Chapter 10
Conservation of Angular Momentum

Conceptual Problems

***1** •

(*a*) True. The cross product of the vectors $\vec{A}$ and $\vec{B}$ is defined to be $\vec{A}\times\vec{B} = AB\sin\phi\,\hat{n}$. If $\vec{A}$ and $\vec{B}$ are parallel, $\sin\phi = 0$.

(*b*) True. By definition, $\vec{\omega}$ is along the axis.

(*c*) True. The direction of a torque exerted by a force is determined by the definition of the cross product.

2 •

Determine the Concept The cross product of the vectors $\vec{A}$ and $\vec{B}$ is defined to be $\vec{A}\times\vec{B} = AB\sin\phi\,\hat{n}$. Hence, the cross product is a maximum when $\sin\phi = 1$. This condition is satisfied provided $\vec{A}$ and $\vec{B}$ are *perpendicular*. $\boxed{(c)\text{ is correct.}}$

3 •

Determine the Concept $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r}\times\vec{p}$. From this definition of the cross product, $\vec{L}$ and $\vec{p}$ are perpendicular; i.e., the angle between them is 90°.

4 •

Determine the Concept $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r}\times\vec{p}$. Because the motion is along a line that passes through point *P*, $r = 0$ and so is L. $\boxed{(b)\text{ is correct.}}$

***5** ••

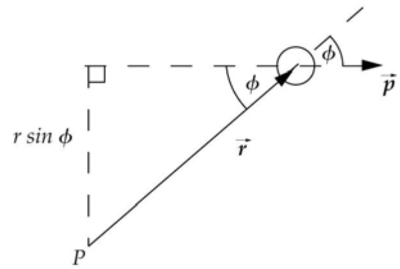
Determine the Concept $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r}\times\vec{p}$.

(*a*) Because $\vec{L}$ is directly proportional to $\vec{p}$: $\boxed{\text{Doubling } \vec{p} \text{ doubles } \vec{L}.}$

(*b*) Because $\vec{L}$ is directly proportional to $\vec{r}$: $\boxed{\text{Doubling } \vec{r} \text{ doubles } \vec{L}.}$

6 ••

Determine the Concept The figure shows a particle moving with constant speed in a straight line (i.e., with constant velocity and constant linear momentum). The magnitude of L is given by $rp\sin\phi = mv(r\sin\phi)$.



Referring to the diagram, note that the distance $r\sin\phi$ from P to the line along which the particle is moving is constant. Hence, $mv(r\sin\phi)$ is constant and so $\boxed{\vec{L} \text{ is constant.}}$

7 •

False. The net torque acting on a rotating system equals the change in the system's angular momentum; i.e., $\tau_{\text{net}} = dL/dt$, where $L = I\omega$. Hence, if τ_{net} is zero, all we can say for sure is that the angular momentum (the product of I and ω) is constant. If I changes, so must ω.

***8** ••

Determine the Concept Yes, you can. Imagine rotating the top half of your body with arms flat at sides through a (roughly) 90° angle. Because the net angular momentum of the system is 0, the bottom half of your body rotates in the opposite direction. Now extend your arms out and rotate the top half of your body back. Because the moment of inertia of the top half of your body is larger than it was previously, the angle which the bottom half of your body rotates through will be smaller, leading to a net rotation. You can repeat this process as necessary to rotate through any arbitrary angle.

9 •

Determine the Concept If L is constant, we know that the *net* torque acting on the system is zero. There may be multiple constant or time-dependent torques acting on the system as long as the net torque is zero. $\boxed{(e) \text{ is correct.}}$

10 ••

Determine the Concept No. In order to do work, a force must act over some distance. In each "inelastic collision" the force of static friction does not act through any distance.

11 ••

Determine the Concept It is easier to crawl radially outward. In fact, a radially inward force is required just to prevent you from sliding outward.

***12** ••

Determine the Concept The pull that the student exerts on the block is at right angles to its motion and exerts no torque (recall that $\vec{\tau} = \vec{r} \times \vec{F}$ and $\tau = rF\sin\theta$). Therefore, we can conclude that the angular momentum of the block is conserved. The student does, however, do work in displacing the block in the direction of the radial force and so the block's energy increases. $\boxed{(b) \text{ is correct.}}$

***13 ••**

Determine the Concept The hardboiled egg is solid inside, so everything rotates with a uniform velocity. By contrast, it is difficult to get the viscous fluid inside a raw egg to start rotating; however, once it is rotating, stopping the shell will not stop the motion of the interior fluid, and the egg may start rotating again after momentarily stopping for this reason.

14 •

False. The relationship $\vec{\boldsymbol{\tau}} = d\vec{\boldsymbol{L}}/dt$ describes the motion of a gyroscope independently of whether it is spinning.

15 •

Picture the Problem We can divide the expression for the kinetic energy of the object by the expression for its angular momentum to obtain an expression for K as a function of I and L.

Express the rotational kinetic energy of the object:

$$K = \tfrac{1}{2} I\omega^2$$

Relate the angular momentum of the object to its moment of inertia and angular velocity:

$$L = I\omega$$

Divide the first of these equations by the second and solve for K to obtain:

$$K = \frac{L^2}{2I} \text{ and so } \boxed{(b) \text{ is correct.}}$$

16 •

Determine the Concept The purpose of the second smaller rotor is to prevent the body of the helicopter from rotating. If the rear rotor fails, the body of the helicopter will tend to rotate on the main axis due to angular momentum being conserved.

