Option Pricing Methods with Monte Carlo Methods

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This paper aims to present the various option pricing models using Monte Carlo Processes, beginning with the classic Black-Scholes-Merton model and building more complex models by adding stochastic variables to recreate the stochastic reality of financial markets. I will then describe the differences between competing models by highlighting each of their key innovations and will follow-up by comparing them in terms of option pricing.

Theory of option pricing and Monte Carlo implementation

An option is a financial asset that gives the right, but not the obligation, to its holder to buy or sell another financial asset (the underlying asset S) at a specified date (the maturity data: T) and at a predetermined price (the strike price: K). There are two types of options: calls and puts. Calls allow the owner to buy the underlying asset at the strike price before or at maturity, while puts allow the owner the right, but not the obligation, to sell the underlying asset at the strike before or at maturity. The pay-off of a call and a put is $\max(S_t - K, 0)$ and $\max(K - S_t, 0)$ respectively. The two most common types of option categories are European and American. These categories differ based on how the owner can exercise the options. European options can be exercised only at the maturity date, whereas American options can be exercised at any time before or at the maturity date. Although there is little difference between European and American options, the ability to exercise them at any time makes pricing American options very different from pricing European options. This paper will be focused on European call options exclusively.

<u>Black Scholes Merton Option Pricing</u>

To price a European option, Black and Scholes (1973) and Merton (1973) built the derivative pricing theory based on geometric Brownian motion (GBM). Consider an asset S whose price follows a GBM as follows

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

In this stochastic differential equation, μ is the expected instantaneous rate of return of $S(t)$, σ is the instantaneous volatility of μ, and $W(t)$ is a standard Wiener process. The price of the contract on the underlying asset at time t is denoted $P(S(t), t)$. The maturity of this contract is T, so at maturity the contract has value $P(S(t), T)$. If the contract is a European option with maturity T and strike K, then its terminal value is

$$P(S(t), T) = max\big(0, K - S(T)\big) \tag{2}$$

for a put, and similarly

$$P(S(t), T) = max(0, S(T) - K) \tag{3}$$

for a call. For a call option on a non-dividend paying asset S with strike K and maturity T, the solution to the Black-Scholes-Merton partial differential equation with boundary condition given in (3) gives the following formula for the price of the call option

$$Call = S(0)N(d_1) - Ke^{(-rT)}N(d_2), \tag{4}$$

Where

$$d_1 = \frac{ln(S(0)/K) + rT}{\sigma\sqrt{(T)}} + \frac{1}{2}\sigma\sqrt{(T)}, \quad d_2 = d_1 - \sigma\sqrt{(T)} \tag{5}$$

And similarly, for a put option

$$Put = Ke^{(-rT)}N(-d_2) - S(0)N(-d_1), \quad (6)$$

The price of an option is modeled on an asset that is uncertain—that is, its price path follows a GBM (a random process) which would result in varying option prices. To address this source of uncertainty, we can make use of Monte Carlo simulations. Monte Carlo simulations are a class of computational algorithms based on random sampling to obtain numerical results for problems that are difficult or impossible to find exact solutions.

This process uses the risk-neutral valuation principle, which states an option can be valued on the assumption that the world is risk neutral. The risk neutral valuation principle means that we use assume that the expected return from all traded assets is the risk-free interest rate and that expected payoffs of derivatives are discounted at the risk-free interest rate (Hull, 2009).

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One can price an option using Monte Carlo simulations as follows: several paths are simulated for the underlying asset under a risk-neutral measure—in this case, the risk-free rate. The value of the option is then found by calculating the average payoff for each path and discounting it at the risk-free rate. As the number of simulated paths increases, the variance of the option price estimate decreases.

Algorithm to price a European Call option using Monte Carlo Processes

Given: $S(0), K, T,$ interest rate r, volatility σ , number of time steps N, number of simulation paths M

$dt = T/N$

$S_0 = S(0)_{Mx1}$

$for\ \ n = 1{:}\,N$

$$S_n = S_{(n-1)} + rS_{(n-1)}dt + \sigma S_{(n-1)}dW$$

end

$Call = e^{(-rT)}mean[max(0, S_N - K)]$

Although practical, the current BSM model is unrealistic in its assumptions of constant return (interest rate), constant volatility, and the asset path following a smooth, continuous path. The interest rate and stocks' volatilities can change daily, and both stock prices and their volatilities exhibit sudden jumps as markets react to news or to economic factors. As such, a more accurate model is needed to account for the stochastic properties of these parameters. Throughout the rest of this paper, I will expand the current option price model by adding stochastic parameters and using Monte Carlo processes to price and compare them to the basic BSM results.

