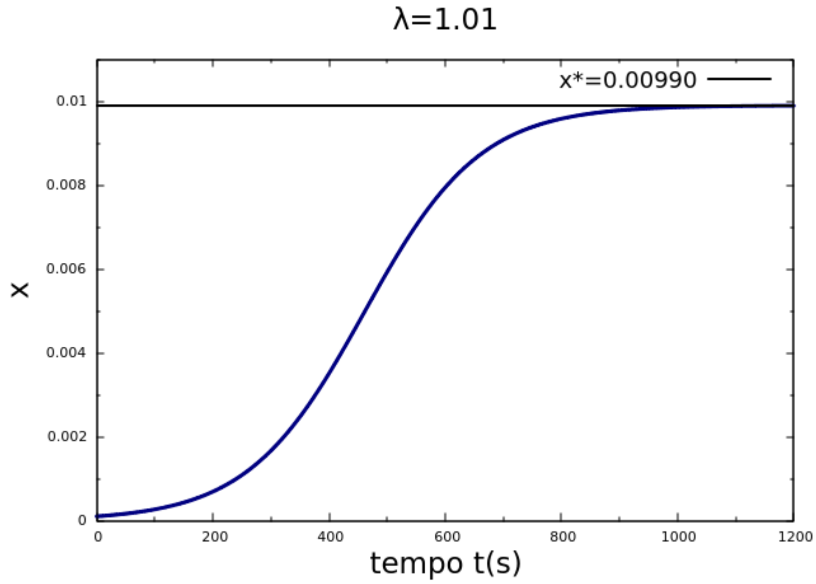

FIG. 1. The graph shows that, as predicted by equation (2), it will converge to a fixed value, which is $x^* = 0.00990$.

Another way to obtain fixed points is through a diagram of $x(t+1)$ by $x(t)$, plotting the points provided by the simulation along with the curves $f_\lambda(x) = \lambda x(1-x)$ and $f(x) = x$.