17 ••

Determine the Concept One can use a right-hand rule to determine the direction of the torque required to turn the angular momentum vector from east to south. Letting the fingers of your right hand point east, rotate your wrist until your fingers point south. Note that your thumb points downward. $\boxed{(b) \text{ is correct.}}$

18 ••

Determine the Concept In turning east, the man redirects the angular momentum vector from north to east by exerting a clockwise torque (viewed from above) on the gyroscope. As a consequence of this torque, the front end of the suitcase will dip downward. $\boxed{(d)\text{ is correct.}}$

19 ••

(a) The lifting of the nose of the plane rotates the angular momentum vector upward. It veers to the right in response to the torque associated with the lifting of the nose.

(b) The angular momentum vector is rotated to the right when the plane turns to the right. In turning to the right, the torque points down. The nose will move downward.

20 ••

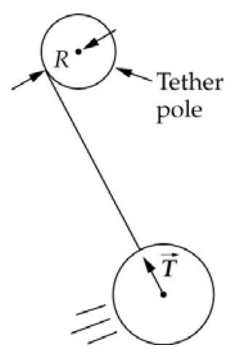
Determine the Concept If $\vec{L}$ points up and the car travels over a hill or through a valley, the force on the wheels on one side (or the other) will increase and car will tend to tip. If $\vec{L}$ points forward and car turns left or right, the front (or rear) of the car will tend to lift. These problems can be averted by having two identical flywheels that rotate on the same shaft in opposite directions.

21 ••

Determine the Concept The rotational kinetic energy of the woman-plus-stool system is given by $K_{\text{rot}} = \frac{1}{2}I\omega^2 = L^2/2I$. Because L is constant (angular momentum is conserved) and her moment of inertia is greater with her arms extended, $\boxed{(b)\text{ is correct.}}$

***22** ••

Determine the Concept Consider the overhead view of a tether pole and ball shown in the adjoining figure. The ball rotates counterclockwise. The torque about the center of the pole is clockwise and of magnitude RT, where R is the pole's radius and T is the tension. So L must decrease and $\boxed{(e)\text{ is correct.}}$



23 ••

Determine the Concept The center of mass of the rod-and-putty system moves in a straight line, and the system rotates about its center of mass.

24 •

(*a*) True. The net external torque acting a system equals the rate of change of the angular momentum of the system; i.e., $\sum_i \vec{\boldsymbol{\tau}}_{i,\text{ext}} = \dfrac{d\vec{\boldsymbol{L}}}{dt}$.

(*b*) False. If the net torque on a body is zero, its angular momentum is *constant* but not necessarily zero.

Estimation and Approximation

***25** ••

Picture the Problem Because we have no information regarding the mass of the skater, we'll assume that her body mass (not including her arms) is 50 kg and that each arm has a mass of 4 kg. Let's also assume that her arms are 1 m long and that her body is cylindrical with a radius of 20 cm. Because the net external torque acting on her is zero, her angular momentum will remain constant during her pirouette.

Express the conservation of her angular momentum during her pirouette:

$$L_\text{i} = L_\text{f}$$

or

$$I_\text{arms out}\omega_\text{arms out} = I_\text{arms in}\omega_\text{arms in} \qquad (1)$$

Express her total moment of inertia with her arms out:

$$I_\text{arms out} = I_\text{body} + I_\text{arms}$$

Treating her body as though it is cylindrical, calculate its moment of inertia of her body, minus her arms:

$$I_\text{body} = \tfrac{1}{2}mr^2 = \tfrac{1}{2}(50\,\text{kg})(0.2\,\text{m})^2 = 1.00\,\text{kg}\cdot\text{m}^2$$

Modeling her arms as though they are rods, calculate their moment of inertia when she has them out:

$$I_\text{arms} = 2\left[\tfrac{1}{3}(4\,\text{kg})(1\,\text{m})^2\right] = 2.67\,\text{kg}\cdot\text{m}^2$$

Substitute to determine her total moment of inertia with her arms out:

$$I_\text{arms out} = 1.00\,\text{kg}\cdot\text{m}^2 + 2.67\,\text{kg}\cdot\text{m}^2 = 3.67\,\text{kg}\cdot\text{m}^2$$

Express her total moment of inertia with her arms in:

$$I_{\text{arms}\,in} = I_\text{body} + I_\text{arms} = 1.00\,\text{kg}\cdot\text{m}^2 + 2\left[(4\,\text{kg})(0.2\,\text{m})^2\right] = 1.32\,\text{kg}\cdot\text{m}^2$$

Solve equation (1) for $\omega_{\text{arms in}}$ and substitute to obtain:

$$\omega_{\text{arms in}} = \frac{I_{\text{arms out}}}{I_{\text{arms in}}}\omega_{\text{arms out}} = \frac{3.67\,\text{kg}\cdot\text{m}^2}{1.32\,\text{kg}\cdot\text{m}^2}(1.5\,\text{rev/s}) = \boxed{4.17\,\text{rev/s}}$$

26 ••

Picture the Problem We can express the period of the earth's rotation in terms of its angular velocity of rotation and relate its angular velocity to its angular momentum and moment of inertia with respect to an axis through its center. We can differentiate this expression with respect to *I* and then use differentials to approximate the changes in *I* and *T*.

