Monte Carlo Bootstrap: current developments.

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1 Introduction

The Conformal Bootstrap offers a way to chart the space of acceptable CFTs.

There has recently been a great revival of interest in the conformal bootstrap program [1, 2] after ref.[3] observed that its applicability extends to Conformal Field Theories (CFTs) in $d > 2$ dimensions. Since ref.[3], considerable progress has been achieved in understanding CFTs in $d \geq 2$ dimensions, both numerically and analytically. Probably the most striking progress has been made in the numerical study of the 3D Ising model, where amazingly precise operator dimensions and OPE coefficients have been determined [4, 5, 6].

2 Bootstrap Basics.

Consider the 4-point function of a scalar primary operator ϕ with scaling dimension Δ_ϕ:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \;=\; \frac{g(z,\bar{z})}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}\,, \tag{2.1}$$

where z and $\bar{z}$ are so called conformally-invariant cross-ratios made up of $x_{ij} \equiv x_i - x_j$. Applying the OPE to the operator pairs $\phi(x_1)\phi(x_2)$ and $\phi(x_3)\phi(x_4)$ in the 4-point function, one can write

$$g(z,\bar{z}) = 1 + \sum_{\Delta,l} p_{\mathcal{O}}\, g_{\Delta,l}(z,\bar{z})\,, \tag{2.2}$$

where the sum runs over all primary operators $\mathcal{O}$ that appear in the $\phi \times \phi$ OPE with Δ and l being respectively their dimension and spin. Notice that l has to be even for identical scalars. For each primary, the sum over all its descendants is encoded in the conformal block function $g_{\Delta,l}(z,\bar{z})$. Reflection positivity (i.e. unitarity) implies bounds on both the squared OPE coefficients $p_{\mathcal{O}}$ and the scaling dimensions Δ:

$$p_{\mathcal{O}} > 0 \;\; \forall \mathcal{O}\,, \quad \Delta \geq l + d - 2 \;\; (l > 0)\,, \quad \Delta \geq \frac{d-2}{2} \;\; (l = 0). \tag{2.3}$$

Imposing crossing symmetry to the 4-point function (2.1) gives, in any number of dimensions,

$$\sum_{\Delta,l} p_{\mathcal{O}}\, \mathcal{F}_{\mathcal{O}}(z,\bar{z}) \;=\; (z\bar{z})^{\Delta_\phi} - ((1-z)(1-\bar{z}))^{\Delta_\phi}\,, \tag{2.4}$$

where

$$\mathcal{F}_{\mathcal{O}}(z,\bar{z}) \;\equiv\; ((1-z)(1-\bar{z}))^{\Delta_\phi} g_{\Delta,l}(z,\bar{z}) - (z\bar{z})^{\Delta_\phi} g_{\Delta,l}(1-z,1-\bar{z})\,. \tag{2.5}$$

3 General Intuition of the MC bootstrap

One can consider 2.4 as a function of the spectrum and the OPE coefficients, which should only be satisfied for physical conformal data. Since any numerical approach involves truncating the OPE up to a maximum dimension Δ^*, it is

important to estimate the remainder of the series in order to bound the amount by which 2.4 might be violated in a given approximation. This was done by Rychkov et al, who found the following error estimate holds (for $\Delta_* \gg 1$):

$$\Big| \sum_{(\Delta \geq \Delta_*), l} p_{\mathcal{O}}\, g_{\Delta,l}(z,\bar{z}) \Big| \leq \mathcal{R}(z,\bar{z})\,, \tag{3.1}$$

where

$$\mathcal{R}(z,\bar{z}) \equiv \frac{(-\log|\rho(z)|)^{-4\Delta_\phi+1}\, 2^{4\Delta_\phi+1}}{\Gamma(4\Delta_\phi)} \frac{\Gamma(4\Delta_\phi, -\Delta_* \log|\rho(z)|)}{(1-|\rho(z)|^2)}\,. \tag{3.2}$$

