2

The Laplace Transform*

2–1 INTRODUCTION

The Laplace transform method is an operational method that can be used advantageously for solving linear differential equations. By use of Laplace transforms, we can convert many common functions, such as sinusoidal functions, damped sinusoidal functions, and exponential functions, into algebraic functions of a complex variable s. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Thus, a linear differential equation can be transformed into an algebraic equation in a complex variable s. If the algebraic equation in s is solved for the dependent variable, then the solution of the differential equation (the inverse Laplace transform of the dependent variable) may be found by use of a Laplace transform table or by use of the partial-fraction expansion technique, which is presented in Section 2–5 and 2–6.

An advantage of the Laplace transform method is that it allows the use of graphical techniques for predicting the system performance without actually solving system differential equations. Another advantage of the Laplace transform method is that, when we solve the differential equation, both the transient component and steady-state component of the solution can be obtained simultaneously.

Outline of the Chapter. Section 2–1 presents introductory remarks. Section 2–2 briefly reviews complex variables and complex functions. Section 2–3 derives Laplace

*This chapter may be skipped if the student is already familiar with Laplace transforms.

transforms of time functions that are frequently used in control engineering. Section 2–4 presents useful theorems of Laplace transforms, and Section 2–5 treats the inverse Laplace transformation using the partial-fraction expansion of $B(s)/A(s)$, where $A(s)$ and $B(s)$ are polynomials in s. Section 2–6 presents computational methods with MATLAB to obtain the partial-fraction expansion of $B(s)/A(s)$, as well as the zeros and poles of $B(s)/A(s)$. Finally, Section 2–7 deals with solutions of linear time-invariant differential equations by the Laplace transform approach.

2–2 REVIEW OF COMPLEX VARIABLES AND COMPLEX FUNCTIONS

Before we present the Laplace transformation, we shall review the complex variable and complex function. We shall also review Euler's theorem, which relates the sinusoidal functions to exponential functions.

Complex Variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex quantity is called a *complex variable*. In the Laplace transformation we use the notation s as a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and ω is the imaginary part.

Complex Function. A complex function $G(s)$, a function of s, has a real part and an imaginary part or

$$G(s) = G_x + jG_y$$

where G_x and G_y are real quantities. The magnitude of $G(s)$ is $\sqrt{G_x^2 + G_y^2}$, and the angle θ of $G(s)$ is $\tan^{-1}(G_y/G_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $G(s)$ is $\bar{G}(s) = G_x - jG_y$.

Complex functions commonly encountered in linear control systems analysis are single-valued functions of s and are uniquely determined for a given value of s.

A complex function $G(s)$ is said to be *analytic* in a region if $G(s)$ and all its derivatives exist in that region. The derivative of an analytic function $G(s)$ is given by

$$\frac{d}{ds}G(s) = \lim_{\Delta s \to 0} \frac{G(s + \Delta s) - G(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta G}{\Delta s}$$

Since $\Delta s = \Delta\sigma + j\Delta\omega$, Δs can approach zero along an infinite number of different paths. It can be shown, but is stated without a proof here, that if the derivatives taken along two particular paths, that is, $\Delta s = \Delta\sigma$ and $\Delta s = j\Delta\omega$, are equal, then the derivative is unique for any other path $\Delta s = \Delta\sigma + j\Delta\omega$ and so the derivative exists.

For a particular path $\Delta s = \Delta\sigma$ (which means that the path is parallel to the real axis).

$$\frac{d}{ds}G(s) = \lim_{\Delta\sigma \to 0}\left(\frac{\Delta G_x}{\Delta\sigma} + j\frac{\Delta G_y}{\Delta\sigma}\right) = \frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma}$$

For another particular path $\Delta s = j\Delta\omega$ (which means that the path is parallel to the imaginary axis).

$$\frac{d}{ds}G(s) = \lim_{j\Delta\omega \to 0}\left(\frac{\Delta G_x}{j\Delta\omega} + j\frac{\Delta G_y}{j\Delta\omega}\right) = -j\frac{\partial G_x}{\partial\omega} + \frac{\Delta G_y}{\partial\omega}$$

If these two values of the derivative are equal,

$$\frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} - j\frac{\partial G_x}{\partial\omega}$$

or if the following two conditions

$$\frac{\partial G_x}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} \quad \text{and} \quad \frac{\partial G_y}{\partial\sigma} = -\frac{\partial G_x}{\partial\omega}$$

are satisfied, then the derivative $dG(s)/ds$ is uniquely determined. These two conditions are known as the Cauchy–Riemann conditions. If these conditions are satisfied, the function $G(s)$ is analytic.

As an example, consider the following $G(s)$:

$$G(s) = \frac{1}{s+1}$$

Then

$$G(\sigma + j\omega) = \frac{1}{\sigma + j\omega + 1} = G_x + jG_y$$

where

$$G_x = \frac{\sigma + 1}{(\sigma+1)^2 + \omega^2} \quad \text{and} \quad G_y = \frac{-\omega}{(\sigma+1)^2 + \omega^2}$$

It can be seen that, except at $s = -1$ (that is, $\sigma = -1$, $\omega = 0$), $G(s)$ satisfies the Cauchy–Riemann conditions:

$$\frac{\partial G_x}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} = \frac{\omega^2 - (\sigma+1)^2}{[(\sigma+1)^2 + \omega^2]^2}$$

$$\frac{\partial G_y}{\partial\sigma} = -\frac{\partial G_x}{\partial\omega} = \frac{2\omega(\sigma+1)}{[(\sigma+1)^2 + \omega^2]^2}$$

Hence $G(s) = 1/(s+1)$ is analytic in the entire s plane except at $s = -1$. The derivative $dG(s)/ds$, except at $s = 1$, is found to be

$$\begin{aligned}\frac{d}{ds}G(s) &= \frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} - j\frac{\partial G_x}{d\omega} \\ &= -\frac{1}{(\sigma + j\omega + 1)^2} = -\frac{1}{(s+1)^2}\end{aligned}$$

Note that the derivative of an analytic function can be obtained simply by differentiating $G(s)$ with respect to s. In this example,

$$\frac{d}{ds}\left(\frac{1}{s+1}\right) = -\frac{1}{(s+1)^2}$$

Points in the s plane at which the function $G(s)$ is analytic are called *ordinary* points, while points in the s plane at which the function $G(s)$ is not analytic are called *singular* points. Singular points at which the function $G(s)$ or its derivatives approach infinity are called *poles*. Singular points at which the function $G(s)$ equals zero are called *zeros*.

If $G(s)$ approaches infinity as s approaches $-p$ and if the function

$$G(s)(s+p)^n, \qquad \text{for } n = 1, 2, 3, \ldots$$

has a finite, nonzero value at $s = -p$, then $s = -p$ is called a pole of order n. If $n = 1$, the pole is called a simple pole. If $n = 2, 3, \ldots$, the pole is called a second-order pole, a third-order pole, and so on.

To illustrate, consider the complex function

$$G(s) = \frac{K(s+2)(s+10)}{s(s+1)(s+5)(s+15)^2}$$

$G(s)$ has zeros at $s = -2$, $s = -10$, simple poles at $s = 0$, $s = -1$, $s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since for large values of s

$$G(s) \doteqdot \frac{K}{s^3}$$

$G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros ($s = -2$, $s = -10$, $s = \infty$, $s = \infty$, $s = \infty$) and five poles ($s = 0$, $s = -1$, $s = -5$, $s = -15$, $s = -15$).

Euler's Theorem. The power series expansions of $\cos\theta$ and $\sin\theta$ are, respectively,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

And so

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we see that

$$\cos\theta + j\sin\theta = e^{j\theta} \tag{2–1}$$

This is known as *Euler's theorem*.

By using Euler's theorem, we can express sine and cosine in terms of an exponential function. Noting that $e^{-j\theta}$ is the complex conjugate of $e^{j\theta}$ and that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

we find, after adding or subtracting these two equations, that

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \tag{2–2}$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \tag{2–3}$$

2–3 LAPLACE TRANSFORMATION

We shall first present a definition of the Laplace transformation and a brief discussion of the condition for the existence of the Laplace transform and then provide examples for illustrating the derivation of Laplace transforms of several common functions.

Let us define

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathscr{L}$ = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called the *inverse Laplace transformation*. The notation for the inverse Laplace transformation is $\mathscr{L}^{-1}$, and the inverse Laplace transform can be found from $F(s)$ by the following inversion integral:

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds, \qquad \text{for } t > 0 \tag{2–4}$$

where c, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

Evaluating the inversion integral appears complicated. In practice, we seldom use this integral for finding $f(t)$. There are simpler methods for finding $f(t)$. We shall discuss such simpler methods in Sections 2–5 and 2–6.

It is noted that in this book the time function $f(t)$ is always assumed to be zero for negative time; that is,

$$f(t) = 0, \qquad \text{for } t < 0$$

Existence of Laplace Transform. The Laplace transform of a function $f(t)$ exists if the Laplace integral converges. The integral will converge if $f(t)$ is sectionally continuous in every finite interval in the range $t > 0$ and if it is of exponential order as t approaches infinity. A function $f(t)$ is said to be of exponential order if a real, positive constant σ exists such that the function

$$e^{-\sigma t}|f(t)|$$

approaches zero as t approaches infinity. If the limit of the function $e^{-\sigma t}|f(t)|$ approaches zero for σ greater than σ_c and the limit approaches infinity for σ less than σ_c, the value σ_c is called the *abscissa of convergence.*

For the function $f(t) = Ae^{-\alpha t}$

$$\lim_{t\to\infty} e^{-\sigma t}|Ae^{-\alpha t}|$$

approaches zero if $\sigma > -\alpha$. The abscissa of convergence in this case is $\sigma_c = -\alpha$. The integral $\int_0^{\infty} f(t)e^{-st}\,dt$ converges only if σ, the real part of s, is greater than the abscissa of convergence σ_c. Thus the operator s must be chosen as a constant such that this integral converges.

In terms of the poles of the function $F(s)$, the abscissa of convergence σ_c corresponds to the real part of the pole located farthest to the right in the s plane. For example, for the following function $F(s)$,

$$F(s) = \frac{K(s + 3)}{(s + 1)(s + 2)}$$

the abscissa of convergence σ_c is equal to -1. It can be seen that for such functions as t, $\sin\omega t$, and $t\sin\omega t$ the abscissa of convergence is equal to zero. For functions like e^{-ct}, te^{-ct}, $e^{-ct}\sin\omega t$, and so on, the abscissa of convergence is equal $-c$. For functions that increase faster than the exponential function, however, it is impossible to find suitable values of the abcissa of convergence. Therefore, such functions as e^{t^2} and te^{t^2} do not possess Laplace transforms.