Express the period of the earth's rotation in terms of its angular velocity of rotation:

$$T = \frac{2\pi}{\omega}$$

Relate the earth's angular velocity of rotation to its angular momentum and moment of inertia:

$$\omega = \frac{L}{I}$$

Substitute to obtain:

$$T = \frac{2\pi I}{L}$$

Find dT/dI:

$$\frac{dT}{dI} = \frac{2\pi}{L} = \frac{T}{I}$$

Solve for dT/T and approximate ΔT:

$$\frac{dT}{T} = \frac{dI}{I} \text{ or } \Delta T \approx \frac{\Delta I}{I}T$$

Substitute for ΔI and I to obtain:

$$\Delta T \approx \frac{\frac{2}{3}mr^2}{\frac{2}{5}M_{\text{E}}R_{\text{E}}^2}T = \frac{5m}{3M_{\text{E}}}T$$

Substitute numerical values and evaluate ΔT:

$$\Delta T = \frac{5(2.3\times10^{19}\,\text{kg})}{3(6\times10^{24}\,\text{kg})}(1\,\text{d}) = 6.39\times10^{-6}\,\text{d} = 6.39\times10^{-6}\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}} = \boxed{0.552\,\text{s}}$$

27 •

Picture the Problem We can use $L = mvr$ to find the angular momentum of the particle. In (*b*) we can solve the equation $L = \sqrt{\ell(\ell+1)}\hbar$ for $\ell(\ell+1)$ and the approximate value of ℓ.

(*a*) Use the definition of angular momentum to obtain:

$$\begin{aligned} L &= mvr \\ &= (2\times10^{-3}\ \text{kg})(3\times10^{-3}\ \text{m/s})(4\times10^{-3}\ \text{m}) \\ &= \boxed{2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}} \end{aligned}$$

(*b*) Solve the equation $L = \sqrt{\ell(\ell+1)}\hbar$ for $\ell(\ell+1)$:

$$\ell(\ell+1) = \frac{L^2}{\hbar^2}$$

Substitute numerical values and evaluate $\ell(\ell+1)$:

$$\begin{aligned} \ell(\ell+1) &= \left(\frac{2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.05\times01^{-34}\ \text{J}\cdot\text{s}}\right)^2 \\ &= \boxed{5.22\times10^{52}} \end{aligned}$$

Because $\ell >> 1$, approximate its value with the square root of $\ell(\ell+1)$:

$$\ell \approx \boxed{2.29\times10^{26}}$$

(*c*) $\boxed{\text{The quantization of angular momentum is not noticed in macroscopic physics because no experiment can differentiate between } \ell = 2\times10^{26} \text{ and } \ell = 2\times10^{26}+1.}$

***28** ••

Picture the Problem We can use conservation of angular momentum in part (*a*) to relate the before-and-after collapse rotation rates of the sun. In part (*b*), we can express the fractional change in the rotational kinetic energy of the sun as it collapses into a neutron star to decide whether its rotational kinetic energy is greater initially or after the collapse.

(*a*) Use conservation of angular momentum to relate the angular momenta of the sun before and after its collapse:

$$I_b\omega_b = I_a\omega_a \qquad (1)$$

Using the given formula, approximate the moment of inertia I_b of the sun before collapse:

$$\begin{aligned} I_b &= 0.059MR_{\text{sun}}^2 \\ &= 0.059(1.99\times10^{30}\ \text{kg})(6.96\times10^5\ \text{km})^2 \\ &= 5.69\times10^{46}\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

Find the moment of inertia I_a of the sun when it has collapsed into a spherical neutron star of radius 10 km and uniform mass distribution:

$$\begin{aligned} I_a &= \tfrac{2}{5}MR^2 \\ &= \tfrac{2}{5}\left(1.99\times10^{30}\text{ kg}\right)\left(10\text{ km}\right)^2 \\ &= 7.96\times10^{37}\text{ kg}\cdot\text{m}^2 \end{aligned}$$

Substitute in equation (1) and solve for ω_a to obtain:

$$\begin{aligned} \omega_a &= \frac{I_b}{I_a}\omega_b = \frac{5.69\times10^{46}\text{ kg}\cdot\text{m}^2}{7.96\times10^{37}\text{ kg}\cdot\text{m}^2}\omega_b \\ &= 7.15\times10^{8}\,\omega_b \end{aligned}$$

Given that ω_b = 1 rev/25 d, evaluate ω_a:

$$\begin{aligned} \omega_a &= 7.15\times10^{8}\left(\frac{1\text{ rev}}{25\text{ d}}\right) \\ &= \boxed{2.86\times10^{7}\text{ rev/d}} \end{aligned}$$

$$\boxed{\begin{array}{l}\text{The additional rotational kinetic energy comes at the expense of}\\ \text{gravitational potential energy, which decreases as the sun gets smaller.}\end{array}}$$

Note that the rotational period decreases by the same factor of I_b/I_a and becomes:

$$T_a = \frac{2\pi}{\omega_a} = \frac{2\pi}{2.86\times10^{7}\dfrac{\text{rev}}{\text{d}}\times\dfrac{2\pi\text{ rad}}{\text{rev}}\times\dfrac{1\text{d}}{24\text{h}}\times\dfrac{1\text{h}}{3600\text{s}}} = 3.02\times10^{-3}\text{ s}$$

(*b*) Express the fractional change in the sun's rotational kinetic energy as a consequence of its collapse and simplify to obtain:

$$\begin{aligned} \frac{\Delta K}{K_b} &= \frac{K_a - K_b}{K_b} = \frac{K_a}{K_b} - 1 \\ &= \frac{\tfrac{1}{2}I_a\omega_a^2}{\tfrac{1}{2}I_b\omega_b^2} - 1 \\ &= \frac{I_a\omega_a^2}{I_b\omega_b^2} - 1 \end{aligned}$$

Substitute numerical values and evaluate $\Delta K/K_b$:

$$\frac{\Delta K}{K_b} = \left(\frac{1}{7.15\times10^{8}}\right)\left(\frac{2.86\times10^{7}\text{ rev/d}}{1\text{ rev/25d}}\right)^2 - 1 = \boxed{7.15\times10^{8}}$$ (i.e., the rotational kinetic energy *increases* by a factor of approximately 7×10^{8}.)

