Injeção de Água quente

Relembrando as equações:

$$\frac{\partial S_1}{\partial t} + \frac{u_t}{\phi}\frac{\partial f_1(S_1,T)}{\partial x} = 0 \quad (1)$$

$$\left[\phi\left(M_{t_1}S_1 + M_{t_2}S_2\right) + (1-\phi)M_{t_s}\right]\frac{\partial T}{\partial t} + \left[u_t(M_{t_1}f_1 + M_{t_2}f_2)\right]\frac{\partial T}{\partial x} = 0 \quad (2)$$

Pela Regra da Cadeia, a Equação 1 pode ser reescrita como:

$$\frac{\partial S_1}{\partial t} + \frac{u_t}{\phi}\left[\frac{\partial f_1}{\partial S_1}\frac{\partial S_1}{\partial x} + \frac{\partial f_1}{\partial T}\frac{\partial T}{\partial x}\right] = 0 \quad (3)$$

As Equações 2 e 3 podem ser representadas na forma matricial:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\partial S_1}{\partial t} \\ \frac{\partial T}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T} \\ 0 & \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1 + M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1 + M_{t_2}S_2\right) + \frac{(1-\phi)}{\phi}M_{t_s}\right]} \end{bmatrix}\begin{bmatrix} \frac{\partial S_1}{\partial x} \\ \frac{\partial T}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4)$$

- **Autovalores (velocidades características):**

A partir da correlação de autovalores e autovetores: $(A - \lambda I)\overrightarrow{r} = 0$:

$$\begin{bmatrix} \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T} \\ 0 & \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1 + M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1 + M_{t_2}S_2\right) + \frac{(1-\phi)}{\phi}M_{t_s}\right]} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} - \lambda & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T} \\ 0 & \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1 + M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1 + M_{t_2}S_2\right) + \frac{(1-\phi)}{\phi}M_{t_s}\right]} - \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

Como:

$$det(A - \lambda I) = 0 \quad (7)$$

então:

$$\left(\frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} - \lambda\right)\left(\frac{u_t}{\phi}\frac{(M_{t_1}f_1 + M_{t_2}f_2)}{\left[(M_{t_1}S_1 + M_{t_2}S_2) + \frac{(1-\phi)}{\phi}M_{t_s}\right]} - \lambda\right) = 0 \quad (8)$$

Assim, os dois autovalores(λ_1eλ_2) serão:

$$\lambda_1 = \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} \tag{9}$$

$$\lambda_2 = \frac{u_t}{\phi}\frac{(M_{t_1}f_1 + M_{t_2}f_2)}{\left[(M_{t_1}S_1 + M_{t_2}S_2) + \frac{(1-\phi)}{\phi}M_{t_s}\right]} \tag{10}$$

- **Autovetores**

Para encontrar os autovetores, substituiremos os autovalores encontrados nas Equações 9 e 10:

- **Para $\boldsymbol{\lambda_1 = \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1}}$:**

$$\begin{bmatrix} \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} - \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T} \\ 0 & \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]} - \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} \end{bmatrix} \begin{bmatrix} r_1^{(1)} \\ r_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\begin{cases} 0r_1^{(1)} + & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T}r_1^{(2)} = 0 \\ 0r_1^{(1)} + & \frac{u_t}{\phi}\left(\frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]} - \frac{\partial f_1}{\partial S_1}\right)r_1^{(2)} = 0 \end{cases} \tag{12}$$

$$\overrightarrow{r_1} = \begin{bmatrix} r_1^{(1)} \\ r_1^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{13}$$

- **Para $\boldsymbol{\lambda_2 = \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]}}$:**

$$\begin{bmatrix} \frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1} - \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]} & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T} \\ 0 & \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]} - \frac{u_t}{\phi}\frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]} \end{bmatrix} \begin{bmatrix} r_2^{(1)} \\ r_2^{(2)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} \frac{u_t}{\phi}\left(\frac{\partial f_1}{\partial S_1} - \frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]}\right)r_2^{(1)} + & \frac{u_t}{\phi}\frac{\partial f_1}{\partial T}r_2^{(2)} = 0 \\ 0r_2^{(1)} & +0r_2^{(2)} = 0 \end{cases} \tag{14}$$