The reader should be cautioned that although e^{t^2}(for $0 \le t \le \infty$) does not possess a Laplace transform, the time function defined by

$$\begin{aligned} f(t) &= e^{t^2}, &\quad &\text{for } 0 \le t \le T < \infty \\ &= 0, && \text{for } t < 0, T < t \end{aligned}$$

does possess a Laplace transform since $f(t) = e^{t^2}$ for only a limited time interval $0 \le t \le T$ and not for $0 \le t \le \infty$. Such a signal can be physically generated. Note that the signals that we can physically generate always have corresponding Laplace transforms.

If a function $f(t)$ has a Laplace transform, then the Laplace transform of $Af(t)$, where A is a constant, is given by

$$\mathscr{L}[Af(t)] = A\mathscr{L}[f(t)]$$

This is obvious from the definition of the Laplace transform. Since Laplace transformation is a linear operation, if functions $f_1(t)$ and $f_2(t)$ have Laplace transforms, $F_1(s)$ and $F_1(s)$, respectively, then the Laplace transform of the function $\alpha f_1(t) + \beta f_2(t)$ is given by

$$\mathscr{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha F_1(s) + \beta F_2(s)$$

In what follows, we derive Laplace transforms of a few commonly encountered functions.

Exponential Function. Consider the exponential function

$$\begin{aligned} f(t) &= 0, &&\text{for } t < 0 \\ &= Ae^{-\alpha t}, &&\text{for } t \geq 0 \end{aligned}$$

where A and α are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt = A\int_0^\infty e^{-(\alpha+s)t}\,dt = \frac{A}{s+\alpha}$$

It is seen that the exponential function produces a pole in the complex plane.

In deriving the Laplace transform of $f(t) = Ae^{-\alpha t}$, we required that the real part of s be greater than $-\alpha$ (the abscissa of convergence). A question may immediately arise as to whether or not the Laplace transform thus obtained is valid in the range where $\sigma < -\alpha$ in the s plane. To answer this question, we must resort to the theory of complex variables. In the theory of complex variables, there is a theorem known as the analytic extension theorem. It states that, if two analytic functions are equal for a finite length along any arc in a region in which both are analytic, then they are equal everywhere in the region. The arc of equality is usually the real axis or a portion of it. By using this theorem the form of $F(s)$ determined by an integration in which s is allowed to have any real positive value greater than the abscissa of convergence holds for any complex values of s at which $F(s)$ is analytic. Thus, although we require the real part of s to be greater than the abscissa of convergence to make the integral $\int_0^\infty f(t)e^{-st}\,dt$ absolutely convergent, once the Laplace transform $F(s)$ is obtained, $F(s)$ can be considered valid throughout the entire s plane except at the poles of $F(s)$.

Step Function. Consider the step function

$$\begin{aligned} f(t) &= 0, &&\text{for } t < 0 \\ &= A, &&\text{for } t > 0 \end{aligned}$$

where A is a constant. Note that it is a special case of the exponential function $Ae^{-\alpha t}$, where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace transform is given by

$$\mathcal{L}[A] = \int_0^\infty Ae^{-st}\,dt = \frac{A}{s}$$

In performing this integration, we assumed that the real part of s was greater than zero (the abscissa of convergence) and therefore that $\lim_{t\to\infty} e^{-st}$ was zero. As stated earlier, the Laplace transform thus obtained is valid in the entire s plane except at the pole $s = 0$.

The step function whose height is unity is called *unit-step* function. The unit-step function that occurs at $t = t_0$ is frequently written as $1(t - t_0)$. The step function of height A that occurs at $t = 0$ can then be written as $f(t) = A1(t)$. The Laplace transform of the unit-step function, which is defined by

$$\begin{aligned} 1(t) &= 0, &&\text{for } t < 0 \\ &= 1, &&\text{for } t > 0 \end{aligned}$$

is $1/s$, or

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

Physically, a step function occurring at $t = 0$ corresponds to a constant signal suddenly applied to the system at time t equals zero.

Ramp Function. Consider the ramp function

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= At, && \text{for } t \geq 0 \end{aligned}$$

where A is a constant. The Laplace transform of this ramp function is obtained as

$$\begin{aligned} \mathcal{L}[At] &= \int_0^\infty Ate^{-st}\,dt = At\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{Ae^{-st}}{-s}\,dt \\ &= \frac{A}{s}\int_0^\infty e^{-st}\,dt = \frac{A}{s^2} \end{aligned}$$

Sinusoidal Function. The Laplace transform of the sinusoidal function

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= A\sin\omega t, && \text{for } t \geq 0 \end{aligned}$$

where A and ω are constants, is obtained as follows. Referring to Equation (2–3), $\sin\omega t$ can be written

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Hence

$$\begin{aligned} \mathcal{L}[A\sin\omega t] &= \frac{A}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}\,dt \\ &= \frac{A}{2j}\frac{1}{s - j\omega} - \frac{A}{2j}\frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2} \end{aligned}$$

Similarly, the Laplace transform of $A\cos\omega t$ can be derived as follows:

$$\mathcal{L}[A\cos\omega t] = \frac{As}{s^2 + \omega^2}$$

Comments. The Laplace transform of any Laplace transformable function $f(t)$ can be found by multiplying $f(t)$ by e^{-st} and then integrating the product from $t = 0$ to $t = \infty$. Once we know the method of obtaining the Laplace transform, however, it is not necessary to derive the Laplace transform of $f(t)$ each time. Laplace transform tables can conveniently be used to find the transform of a given function $f(t)$. Table 2–1 shows Laplace transforms of time functions that will frequently appear in linear control systems analysis.

Table 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \qquad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \qquad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \qquad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \qquad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

(*continues on next page*)

Table 2–1 *(continued)*

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1,\quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1,\quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega} t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

In the following discussion we present Laplace transforms of functions as well as theorems on the Laplace transformation that are useful in the study of linear control systems.

Translated Function. Let us obtain the Laplace transform of the translated function $f(t - \alpha)1(t - \alpha)$, where $\alpha \geq 0$. This function is zero for $t < \alpha$. The functions $f(t)1(t)$ and $f(t - \alpha)1(t - \alpha)$ are shown in Figure 2–1.

By definition, the Laplace transform of $f(t - \alpha)1(t - \alpha)$ is

$$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = \int_0^\infty f(t - \alpha)1(t - \alpha)e^{-st}\,dt$$

By changing the independent variable from t to τ, where $\tau = t - \alpha$, we obtain

$$\int_0^\infty f(t - \alpha)1(t - \alpha)e^{-st}\,dt = \int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau$$

Since in this book we always assume that $f(t) = 0$ for $t < 0$, $f(\tau)1(\tau) = 0$ for $\tau < 0$. Hence we can change the lower limit of integration from $-\alpha$ to 0. Thus

$$\begin{aligned}\int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau &= \int_0^\infty \phi(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}e^{-\alpha s}\,d\tau \\ &= e^{-\alpha s}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-\alpha s}F(s)\end{aligned}$$

where

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

And so

$$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = e^{-\alpha s}F(s), \qquad \text{for } \alpha \geq 0$$

This last equation states that the translation of the time function $f(t)1(t)$ by α (where $\alpha \geq 0$) corresponds to the multiplication of the transform $F(s)$ by $e^{-\alpha s}$.

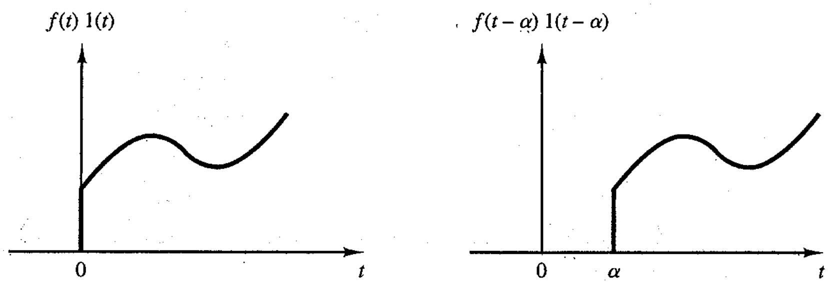

Figure 2–1 Function $f(t)1(t)$ and translated function $f(t - \alpha)1(t - \alpha)$.

Pulse Function. Consider the pulse function

$$f(t) = \frac{A}{t_0}, \qquad \text{for } 0 < t < t_0$$
$$= 0, \qquad \text{for } t < 0,\, t_0 < t$$

where A and t_0 are constants.

The pulse function here may be considered a step function of height A/t_0 that begins at $t = 0$ and that is superimposed by a negative step function of height A/t_0 beginning at $t = t_0$; that is,

$$f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$$

Then the Laplace transform of $f(t)$ is obtained as

$$\begin{aligned}
\mathscr{L}[f(t)] &= \mathscr{L}\left[\frac{A}{t_0} 1(t)\right] - \mathscr{L}\left[\frac{A}{t_0} 1(t - t_0)\right] \\
&= \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0} \\
&= \frac{A}{t_0 s}\left(1 - e^{-st_0}\right) \qquad (2\text{–}5)
\end{aligned}$$

Impulse Function. The impulse function is a special limiting case of the pulse function. Consider the impulse function

$$g(t) = \lim_{t_0 \to 0} \frac{A}{t_0}, \qquad \text{for } 0 < t < t_0$$
$$= 0, \qquad \text{for } t < 0,\, t_0 < t$$

Since the height of the impulse function is A/t_0 and the duration is t_0, the area under the impulse is equal to A. As the duration t_0 approaches zero, the height A/t_0 approaches infinity, but the area under the impulse remains equal to A. Note that the magnitude of the impulse is measured by its area.

Referring to Equation (2–5), the Laplace transform of this impulse function is shown to be

$$\begin{aligned}
\mathscr{L}[g(t)] &= \lim_{t_0 \to 0}\left[\frac{A}{t_0 s}\left(1 - e^{-st_0}\right)\right] \\
&= \lim_{t_0 \to 0} \frac{\dfrac{d}{dt_0}\left[A\left(1 - e^{-st_0}\right)\right]}{\dfrac{d}{dt_0}\left(t_0 s\right)} = \frac{As}{s} = A
\end{aligned}$$

Thus the Laplace transform of the impulse function is equal to the area under the impulse.

The impulse function whose area is equal to unity is called the *unit-impulse function* or the *Dirac delta function*. The unit-impulse function occurring at $t = t_0$ is usually denoted by $\delta(t - t_0)$. $\delta(t - t_0)$ satisfies the following:

$$\delta(t - t_0) = 0, \qquad \text{for } t \neq t_0$$

$$\delta(t - t_0) = \infty, \qquad \text{for } t = t_0$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

It should be mentioned that an impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems. If, however, the magnitude of a pulse input to a system is very large and its duration is very short compared to the system time constants, then we can approximate the pulse input by an impulse function. For instance, if a force or torque input $f(t)$ is applied to a system for a very short time duration, $0 < t < t_0$, where the magnitude of $f(t)$ is sufficiently large so that the integral $\int_0^{t_0} f(t)dt$ is not negligible, then this input can be considered an impulse input. (Note that when we describe the impulse input the area or magnitude of the impulse is most important, but the exact shape of the impulse is usually immaterial.) The impulse input supplies energy to the system in an infinitesimal time.