29 ••

Picture the Problem We can solve $I = CMR^2$ for C and substitute numerical values in order to determine an experimental value of C for the earth. We can then compare this value to those for a spherical shell and a sphere in which the mass is uniformly distributed to decide whether the earth's mass density is greatest near its core or near its crust.

(*a*) Express the moment of inertia of the earth in terms of the constant C:

$$I = CMR^2$$

Solve for C to obtain:

$$C = \frac{I}{MR^2}$$

Substitute numerical values and evaluate C:

$$C = \frac{8.03\times10^{37}\ \text{kg}\cdot\text{m}^2}{\left(5.98\times10^{24}\ \text{kg}\right)\left(6370\,\text{km}\right)^2} = \boxed{0.331}$$

(*b*) If all of the mass were in the crust, the moment of inertia of the earth would be that of a thin spherical shell:

$$I_{\text{spherical shell}} = \tfrac{2}{3}MR^2$$

If the mass of the earth were uniformly distributed throughout its volume, its moment of inertia would be:

$$I_{\text{solid sphere}} = \tfrac{2}{5}MR^2$$

Because experimentally $C < 2/5 = 0.4$, the mass density must be greater near the center of the earth.

***30** ••

Picture the Problem Let's estimate that the diver with arms extended over head is about 2.5 m long and has a mass $M = 80$ kg. We'll also assume that it is reasonable to model the diver as a uniform stick rotating about its center of mass. From the photo, it appears that he sprang about 3 m in the air, and that the diving board was about 3 m high. We can use these assumptions and estimated quantities, together with their definitions, to estimate ω and L.

Express the diver's angular velocity ω and angular momentum L:

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad (1)$$

and

$$L = I\omega \qquad (2)$$

Using a constant-acceleration equation, express his time in the air:

$$\Delta t = \Delta t_{\text{rise 3 m}} + \Delta t_{\text{fall 6 m}} = \sqrt{\frac{2\Delta y_{\text{up}}}{g}} + \sqrt{\frac{2\Delta y_{\text{down}}}{g}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \sqrt{\frac{2(3\,\text{m})}{9.81\,\text{m/s}^2}} + \sqrt{\frac{2(6\,\text{m})}{9.81\,\text{m/s}^2}} = 1.89\,\text{s}$$

Estimate the angle through which he rotated in 1.89 s:

$$\Delta\theta \approx 0.5\,\text{rev} = \pi\ \text{rad}$$

Substitute in equation (1) and evaluate ω:

$$\omega = \frac{\pi\,\text{rad}}{1.89\,\text{s}} = \boxed{1.66\,\text{rad/s}}$$

Use the "stick rotating about an axis through its center of mass" model to approximate the moment of inertia of the diver:

$$I = \tfrac{1}{12}ML^2$$

Substitute in equation (2) to obtain:

$$L = \tfrac{1}{12}ML^2\omega$$

Substitute numerical values and evaluate L:

$$L = \tfrac{1}{12}(80\,\text{kg})(2.5\,\text{m})^2(1.66\,\text{rad/s})$$
$$= 69.2\,\text{kg}\cdot\text{m}^2/\text{s} \approx \boxed{70\,\text{kg}\cdot\text{m}^2/\text{s}}$$

Remarks: We can check the reasonableness of this estimation in another way. Because he rose about 3 m in the air, the initial impulse acting on him must be about 600 kg·m/s (i.e., $I = \Delta p = Mv_i$). If we estimate that the lever arm of the force is roughly ℓ = 1.5 m, and the angle between the force exerted by the board and a line running from his feet to the center of mass is about 5°, we obtain $L = I\ell\sin 5° \approx 78$ kg·m^2/s, which is not too bad considering the approximations made here.

31 ••

Picture the Problem First we assume a spherical diver whose mass M = 80 kg and whose diameter, when curled into a ball, is 1 m. We can estimate his angular velocity when he has curled himself into a ball from the ratio of his angular momentum to his moment of inertia. To estimate his angular momentum, we'll guess that the lever arm ℓ of the force that launches him from the diving board is about 1.5 m and that the angle between the force exerted by the board and a line running from his feet to the center of mass is about 5°.

Express the diver's angular velocity ω when he curls himself into a ball in mid-dive:

$$\omega = \frac{L}{I} \qquad (1)$$

Using a constant-acceleration equation, relate the speed with which he left the diving board v_0 to his maximum height Δy and our estimate of his angle with the vertical direction:

$$0 = v_{0y}^2 + 2a_y\Delta y$$
where
$$v_{0y} = v_0\cos 5°$$

Solve for v_0:

$$v_0 = \sqrt{\frac{2g\Delta y}{\cos^2 5°}}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{\sqrt{2(9.81\,\text{m/s}^2)(3\,\text{m})}}{\cos 5°} = 7.70\,\text{m/s}$$