This motivates trying to find conformal data numerically using some sort of gradient descent. In this case, one can take the truncated Bootstrap equations as a function of a discrete spectrum and OPE coefficients and minimize it with a Monte Carlo simulation. Concretely, one can rewrite the bootstrap equations as

$$\sum_{\Delta<\Delta^*, l} p_{\cdot,l}\, \mathcal{F}_{\mathcal{O}}(z,\bar{z}) \;=\; (z\bar{z})^{\Delta_\phi} - ((1-z)(1-\bar{z}))^{\Delta_\phi} + \mathcal{E}(z,\bar{z}), \tag{3.3}$$

where l is bounded in turn by 2.3, and

$$|\mathcal{E}(z,\bar{z})| \leq \mathcal{E}_{\max}(z,\bar{z}) \equiv v^{\Delta_\phi}\mathcal{R}(z,\bar{z}) + u^{\Delta_\phi}\mathcal{R}(1-z,1-\bar{z}). \tag{3.4}$$

Since these equations have to be satisfied everywhere outside the branch cut at $[1,\infty)$, one can take a finite set of points $\{z_i\}_{i=1}^{N_z}$ and write 3.3 as

$$\mathcal{M}\cdot\vec{\rho} = \vec{\sigma} + \vec{\epsilon} \tag{3.5}$$

where

$$\vec{\sigma} \equiv \begin{pmatrix} |z_1|^{2\Delta_\phi} - |1-z_1|^{2\Delta_\phi} \\ |z_2|^{2\Delta_\phi} - |1-z_2|^{2\Delta_\phi} \\ \vdots \end{pmatrix} \quad \text{and} \quad \vec{\epsilon} \equiv \begin{pmatrix} \mathcal{E}(z_1,\bar{z}_1,\Delta_*,\Delta_\phi) \\ \mathcal{E}(z_2,\bar{z}_2,\Delta_*,\Delta_\phi) \\ \vdots \end{pmatrix}. \tag{3.6}$$

and

$$\mathcal{M} \equiv \begin{pmatrix} \mathcal{F}_{\Delta_1,l_1}(z_1,\bar{z}_1) & \mathcal{F}_{\Delta_2,l_2}(z_1,\bar{z}_1) & \cdots \\ \mathcal{F}_{\Delta_1,l_1}(z_2,\bar{z}_2) & \mathcal{F}_{\Delta_2,l_2}(z_2,\bar{z}_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.7}$$

The straightforward path is to define a cost function

$$C \sim |\mathcal{M}\cdot\vec{\rho} - \vec{\sigma}|$$

and minimize it stochastically. This naive approach proved to be untractable in the early stages of the research, mostly because it became exponentially time consuming to look for more operators. Thus, we decided to split the optimization in two steps. One that finds an approximate spectrum without worrying about the OPE coefficients, and then solving for the complete conformal data with a more refined Monte Carlo. The first step was motivated by Gliozzi's determinant bootstrap, which we review in the following section.

4 Review of Gliozzi's paper

In 2013, Gliozzi realized that, for N_z bigger than the number of operators one considered in the OPE, 3.5 represented an over-determined Linear System. Thus, in order for a solution (that is, a set of OPE coefficients satisfying crossing) to exist, each minor should vanish.

5 LogDet implementation

5.1 naive approach

Given a putative spectrum, the determinant of 3.7 should get closer to zero as we approach a physically sensible set of operator dimensions. Since this quantity could range over many orders of magnitude, we decided to square it and then take the log of it. This proved to be numerically troubling, since the behaviour of the logarithm around 0 is far from differentiable and the landscape was thus full of spurious minima out of which the Metropolis algorithm could not get out.

