Math 171 - Final Project
The Cramér–Lundberg Model

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1 Introduction

One of the first applications of Poisson processes, proposed by Swedish mathematician Filip Lundberg in 1903 in [4], was modeling insurance claims. In this project, I will focus on the Cramér–Lundberg model. First, we will introduce the aforementioned model, which describes an insurance company who experiences two opposing cash flows: incoming cash premiums and outgoing claims. Second, we will study the ruin probability associated with this model. Third, we will obtain an exact formula for the ruin probability in infinite time when claims are exponentially distributed. Lastly, we will perform a Monte Carlo experiment to estimate the ruin probability for Pareto distributions, which is a distribution actually used to model insurance claims. The R code used in this project to produce the plots and to run the Monte Carlo simulation is available here:
https://github.com/marcoscialanga/171FinalProject.

2 The Cramér–Lundberg model

The Cramér–Lundberg model keeps track of the resources of an insurance company in the following, simplified, way. We define the **surplus** of an insurance company at time t as:

$$U(t) = u + P(t) - S(t), \qquad t \geq 0,$$

where:

- u are the resources available to the company at time $t = 0$;
- $P(t)$ denotes the premiums collected at time t;

- $S(t)$ is the sum of the claims paid up to time t.

We will assume that premiums arrive at a constant rate c. Our analysis will then focus mainly on the stochastic process $S(t)$.

$S(t)$ is a compound Poisson process where at each arrival we associate independent and identically distributed claim sizes X_i. We have:

$$S(t) = \sum_{i=1}^{n} X_i,$$

where $\{X_i\}_{i=1}^{n}$ are the claims arrived before time t.

Thus, $S(t)$ is completely determined by the **frequency** of claims, which itself is determined by the rate λ of our underlying Poisson process, and by the **severity** of each claim, determined by the distribution of each X_i.

Below are the plots of two simulations of the Cramér–Lundberg Model with independent and exponentially distributed claims X_i with the same mean. In the second plot, after initially going up, the surplus falls below zero around $t = 175$. Insurance companies are particularly interested in the probability of surplus reaching a lower bound. We will focus on this topic for the rest of the project.