Approximate the impulse acting on the diver to launch him with the speed v_0:

$$I = \Delta p = Mv_0$$

Letting ℓ represent the lever arm of the force acting on the diver as he leaves the diving board, express his angular momentum:

$$L = I\ell \sin 5° = Mv_0\ell \sin 5°$$

Use the "uniform sphere" model to approximate the moment of inertia of the diver:

$$I = \tfrac{2}{5}MR^2$$

Substitute in equation (1) to obtain:

$$\omega = \frac{Mv_0\ell \sin 5°}{\frac{2}{5}MR^2} = \frac{5v_0\ell \sin 5°}{2R^2}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{5(7.70\,\text{m/s})(1.5\,\text{m})\sin 5°}{2(0.5\,\text{m})^2} = \boxed{10.1\,\text{rad/s}}$$

***32 ••**

Picture the Problem We'll assume that he launches himself at an angle of 45° with the horizontal with his arms spread wide, and then pulls them in to increase his rotational speed during the jump. We'll also assume that we can model him as a 2-m long cylinder with an average radius of 0.15 m and a mass of 60 kg. We can then find his take-off speed and "air time" using constant-acceleration equations, and use the latter, together with the definition of rotational velocity, to find his initial rotational velocity. Finally, we can apply conservation of angular momentum to find his initial angular momentum.

Using a constant-acceleration equation, relate his takeoff speed v_0 to his maximum elevation Δy:

$$v^2 = v_{0y}^2 + 2a_y\Delta y$$

or, because $v_{0y} = v_0\sin 45°$, $v = 0$, and $a_y = -g$,

$$0 = v_0^2 \sin^2 45° - 2g\Delta y$$

Solve for v_0 to obtain:

$$v_0 = \sqrt{\frac{2g\Delta y}{\sin^2 45°}} = \frac{\sqrt{2g\Delta y}}{\sin 45°}$$

Substitute numerical values and evaluate v_0:

$$v_0 = \frac{\sqrt{2(9.81\,\text{m/s}^2)(0.6\,\text{m})}}{\sin 45°} = \boxed{4.85\,\text{m/s}}$$

Use its definition to express Goebel's angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Use a constant-acceleration equation to express Goebel's "air time" Δt:

$$\Delta t = 2\Delta t_{\text{rise}\,0.6\,\text{m}} = 2\sqrt{\frac{2\Delta y}{g}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = 2\sqrt{\frac{2(0.6\,\text{m})}{9.81\,\text{m/s}^2}} = 0.699\,\text{s}$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{4\,\text{rev}}{0.699\,\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}} = \boxed{36.0\,\text{rad/s}}$$

Use conservation of angular momentum to relate his take-off angular velocity ω_0 to his average angular velocity ω as he performs a quadruple Lutz:

$$I_0\omega_0 = I\omega$$

Assuming that he can change his angular momentum by a factor of 2 by pulling his arms in, solve for and evaluate ω_0:

$$\omega_0 = \frac{I}{I_0}\omega = \frac{1}{2}(36\,\text{rad/s}) = \boxed{18.0\,\text{rad/s}}$$

Express his take-off angular momentum:

$$L_0 = I_0\omega_0$$

Assuming that we can model him as a solid cylinder of length ℓ with an average radius r and mass m, express his moment of inertia with arms drawn in (his take-off configuration):

$$I_0 = 2\left(\tfrac{1}{2}mr^2\right) = mr^2$$

where the factor of 2 represents our assumption that he can double his moment of inertia by extending his arms.

Substitute to obtain:

$$L_0 = mr^2\omega_0$$

Substitute numerical values and evaluate L_0:

$$L_0 = (60\,\text{kg})(0.15\,\text{m})^2(18\,\text{rad/s}) = \boxed{24.3\,\text{kg}\cdot\text{m}^2/\text{s}}$$

Vector Nature of Rotation

33 •

Picture the Problem We can express $\vec{F}$ and $\vec{r}$ in terms of the unit vectors $\hat{i}$ and $\hat{j}$ and then use the definition of the cross product to find $\vec{\tau}$.

Express $\vec{F}$ in terms of F and the unit vector $\hat{i}$:

$$\vec{F} = -F\hat{i}$$

Express $\vec{r}$ in terms of R and the unit vector $\hat{j}$:

$$\vec{r} = R\hat{j}$$

Calculate the cross product of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r} \times \vec{F} = FR\left(\hat{j} \times -\hat{i}\right) = FR\left(\hat{i} \times \hat{j}\right) = \boxed{FR\hat{k}}$$

34 •

Picture the Problem We can find the torque is the cross product of $\vec{r}$ and $\vec{F}$.

Compute the cross product of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r} \times \vec{F} = \left(x\hat{i} + y\hat{j}\right)\left(-mg\hat{j}\right) = -mgx\left(\hat{i} \times \hat{j}\right) - mgy\left(\hat{j} \times \hat{j}\right) = \boxed{-mgx\hat{k}}$$

35 •

Picture the Problem The cross product of the vectors $\vec{A} = A_x\hat{i} + A_y\hat{j}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j}$ is given by

$$\begin{aligned}\vec{A} \times \vec{B} &= A_xB_x\left(\hat{i} \times \hat{i}\right) + A_xB_y\left(\hat{i} \times \hat{j}\right) + A_yB_x\left(\hat{j} \times \hat{i}\right) + A_yB_y\left(\hat{j} \times \hat{j}\right)\\ &= A_xB_x(0) + A_xB_y\left(\hat{k}\right) + A_yB_x\left(-\hat{k}\right) + A_yB_y(0)\\ &= A_xB_y\left(\hat{k}\right) + A_yB_x\left(-\hat{k}\right)\end{aligned}$$