The concept of the impulse function is quite useful in differentiating discontinuous functions. The unit-impulse function $\delta(t - t_0)$ can be considered the derivative of the unit-step function $1(t - t_0)$ at the point of discontinuity $t = t_0$ or

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

Conversely, if the unit-impulse function $\delta(t - t_0)$ is integrated, the result is the unit-step function $1(t - t_0)$. With the concept of the impulse function we can differentiate a function containing discontinuities, giving impulses, the magnitudes of which are equal to the magnitude of each corresponding discontinuity.

Multiplication of $f(t)$ by $e^{-\alpha t}$. If $f(t)$ is Laplace transformable, its Laplace transform being $F(s)$, then the Laplace transform of $e^{-\alpha t}f(t)$ is obtained as

$$\mathscr{L}[e^{-\alpha t}f(t)] = \int_0^{\infty} e^{-\alpha t}f(t)e^{-st}\, dt = F(s + \alpha) \tag{2–6}$$

We see that the multiplication of $f(t)$ by $e^{-\alpha t}$ has the effect of replacing s by $(s + \alpha)$ in the Laplace transform. Conversely, changing s to $(s + \alpha)$ is equivalent to multiplying $f(t)$ by $e^{-\alpha t}$. (Note that α may be real or complex.)

The relationship given by Equation (2–6) is useful in finding the Laplace transforms of such functions as $e^{-\alpha t}\sin \omega t$ and $e^{-\alpha t}\cos \omega t$. For instance, since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} = F(s), \qquad \mathscr{L}[\cos \omega] = \frac{s}{s^2 + \omega^2} = G(s)$$

it follows from Equation (2–6) that the Laplace transforms of $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$ are given, respectively, by

$$\mathscr{L}\left[e^{-\alpha t}\sin\omega t\right] = F(s+\alpha) = \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$\mathscr{L}\left[e^{-\alpha t}\cos\omega t\right] = G(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

Change of Time Scale. In analyzing physical systems, it is sometimes desirable to change the time scale or normalize a given time function. The result obtained in terms of normalized time is useful because it can be applied directly to different systems having similar mathematical equations.

If t is changed into t/α, where α is a positive constant, then the function $f(t)$ is changed into $f(t/\alpha)$. If we denote the Laplace transform of $f(t)$ by $F(s)$, then the Laplace transform of $f(t/\alpha)$ may be obtained as follows:

$$\mathscr{L}\left[f\left(\frac{t}{\alpha}\right)\right] = \int_0^\infty f\left(\frac{t}{\alpha}\right)e^{-st}\,dt$$

Letting $t/\alpha = t_1$ and $\alpha s = s_1$, we obtain

$$\begin{aligned}\mathscr{L}\left[f\left(\frac{t}{\alpha}\right)\right] &= \int_0^\infty f(t_1)e^{-s_1t_1}\,d(\alpha t_1)\\ &= \alpha\int_0^\infty f(t_1)e^{-s_1t_1}\,dt_1\\ &= \alpha F(s_1)\end{aligned}$$

or

$$\mathscr{L}\left[f\left(\frac{t}{\alpha}\right)\right] = \alpha F(\alpha s)$$

As an example, consider $f(t) = e^{-t}$ and $f(t/5) = e^{-0.2t}$. We obtain

$$\mathscr{L}[f(t)] = \mathscr{L}[e^{-t}] = F(s) = \frac{1}{s+1}$$

Hence

$$\mathscr{L}\left[f\left(\frac{t}{5}\right)\right] = \mathscr{L}[e^{-0.2t}] = 5F(5s) = \frac{5}{5s+1}$$

This result can be verified easily by taking the Laplace transform of $e^{-0.2t}$ directly as follows:

$$\mathscr{L}[e^{-0.2t}] = \frac{1}{s+0.2} = \frac{5}{5s+1}$$

Comments on the Lower Limit of the Laplace Integral. In some cases, $f(t)$ possesses an impulse function at $t = 0$. Then the lower limit of the Laplace integral must be clearly specified as to whether it is 0− or 0+, since the Laplace transforms of $f(t)$ differ for these two lower limits. If such a distinction of the lower limit of the Laplace integral is necessary, we use the notations

$$\mathcal{L}_+[f(t)] = \int_{0+}^{\infty} f(t)e^{-st}\,dt$$

$$\mathcal{L}_-[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt = \mathcal{L}_+[f(t)] + \int_{0-}^{0+} f(t)e^{-st}\,dt$$

If $f(t)$ involves an impulse function at $t = 0$, then

$$\mathcal{L}_+[f(t)] \neq \mathcal{L}_-[f(t)]$$

since

$$\int_{0-}^{0+} f(t)e^{-st}\,dt \neq 0$$

for such a case. Obviously, if $f(t)$ does not possess an impulse function at $t = 0$ (that is, if the function to be transformed is finite between $t = 0-$ and $t = 0+$), then

$$\mathcal{L}_+[f(t)] = \mathcal{L}_-[f(t)]$$

2–4 LAPLACE TRANSFORM THEOREMS

This section presents several theorems on Laplace transformation that are important in control engineering.

Real Differentiation Theorem. The Laplace transform of the derivative of a function $f(t)$ is given by

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0) \tag{2–7}$$

where $f(0)$ is the initial value of $f(t)$ evaluated at $t = 0$. [Here we assumed $f(0-) = f(0+) = f(0)$.]

For a given function $f(t)$, the values of $f(0+)$ and $f(0-)$ may be the same or different, as illustrated in Figure 2–2. The distinction between $f(0+)$ and $f(0-)$ is important

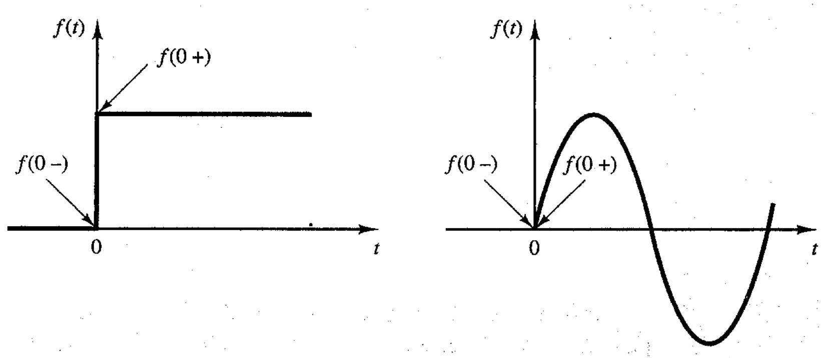

Figure 2–2
Step function and sine function indicating initial values at $t = 0-$ and $t = 0+$.

when $f(t)$ has a discontinuity at $t = 0$ because in such a case $df(t)/dt$ will involve an impulse function at $t = 0$. If $f(0+) \neq f(0-)$, Equation (2–7) must be modified to

$$\mathcal{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

$$\mathcal{L}_-\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0-)$$

To prove the real differentiation theorem, Equation (2–7), we proceed as follows. Integrating the Laplace integral by parts gives

$$\int_0^\infty f(t)e^{-st}\,dt = f(t)\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \left[\frac{d}{dt}f(t)\right]\frac{e^{-st}}{-s}\,dt$$

Hence

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\left[\frac{d}{dt}f(t)\right]$$

It follows that

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

Similarly, we obtain the following relationship for the second derivative of $f(t)$:

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

where $\dot{f}(0)$ is the value of $df(t)/dt$ evaluated at $t = 0$. To derive this equation, define

$$\frac{d}{dt}f(t) = g(t)$$

Then

$$\begin{aligned}\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] &= \mathcal{L}\left[\frac{d}{dt}g(t)\right] = s\mathcal{L}[g(t)] - g(0)\\ &= s\mathcal{L}\left[\frac{d}{dt}f(t)\right] - \dot{f}(0)\\ &= s^2F(s) - sf(0) - \dot{f}(0)\end{aligned}$$

Similarly, for the nth derivative of $f(t)$, we obtain

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - s\overset{(n-2)}{f}(0) - \overset{(n-1)}{f}(0)$$

where $f(0), \dot{f}(0), \ldots, \overset{(n-1)}{f}(0)$ represent the values of $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, respectively, evaluated at $t = 0$. If the distinction between $\mathcal{L}_+$ and $\mathcal{L}_-$ is necessary, we substitute $t = 0+$ or $t = 0-$ into $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, depending on whether we take $\mathcal{L}_+$ or $\mathcal{L}_-$.

Note that, in order for Laplace transforms of derivatives of $f(t)$ to exist, $d^nf(t)/dt^n$ $(n = 1, 2, 3, \ldots)$ must be Laplace transformable.

Note also that, if all the initial values of $f(t)$ and its derivatives are equal to zero, then the Laplace transform of the nth derivative of $f(t)$ is given by $s^nF(s)$.

EXAMPLE 2–1 Consider the cosine function:

$$\begin{aligned} g(t) &= 0, \qquad && \text{for } t < 0 \\ &= \cos\omega t, && \text{for } t \geq 0 \end{aligned}$$

The Laplace transform of this cosine function can be obtained directly as in the case of the sinusoidal function considered earlier. The use of the real differentiation theorem, however, will be demonstrated here by deriving the Laplace transform of the cosine function from the Laplace transform of the sine function. If we define

$$\begin{aligned} f(t) &= 0, \qquad && \text{for } t < 0 \\ &= \sin\omega t, && \text{for } t \geq 0 \end{aligned}$$

then

$$\mathscr{L}[\sin\omega t] = F(s) = \frac{\omega}{s^2 + \omega^2}$$

The Laplace transform of the cosine function is obtained as

$$\begin{aligned} \mathscr{L}[\cos\omega t] &= \mathscr{L}\left[\frac{1}{\omega}\left(\frac{d}{dt}\sin\omega t\right)\right] = \frac{1}{\omega}[sF(s) - f(0)] \\ &= \frac{1}{\omega}\left[\frac{s\omega}{s^2 + \omega^2} - 0\right] = \frac{s}{s^2 + \omega^2} \end{aligned}$$

Final-Value Theorem. The final-value theorem relates the steady-state behavior of $f(t)$ to the behavior of $sF(s)$ in the neighborhood of $s = 0$. This theorem, however, applies if and only if $\lim_{t\to\infty} f(t)$ exists [which means that $f(t)$ settles down to a definite value for $t \to \infty$]. If all poles of $sF(s)$ lie in the left half s plane, $\lim_{t\to\infty} f(t)$ exists. But if $sF(s)$ has poles on the imaginary axis or in the right half s plane, $f(t)$ will contain oscillating or exponentially increasing time functions, respectively, and $\lim_{t\to\infty} f(t)$ will not exist. The final-value theorem does not apply to such cases. For instance, if $f(t)$ is the sinusoidal function $\sin\omega t$, $sF(s)$ has poles at $s = \pm j\omega$ and $\lim_{t\to\infty} f(t)$ does not exist. Therefore, this theorem is not applicable to such a function.