Stochastic volatility

I propose to augment the current BSM model by including non-constant volatilities in the underlying assets. Stock returns often have relatively constant volatilities; however, stocks will often enter turbulent periods during which volatility will vary drastically. To account for this, I will focus on the Hull-White (1987) and Heston (1993) stochastic volatility models. Both models have stochastic volatility that follow a mean reverting process as follows

$$d\sigma^2 = \kappa\big(\theta - \sigma^2(t)\big)dt + \sigma_\sigma\sqrt{\sigma^2}dZ(t) \tag{7}$$

and a stochastic path for the stock that follows the usual form

$$dS(t) = \mu S(t)dt + \sigma(t)^{\alpha} S(t)dW(t) \tag{8}$$

$$(dW, dZ) = \rho \tag{9}$$

Where σ_σ is the volatility of the volatility and determines the variance of σ^2. dW and dZ are Wiener processes. Both models start with a spot variance rate V_0, which is a historical value of the underlying asset's volatility, and account for the mean-reversion property of the volatility. The variance reverts to its long run average θ at rate κ as t tends to infinity. In the Hull-White model, $\alpha = 1$ whereas it is .5 in the Heston Model. Further, Hull-White set ρ to zero whereas Heston does not.

Hull and White show that when the stochastic volatility is uncorrelated with the underlying asset, the price of a European option is the BSM price integrated over the probability distribution of the life of the option (Hull, 2009). Hull and White use this to show that a call option on a security with stochastic volatility is uncorrelated with the security price. In this case, the BSM model over-values at-the-money (ATM) options and undervalues deep-in and out-of-the-money options (ITM and OTM, respectively). While the Hull-White model mostly assumes zero correlation between stochastic volatility and the underlying asset, the Heston model does not. Heston argues that the correlation parameter ρ between volatility and the underlying asset affects the skewness of spot returns. Skewness in the distribution of underlying returns affects the pricing of ITM relative to OTM options. A negative correlation decreases the prices of OTM options relative to ITM options, while positive correlation has the opposite effect (Heston, 1993).

Hull-White point out that the choice of the volatility of the volatility is not obvious. They argue that it can be estimated by examining the changes in volatilities implied by option prices (implied volatility). Alternatively, it can be estimated from changes in estimates of the actual variance. Hull-White argue that using the implied volatility is a best an indirect method and that it loses accuracy since changes in it are to some extent the result of pricing errors in the options. Further, estimates of the actual variances require large amounts of data which is not always feasible.

Algorithm to price a call using stochastic volatility

Given: $S(0), V(0), K, T,$ interest rate r, volatility σ ,

number of time steps N, number of simulation paths M

$dt = T/N$

$S_0 = S(0)_{Mx1}$

$V_0 = V(0)_{Mx1}$

$for\ \ n = 1{:}N$

$$V_n = \kappa(\theta - V_{n-1})dt + \sigma_\sigma\sqrt{\mathrm{V_{n-1}}}dZ_t$$

$$S_n = S_{(n-1)} + rS_{(n-1)}dt + \sqrt{}(V_n dt)\ S_n dW_t$$

end

$Call = e^{(-rT)} mean[max(0, S_N - K)]$

Jumps in stock prices

The BSM model assumption that the underlying stock returns follow a stochastic process with a continuous path is not a very accurate reflection of how financial markets operate. Stocks are often characterized by rapid jumps and drops as the market reacts to sudden news and economic factors. To accommodate for this stylized fact of stock returns, I introduce Merton's (1975) discontinuous stock return model. Merton's innovation was to allow for stocks to jump or to have 'vibrations,' and distinguishes between 'normal' and 'abnormal' vibrations in a stock's price. The normal vibrations are marginal changes in stock price that come from regular demand and supply imbalances or small news. These vibrations are modeled by a standard GBM with a constant variance per unit time and have continuous sample paths. The 'abnormal' vibrations are the result of sudden arrivals of new, important information regarding the stock that is usually specific to the firm or to the industry. These vibrations cause more than a marginal effect on the price. 'Jumps' are modeled by a Poisson-driven process where the Poisson-distributed event is the arrival of important news about the stock that has a non-marginal effect. The Merton model takes the following form