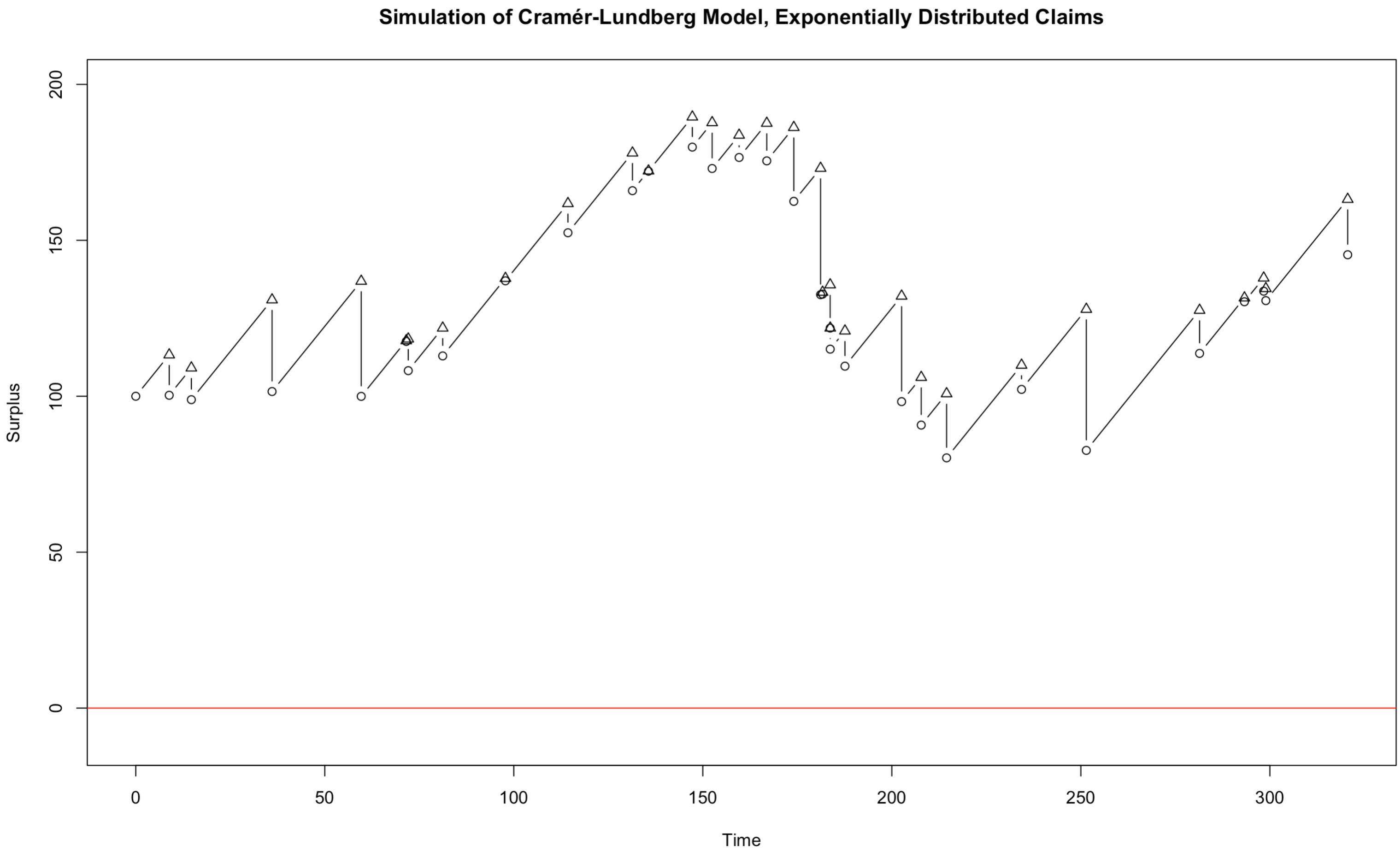

Figure 1: A simulation of the the Cramér–Lundberg Model. In this case, the surplus never fell below zero in the first 300 units of time. Note the constant increase of the surplus between claim arrivals, caused by $P(t)$, and the immediate drops (of random sizes) once each claim arrives.