(*a*) Find $\vec{A} \times \vec{B}$ for $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$:

$$\begin{aligned}\vec{A} \times \vec{B} &= 4\hat{i} \times \left(6\hat{i} + 6\hat{j}\right)\\ &= 24\left(\hat{i} \times \hat{i}\right) + 24\left(\hat{i} \times \hat{j}\right)\\ &= 24(0) + 24\hat{k} = \boxed{24\hat{k}}\end{aligned}$$

(*b*) Find $\vec{A} \times \vec{B}$ for $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{k}$:

$$\begin{aligned}\vec{A} \times \vec{B} &= 4\hat{i} \times \left(6\hat{i} + 6\hat{k}\right)\\ &= 24\left(\hat{i} \times \hat{i}\right) + 24\left(\hat{i} \times \hat{k}\right)\\ &= 24(0) + 24\left(-\hat{j}\right) = \boxed{-24\hat{j}}\end{aligned}$$

(*c*) Find $\vec{A} \times \vec{B}$ for $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$:

$$\begin{aligned}\vec{A} \times \vec{B} &= \left(2\hat{i} + 3\hat{j}\right) \times \left(3\hat{i} + 2\hat{j}\right)\\ &= 6\left(\hat{i} \times \hat{i}\right) + 4\left(\hat{i} \times \hat{j}\right) + 9\left(\hat{j} \times \hat{i}\right)\\ &\quad + 6\left(\hat{j} \times \hat{j}\right)\\ &= 6(0) + 4\left(\hat{k}\right) + 9\left(-\hat{k}\right) + 6(0)\\ &= \boxed{-5\hat{k}}\end{aligned}$$

***36 •**

Picture the Problem The magnitude of $\vec{A}\times\vec{B}$ is given by $|AB\sin\theta|$.

Equate the magnitudes of $\vec{A}\times\vec{B}$ and $\vec{A}\cdot\vec{B}$:

$$|AB\sin\theta| = |AB\cos\theta|$$

$$\therefore |\sin\theta| = |\cos\theta|$$

or

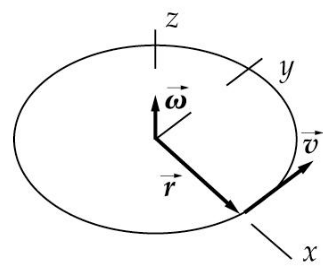
$$\tan\theta = \pm 1$$

Solve for θ to obtain:

$$\theta = \tan^{-1}\pm 1 = \boxed{\pm 45^\circ \text{ or } \pm 135^\circ}$$

37 ••

Picture the Problem Let $\vec{r}$ be in the *xy* plane. Then $\vec{\omega}$ points in the positive *z* direction. We can establish the results called for in this problem by forming the appropriate cross products and by differentiating $\vec{v}$.



(*a*) Express $\vec{\omega}$ using unit vectors:

$$\vec{\omega} = \omega\hat{k}$$

Express $\vec{r}$ using unit vectors:

$$\vec{r} = r\hat{i}$$

Form the cross product of $\vec{\omega}$ and $\vec{r}$:

$$\vec{\omega}\times\vec{r} = \omega\hat{k}\times r\hat{i} = r\omega\left(\hat{k}\times\hat{i}\right) = r\omega\hat{j}$$
$$= v\hat{j}$$

$$\therefore \boxed{\vec{v} = \vec{\omega}\times\vec{r}}$$

(*b*) Differentiate $\vec{v}$ with respect to *t* to express $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right)$$
$$= \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$
$$= \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\vec{v}$$
$$= \vec{a}_t + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$
$$= \vec{a}_t + \vec{a}_c$$

where $\vec{a}_c = \boxed{\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)}$

and $\vec{a}_t$ and $\vec{a}_c$ are the tangential and centripetal accelerations, respectively.

38 ••

Picture the Problem Because $B_z = 0$, we can express $\vec{B}$ as $\vec{B} = B_x\hat{i} + B_y\hat{j}$ and form its cross product with $\vec{A}$ to determine B_x and B_y.

Express $\vec{B}$ in terms of its components: $\vec{B} = B_x\hat{i} + B_y\hat{j}$ (1)

Express $\vec{A}\times\vec{B}$: $\vec{A}\times\vec{B} = 4\hat{i}\times\left(B_x\hat{i} + B_y\hat{j}\right) = 4B_y\hat{k} = 12\hat{k}$

Solve for B_y: $B_y = 3$

Relate B to B_x and B_y: $B^2 = B_x^2 + B_y^2$

Solve for and evaluate B_x: $B_x = \sqrt{B^2 - B_y^2} = \sqrt{5^2 - 3^2} = 4$

Substitute in equation (1): $\vec{B} = \boxed{4\hat{i} + 3\hat{j}}$

39 •

Picture the Problem We can write $\vec{B}$ in the form $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ and use the dot product of $\vec{A}$ and $\vec{B}$ to find B_y and their cross product to find B_x and B_z.

Express $\vec{B}$ in terms of its components: $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ (1)

Evaluate $\vec{A}\cdot\vec{B}$: $\vec{A}\cdot\vec{B} = 3B_y = 12$

and

$B_y = 4$

Evaluate $\vec{A}\times\vec{B}$:

$$\vec{A}\times\vec{B} = 3\hat{j}\times\left(B_x\hat{i} + 4\hat{j} + B_z\hat{k}\right) = -3B_x\hat{k} + 3B_z\hat{i}$$

Because $\vec{A}\times\vec{B} = 9\hat{i}$: $B_x = 0$ and $B_z = 3$.