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) + S(t)(Y-1)dN(t) \tag{10}$$

Where W(t) is the GBM, $N(t)$ is the Poisson-distributed number of jumps with intensity λdt and $Y-1$ is the random variable jump size in the stock price if the Poisson event occurs. Y is log-normally distributed. dW and dN are assumed to be independent.

$\underline{Algorithm\text{m to price a call using jumps in stock price}}$

Given: $S(0), K, T, r, \sigma$, λ , time steps N, simulation paths M

$dt = T/N$

$S_0 = S(0)_{Mx1}$

$PoissonProcess = Poison(\frac{\lambda}{N}, 1, N)$

$J_0 = zeros_{Mx1}$

$for\ \ n = 1{:}N$

$\quad \mathrm{J_n} = \sum \mathrm{PoissonProcess}$

end

$for\ \ n = 1{:}N$

$\quad S_n = S_{(n-1)} + rS_{(n-1)}dt + \sigma S_{n-1}dW_t + S_{n-1}(Y-1)J_n$

end

$Call = e^{(-rT)}mean[max(0, S_N - K)]$

Stochastic Volatility with Jumps

Now that we have a way to model stochastic volatility and stock jumps, it is the natural solution to combine the two. Bates model (1996) is essentially a combination of Merton's option pricing with discontinuous returns and Heston's stochastic volatility models. This model has log-normally distributed stock jumps, stochastic volatility, and correlation between the underlying asset and volatilities GBM. Bates Model takes the following form:

$$dS(t) = \mu S(t)dt + \sqrt{\mathrm{V(t)}}S(t)dW(t) + S(t)(Y-1)dN(t) \tag{11}$$

$$d\mathrm{V(t)} = \kappa\big(\theta - V(t)\big)dt + \sigma_\sigma\sqrt{\mathrm{V(t)}}dZ(t) \tag{12}$$

$$S(0) = S_0$$

$$V(0) = V_0$$

$$N(0) = N_0$$

$$(dW, dZ) = \rho dt$$

$$(dW, N) = (dW, Y) = 0$$

Where all the parameters are exactly as those found in the Merton and Heston model.

Bates innovation was to have a more complete model that accounted for more of stocks stochastic nature. By allowing for skewed and/or leptokurtic distribution, Bates model improves all other model's ability to fit option prices. The model was better than the other models at pricing ITM and OTM 0 to 3 month call and put options. Further, Bates compares his model with Heston's Stochastic volatility model on their consistencies with option prices and implied volatility. Bate's finds that Heston's model is consistent only for extreme and implausible levels of the volatility of variance whereas Bate's model is more realistic since it incorporates "jump fears" which explain more of the volatility smile (Bates, 1996).

Algorithm to price a call using jumps in stock prices and stochastic volatility

Given: $S(0), K, T, r, \sigma, \lambda$, time steps N, simulation paths M

$dt = T/N$

$S_0 = S(0)_{Mx1}$

$V_0 = V(0)_{Mx1}$

$PoissonProcess = Poison(\frac{\lambda}{N}, 1, N)$

$J_0 = zeros_{Mx1}$

$for\ \ n = 1{:}N$

$\quad \mathrm{J_n} = \sum \mathrm{PoissonProcess}$

end

$for\ \ n = 1{:}N$

$\quad V_n = \kappa(\theta - V_{n-1})dt + \sigma_\sigma \sqrt{\mathrm{V_{n-1}}} dZ_t$