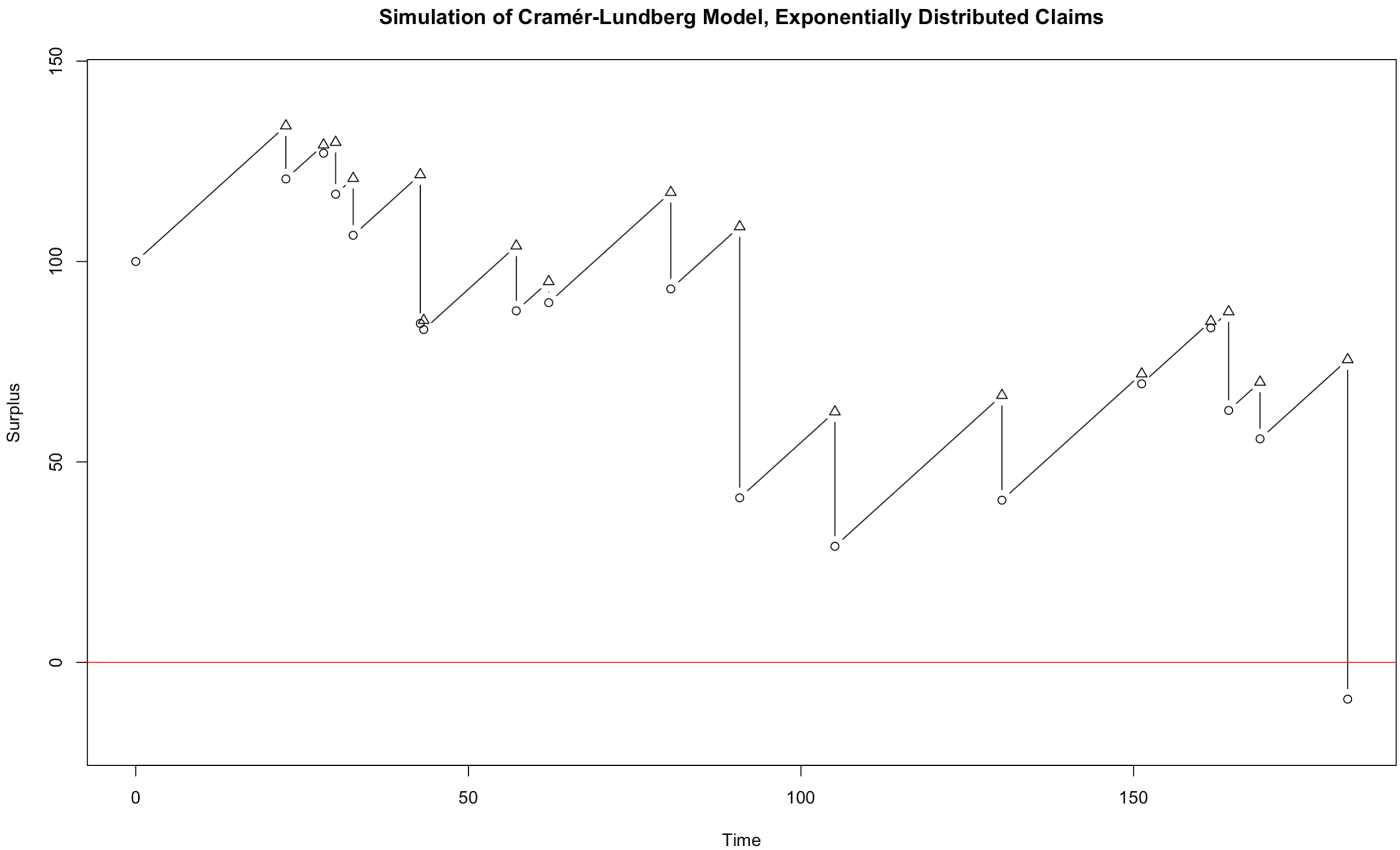

Figure 2: Another simulation of the the Cramér–Lundberg Model. In this case, the surplus fell below zero before the 200th unit of time.

3 Ruin Probability of the Cramér–Lundberg Model

We now focus on the **ruin probability in infinite time** of the model that we introduced in section 2. Ruin occurs if the surplus reaches a certain lower bound. For our purposes, we will set 0 as the lower bound necessary for ruin to occur. It is intuitive that the ruin probability in infinite time will represent an upper bound for the ruin probability in any finite time.
Let $T = inf\{t \mid U(t) \leq 0\}$. Then, define the ruin probability in infinite time, conditioned on the initial surplus u, as a function

$$\psi : [0, \infty) \to [0, 1]$$

$$\psi(u) := \mathbb{P}(T < \infty | \ U(t) = u),$$

with the convention that $T = \infty$ if $U(t) > 0 \ \forall t$.

Naturally, this probability will be heavily affected by the distribution of each claim X_i. Here we assume that our X_i are independent and identically distributed random variables. Even with this assumption, computing the exact ruin probability is unfeasible for most distributions. In section 3.1, we will cover the case where X_i are independent exponential random variables with the same mean. In this case, we will be able to compute $\psi(u)$ exactly, following a derivation from [1]. In section 3.2, we will instead perform a Monte Carlo simulation to compute the ruin probability for a Pareto distribution, a more realistic distribution for insurance claims, due to its flexibility [3].

3.1 Exponential Claims

In this section we will present an exact formula for the ruin probability of the Cramér–Lundberg Model when X_i are independent exponentially distributed with the same mean.

Let $S(t)$ consist of a Poisson process of rate λ and independent and exponentially distributed claims X_i with parameter μ and mean $\frac{1}{\mu}$. Let c be the constant rate at which the insurance company collects premiums. Thus, $P(t) = ct$.

We will now define two quantities that will be useful in our analysis.

First, rewrite $c = (1+\theta)\lambda\mathbb{E}[X]$, where $\theta > 0$ and $\mathbb{E}[X]$ is the mean of each X_i. If $\theta \leq 0$, it is intuitive that $\psi(u) = 1$.

θ is called the **relative security loading**. Clearly, as θ increases and all the other variables remain fixed, $\psi(u)$ decreases, as premiums are arriving at a faster rate.