The final-value theorem may be stated as follows. If $f(t)$ and $df(t)/dt$ are Laplace transformable, if $F(s)$ is the Laplace transform of $f(t)$, and if $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

To prove the theorem, we let s approach zero in the equation for the Laplace transform of the derivative of $f(t)$ or

$$\lim_{s\to 0} \int_0^\infty \left[\frac{d}{dt} f(t)\right] e^{-st}\, dt = \lim_{s\to 0} [sF(s) - f(0)]$$

Since $\lim_{s\to 0} e^{-st} = 1$, we obtain

$$\begin{aligned} \int_0^\infty \left[\frac{d}{dt} f(t)\right] dt &= f(t)\Big|_0^\infty = f(\infty) - f(0) \\ &= \lim_{s\to 0} sF(s) - f(0) \end{aligned}$$

from which

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The final-value theorem states that the steady-state behavior of $f(t)$ is the same as the behavior of $sF(s)$ in the neighborhood of $s = 0$. Thus, it is possible to obtain the value of $f(t)$ at $t = \infty$ directly from $F(s)$.

EXAMPLE 2–2 Given

$$\mathscr{L}[f(t)] = F(s) = \frac{1}{s(s+1)}$$

what is $\lim_{t\to\infty} f(t)$?

Since the pole of $sF(s) = 1/(s+1)$ lies in the left half s plane, $\lim_{t\to\infty} f(t)$ exists. So the final-value theorem is applicable in this case.

$$\lim_{t\to\infty} f(t) = f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s(s+1)} = \lim_{s\to 0} \frac{1}{s+1} = 1$$

In fact, this result can easily be verified, since

$$f(t) = 1 - e^{-t}, \qquad \text{for } t \geq 0$$

Initial-Value Theorem. The initial-value theorem is the counterpart of the final-value theorem. By using this theorem, we are able to find the value of $f(t)$ at $t = 0+$ directly from the Laplace transform of $f(t)$. The initial-value theorem does not give the value of $f(t)$ at exactly $t = 0$ but at a time slightly greater than zero.

The initial-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are both Laplace transformable and if $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0+) = \lim_{s\to\infty} sF(s)$$

To prove this theorem, we use the equation for the $\mathscr{L}_+$ transform of $df(t)/dt$:

$$\mathscr{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

For the time interval $0+ \leq t \leq \infty$, as s approaches infinity, e^{-st} approaches zero. (Note that we must use $\mathscr{L}_+$ rather than $\mathscr{L}_-$ for this condition.) And so

$$\lim_{s\to\infty} \int_{0+}^{\infty}\left[\frac{d}{dt}f(t)\right]e^{-st}\,dt = \lim_{s\to\infty}\left[sF(s) - f(0+)\right] = 0$$

or

$$f(0+) = \lim_{s\to\infty} sF(s)$$

In applying the initial-value theorem, we are not limited as to the locations of the poles of $sF(s)$. Thus the initial-value theorem is valid for the sinusoidal function.

It should be noted that the initial-value theorem and the final-value theorem provide a convenient check on the solution, since they enable us to predict the system behavior in the time domain without actually transforming functions in s back to time functions.

Real-Integration Theorem. If $f(t)$ is of exponential order and $f(0-) = f(0+) = f(0)$, then the Laplace transform of $\int f(t)\,dt$ exists and is given by

$$\mathscr{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} \tag{2–8}$$

where $F(s) = \mathscr{L}[f(t)]$ and $f^{-1}(0) = \int f(t)\,dt$ evaluated at $t = 0$.

Note that if $f(t)$ involves an impulse function at $t = 0$, then $f^{-1}(0+) \neq f^{-1}(0-)$. So if $f(t)$ involves an impulse function at $t = 0$, we must modify Equation (2–8) as follows:

$$\mathscr{L}_+\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0+)}{s}$$

$$\mathscr{L}_-\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0-)}{s}$$

The real-integration theorem given by Equation (2–8) can be proved in the following way. Integration by parts yields

$$\begin{aligned}
\mathscr{L}\left[\int f(t)\,dt\right] &= \int_0^\infty \left[\int f(t)\,dt\right] e^{-st}\,dt \\
&= \left[\int f(t)\,dt\right]\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty f(t)\frac{e^{-st}}{-s}\,dt \\
&= \frac{1}{s}\int f(t)\,dt\bigg|_{t=0} + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt \\
&= \frac{f^{-1}(0)}{s} + \frac{F(s)}{s}
\end{aligned}$$

and the theorem is proved.

We see that integration in the time domain is converted into division in the s domain. If the initial value of the integral is zero, the Laplace transform of the integral of $f(t)$ is given by $F(s)/s$.

The preceding real-integration theorem given by Equation (2–8) can be modified slightly to deal with the definite integral of $f(t)$. If $f(t)$ is of exponential order, the Laplace transform of the definite integral $\int_0^t f(t)\,dt$ is given by

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \tag{2–9}$$

where $F(s) = \mathscr{L}[f(t)]$. This is also referred to as the real-integration theorem. Note that if $f(t)$ involves an impulse function at $t = 0$ then $\int_{0+}^t f(t)\,dt \neq \int_{0-}^t f(t)\,dt$ and the following distinction must be observed:

$$\mathscr{L}_+\left[\int_{0+}^t f(t)\,dt\right] = \frac{\mathscr{L}_+[f(t)]}{s}$$

$$\mathscr{L}_-\left[\int_{0-}^t f(t)\,dt\right] = \frac{\mathscr{L}_-[f(t)]}{s}$$

To prove Equation (2–9), first note that

$$\int_0^t f(t)\,dt = \int f(t)\,dt - f^{-1}(0)$$

where $f^{-1}(0)$ is equal to $\int f(t)\,dt$ evaluated at $t = 0$ and is a constant. Hence

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \mathscr{L}\left[\int f(t)\,dt\right] - \mathscr{L}[f^{-1}(0)]$$

Noting that $f^{-1}(0)$ is a constant so that

$$\mathscr{L}[f^{-1}(0)] = \frac{f^{-1}(0)}{s}$$

we obtain

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} - \frac{f^{-1}(0)}{s} = \frac{F(s)}{s}$$

Complex-Differentiation Theorem. If $f(t)$ is Laplace transformable, then, except at poles of $F(s)$,

$$\mathscr{L}[tf(t)] = -\frac{d}{ds}F(s)$$

where $F(s) = \mathscr{L}[f(t)]$. This is known as the complex-differentiation theorem. Also,

$$\mathscr{L}[t^2 f(t)] = \frac{d^2}{ds^2}F(s)$$

In general,

$$\mathscr{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \qquad \text{for } n = 1, 2, 3, \ldots$$

To prove the complex-differentiation theorem, we proceed as follows:

$$\begin{aligned}\mathscr{L}[tf(t)] &= \int_0^\infty tf(t)e^{-st}\,dt = -\int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt \\ &= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = -\frac{d}{ds}F(s)\end{aligned}$$

Hence the theorem. Similarly, by defining $tf(t) = g(t)$, the result is

$$\begin{aligned}\mathscr{L}[t^2 f(t)] = \mathscr{L}[tg(t)] &= -\frac{d}{ds}G(s) = -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right] \\ &= (-1)^2\frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}F(s)\end{aligned}$$

Repeating the same process, we obtain

$$\mathscr{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \qquad \text{for } n = 1, 2, 3, \ldots$$

Convolution Integral. Consider the Laplace transform of

$$\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$$

This integral is often written as

$$f_1(t) * f_2(t)$$

The mathematical operation $f_1(t) * f_2(t)$ is called *convolution*. Note that if we put $t - \tau = \xi$, then

$$\begin{aligned}\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau &= -\int_t^0 f_1(\xi)f_2(t-\xi)\,d\xi \\ &= \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\end{aligned}$$

Hence

$$\begin{aligned}f_1(t) * f_2(t) &= \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau \\ &= \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau \\ &= f_2(t) * f_1(t)\end{aligned}$$

If $f_1(t)$ and $f_2(t)$ are piecewise continuous and of exponential order, then the Laplace transform of

$$\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$$

can be obtained as follows:

$$\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s) \tag{2–10}$$

where

$$\begin{aligned}F_1(s) &= \int_0^\infty f_1(t)e^{-st}\,dt = \mathscr{L}[f_1(t)] \\ F_2(s) &= \int_0^\infty f_2(t)e^{-st}\,dt = \mathscr{L}[f_2(t)]\end{aligned}$$

To prove Equation (2–10) note that $f_1(t-\tau)1(t-\tau) = 0$ for $\tau > t$. Hence

$$\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau = \int_0^\infty f_1(t-\tau)1(t-\tau)f_2(\tau)\,d\tau$$

Then

$$\begin{aligned}\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] &= \mathscr{L}\left[\int_0^\infty f_1(t-\tau)1(t-\tau)f_2(\tau)\,d\tau\right] \\ &= \int_0^\infty e^{-st}\left[\int_0^\infty f_1(t-\tau)1(t-\tau)f_2(\tau)\,d\tau\right]dt\end{aligned}$$

Substituting $t - \tau = \lambda$ in this last equation and changing the order of integration, which is valid in this case because $f_1(t)$ and $f_2(t)$ are Laplace transformable, we obtain

$$\begin{aligned}\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] &= \int_0^\infty f_1(t-\tau)1(t-\tau)e^{-st}\,dt \int_0^\infty f_2(\tau)\,d\tau \\ &= \int_0^\infty f_1(\lambda)e^{-s(\lambda+\tau)}\,d\lambda \int_0^\infty f_2(\tau)\,d\tau \\ &= \int_0^\infty f_1(\lambda)e^{-s\lambda}\,d\lambda \int_0^\infty f_2(\tau)e^{-s\tau}\,d\tau \\ &= F_1(s)F_2(s)\end{aligned}$$

This last equation gives the Laplace transform of the convolution integral. Conversely, if the Laplace transform of a function is given by a product of two Laplace transform functions, $F_1(s)F_2(s)$, then the corresponding time function (the inverse Laplace transform) is given by the convolution integral $f_1(t) * f_2(t)$.