5.2 Cross smearing

In order to smooth this behaviour out, we implemented an empirical renormalization that consists in computing the value of the determinant at adjacent points to the main one. Thus

$$\begin{aligned} S(\Delta_i, l_i) &= \log \mathrm{Det}^2 \mathcal{M}(\{\Delta_i, l_i\}) \\ \rightarrow S_{\mathrm{cross}}(\{\Delta_i, l_i\}) &= \log \mathrm{Det}^2 \mathcal{M}(\{\Delta_i, l_i\}) + \sum_i \left(\log \mathrm{Det}^2 \mathcal{M}(\{\Delta_i + \delta, l_i\}) + \log \mathrm{Det}^2 \mathcal{M}(\{\Delta_i - \delta, l_i\})\right) \end{aligned}$$

This action was much more efficient at finding the correct dimensions for the free theory, and numerical exploration showed that the basin of attraction for each given operator's dimension was very sharp, as long as the other ones were kept at the exact value. Small perturbations destroyed this basin rather quickly, and thus it was hard to get the MC routine to solve for a large number of operators (~ 7)at the same time.

5.3 Recursive with check

The next breakthrough in the routine came when we decided to start with few (~ 4) operators and add the next operators one by one. To avoid spoiling the previous solution, only the new operator was varied for a given number of steps before we allowed the whole spectrum to be optimized. This gave a very efficient routine with a large radius of convergence.

This recursive step was further improved by using 3.4 to bound 3.5 as

$$|\mathcal{M} \cdot \vec{\rho} - \vec{\sigma}| < \vec{\epsilon}_{\max}. \tag{5.1}$$

Then, the result of each end state of the MC could be tested to see if it satisfied Crossing Symmetry with the precision correspoinding to the Δ^* considered. Otherwise, it changed the seed of the pseudorandom generator and started again the MC.

6 χ^2 refinement

6.1 cWLS as a way to determine OPEs

Since the dependence of 5.1 on $\vec{\rho}$ is linear, for $N_z > N_{Op}$ it is possible to find the set of OPE coefficients that best satisfy crossing via the Weighted Least Squares method[1]. For a generic spectrum obtained via the method described in section **??**, this naive set of OPE coefficients will have some negative components, which stems from the fact that our spectrum is only approximate. Since the error estimate depends crucially on the positivity of the OPE coefficients, we need to find the solution that minimizes 5.1 and has all entries non-negative. This is attained by setting the first negative OPE coefficient (from back to forth, if $\Delta_i < \Delta_{i+1}$) and then running a WLS routine with the remaining entries. If this solution still has negative entries, the proccedure is repeated until an acceptable $\vec{\rho}$ is obtained.

[1] Should we add a reference here, other than Wikipedia?

It is an interesting fact observed in the cases studied so far, that this constrained solution doesn't satisfy crossing with the error estimate corresponding to the original Δ^*, but once we consider the zeros in $\vec{\rho}$, which effectively remove higher dimensional operators from the OPE, 5.1 is satisfied.

For properly chosen points in z, not all the inequalties in 5.1 will be satisfied, one can then take this number and try to minimize it with another Monte Carlo routine.

7 Summary of results

7.1 Free CFT in 4d

7.2 GFT in 1d

7.3 GFT in 4d

References

[1] S. Ferrara, A. F. Grillo and R. Gatto, "Tensor representations of conformal algebra and conformally covariant operator product expansion," Annals Phys. **76** (1973) 161.

[2] A. M. Polyakov, "Nonhamiltonian approach to conformal quantum field theory," Zh. Eksp. Teor. Fiz. **66** (1974) 23.

[3] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, "Bounding scalar operator dimensions in 4D CFT," JHEP **0812** (2008) 031 [arXiv:0807.0004 [hep-th]].

[4] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, "Solving the 3D Ising Model with the Conformal Bootstrap," Phys. Rev. D **86** (2012) 025022 doi:10.1103/PhysRevD.86.025022 [arXiv:1203.6064 [hep-th]].

[5] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, "Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents," J. Stat. Phys. **157** (2014) 869 doi:10.1007/s10955-014-1042-7 [arXiv:1403.4545 [hep-th]].

[6] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, "Precision Islands in the Ising and $O(n)$ Models," arXiv:1603.04436 [hep-th].

[7] F. Kos, D. Poland and D. Simmons-Duffin, "Bootstrapping the $O(n)$ vector models," JHEP **1406** (2014) 091 doi:10.1007/JHEP06(2014)091 [arXiv:1307.6856 [hep-th]].