We now define the **adjustment coefficient**, when it exists, as the unique strictly positive solution R of the equation:

$$1 + (1+\theta)\mathbb{E}[X] \cdot r = M_X(r),$$

where M_X and $\mathbb{E}[X]$ denote, respectively, the moment generating function and the mean of each X_i.

In our simplified case, where $X_i \sim Expo(\mu)$, we only need some basic algebra to get:

$$R = \frac{\theta\mu}{1+\theta}.$$

The adjustment coefficient is used to prove the following formula, valid for any distribution of X_i:

$$\psi(u) = \frac{exp(-Ru)}{\mathbb{E}[exp(-RU(T)) \mid T < \infty]}. \tag{1}$$

For the motivation behind the definition of adjustment coefficient and for the proof of (1), we refer to [1].

Note that, since $U(T)$ is always negative in our setup, the denominator is always greater than one. This means that the numerator is an upper bound for the ruin probability for any distribution of the claims X_i. This is also called **Lundberg's inequality**.

In most cases, a closed form evaluation of the denominator is not attainable. Sometimes, it is not even possible to solve for R. However, there are some exceptions.

Let's look at the case where $X_i \sim Expo(\mu)$. We already computed the adjustment coefficient for this distribution. Now we compute the conditional distribution of $U(T)$ given that ruin occurs.

Let u' be the surplus just before time T. For ruin to occur, we need the last claim, call it X', to satisfy $X > u'$. $-U(T) > y$ is equivalent to $X > u' + y$. Now we can quickly compute:

$$\mathbb{P}(-U(T) > y \mid T < \infty) = \frac{P(X > u' + y)}{P(X > u')} = e^{-\mu y}.$$

Then, the pdf of $-U(T)$ conditional on $T < \infty$ is given by:

$$\frac{d}{dy}(1 - e^{-\mu y}) = \mu e^{\mu y}.$$

Therefore in this case,

$$\mathbb{E}[exp(-RU(T)) \mid T < \infty] = \mu \int_0^\infty e^{-\mu y} e^{Ry} dy = \frac{\mu}{\mu - R}$$

Then, (1) becomes:

$$\psi(u) = \frac{1}{1+\theta} exp(\frac{-\theta\mu u}{(1+\theta)}).$$

Note that, when $u = 0$:

$$\psi(u) = \frac{1}{1+\theta}.$$

This is somewhat surprising: if $X_i \sim Expo(\mu)$ and the initial capital is zero, the probability of ruin depends only on the relative security loading, neither on the rate of the Poisson process, nor on the expected amount to pay for each claim.

A perhaps even more surprising fact is that the above is true for all claim distributions, not only for the exponential, as we can see in [2].

3.2 Pareto Claims

Now we will focus our attention on the case where $X_i \sim Pareto(\alpha, x_0)$. This distribution is actually used to model different types of insurance [3]. For the Pareto distribution, we can't obtain a nice formula for the ruin probability as we did in section 3.1. Thus, we will perform a Monte Carlo experiment to approximate $\psi(u)$ given certain parameters.

For this experiment, we aim to simulate claims regarding car insurances. We use a Pareto distribution with $\alpha = 2.2$ and $x_0 = 2000$. The value of α was chosen as suggested in [3] to model car insurances, the value of x_0 was chosen so that the expected amount of a claim was around 3500, which, according to [5], is about the expected car collision claim size for physical damage in the United States. Furthermore, we chose a Poisson rate of 10, $c = 2000$ or, equivalently, $\theta = 4.54$. We ran the experiment for u going from 0 to 10000, with a step size of 200. We simulated the cash flow of the insurance company 1000 times for each value of u until a time of 10000 units. Increasing both the number of experiments and the time for which each experiment is run will naturally give more precise results, as we can't simulate infinite times. Below is the plot of different ruin probabilities that we approximated.

Notice the overall exponential decay of $\psi(u)$ as u increases, as predicted by (1). Furthermore, note that for $u = 0$, $\psi(u)$ is approximately 0.18. This is very close to the value $\frac{1}{1+\theta}$, offering more evidence of the fact stated above (without proof), that when $u = 0$, $\psi(u)$ only depends on θ, and not on the distribution of the claims X_i.