Substitute in equation (1) to obtain: $\vec{B} = \boxed{4\hat{j} + 3\hat{k}}$

40 ••

Picture the Problem The dot product of $\vec{A}$ with the cross product of $\vec{B}$ and $\vec{C}$ is a scalar quantity and can be expressed in determinant form as $\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$. We can expand this determinant by minors to show that it is equivalent to $\vec{A}\cdot(\vec{B}\times\vec{C})$, $\vec{C}\cdot(\vec{A}\times\vec{B})$, and $\vec{B}\cdot(\vec{C}\times\vec{A})$.

The dot product of $\vec{A}$ with the cross product of $\vec{B}$ and $\vec{C}$ is a scalar quantity and can be expressed in determinant form as:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Expand the determinant by minors to obtain:

$$\begin{aligned}\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} &= a_xb_yc_z - a_xb_zc_y \\ &\quad + a_yb_zc_x - a_yb_xc_z \qquad (1) \\ &\quad + a_zb_xc_y - a_zb_yc_x\end{aligned}$$

Evaluate the cross product of $\vec{B}$ and $\vec{C}$ to obtain:

$$\begin{aligned}\vec{B}\times\vec{C} &= (b_yc_z - b_zc_y)\hat{\boldsymbol{i}} \\ &\quad + (b_zc_x - b_xc_z)\hat{\boldsymbol{j}} + (b_xc_y - b_yc_x)\hat{\boldsymbol{k}}\end{aligned}$$

Form the dot product of $\vec{A}$ with $\vec{B}\times\vec{C}$ to obtain:

$$\begin{aligned}\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) &= a_xb_yc_z - a_xb_zc_y \\ &\quad + a_yb_zc_x - a_yb_xc_z \qquad (2) \\ &\quad + a_zb_xc_y - a_zb_yc_x\end{aligned}$$

Because (1) and (2) are the same, we can conclude that:

$$\boxed{\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}}$$

Proceed as above to establish that:

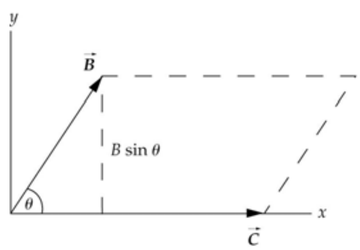
$$\boxed{\vec{C}\cdot(\vec{A}\times\vec{B}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}}$$

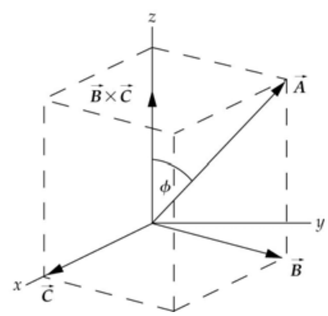
and

$$\boxed{\vec{\boldsymbol{B}}\cdot(\vec{\boldsymbol{C}}\times\vec{\boldsymbol{A}}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}}$$

41 ••

Picture the Problem Let, without loss of generality, the vector $\vec{\boldsymbol{C}}$ lie along the x axis and the vector $\vec{\boldsymbol{B}}$ lie in the xy plane as shown below to the left. The diagram to the right shows the parallelepiped spanned by the three vectors. We can apply the definitions of the cross- and dot-products to show that $\vec{\boldsymbol{A}}\cdot\left(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}\right)$ is the volume of the parallelepiped.





Express the cross-product of $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$:

$$\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}} = (BC\sin\theta)\left(-\hat{\boldsymbol{k}}\right)$$

and

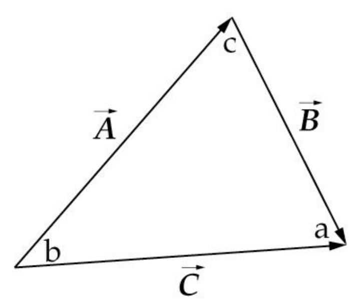
$$\left|\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}\right| = (B\sin\theta)C$$
$$= \text{area of the parallelogram}$$

Form the dot-product of $\vec{\boldsymbol{A}}$ with the cross-product of $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ to obtain:

$$\begin{aligned}\vec{\boldsymbol{A}}\cdot\left(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}\right) &= A(B\sin\theta)C\cos\phi \\ &= (BC\sin\theta)(A\cos\phi) \\ &= (\text{area of base})(\text{height}) \\ &= \boxed{V_{\text{parallelepiped}}}\end{aligned}$$

***42** ••

Picture the Problem Draw the triangle using the three vectors as shown below. Note that $\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}=\vec{\boldsymbol{C}}$. We can find the magnitude of the cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ and of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{C}}$ and then use the cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{C}}$, using $\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}=\vec{\boldsymbol{C}}$, to show that $AC\sin b = AB\sin c$ or $B/\sin b = C/\sin c$. Proceeding similarly, we can extend the law of sines to the third side of the triangle and the angle opposite it.



Express the magnitude of the cross product of $\vec{A}$ and $\vec{B}$:

$$\left|\vec{A}\times\vec{B}\right| = AB\sin c$$

Express the magnitude of the cross product of $\vec{A}$ and $\vec{C}$:

$$\left|\vec{A}\times\vec{C}\right| = AC\sin b$$

Form the cross product of $\vec{A}$ with $\vec{C}$ to obtain:

$$\begin{aligned}\vec{A}\times\vec{C} &= \vec{A}\times\left(\vec{A}+\vec{B}\right)\\ &= \vec{A}\times\vec{A}+\vec{A}\times\vec{B}\\ &= \vec{A}\times\vec{B}\end{aligned}$$

because $\vec{A}\times\vec{A}=0$.