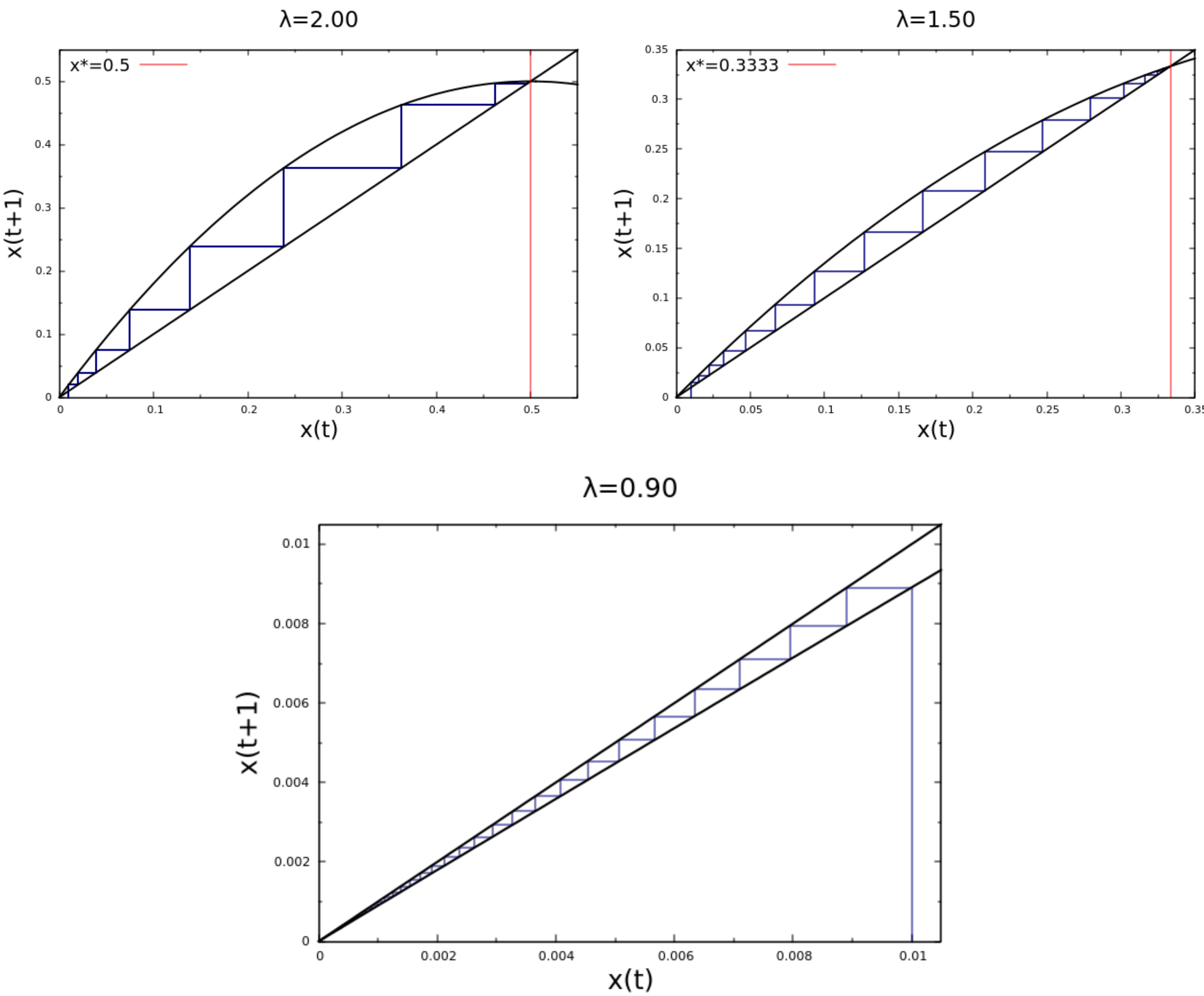

FIG. 2. Diagrams for values $\lambda > 1$ e $\lambda < 1$.

As can be seen in the diagrams (3), the fixed points are in agreement with the analytical solutions. In addition, for $\lambda < 1$, regardless of the initial value $x(0)$ chosen, $x^* = 0$ will always be observed, while for $\lambda > 1$, we have the case where $x^* = \frac{\lambda - 1}{\lambda}$. In particular, with $0 < \lambda < 3$, the fixed points merge with the attractor points. For $\lambda > 3$, the result is invalid, and we must look for another approach, as it is outside the analytical predictions made earlier, resulting in the following diagrams:

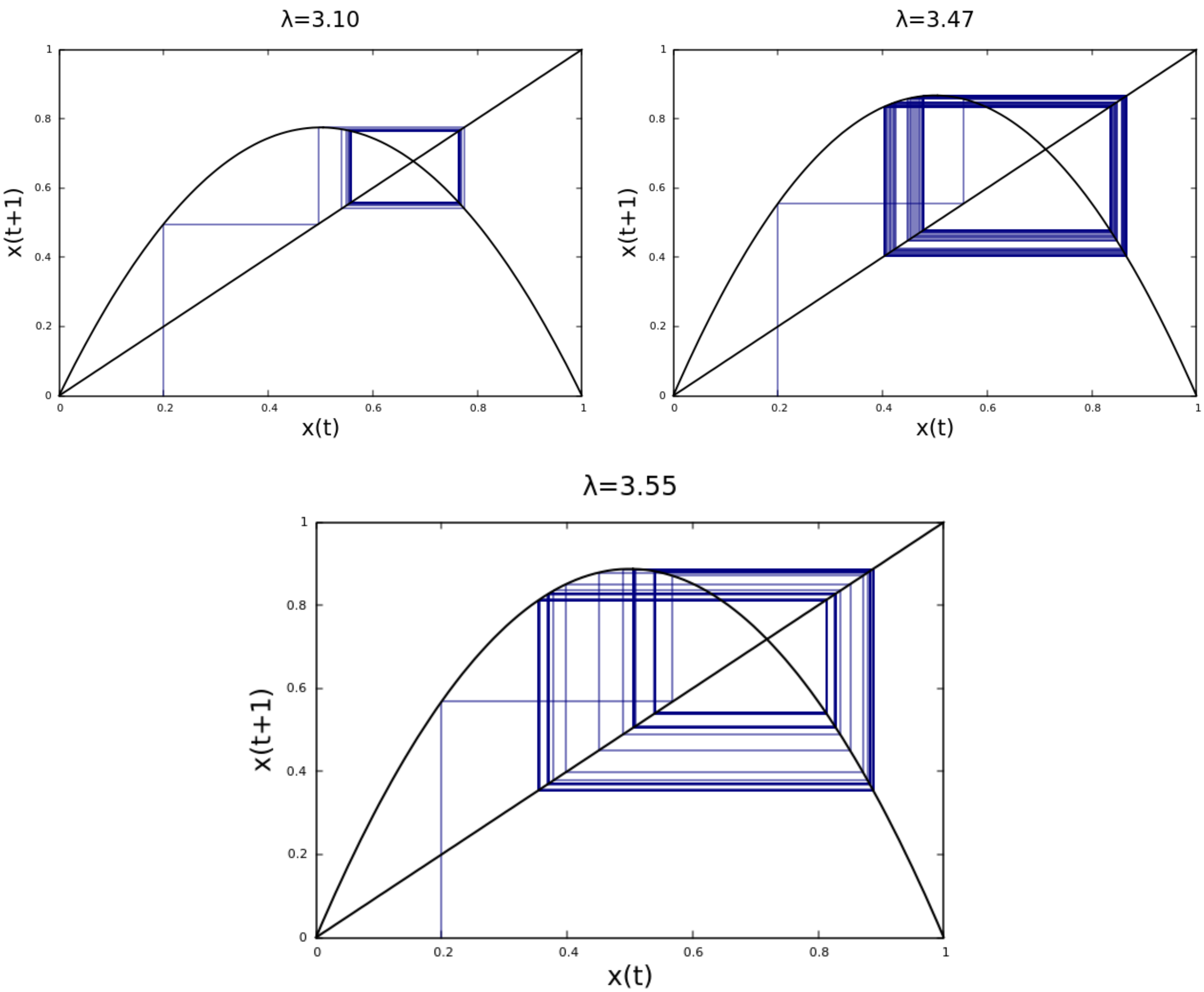

FIG. 3. Diagramas para valores $\lambda > 3$.

In the new method, when the map reaches the supposed steady-state x^*, it will no longer converge to a single value but will oscillate between two or more values periodically, as illustrated by the graphs (4), (5), and (6). Mathematically, the way to identify these attractor points is by substituting the logistic map with a composite map. In the case of 2 attractors, the composite map is:

$$x(t+1) = f_\lambda(f_\lambda(x)) = f_\lambda^{(2)}(x), \tag{3}$$

where

$$f_\lambda(x) = \lambda x(t)\,(1 - x(t)) \tag{4}$$

is the original logistic map. The same can be done in the case where λ has more than two attractors, following the same idea. From simulations, we have for more than two attractors:

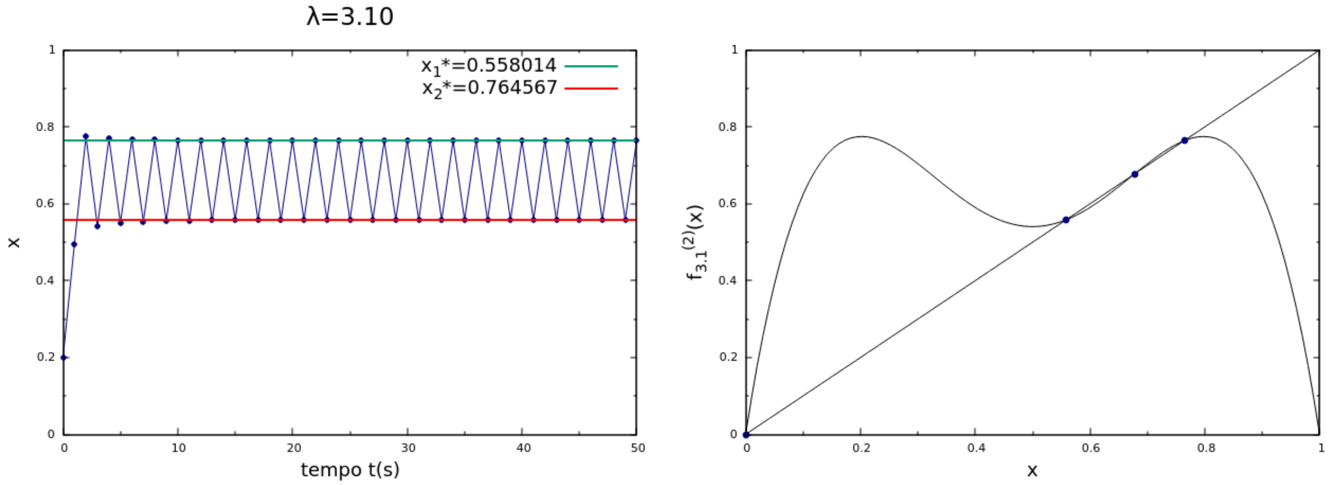

FIG. 4. The left graph shows 2 attractors at x^* and the one on the right confirms this, plotting $f_\lambda^{(2)}(x)$ along with the line x and verifying that the intersection points provide us with the attractors that were observed. In addition to the attractor points, we have an unstable fixed point at $x^* = 1 - 1/\lambda = 0.711815$ and $x^* = 0$.

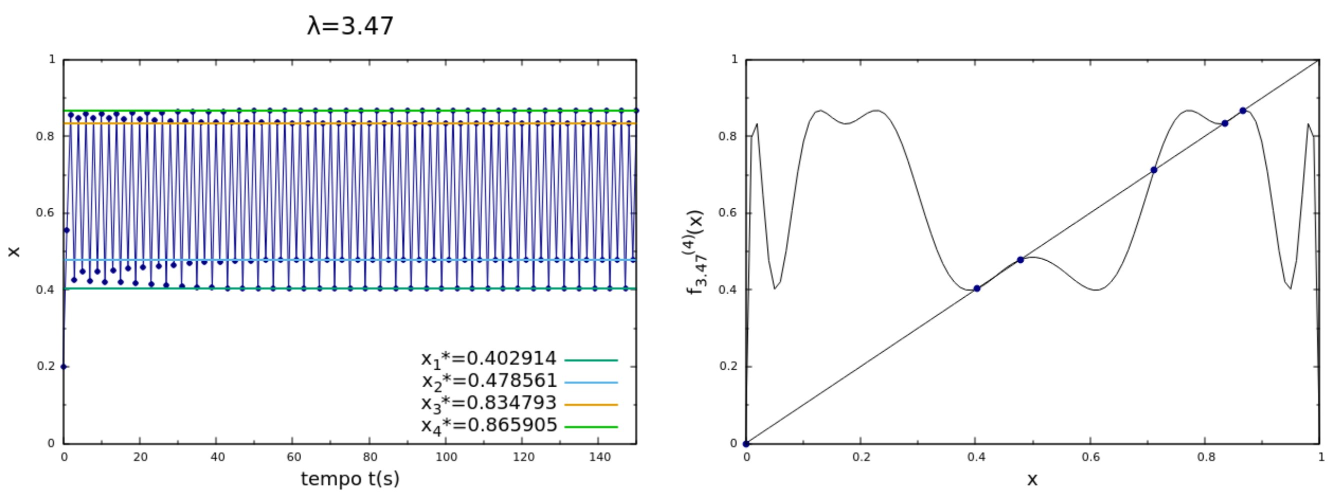

FIG. 5. The left graph shows 4 attractors at x^* and the one on the right confirms this, by plotting $f_\lambda^{(4)}(x)$ together with the line x, and verifying that the intersection points provide the observed attractors. In addition to the attractors, we have an unstable fixed point at $x^= 1 - 1/\lambda = 0.711815$ and $x^* = 0$.