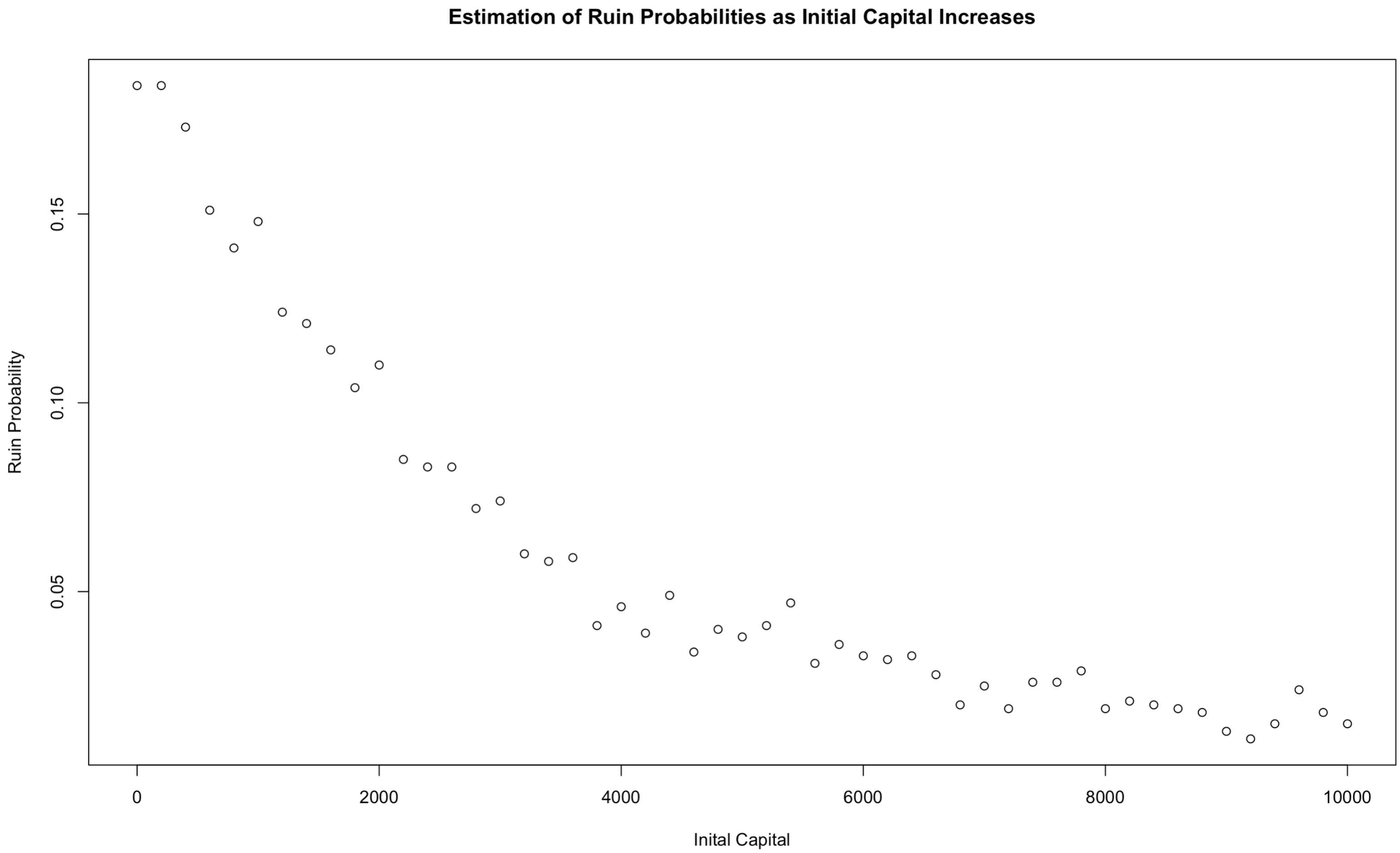

Figure 3: Monte Carlo estimation of ruin probability for claims distributed as $Pareto(2.2, 2000)$, in 10000 units of time, with $\lambda = 10$, for different values of u. Note the trend of exponential decay as u increases, as predicted by (1).

References

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[4] Filip Lundberg. *1. Approximerad framställning af sannolikhetsfunktionen: 2. Återförsäkring af kollektivrisker. Akademisk afhandling.* Almqvist &Wiksells, 1903.

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