Laplace Transform of Product of Two Time Functions. The Laplace transform of the product of two Laplace transformable functions $f(t)$ and $g(t)$ can be given by

$$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp \tag{2–11}$$

To show this, we may proceed as follows: The Laplace transform of the product of $f(t)$ and $g(t)$ can be written as

$$\mathscr{L}[f(t)g(t)] = \int_0^\infty f(t)g(t)e^{-st}\,dt \tag{2–12}$$

Note that the inversion integral is

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds, \qquad \text{for } t > 0$$

where c is the abscissa of convergence for $F(s)$. Thus,

$$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_0^\infty \int_{c-j\infty}^{c+j\infty} F(p)e^{pt}\,dp\; g(t)e^{-st}\,dt$$

Because of the uniform convergence of the integrals considered, we may invert the order of integration:

$$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,dp \int_0^\infty g(t)e^{-(s-p)t}\,dt$$

Noting that

$$\int_0^\infty g(t)e^{-(s-p)t}\,dt = G(s-p)$$

we obtain

$$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp \tag{2–13}$$

Summary. Table 2–2 summarizes properties and theorems of the Laplace transforms. Most of them have been derived or proved in this section.

Table 2–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathscr{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$ if $\int_0^\infty f(t)\,dt$ exists
10	$\mathscr{L}[e^{-\alpha t}f(t)] = F(s+a)$
11	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
12	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$
15	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$ if $\lim_{t\to 0}\frac{1}{t}f(t)$ exists
16	$\mathscr{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$

2–5 INVERSE LAPLACE TRANSFORMATION

As noted earlier, the inverse Laplace transform can be obtained by use of the inversion integral given by Equation (2–4). However, the inversion integral is complicated and, therefore, its use is not recommended for finding inverse Laplace transforms of commonly encountered functions in control engineering.

A convenient method for obtaining inverse Laplace transforms is to use a table of Laplace transforms. In this case, the Laplace transform must be in a form immediately recognizable in such a table. Quite often the function in question may not appear in tables of Laplace transforms available to the engineer. If a particular transform $F(s)$ cannot be found in a table, then we may expand it into partial fractions and write $F(s)$ in terms of simple functions of s for which the inverse Laplace transforms are already known.

Note that these simpler methods for finding inverse Laplace transforms are based on the fact that the unique correspondence of a time function and its inverse Laplace transform holds for any continuous time function.

Partial-Fraction Expansion Method for Finding Inverse Laplace Transforms. For problems in control systems analysis, $F(s)$, the Laplace transform of $f(t)$, frequently occurs in the form

$$F(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are polynomials in s. In the expansion of $F(s) = B(s)/A(s)$ into a partial-fraction form, it is important that the highest power of s in $A(s)$ be greater than the highest power of s in $B(s)$. If such is not the case, the numerator $B(s)$ must be divided by the denominator $A(s)$ in order to produce a polynomial in s plus a remainder (a ratio of polynomials in s whose numerator is of lower degree than the denominator).

If $F(s)$ is broken up into components,

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

and if the inverse Laplace transforms of $F_1(s), F_2(s), \ldots, F_n(s)$ are readily available, then

$$\begin{aligned}\mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}[F_1(s)] + \mathscr{L}^{-1}[F_2(s)] + \cdots + \mathscr{L}^{-1}[F_n(s)] \\ &= f_1(t) + f_2(t) + \cdots + f_n(t)\end{aligned}$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are the inverse Laplace transforms of $F_1(s), F_2(s), \ldots, F_n(s)$, respectively. The inverse Laplace transform of $F(s)$ thus obtained is unique except possibly at points where the time function is discontinuous. Whenever the time function is continuous, the time function $f(t)$ and its Laplace transform $F(s)$ have a one-to-one correspondence.

The advantage of the partial-fraction expansion approach is that the individual terms of $F(s)$, resulting from the expansion into partial-fraction form, are very simple functions of s; consequently, it is not necessary to refer to a Laplace transform table if we memorize several simple Laplace transform pairs. It should be noted, however, that in applying the partial-fraction expansion technique in the search for the inverse Laplace transform of $F(s) = B(s)/A(s)$ the roots of the denominator polynomial $A(s)$ must be obtained in advance. That is, this method does not apply until the denominator polynomial has been factored.

Partial-Fraction Expansion when $F(s)$ Involves Distinct Poles Only. Consider $F(s)$ written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}, \qquad \text{for } m < n$$

where $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_m$ are either real or complex quantities, but for each complex p_i or z_i there will occur the complex conjugate of p_i or z_i, respectively. If $F(s)$ involves distinct poles only, then it can be expanded into a sum of simple partial fractions as follows:

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \tag{2–14}$$

where a_k $(k = 1, 2, \ldots, n)$ are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation (2–14) by $(s + p_k)$ and letting $s = -p_k$, which gives

$$\begin{aligned}\left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} &= \left[\frac{a_1}{s + p_1}(s + p_k) + \frac{a_2}{s + p_2}(s + p_k)\right. \\ &\qquad \left. + \cdots + \frac{a_k}{s + p_k}(s + p_k) + \cdots + \frac{a_n}{s + p_n}(s + p_k)\right]_{s=-p_k} \\ &= a_k\end{aligned}$$

We see that all the expanded terms drop out with the exception of a_k. Thus the residue a_k is found from

$$a_k = \left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \tag{2–15}$$

Note that, since $f(t)$ is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates, a_1 or a_2, needs to be evaluated because the other is known automatically.

Since

$$\mathcal{L}^{-1}\left[\frac{a_k}{s + p_k}\right] = a_k e^{-p_k t}$$

$f(t)$ is obtained as

$$f(t) = \mathcal{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t}, \qquad \text{for } t \geq 0$$

EXAMPLE 2–3 Find the inverse Laplace transform of

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2}$$

where a_1 and a_2 are found by using Equation (2–15):

$$a_1 = \left[(s + 1)\frac{s + 3}{(s + 1)(s + 2)}\right]_{s=-1} = \left[\frac{s + 3}{s + 2}\right]_{s=-1} = 2$$

$$a_2 = \left[(s + 2)\frac{s + 3}{(s + 1)(s + 2)}\right]_{s=-2} = \left[\frac{s + 3}{s + 1}\right]_{s=-2} = -1$$

Thus

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}[F(s)] \\ &= \mathscr{L}^{-1}\left[\frac{2}{s + 1}\right] + \mathscr{L}^{-1}\left[\frac{-1}{s + 2}\right] \\ &= 2e^{-t} - e^{-2t}, \qquad \text{for } t \geq 0 \end{aligned}$$

EXAMPLE 2–4 Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)}$$

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator.

$$G(s) = s + 2 + \frac{s + 3}{(s + 1)(s + 2)}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is 1 and that the Laplace transform of $d\delta(t)/dt$ is s. The third term on the right-hand side of this last equation is $F(s)$ in Example 2–3. So the inverse Laplace transform of $G(s)$ is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t}, \qquad \text{for } t \geq 0-$$

EXAMPLE 2–5 Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

If the function $F(s)$ involves a pair of complex-conjugate poles, it is convenient not to expand $F(s)$ into the usual partial fractions but to expand it into the sum of a damped sine and a damped cosine function.

Noting that $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$, rewritten thus,

$$\mathscr{L}[e^{-\alpha t}\sin\omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

$$\mathscr{L}[e^{-\alpha t}\cos\omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

the given $F(s)$ can be written as a sum of a damped sine and a damped cosine function.

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2}$$

$$= 5\frac{2}{(s + 1)^2 + 2^2} + 2\frac{s + 1}{(s + 1)^2 + 2^2}$$

It follows that

$$f(t) = \mathscr{L}^{-1}[F(s)]$$

$$= 5\mathscr{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] + 2\mathscr{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right]$$

$$= 5e^{-t}\sin 2t + 2e^{-t}\cos 2t, \qquad \text{for } t \geq 0$$

Partial-Fraction Expansion when $F(s)$ Involves Multiple Poles. Instead of discussing the general case, we shall use an example to show how to obtain the partial-fraction expansion of $F(s)$.

Consider the following $F(s)$:

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

The partial-fraction expansion of this $F(s)$ involves three terms,

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + 1} + \frac{b_2}{(s + 1)^2} + \frac{b_3}{(s + 1)^3}$$

where b_3, b_2, and b_1 are determined as follows. By multiplying both sides of this last equation by $(s + 1)^3$, we have

$$(s + 1)^3\frac{B(s)}{A(s)} = b_1(s + 1)^2 + b_2(s + 1) + b_3 \tag{2–16}$$

Then letting $s = -1$, Equation (2–16) gives

$$\left[(s + 1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

Also, differentiation of both sides of Equation (2–16) with respect to s yields

$$\frac{d}{ds}\left[(s + 1)^3\frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s + 1) \tag{2–17}$$

If we let $s = -1$ in Equation (2–17), then

$$\frac{d}{ds}\left[(s + 1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

By differentiating both sides of Equation (2–17) with respect to s, the result is

$$\frac{d^2}{ds^2}\left[(s + 1)^3\frac{B(s)}{A(s)}\right] = 2b_1$$

From the preceding analysis it can be seen that the values of b_3, b_2, and b_1 are found systematically as follows:

$$\begin{aligned}
b_3 &= \left[(s+1)^3 \frac{B(s)}{A(s)} \right]_{s=-1} \\
&= (s^2 + 2s + 3)_{s=-1} \\
&= 2 \\
b_2 &= \left\{ \frac{d}{ds} \left[(s+1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} \\
&= \left[\frac{d}{ds} (s^2 + 2s + 3) \right]_{s=-1} \\
&= (2s + 2)_{s=-1} \\
&= 0 \\
b_1 &= \frac{1}{2!} \left\{ \frac{d^2}{ds^2} \left[(s+1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} \\
&= \frac{1}{2!} \left[\frac{d^2}{ds^2} (s^2 + 2s + 3) \right]_{s=-1} \\
&= \frac{1}{2}(2) = 1
\end{aligned}$$

We thus obtain

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\
&= e^{-t} + 0 + t^2 e^{-t} \\
&= (1 + t^2)e^{-t}, \qquad \text{for } t \geq 0
\end{aligned}$$

Comments. For complicated functions with denominators involving higher-order polynomials, partial-fraction expansion may be quite time consuming. In such a case, use of MATLAB is recommended. (See Section 2–6.)

2–6 PARTIAL-FRACTION EXPANSION WITH MATLAB

MATLAB has a command to obtain the partial-fraction expansion of $B(s)/A(s)$. It also has a command to obtain the zeros and poles of $B(s)/A(s)$.

We shall first present the MATLAB approach to obtain the partial-fraction expansion of $B(s)/A(s)$. Then we discuss the MATLAB approach to obtain the zeros and poles of $B(s)/A(s)$.

Partial-Fraction Expansion with MATLAB. Consider the following function $B(s)/A(s)$:

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

where some of a_i and b_j may be zero. In MATLAB row vectors num and den specify the coefficients of the numerator and denominator of the transfer function. That is,

```
num = [b0  b1  ...  bn]
den = [1   a1  ...  an]
```

The command

```
[r,p,k] = residue(num,den)
```

finds the residues (r), poles (p), and direct terms (k) of a partial-fraction expansion of the ratio of two polynomials $B(s)$ and $A(s)$.