$\quad S_n = S_{(n-1)} + rS_{(n-1)}dt + \sqrt{V_{n-1}} S_{n-1} dW_t + S_{n-1}(Y-1)J_n$

end

$Call = e^{(-rT)} mean[max(0, S_N - K)]$

<u>Interest Rate models</u>

The final parameter in the BSM model that is assumed as constant is the interest rate. This assumption, however, is not realistic since interest rate fluctuates in the same manner that the other parameters do. What differentiates interest rate from the other variables is that it is used to price options when discounting price to the present value. It is also essential in the valuation of bonds and interest rate derivatives. In their works on the term structure of interest rates, Vasicek (1977) and Cox, Ingersoll and Ross (1985), and Hull and White (1993) provide models on the stochastic nature of the interest rate. Although all these models account for the stochasticity of the interest rate, they differ in how the rate is derived. In Vasicek's model, interest rate is derived from the term structure of interest rates, $R(t,T) = -\log \frac{P(t,T)}{T-t}$ where $P(t,T)$ is the price at time t of a discount bond maturing at time s, $t \le s$, with unit maturity value $P(s,s) = 1$. The interest rate is the interest rate for an infinitesimally short amount of time. At any time t, the current value $r(t)$ of the spot rate is the instantaneous rate of change of the loan value. The Vasicek model is modeled by a mean reverting Ornstein-Uhlenback process:

$$dr(t) = \kappa\big(\theta - r(t)\big)dt + \sigma dW(t) \tag{13}$$

Where dW is a Wiener process, θ is the long-run interest rate and κ is the reversion rate. The interest rate is normally distributed, and as such it can take on negative value.

A general equilibrium framework is developed in the CIR model, which rules that if the change in production opportunities—described by a single state variable Y—is assumed to follow an SDE like the BSM model, the same will apply for the interest rate, and includes the square-root diffusion described as:

$$dr(t) = \kappa\big(\theta - r(t)\big)dt + \sigma\sqrt{r(t)}dW(t) \tag{14}$$

Similar to the Vasicek model, this model follows a mean reverting process with $r(t)$ pulled towards the long-term value θ at rate κ . Firstly, in contrast to the Vasicek model, the CIR models diffusion term $\sqrt{r(t)}$ decreases to zero as $r(t)$ approaches the origin, preventing the interest rate from taking on negative values. Secondly, the absolute variance of the interest rate increases when the interest rate itself increases. Thirdly, if the interest rate reaches zero, it can subsequently become positive. These are three empirically relevant properties of the model, since it was previously assumed that interest rates could not become negative. The interest rate follows a non-central chi-squared distribution.

In the Hull-White model, the interest rate follows a similar SDE path as Vasicek's model, which takes the form:

$$dr(t) = \kappa\big(\theta(t) - r(t)\big)dt + \sigma dW(t) \tag{15}$$

where a time varying long run interest rate $\theta(t)$ is included that provides a richer pattern of term structure movements and a richer pattern of volatilities than other models (Hull, 2009).

$\underline{Algorithm\text{ to price a call with stochastic interest rate}}$

Given: $S(0)$, $\mathrm{r}(0)$, K, T, volatility σ , number of time steps N, number of simulation paths M

$dt = T/N$

$S_0 = S(0)_{Mx1}$

$r_0 = \mathrm{r}(0)_{Mx1}$

$for\ \ n = 1{:}\,N$

$$r_n = \kappa(\theta - r_{n-1})dt + \sigma_r\sqrt{r_{\mathrm{n}-1}}dZ_t$$

$$S_n = S_{(n-1)} + r_{n-1}S_{n-1}dt + \sigma \mathrm{S}_{\mathrm{n}-1}dW_t$$

end

$Call = e^{(-rT)}mean[max(0, S_N - K)]$

In addition to enriching the BSM model by adding stochastic interest rate, this code can be used to price a call option on a risk-free zero-coupon bond. The price of a zero-coupon bond with maturity time T is $P(t, T)$, which satisfies a PDE that is not derived here. It is subject to constraints that are unique to bonds—for example, the bond's volatility shrinks to zero as it approaches maturity. It is given by:

$$P(t, T) = A(t, T)\mathrm{e}^{-\mathrm{B(t,T)r(t)}} \tag{16}$$

Where $A(t, T)$ and $B(t, T)$ are functions of time and parameters of the model which differ under the Vasicek and CIR models. In this paper we will briefly examine a European style plain vanilla call option on bonds. The option price will follow the basic Monte Carlo valuation procedure with terminal condition:

$$Call(r, t; s, K) = \max[P(r, T, s) - K, 0] \tag{17}$$

where the value of the call option at time t on a zero-coupon bond with maturity s, strike K and with expiration date T . Further, $s \geq T \geq t$, and the strike is restricted to be less than $A(T,s)$, the maximum possible bond price at maturity. Otherwise, the option would never be exercised and would thus be worthless (Cox et al, 1985). The CIR paper points out that the call option is an increasing function of maturity, and numerical analysis indicates that an increase in the interest rate will decrease the options value, since the interest rate will depress the price of the underlying bond.