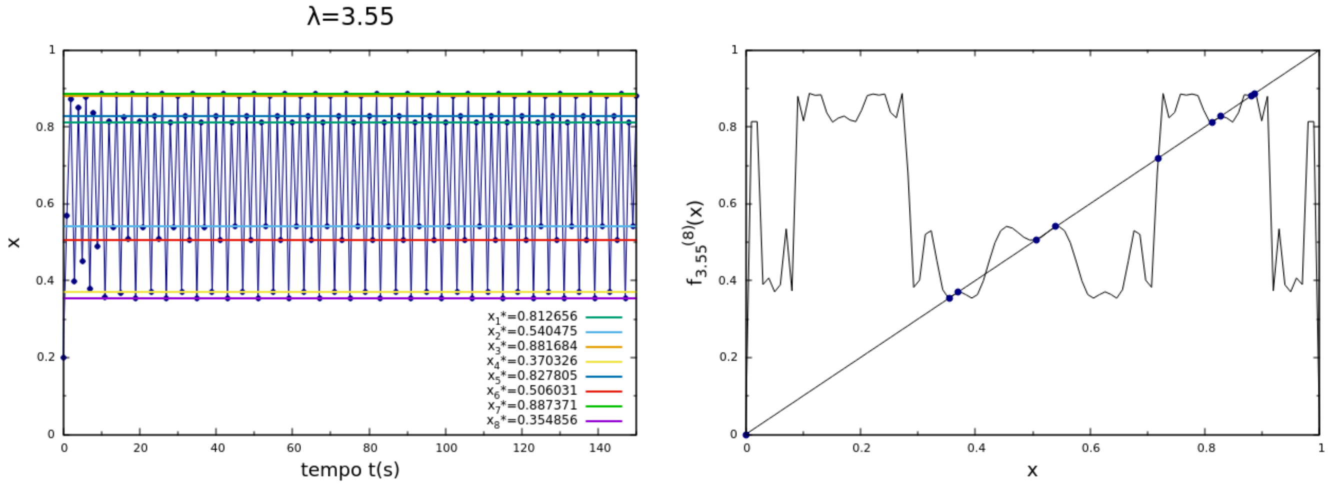

FIG. 6. The left graph shows 4 attractors in x^* and the one on the right confirms this, with $f_\lambda^{(8)}(x)$ plotted alongside the line x, and it is verified that the points of intersection give us the observed attractors. In addition to the attractors, there is an unstable fixed point at $x^* = 1 - 1/\lambda = 0.711815$ and $x^* = 0$..

As λ approaches 1, it takes longer iterations to reach the value $x^* = 0$, until for $\lambda = 1$ the function that describes the behavior of the map will no longer be an exponential decay, but an algebraic decay. This classifies the critical point $\lambda = 1$, as can be seen in the graphs (7).