The partial-fraction expansion of $B(s)/A(s)$ is given by

$$\frac{B(s)}{A(s)} = \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \cdots + \frac{r(n)}{s - p(n)} + k(s) \tag{2–18}$$

Comparing Equations (2–14) and (2–18), we note that $p(1) = -p_1$, $p(2) = -p_2, \ldots,$ $p(n) = -p_n$; $r(1) = a_1$, $r(2) = a_2, \ldots, r(n) = a_n$. [$k(s)$ is a direct term.]

EXAMPLE 2–6 Consider the following transfer function,

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

For this function,

```
num = [2  5  3  6]
den = [1  6  11  6]
```

The command

```
[r,p,k] = residue(num,den)
```

gives the following result:

```
[r,p,k] = residue(num,den)

r =

    -6.0000
    -4.0000
     3.0000

p =

    -3.0000
    -2.0000
    -1.0000

k =

     2
```

(Note that the residues are returned in column vector r, the pole locations in column vector p, and the direct term in row vector k.) This is the MATLAB representation of the following partial-fraction expansion of $B(s)/A(s)$:

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

$$= \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1} + 2$$

The residue command can also be used to form the polynomials (numerator and denominator) from its partial-fraction expansion. That is, the command

```
[num,den] = residue(r,p,k)
```

where r, p, and k are as given in the previous MATLAB output, converts the partial-fraction expansion back to the polynomial ratio $B(s)/A(s)$, as follows:

```
[num,den] = residue(r,p,k);
printsys(num,den,'s')
num/den =
  2s^3 + 5s^2 + 3s + 6
  --------------------
  s^3 + 6s^2 + 11s + 6
```

The command

```
printsys(num,den,'s')
```

prints the num/den in terms of the ratio of polynomials in s.

Note that if $p(j) = p(j+1) = \cdots = p(j+m-1)$ [that is, $p_j = p_{j+1} = \cdots = p_{j+m-1}$], the pole $p(j)$ is a pole of multiplicity m. In such a case, the expansion includes terms of the form

$$\frac{r(j)}{s - p(j)} + \frac{r(j+1)}{[s - p(j)]^2} + \cdots + \frac{r(j+m-1)}{[s - p(j)]^m}$$

For details, see Example 2–7.

EXAMPLE 2–7 Expand the following $B(s)/A(s)$ into partial-fractions with MATLAB.

$$\frac{B(s)}{A(s)} = \frac{s^2 + 2s + 3}{(s+1)^3} = \frac{s^2 + 2s + 3}{s^3 + 3s^2 + 3s + 1}$$

For this function, we have

```
num = [0 1 2 3]
den = [1 3 3 1]
```

The command

```
[r,p,k] = residue(num,den)
```

gives the result shown on the next page. It is the MATLAB representation of the following partial-fraction expansion of $B(s)/A(s)$:

$$\frac{B(s)}{A(s)} = \frac{1}{s+1} + \frac{0}{(s+1)^2} + \frac{2}{(s+1)^3}$$

```
num = [0 1 2 3];
den = [1 3 3 1];
[r,p,k] = residue(num,den)

r =

    1.0000
    0.0000
    2.0000

p =

   -1.0000
   -1.0000
   -1.0000

k =

    []
```

Note that the direct term k is zero.

To obtain the original function $B(s)/A(s)$ from r, p, and k, enter the following program to the computer:

```
[num,den] = residue(r,p,k);
printsys(num,den,'s')
```

Then the computer will show the num/den as follows:

```
num/den =   s^2 + 2s + 3
          ----------------------
          s^3 + 3s^2 + 3s + 1
```

Finding Zeros and Poles of $B(s)/A(s)$ with MATLAB. MATLAB has a command

[z,p,K] = tf2zp(num,den)

to obtain the zeros, poles, and gain K of $B(s)/A(s)$.

Consider the system defined by

$$\frac{B(s)}{A(s)} = \frac{4s^2 + 16s + 12}{s^4 + 12s^3 + 44s^2 + 48s}$$

To obtain the zeros (z), poles (p), and gain (K), enter the following MATLAB program into the computer:

```
num = [0   0   4   16   12];
den = [1   12   44   48   0];
[z,p,K] = tf2zp(num,den)
```

Then the computer will produce the following output on the screen:

```
z =

     -3
     -1

p =

      0
     -6.0000
     -4.0000
     -2.0000

K =

    4
```

The zeros are at $s = -3$ and -1. The poles are at $s = 0, -6, -4$, and -2. The gain K is 4.

If the zeros, poles, and gain K are given, then the following MATLAB program will yield the original num/den.

```
z = [-1; -3];
p = [0; -2; -4; -6];
K = 4;
[num,den] = zp2tf(z,p,K);
printsys(num,den,'s')

num/den =

         4s^2 + 16s + 12
    ------------------------
    s^4 + 12s^3 + 44s^2 + 48s
```

2-7 SOLVING LINEAR, TIME-INVARIANT, DIFFERENTIAL EQUATIONS

In this section we are concerned with the use of the Laplace transform method in solving linear, time-invariant, differential equations.

The Laplace transform method yields the complete solution (complementary solution and particular solution) of linear, time-invariant, differential equations. Classical methods for finding the complete solution of a differential equation require the evaluation of the integration constants from the initial conditions. In the case of the Laplace transform method, however, this requirement is unnecessary because the initial conditions are automatically included in the Laplace transform of the differential equation.

If all initial conditions are zero, then the Laplace transform of the differential equation is obtained simply by replacing d/dt with s, d^2/dt^2 with s^2, and so on.

In solving linear, time-invariant, differential equations by the Laplace transform method, two steps are involved.

1. By taking the Laplace transform of each term in the given differential equation, convert the differential equation into an algebraic equation in s and obtain the expression for the Laplace transform of the dependent variable by rearranging the algebraic equation.
2. The time solution of the differential equation is obtained by finding the inverse Laplace transform of the dependent variable.

In the following discussion, two examples are used to demonstrate the solution of linear, time-invariant, differential equations by the Laplace transform method.

EXAMPLE 2–8 Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \qquad x(0) = a, \qquad \dot{x} = b$$

where a and b are constants.

By writing the Laplace transform of $x(t)$ as $X(s)$ or

$$\mathscr{L}[x(t)] = X(s)$$

we obtain

$$\mathscr{L}[\dot{x}] = sX(s) - x(0)$$

$$\mathscr{L}[\ddot{x}] = s^2X(s) - sx(0) - \dot{x}(0)$$

And so the given differential equation becomes

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 3[sX(s) - x(0)] + 2X(s) = 0$$

By substituting the given initial conditions into this last equation, we obtain

$$[s^2X(s) - as - b] + 3[sX(s) - a] + 2X(s) = 0$$

or

$$(s^2 + 3s + 2)X(s) = as + b + 3a$$

Solving for $X(s)$, we have

$$X(s) = \frac{as + b + 3a}{s^2 + 3s + 2} = \frac{as + b + 3a}{(s + 1)(s + 2)} = \frac{2a + b}{s + 1} - \frac{a + b}{s + 2}$$

The inverse Laplace transform of $X(s)$ gives

$$x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{2a + b}{s + 1}\right] - \mathscr{L}^{-1}\left[\frac{a + b}{s + 2}\right]$$

$$= (2a + b)e^{-t} - (a + b)e^{-2t}, \qquad \text{for } t \geq 0$$

which is the solution of the given differential equation. Notice that the initial conditions a and b appear in the solution. Thus $x(t)$ has no undetermined constants.

EXAMPLE 2–9 Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Noting that $\mathscr{L}[3] = 3/s$, $x(0) = 0$, and $\dot{x}(0) = 0$, the Laplace transform of the differential equation becomes

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

Solving for $X(s)$, we find

$$X(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{s + 2}{s^2 + 2s + 5}$$
$$= \frac{3}{5}\frac{1}{s} - \frac{3}{10}\frac{2}{(s + 1)^2 + 2^2} - \frac{3}{5}\frac{s + 1}{(s + 1)^2 + 2^2}$$

Hence the inverse Laplace transform becomes

$$x(t) = \mathscr{L}^{-1}[X(s)]$$
$$= \frac{3}{5}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{10}\mathscr{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] - \frac{3}{5}\mathscr{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right]$$
$$= \frac{3}{5} - \frac{3}{10}e^{-t}\sin 2t - \frac{3}{5}e^{-t}\cos 2t, \qquad \text{for } t \geq 0$$

which is the solution of the given differential equation.

EXAMPLE PROBLEMS AND SOLUTIONS

A–2–1. Find the poles of the following $F(s)$:

$$F(s) = \frac{1}{1 - e^{-s}}$$

Solution. The poles are found from

$$e^{-s} = 1$$

or

$$e^{-(\sigma + j\omega)} = e^{-\sigma}(\cos\omega - j\sin\omega) = 1$$

From this it follows that $\sigma = 0, \omega = \pm 2n\pi$ $(n = 0, 1, 2, \ldots)$. Thus, the poles are located at

$$s = \pm j2n\pi \qquad (n = 0, 1, 2, \ldots)$$

A–2–2. Find the Laplace transform of $f(t)$ defined by

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= te^{-3t}, && \text{for } t \geq 0 \end{aligned}$$

Solution. Since

$$\mathscr{L}[t] = G(s) = \frac{1}{s^2}$$

referring to Equation (2–6), we obtain

$$F(s) = \mathscr{L}[te^{-3t}] = G(s + 3) = \frac{1}{(s + 3)^2}$$

A–2–3. What is the Laplace transform of

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= \sin(\omega t + \theta), && \text{for } t \geq 0 \end{aligned}$$

where θ is a constant?

Solution. Noting that

$$\sin(\omega t + \theta) = \sin\omega t\cos\theta + \cos\omega t\sin\theta$$

we have

$$\begin{aligned}\mathscr{L}[\sin(\omega t + \theta)] &= \cos\theta\,\mathscr{L}[\sin\omega t] + \sin\theta\,\mathscr{L}[\cos\omega t]\\ &= \cos\theta\frac{\omega}{s^2+\omega^2} + \sin\theta\frac{s}{s^2+\omega^2}\\ &= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\end{aligned}$$

A–2–4. Find the Laplace transform $F(s)$ of the function $f(t)$ shown in Figure 2–3. where $f(t) = 0$ for $t < 0$ and $2a \le t$. Also find the limiting value of $F(s)$ as a approaches zero.