$$\begin{cases} \left(\frac{\partial f_1}{\partial S_1} - \frac{\left(M_{t_1}f_1+M_{t_2}f_2\right)}{\left[\left(M_{t_1}S_1+M_{t_2}S_2\right)+\frac{(1-\phi)}{\phi}M_{t_s}\right]}\right)r_2^{(1)} + & \frac{\partial f_1}{\partial T}r_2^{(2)} = 0 \\ 0r_2^{(1)} & +0r_2^{(2)} = 0 \end{cases} \tag{15}$$

$$\overrightarrow{r_2} = \begin{bmatrix} r_2^{(1)} \\ r_2^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial T} \\ \frac{\left(M_{t_1} f_1 + M_{t_2} f_2\right)}{\left[\left(M_{t_1} S_1 + M_{t_2} S_2\right) + \frac{(1-\phi)}{\phi} M_{t_s}\right]} - \frac{\partial f_1}{\partial S_1} \end{bmatrix} \tag{16}$$

Condições de Choque:

- $\frac{\partial S_1}{\partial t} + \frac{u_t}{\phi}\frac{\partial f_1(S_1,T)}{\partial x} = 0 \rightarrow \frac{\partial S_1}{\partial t} + \frac{\partial \frac{u_t}{\phi} f_1(S_1,T)}{\partial x} = 0$

$$D = \frac{[\frac{u_t}{\phi} f_1(S_1,T)]}{[S]} \rightarrow D = \frac{u_t}{\phi}\frac{[f_1]}{[S]} \rightarrow D = \frac{u_t}{\phi}\frac{(f_1^+ - f_1^-)}{(S^+ - S^-)}$$

$\frac{\partial}{\partial t}\left[\phi\left(\rho_1 S_1 H_1 + \rho_2 S_2 H_2\right) + (1-\phi)\rho_s H_s\right] + \frac{\partial}{\partial x}\cdot\left(\rho_1 \overrightarrow{u_1} H_1 + \rho_2 \overrightarrow{u_2} H_2\right) = 0$

$\frac{\partial}{\partial t}\left[\phi\left(\rho_1 S_1 C_1 T + \rho_2 S_2 C_2 T\right) + (1-\phi)\rho_s C_S T\right] + \frac{\partial}{\partial x}\cdot\left(\rho_1 \overrightarrow{u_1} C_1 T + \rho_2 \overrightarrow{u_2} C_2 T\right) = 0$

$\frac{\partial}{\partial t}\left[\phi\left(M_{t_1} S_1 T + M_{t_2} S_2 T\right) + (1-\phi) M_{ts} T\right] + \frac{\partial}{\partial x}\cdot\left(u_t\left(M_{t_1} f_1 T + M_{t_2} f_2 T\right)\right) = 0$

$\frac{\partial}{\partial t}\left[\phi\left(M_{t_1} S_1 T + M_{t_2}(1-S_1) T\right) + (1-\phi) M_{ts} T\right] + \frac{\partial}{\partial x}\cdot\left(u_t\left(M_{t_1} f_1 T + M_{t_2}(1-f_1) T\right)\right) = 0$

$\frac{\partial}{\partial t}\left[\phi\left(M_{t_1} S_1 T - M_{t_2} S_1 T + M_{t_2} T\right) + (1-\phi) M_{ts} T\right] + \frac{\partial}{\partial x}\cdot\left(u_t\left(M_{t_1} f_1 T + M_{t_2} T - M_{t_2} T f_1\right)\right) = 0$

$$D = \frac{u_t}{\phi}\frac{\left((M_{t_1} - M_{t_2})(f_1^+ - f_1^-) + M_{t_2}\right)(T^+ - T^-)}{\left(((M_{t_1} - M_{t_2})(S^+ - S^-) + M_{t_2})(T^+ - T^-) + \frac{(1-\phi)}{\phi} M_{ts}(T^+ - T^-)\right)}$$

Famílias de Rarefação:

primeira família de rarefação:

- $\alpha_1 = \left(\frac{\partial \lambda_1}{\partial S_1} r_1^{(1)} + \frac{\partial \lambda_1}{\partial T} r_1^{(2)}\right)^{-1}$