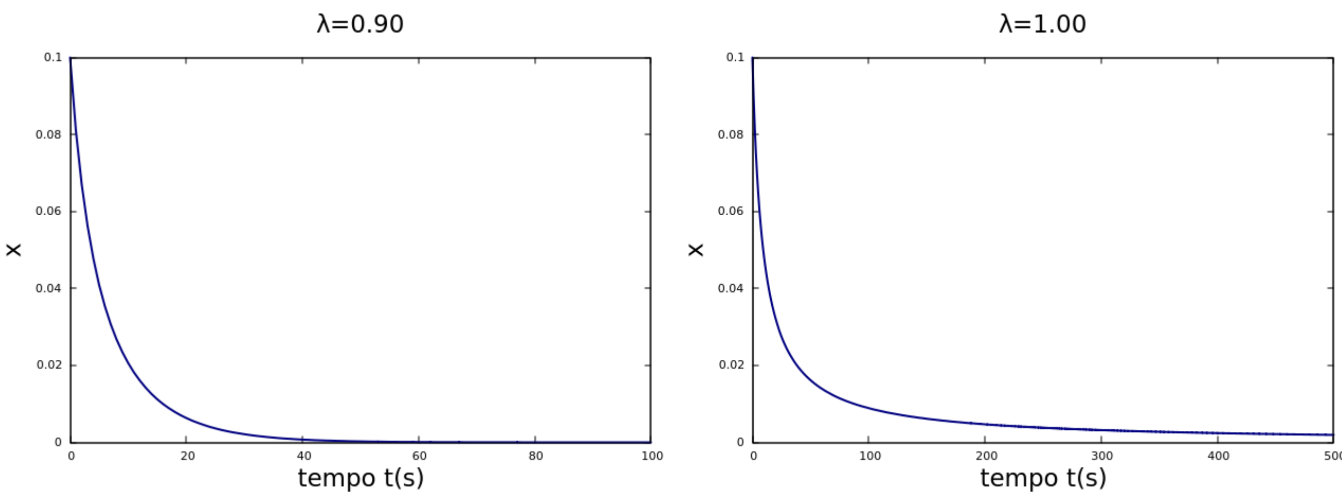

FIG. 7. Gráficos para ilustrar o caso $\lambda < 1$ a esquerda e $\lambda = 1$ a direita.

To confirm the types of curves that describe the cases $\lambda < 1$ and $\lambda = 1$, I will plot the above graphs on a log-log scale and log scale, respectively.

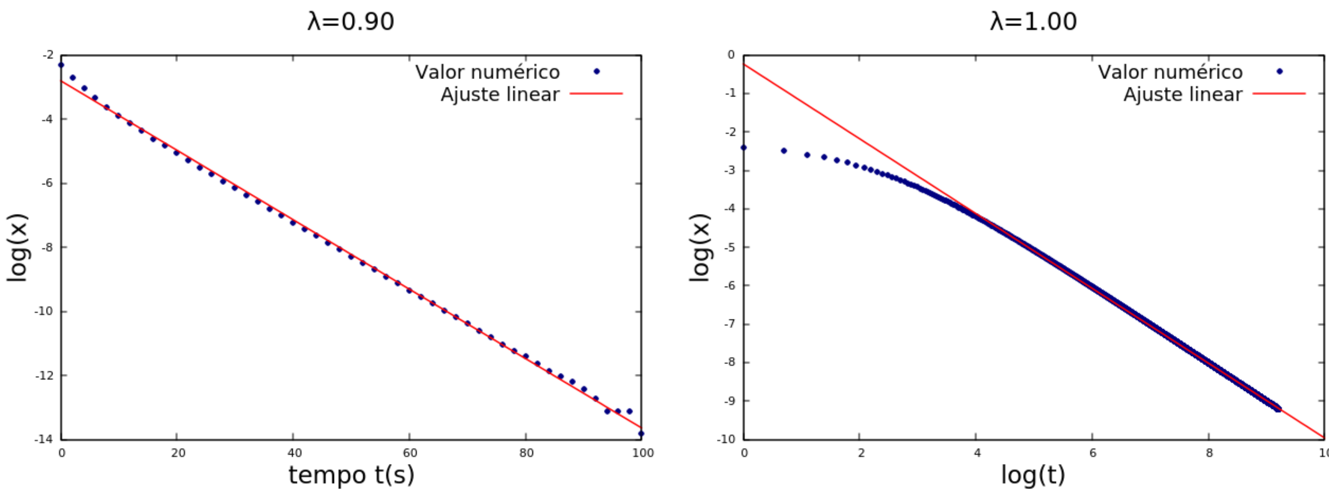

FIG. 8. Gráficos na escala log para $\lambda < 1$ e log por log para $\lambda = 1$ com as devidas retas de ajuste para cada caso.

For $\lambda = 0.9$, performing linear regression on the graph, we see that the fitting line is of the form $\log x = at + b$, with $a = -0.108406$ and $b = -2.80324$. Exponentiating this equation, we obtain the behavior of the curve without the need to solve a differential equation.

$$x \sim e^{-0.11t} \tag{5}$$

Doing the same for $\lambda = 1$ with the fitting line of the form $\log x = a \log t + b$, with $a = -0.987588$ and $b = -0.109602$, we have:

$$x \sim t^{-0.99t} \tag{6}$$

This means that the critical decay follows a power law of the form $t^{-\alpha}$, with $\alpha \approx 1$ being the universality class to which this problem belongs.