Solution. The function $f(t)$ can be written

$$f(t) = \frac{1}{a^2}1(t) - \frac{2}{a^2}1(t-a) + \frac{1}{a^2}1(t-2a)$$

Then

$$\begin{aligned}F(s) &= \mathscr{L}[f(t)]\\ &= \frac{1}{a^2}\mathscr{L}[1(t)] - \frac{2}{a^2}\mathscr{L}[1(t-a)] + \frac{1}{a^2}\mathscr{L}[1(t-2a)]\\ &= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as}\\ &= \frac{1}{a^2 s}\left(1 - 2e^{-as} + e^{-2as}\right)\end{aligned}$$

As a approaches zero, we have

$$\begin{aligned}\lim_{a\to 0}F(s) &= \lim_{a\to 0}\frac{1-2e^{-as}+e^{-2as}}{a^2 s} = \lim_{a\to 0}\frac{\dfrac{d}{da}\left(1-2e^{-as}+e^{-2as}\right)}{\dfrac{d}{da}(a^2 s)}\\ &= \lim_{a\to 0}\frac{2se^{-as}-2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as}-e^{-2as}}{a}\\ &= \lim_{a\to 0}\frac{\dfrac{d}{da}\left(e^{-as}-e^{-2as}\right)}{\dfrac{d}{da}(a)} = \lim_{a\to 0}\frac{-se^{-as}+2se^{-2as}}{1}\\ &= -s + 2s = s\end{aligned}$$

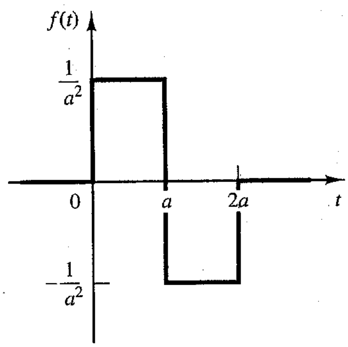

Figure 2–3
Function $f(t)$.

A–2–5. Find the initial value of $df(t)/dt$ when the Laplace transform of $f(t)$ is given by

$$F(s) = \mathscr{L}[f(t)] = \frac{2s + 1}{s^2 + s + 1}$$

Solution. Using the initial-value theorem,

$$\lim_{t \to 0+} f(t) = f(0+) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{s(2s + 1)}{s^2 + s + 1} = 2$$

Since the $\mathscr{L}_+$ transform of $df(t)/dt = g(t)$ is given by

$$\begin{aligned} \mathscr{L}_+[g(t)] &= sF(s) - f(0+) \\ &= \frac{s(2s + 1)}{s^2 + s + 1} - 2 = \frac{-s - 2}{s^2 + s + 1} \end{aligned}$$

the initial value of $df(t)/dt$ is obtained as

$$\begin{aligned} \lim_{t \to 0+} \frac{df(t)}{dt} &= g(0+) = \lim_{s \to \infty} s[sF(s) - f(0+)] \\ &= \lim_{s \to \infty} \frac{-s^2 - 2s}{s^2 + s + 1} = -1 \end{aligned}$$

A–2–6. The derivative of the unit-impulse function $\delta(t)$ is called a *unit-doublet* function. (Thus, the integral of the unit-doublet function is the unit-impulse function.) Mathematically, an example of the unit-doublet function, which is usually denoted by $u_2(t)$, may be given by

$$u_2(t) = \lim_{t_0 \to 0} \frac{1(t) - 2[1(t - t_0)] + 1(t - 2t_0)}{t_0^2}$$

Obtain the Laplace transform of $u_2(t)$.

Solution. The Laplace transform of $u_2(t)$ is given by

$$\begin{aligned} \mathscr{L}[u_2(t)] &= \lim_{t_0 \to 0} \frac{1}{t_0^2}\left(\frac{1}{s} - \frac{2}{s}e^{-t_0 s} + \frac{1}{s}e^{-2t_0 s}\right) \\ &= \lim_{t_0 \to 0} \frac{1}{t_0^2 s}\left[1 - 2\left(1 - t_0 s + \frac{t_0^2 s^2}{2} + \cdots\right) + \left(1 - 2t_0 s + \frac{4t_0^2 s^2}{2} + \cdots\right)\right] \\ &= \lim_{t_0 \to 0} \frac{1}{t_0^2 s}\left[t_0^2 s^2 + (\text{higher-order terms in } t_0 s)\right] = s \end{aligned}$$

A–2–7. Find the Laplace transform of $f(t)$ defined by

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= t^2 \sin\omega t, && \text{for } t \geq 0 \end{aligned}$$

Solution. Since

$$\mathscr{L}[\sin\omega t] = \frac{\omega}{s^2 + \omega^2}$$

applying the complex-differentiation theorem

$$\mathscr{L}[t^2 f(t)] = \frac{d^2}{ds^2} F(s)$$

to this problem, we have

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2+\omega^2)^3}$$

A–2–8. Prove that if $f(t)$ is of exponential order and if $\int_0^\infty f(t)\,dt$ exists [which means that $\int_0^\infty f(t)\,dt$ assumes a definite value] then

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

Solution. Note that

$$\int_0^\infty f(t)\,dt = \lim_{t\to\infty}\int_0^t f(t)\,dt$$

Referring to Equation (2–9),

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Since $\int_0^\infty f(t)\,dt$ exists, by applying the final-value theorem to this case,

$$\lim_{t\to\infty}\int_0^t f(t)\,dt = \lim_{s\to 0} s\frac{F(s)}{s}$$

or

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

A–2–9. Prove that if $f(t)$ is a periodic function with period T, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}}$$

Solution.

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt$$

By changing the independent variable from t to τ, where $\tau = t - nT$,

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\int_0^T f(\tau)e^{-s\tau}\,d\tau$$

where we used the fact that $f(\tau + nT) = f(\tau)$ because the function $f(t)$ is periodic with period T. Noting that

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-nTs} &= 1 + e^{-Ts} + e^{-2Ts} + \cdots \\ &= 1 + e^{-Ts}(1 + e^{-Ts} + e^{-2Ts} + \cdots) \\ &= 1 + e^{-Ts}\left(\sum_{n=0}^{\infty} e^{-nTs}\right)\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}}$$

It follows that

$$\mathscr{L}[f(t)] = \frac{\displaystyle\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}}$$

A–2–10. What is the Laplace transform of the periodic function shown in Figure 2–4?

Solution. Note that

$$\begin{aligned}
\int_0^T f(t)e^{-st}\,dt &= \int_0^{T/2} e^{-st}\,dt + \int_{T/2}^{T} (-1)e^{-st}\,dt \\
&= \frac{e^{-st}}{-s}\Bigg|_0^{T/2} - \frac{e^{-st}}{-s}\Bigg|_{T/2}^{T} \\
&= \frac{e^{-(1/2)Ts} - 1}{-s} + \frac{e^{-Ts} - e^{-(1/2)Ts}}{s} \\
&= \frac{1}{s}\left[e^{-Ts} - 2e^{-(1/2)Ts} + 1\right] \\
&= \frac{1}{s}\left[1 - e^{-(1/2)Ts}\right]^2
\end{aligned}$$

Referring to Problem A–2–9, we have

$$\begin{aligned}
F(s) &= \frac{\displaystyle\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}} = \frac{(1/s)\left[1 - e^{-(1/2)Ts}\right]^2}{1 - e^{-Ts}} \\
&= \frac{1 - e^{-(1/2)Ts}}{s\left[1 + e^{-(1/2)Ts}\right]} = \frac{1}{s}\tanh\frac{Ts}{4}
\end{aligned}$$

A–2–11. Find the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Solution. Since

$$s^2 + 2s + 2 = (s + 1 + j1)(s + 1 - j1)$$

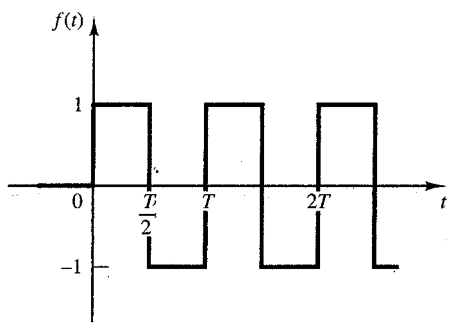

Figure 2–4
Periodic function (square wave).

we notice that $F(s)$ involves a pair of complex-conjugate poles, and so we expand $F(s)$ into the form

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2s + 2}$$

where a_1, a_2, and a_3 are determined from

$$1 = a_1(s^2 + 2s + 2) + (a_2 s + a_3)s$$

By comparing coefficients of s^2, s, and s^0 terms on both sides of this last equation, respectively, we obtain

$$a_1 + a_2 = 0, \qquad 2a_1 + a_3 = 0, \qquad 2a_1 = 1$$

from which

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1$$

Therefore,

$$\begin{aligned} F(s) &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{s^2 + 2s + 2} \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+1)^2 + 1^2} - \frac{1}{2}\frac{s+1}{(s+1)^2 + 1^2} \end{aligned}$$

The inverse Laplace transform of $F(s)$ gives

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t, \qquad \text{for } t \geq 0$$

A–2–12. Obtain the inverse Laplace transform of

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}$$

Solution.

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)} = \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

where

$$\begin{aligned} a_1 &= \left.\frac{5(s+2)}{s^2(s+3)}\right|_{s=-1} = \frac{5}{2} \\ a_2 &= \left.\frac{5(s+2)}{s^2(s+1)}\right|_{s=-3} = \frac{5}{18} \\ b_2 &= \left.\frac{5(s+2)}{(s+1)(s+3)}\right|_{s=0} = \frac{10}{3} \\ b_1 &= \frac{d}{ds}\left[\frac{5(s+2)}{(s+1)(s+3)}\right]_{s=0} \\ &= \left.\frac{5(s+1)(s+3) - 5(s+2)(2s+4)}{(s+1)^2(s+3)^2}\right|_{s=0} = -\frac{25}{9} \end{aligned}$$

Thus

$$F(s) = -\frac{25}{9}\frac{1}{s} + \frac{10}{3}\frac{1}{s^2} + \frac{5}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s+3}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = -\frac{25}{9} + \frac{10}{3}t + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t}, \qquad \text{for } t \geq 0$$

A–2–13. Find the inverse Laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 5}{s(s+1)}$$

Solution. Since the numerator polynomial is of higher degree than the denominator polynomial, by dividing the numerator by the denominator until the remainder is a fraction, we obtain

$$F(s) = s^2 + s + 2 + \frac{2s+5}{s(s+1)} = s^2 + s + 2 + \frac{a_1}{s} + \frac{a_2}{s+1}$$

where

$$a_1 = \frac{2s+5}{s+1}\bigg|_{s=0} = 5$$

$$a_2 = \frac{2s+5}{s}\bigg|_{s=-1} = -3$$

It follows that

$$F(s) = s^2 + s + 2 + \frac{5}{s} - \frac{3}{s+1}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{d^2}{dt^2}\delta(t) + \frac{d}{dt}\delta(t) + 2\delta(t) + 5 - 3e^{-t}, \qquad \text{for } t \geq 0-$$

A–2–14. Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + \omega^2)}$$

Solution.

$$F(s) = \frac{1}{s(s^2+\omega^2)} = \frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2+\omega^2}$$

Hence the inverse Laplace transform of $F(s)$ is obtained as

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{\omega^2}(1 - \cos\omega t), \qquad \text{for } t \geq 0$$

A–2–15. Obtain the inverse Laplace transform of the following $F(s)$. [Use MATLAB to find the partial-fraction expansion of $F(s)$.]

$$F(s) = \frac{s^5 + 8s^4 + 23s^3 + 35s^2 + 28s + 3}{s^3 + 6s^2 + 8s}$$

Solution. The following MATLAB program will produce the partial-fraction expansion of $F(s)$:

```
num = [1  8  23  35  28  3];
den = [0  0  1  6  8  0];
[r,p,k] = residue(num,den)

r =

    0.3750
    0.2500
    0.3750

p =

    -4
    -2
     0

k =

   1  2  3
```

Note that k = [1 2 3] means that $F(s)$ involves $s^2 + 2s + 3$ as shown below:

$$F(s) = s^2 + 2s + 3 + \frac{0.375}{s+4} + \frac{0.25}{s+2} + \frac{0.375}{s}$$

Hence, the inverse Laplace transform of $F(s)$ is given by

$$f(t) = \frac{d^2}{dt^2}\delta(t) + 2\frac{d}{dt}\delta(t) + 3\delta(t) + 0.375e^{-4t} + 0.25e^{-2t} + 0.375, \qquad \text{for } t \geq 0-$$

A–2–16. Given the zero(s), pole(s), and gain K of $B(s)/A(s)$, obtain the function $B(s)/A(s)$ using MATLAB. Consider the three cases below.