$\alpha_1 = \left(\frac{\partial}{\partial S_1}\left(\frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1}\right)\cdot 1 + \frac{\partial}{\partial T}\left(\frac{u_t}{\phi}\frac{\partial f_1}{\partial S_1}\right)\cdot 0\right)^{-1} \rightarrow \left(\frac{u_t}{\phi}\frac{\partial^2 f_1}{\partial S_1^2}\right)^{-1} = \frac{1}{\frac{u_t}{\phi}\frac{\partial^2 f_1}{\partial S_1^2}}$

$\frac{du}{d\xi} = \alpha \overrightarrow{r}$

$\frac{dS}{d\xi} = \frac{1}{\frac{u_t}{\phi}\frac{\partial^2 f_1}{\partial S_1^2}}$

$\frac{dT}{d\xi} = 0$

$\frac{dT}{dS} = 0$

Segunda família de rarefação:

- $\alpha_2 = \left(\frac{\partial \lambda_2}{\partial S_2} r_2^{(1)} + \frac{\partial \lambda_2}{\partial T} r_2^{(2)} \right)^{-1}$

$$\alpha_2 = \Bigg[\frac{\partial f_1}{\partial T} \Bigg[\frac{\partial}{\partial S_1} \Bigg(\frac{u_t}{\phi} \frac{\big(M_{t_1} f_1 + M_{t_2} f_2\big)}{\big[\big(M_{t_1} S_1 + M_{t_2} S_2\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} \Bigg) \Bigg] + \Bigg(\frac{\big(M_{t_1} f_1 + M_{t_2} f_2\big)}{\big[\big(M_{t_1} S_1 + M_{t_2} S_2\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} - \frac{\partial f_1}{\partial S_1} \Bigg) \Bigg[\frac{\partial}{\partial T} \Bigg(\frac{u_t}{\phi} \frac{\big(M_{t_1} f_1}{\big[\big(M_{t_1} S_1 + M_{t_2}}$$

$$\Bigg[\frac{\partial f_1}{\partial T} \Bigg[\frac{\partial}{\partial S_1} \Bigg(\frac{u_t}{\phi} \frac{\big(M_{t_1} f_1 + M_{t_2}(1-f_1)\big)}{\big[\big(M_{t_1} S_1 + M_{t_2}(1-S_1)\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} \Bigg) \Bigg] + \Bigg(\frac{\big(M_{t_1} f_1 + M_{t_2}(1-f_1)\big)}{\big[\big(M_{t_1} S_1 + M_{t_2}(1-S_1)\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} - \frac{\partial f_1}{\partial S_1} \Bigg) \Bigg[\frac{\partial}{\partial T} \Bigg(\frac{u_t}{\phi} \frac{\big(M_{t_1}}{\big[\big(M_{t_1} S_1 + M}$$

$$\Bigg[\frac{\partial f_1}{\partial T} \Bigg[\frac{\partial}{\partial S_1} \Bigg(\frac{u_t}{\phi} \frac{\big(M_{t_1} f_1 + M_{t_2} - M_{t_2} f_1\big)}{\big[\big(M_{t_1} S_1 + M_{t_2} - M_{t_2} S_1\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} \Bigg) \Bigg] + \Bigg(\frac{\big(M_{t_1} f_1 + M_{t_2} - M_{t_2} f_1\big)}{\big[\big(M_{t_1} S_1 + M_{t_2} - M_{t_2} S_1\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} - \frac{\partial f_1}{\partial S_1} \Bigg) \Bigg[\frac{\partial}{\partial T} \Bigg(\frac{u_t}{\phi} \frac{\big(M}{\big[\big(M_{t_1} S_1}$$

$$\Bigg[\underbrace{\frac{\partial f_1}{\partial T} \Bigg[\frac{\partial}{\partial S_1} \Bigg(\frac{u_t}{\phi} \frac{(f_1(M_{t_1} - M_{t_2}) + M_{t_2})}{\big[(S_1(M_{t_1} - M_{t_2}) + M_{t_2}) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} \Bigg) \Bigg]}_{I} + \Bigg(\frac{(f_1(M_{t_1} - M_{t_2}) + M_{t_2})}{S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}} - \frac{\partial f}{\partial S}$$

derivando,

parte I: $\frac{\partial f_1}{\partial T} \underbrace{\left[\frac{\partial}{\partial S_1} \left(\frac{u_t}{\phi} \frac{\big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\big)}{\big[\big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} \right) \right]}_{I}$