I. DIAGRAMA DE BIFURCAÇÕES

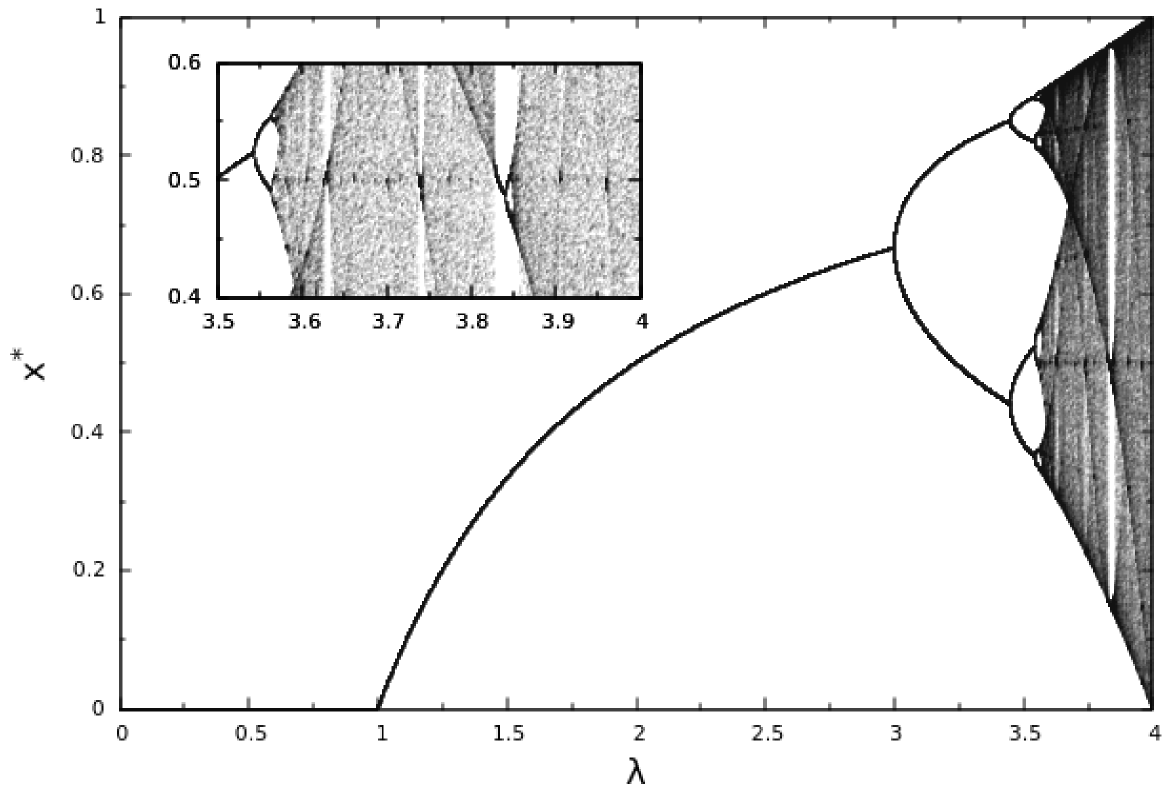

FIG. 9. Diagram of the attractor points x^* as a function of λ, starting with $x(0) = 0.20$. The simulation to obtain this diagram was performed with 1000 iterations for each λ, with increments of 10^{-4} starting from $\lambda = 0$ up to $\lambda = 4$.

Now let's analyze each scenario of this diagram. In the region for $0 < \lambda < 1$, the fixed point is $x^* = 0$, configuring an absorbing state of the system. At $\lambda = 1$, we have the critical point that separates two different phases: the phase where we have only the absorbing state ($x^* = 0$) and the phase where we have an active state ($x^* > 0$). For $1 < \lambda < 3$, the fixed point takes values of the form $x^* = 1 - \frac{1}{\lambda}$. For $\lambda > 3$, no fixed point is stable and can assume two or more values. Finally, for $3.5699 < \lambda < 4$, we have the so-called chaos, with the zoom window showing periodic behavior within the chaos.