(1) There is no zero. Poles are at $-1 + 2j$ and $-1 - 2j$. $K = 10$.

(2) A zero is at 0. Poles are at $-1 + 2j$ and $-1 - 2j$. $K = 10$.

(3) A zero is at -1. Poles are at -2, -4 and -8. $K = 12$.

Solution. MATLAB programs to obtain $B(s)/A(s)$ = num/den for the three cases are shown below.

```
z = [];
p = [-1+2*j; -1-2*j];
K = 10;
[num,den] = zp2tf(z,p,K);
printsys(num,den)

num/den =

       10
   ------------
   s^2 + 2s + 5
```

```
z = [0];
p = [-1+2*j; -1-2*j];
K = 10;
[num,den] = zp2tf(z,p,K);
printsys(num,den)

num/den =

       10s
   ------------
   s^2 + 2s + 5
```

```
z = [-1];
p = [-2; -4; -8];
K = 12;
[num,den] = zp2tf(z,p,K);
printsys(num,den)

num/den =

           12s + 12
   ------------------------
   s^3 + 14s^2 + 56s + 64
```

A–2–17. Solve the following differential equation:

$$\ddot{x} + 2\dot{x} + 10x = t^2, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Solution. Noting that the initial conditions are zeros, the Laplace transform of the equation becomes as follows:

$$s^2X(s) + 2sX(s) + 10X(s) = \frac{2}{s^3}$$

Hence

$$X(s) = \frac{2}{s^3(s^2 + 2s + 10)}$$

We need to find the partial-fraction expansion of $X(s)$. Since the denominator involves a triple pole, it is simpler to use MATLAB to obtain the partial-fraction expansion. The following MATLAB program may be used:

```
num = [0  0  0  0  0  2];
den = [1  2  10  0  0  0];
[r,p,k] = residue(num,den)

r =

     0.0060- 0.0087i
     0.0060+ 0.0087i
    -0.0120
    -0.0400
     0.2000

p =

    -1.0000+ 3.0000i
    -1.0000- 3.0000i
     0
     0
     0

k =

     []
```


From the MATLAB output, we find

$$X(s) = \frac{0.006 - 0.0087j}{s + 1 - 3j} + \frac{0.006 + 0.0087j}{s + 1 + 3j} + \frac{-0.012}{s} + \frac{-0.04}{s^2} + \frac{0.2}{s^3}$$

Combining the first two terms on the right-hand side of the equation, we get

$$X(s) = \frac{0.012(s + 1) + 0.0522}{(s + 1)^2 + 3^2} - \frac{0.012}{s} - \frac{0.04}{s^2} + \frac{0.2}{s^3}$$

The inverse Laplace transform of $X(s)$ gives

$$x(t) = 0.012e^{-t}\cos 3t + 0.0174e^{-t}\sin 3t - 0.012 - 0.04t + 0.1t^2, \qquad \text{for } t \geq 0$$

B–2–1. Find the Laplace transforms of the following functions:

(a)
$$f_1(t) = 0, \qquad \text{for } t < 0$$
$$= e^{-0.4t}\cos 12t, \qquad \text{for } t \geq 0$$

(b)
$$f_2(t) = 0, \qquad \text{for } t < 0$$
$$= \sin\left(4t + \frac{\pi}{3}\right), \qquad \text{for } t \geq 0$$

B–2–2. Find the Laplace transforms of the following functions:

(a)
$$f_1(t) = 0, \qquad \text{for } t < 0$$
$$= 3\sin(5t + 45^\circ), \qquad \text{for } t \geq 0$$

(b)
$$f_2(t) = 0, \qquad \text{for } t < 0$$
$$= 0.03(1 - \cos 2t), \qquad \text{for } t \geq 0$$

B–2–3. Obtain the Laplace transform of the function defined by

$$f(t) = 0, \qquad \text{for } t < 0$$
$$= t^2 e^{-at}, \qquad \text{for } t \geq 0$$

B–2–4. Obtain the Laplace transforms of the following functions:

(a)
$$f(t) = 0, \qquad \text{for } t < 0$$
$$= \sin\omega t \cdot \cos\omega t, \qquad \text{for } t \geq 0$$

(b)
$$f(t) = 0, \qquad \text{for } t < 0$$
$$= te^{-t}\sin 5t, \qquad \text{for } t \geq 0$$

B–2–5. Obtain the Laplace transform of the function defined by

$$f(t) = 0, \qquad \text{for } t < 0$$
$$= \cos 2\omega t \cdot \cos 3\omega t, \qquad \text{for } t \geq 0$$

B–2–6. What is the Laplace transform of the function $f(t)$ shown in Figure 2–5?

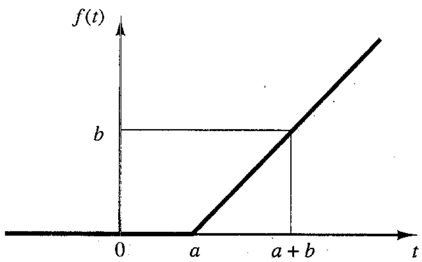

Figure 2–5
Function $f(t)$.

B–2–7. Obtain the Laplace transform of the function $f(t)$ shown in Figure 2–6.

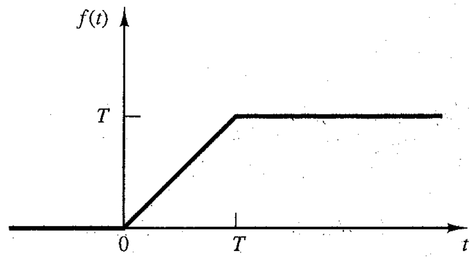

Figure 2–6
Function $f(t)$

B–2–8. Find the Laplace transform of the function $f(t)$ shown in Figure 2–7. Also, find the limiting value of $\mathcal{L}[f(t)]$ as a approaches zero.

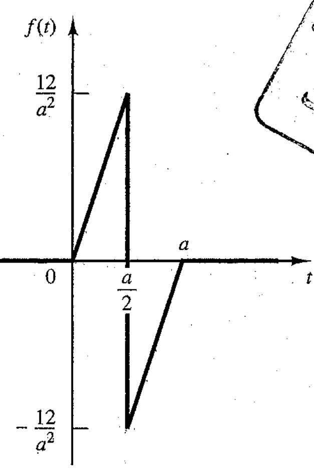

Figure 2–7
Function $f(t)$.

B–2–9. By applying the final-value theorem, find the final value of $f(t)$ whose Laplace transform is given by

$$F(s) = \frac{10}{s(s+1)}$$

Verify this result by taking the inverse Laplace transform of $F(s)$ and letting $t \to \infty$.

B–2–10. Given

$$F(s) = \frac{1}{(s+2)^2}$$

determine the values of $f(0+)$ and $\dot{f}(0+)$. (Use the initial-value theorem.)

B–2–11. Find the inverse Laplace transform of

$$F(s) = \frac{s+1}{s(s^2+s+1)}$$

B–2–12. Obtain the inverse Laplace transform of the following function:

$$F(s) = \frac{5e^{-s}}{s+1}$$

B–2–13. Find the inverse Laplace transforms of the following functions:

(a) $$F_1(s) = \frac{6s+3}{s^2}$$

(b) $$F_2(s) = \frac{5s+2}{(s+1)(s+2)^2}$$

B–2–14. Find the inverse Laplace transforms of the following functions:

(a) $$F_1(s) = \frac{1}{s^2(s^2+\omega^2)}$$

(b) $$F_2(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad (0 < \zeta < 1)$$

B–2–15. Obtain the partial-fraction expansion of the following function with MATLAB:

$$F(s) = \frac{10(s+2)(s+4)}{(s+1)(s+3)(s+5)^2}$$

Then, obtain the inverse Laplace transform of $F(s)$.

B–2–16. Consider the following function $F(s)$:

$$F(s) = \frac{s^4+5s^3+6s^2+9s+30}{s^4+6s^3+21s^2+46s+30}$$

Using MATLAB, obtain the partial-fraction expansion of $F(s)$. Then, obtain the inverse Laplace transform of $F(s)$.

B–2–17. A function $B(s)/A(s)$ consists of the following zeros, poles, and gain K:

$$\begin{aligned}&\text{zeros at } s=-1,\ s=-2\\ &\text{poles at } s=0,\ s=-4,\ s=-6\\ &\text{gain } K=5\end{aligned}$$

Obtain the expression for $B(s)/A(s) = \text{num/den}$ with MATLAB.

B–2–18. What is the solution of the following differential equation?

$$2\ddot{x}+7\dot{x}+3x=0, \qquad x(0)=3, \qquad \dot{x}(0)=0$$

B–2–19. Solve the differential equation

$$\dot{x}+2x=\delta(t), \qquad x(0-)=0$$

B–2–20. Solve the following differential equation:

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=0, \qquad x(0)=a, \qquad \dot{x}(0)=b$$

where a and b are constants.

B–2–21. Obtain the solution of the differential equation

$$\dot{x}+ax=A\sin\omega t, \qquad x(0)=b$$

B–2–22. Obtain the solution of the differential equation

$$\ddot{x}+3\dot{x}+6x=0, \qquad x(0)=0, \qquad \dot{x}(0)=3$$

B–2–23. Solve the following differential equation:

$$\ddot{x}+2\dot{x}+10x=e^{-t}, \qquad x(0)=0, \qquad \dot{x}(0)=0$$

The foreing function e^{-t} is given at $t = 0$ when the system is at rest.