Algorithm to price a call option on a zero coupon bond with stochastic interest rate

Given: $S(0), \mathrm{r}(0), K, T,$ volatility σ , number of time steps $N,$ number of simulation paths M

$dt = T/N$

$S_0 = S(0)_{Mx1}$

$r_0 = \mathrm{r}(0)_{Mx1}$

$for \;\; n = 1{:}N$

$$r_n = \kappa(\theta - r_{n-1})dt + \sigma_r\sqrt{r_{\mathrm{n}-1}}dZ_t$$

end

$P(t,T) = A(t,T)\exp(-B(t,T)r_N)$

$Call = e^{(-rT)}[max(0, P - K)]$

Firm asset value models

Credit risk is the risk of financial losses due to changes in the credit quality of a firm, with the most extreme case being a default event. Credit risk includes the possibility that a debtor will fail to repay a loan, that a bond issuer will miss a coupon payment, or that a counterparty to a swap will fail to make an interest payment (Glasserman, 2003). The difference between yields on corporate debt subject to default risk and government bonds free of such risk is known as credit spreads (D'Amato, 2003). The "credit spread puzzle" is the difficulty in explaining the relationship between credit spreads and the compensation for various forms of credit risk (D'Amato). In their structural credit risk model, Elkahmhi, Ericsson, and Jiang (2011) model the firm's unlevered asset value as a stochastic process with a stochastic volatility that follows the CIR model shown below:

$$dX(t) = (\mu - \delta)X(t)dt + \sqrt{V(t)X(t)}dW(t) \tag{18}$$

$$dV(t) = \kappa\big(\theta - V(t)\big)dt + \sigma\sqrt{V(t)}dZ(t) \tag{19}$$

Where, $X(t)$ is the firms unlevered asset value, δ is the firms payout ration, the assets variance follows the CIR model and $E(dWdZ) = \rho dt.$ Ericsson et al show point out that asset risk volatility, dependence between the levels of risk and asset value and volatility risk premia are the main credit risks that influence spreads. However, only volatility risk premium is economically significant as it matches historical spreads and equity volatilities.

Numerical Results: Pricing a European Call Option Under Various Models

							Models			
				Constant parameters	Stochastic Volatility		Discontinuous stock price	Stochastic and discontinuous volatility	Stochastic Interest rate	
	Parameters		BSM	Monte Carlo	Heston	Hull-White	Merton	Bates	Vasicek	CIR
$S(0)$ \$	K \$	T $years$	$Price$ \$	$Price$ \$	$Price$ \$	$Price$ \$	$Price$ \$	$Price$ \$	$Price$ \$	$Price$ \$
90	100	1	5.1	4.95	3.34	4.01	4.41	7.23	4.96	5.09
90	100	2	9.92	9.98	5.36	7.33	5.03	10.49	9.31	9.81
100	100	1	10.44	10.39	7.67	6.45	6.78	12.30	10.27	10.41
100	100	2	16.12	16.19	8.61	11.50	11.47	13.89	15.83	16.01
110	100	1	17.66	17.61	9.77	12.89	13.11	16.02	16.54	17.73
110	100	2	23.57	23.12	16.18	15.67	14.83	19.27	23.31	23.50
σ_S			.2	.2			.3		.2	.2
k					1.25	1.25		1.25	.25	.25
θ					.04	.04		.04	.05	.05
r_0			.05	.05	.05	.05	.01		.05	.05
σ_r								0	.05	.05
v_0					.2	.2	.02	.2		
ρ					-.2	-.2			-.2	-.2
σ_σ					.2	.2	.1			
μ_J							0			
σ_J							.34			
λ							3	15		

Numerical results of pricing a European Call options at various moneyness and maturities under the various models

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