$$\left(\frac{\Big[\Big(\frac{\partial f_1}{\partial S_1}(M_{t_1} - M_{t_2})\Big)\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}\Big)\Big] - \Big[\Big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big)(M_{t_1} - M_{t_2})\Big]}{\Big[\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big) + \frac{(1-\phi)}{\phi} M_{t_s}\Big]^2} \right) \frac{u_t}{\phi} \frac{\partial f_1}{\partial T}$$

parte II: $\left(\frac{\big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\big)}{S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}} - \frac{\partial f_1}{\partial S_1} \right) \left[\frac{\partial}{\partial T} \left(\frac{u_t}{\phi} \frac{\big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\big)}{\big[\big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\big) + \frac{(1-\phi)}{\phi} M_{t_s}\big]} \right) \right]$

$$\left(\frac{u_t}{\phi} \frac{\Big[\Big(\frac{\partial f_1}{\partial T}(M_{t_1} - M_{t_2})\Big)\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}\Big)\Big] - \Big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big) \cdot 0}{\Big[\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big) + \frac{(1-\phi)}{\phi} M_{t_s}\Big]^2} \right) \left(\frac{\big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\big)}{S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}} - \frac{\partial f}{\partial S} \right.$$

$$\left(\frac{u_t}{\phi} \frac{\Big[\Big(\frac{\partial f_1}{\partial T}(M_{t_1} - M_{t_2})\Big)\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}\Big)\Big]}{\Big[\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big) + \frac{(1-\phi)}{\phi} M_{t_s}\Big]^2} \right) \left(\frac{\big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\big)}{S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}} - \frac{\partial f_1}{\partial S_1} \right)$$

juntando as partes,

$$\alpha_2^{-1} = \left(\frac{\Big[\Big(\frac{\partial f_1}{\partial S_1}(M_{t_1} - M_{t_2})\Big)\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}\Big)\Big] - \Big[\Big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big)(M_{t_1} - M_{t_2})\Big]}{\Big[\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big) + \frac{(1-\phi)}{\phi} M_{t_s}\Big]^2} \right) \frac{u_t}{\phi} \frac{\partial f_1}{\partial T}$$
$$+ \left(\frac{u_t}{\phi} \frac{\Big[\Big(\frac{\partial f_1}{\partial T}(M_{t_1} - M_{t_2})\Big)\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}\Big)\Big]}{\Big[\Big(S_1(M_{t_1} - M_{t_2}) + M_{t_2}\Big) + \frac{(1-\phi)}{\phi} M_{t_s}\Big]^2} \right) \left(\frac{\big(f_1(M_{t_1} - M_{t_2}) + M_{t_2}\big)}{S_1(M_{t_1} - M_{t_2}) + M_{t_2} + \frac{(1-\phi)}{\phi} M_{t_s}} - \frac{\partial f_1}{\partial S_1} \right)$$

$$\frac{dS}{d\xi} = \frac{\partial f_1}{\partial T} \alpha_2$$

$$\frac{dT}{d\xi} = \left(\frac{(M_{t_1} f_1 + M_{t_2} f_2)}{\Big[(M_{t_1} S_1 + M_{t_2} S_2) + \frac{(1-\phi)}{\phi} M_{t_s}\Big]} - \frac{\partial f_1}{\partial S_1} \right) \alpha_2$$

$$\frac{dT}{dS} = \frac{\left(\frac{\left(M_{t_1} f_1 + M_{t_2} f_2\right)}{\left[\left(M_{t_1} S_1 + M_{t_2} S_2\right) + \frac{(1-\phi)}{\phi} M_{t_s}\right]} - \frac{\partial f_1}{\partial S_1} \right) \alpha_2}{\frac{\partial f_1}{\partial T} \alpha_2}$$

$$\frac{dT}{dS} = \frac{\frac{\left(M_{t_1} f_1 + M_{t_2} f_2\right)}{\left[\left(M_{t_1} S_1 + M_{t_2} S_2\right) + \frac{(1-\phi)}{\phi} M_{t_s}\right]} - \frac{\partial f_1}{\partial S_1}}{\frac{\partial f_1}{\partial T}}$$