Because $\vec{A}\times\vec{C}=\vec{A}\times\vec{B}$:

$$\left|\vec{A}\times\vec{C}\right| = \left|\vec{A}\times\vec{B}\right|$$

and

$$AC\sin b = AB\sin c$$

Simplify and rewrite this expression to obtain:

$$\boxed{\frac{B}{\sin b}=\frac{C}{\sin c}}$$

Proceed similarly to extend this result to the law of sines:

$$\boxed{\frac{A}{\sin a}=\frac{B}{\sin b}=\frac{C}{\sin c}}$$

Angular Momentum

43 •

Picture the Problem $\vec{L}$ and $\vec{p}$ are related according to $\vec{L}=\vec{r}\times\vec{p}$. If $\vec{L}=0$, then examination of the magnitude of $\vec{r}\times\vec{p}$ will allow us to conclude that $\sin\phi=0$ and that the particle is moving either directly toward the point, directly away from the point, or through the point.

Because $\vec{L}=0$:

$$\vec{r}\times\vec{p}=\vec{r}\times m\vec{v}=m\vec{r}\times\vec{v}=0$$

or

$$\vec{r}\times\vec{v}=0$$

Express the magnitude of $\vec{r}\times\vec{v}$:

$$\left|\vec{r}\times\vec{v}\right| = rv\sin\phi = 0$$

Because neither r nor v is zero:

$$\sin\phi=0$$

where ϕ is the angle between $\vec{r}$ and $\vec{v}$.

Solve for ϕ:

$$\phi=\sin^{-1}0=\boxed{0° \text{ or } 180°}$$

44 •

Picture the Problem We can use their definitions to calculate the angular momentum and moment of inertia of the particle and the relationship between L, I, and ω to determine its angular speed.

(*a*) Express and evaluate the magnitude of $\vec{L}$:

$$L = mvr = (2\,\text{kg})(3.5\,\text{m/s})(4\,\text{m}) = \boxed{28.0\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) Express the moment of inertia of the particle with respect to an axis through the center of the circle in which it is moving:

$$I = mr^2 = (2\,\text{kg})(4\,\text{m})^2 = \boxed{32\,\text{kg}\cdot\text{m}^2}$$

(*c*) Relate the angular speed of the particle to its angular momentum and solve for and evaluate ω:

$$\omega = \frac{L}{I} = \frac{28.0\,\text{kg}\cdot\text{m}^2/\text{s}}{32\,\text{kg}\cdot\text{m}^2} = \boxed{0.875\,\text{rad/s}^2}$$

45 •

Picture the Problem We can use the definition of angular momentum to calculate the angular momentum of this particle and the relationship between its angular momentum and angular speed to describe the variation in its angular speed with time.

(*a*) Express the angular momentum of the particle as a function of its mass, speed, and distance of its path from the reference point:

$$L = rmv\sin\theta = (6\,\text{m})(2\,\text{kg})(4.5\,\text{m/s})\sin 90° = \boxed{54.0\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) Because $L = mr^2\omega$:

$$\omega \propto \frac{1}{r^2}$$ and

$\boxed{\omega \text{ increases as the particle approaches the point and decreases as it recedes.}}$

***46** ••

Picture the Problem We can use the formula for the area of a triangle to find the area swept out at $t = t_1$, add this area to the area swept out in time dt, and then differentiate this expression with respect to time to obtain the given expression for dA/dt.

Express the area swept out at $t = t_1$:

$$A_1 = \tfrac{1}{2}br_1\cos\theta_1 = \tfrac{1}{2}bx_1$$

where θ_1 is the angle between $\vec{r}_1$ and $\vec{v}$ and

x_1 is the component of $\vec{r}_1$ in the direction of $\vec{v}$.

Express the area swept out at $t = t_1 + dt$:

$$A = A_1 + dA = \tfrac{1}{2}b(x_1 + dx)$$
$$= \tfrac{1}{2}b(x_1 + vdt)$$

Differentiate with respect to t:

$$\frac{dA}{dt} = \tfrac{1}{2}b\frac{dx}{dt} = \tfrac{1}{2}bv = \text{constant}$$

Because $r\sin\theta = b$:

$$\tfrac{1}{2}bv = \tfrac{1}{2}(r\sin\theta)v = \frac{1}{2m}(rp\sin\theta)$$
$$= \boxed{\frac{L}{2m}}$$

47 ••

Picture the Problem We can find the total angular momentum of the coin from the sum of its spin and orbital angular momenta.

(*a*) Express the spin angular momentum of the coin:

$$L_{\text{spin}} = I_{\text{cm}}\omega_{\text{spin}}$$

From Problem 9-44:

$$I = \tfrac{1}{4}MR^2$$

Substitute for I to obtain:

$$L_{\text{spin}} = \tfrac{1}{4}MR^2\omega_{\text{spin}}$$

Substitute numerical values and evaluate L_{spin}:

$$L_{\text{spin}} = \tfrac{1}{4}(0.015\,\text{kg})(0.0075\,\text{m})^2$$
$$\times\left(10\frac{\text{rev}}{\text{s}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\right)$$
$$= \boxed{1.33\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) Express and evaluate the total angular momentum of the coin:

$$L = L_{\text{orbit}} + L_{\text{spin}} = 0 + L_{\text{spin}}$$
$$= \boxed{1.33\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*c*) From Problem 10-14:

$$L_{\text{orbit}} = 0$$

and

$$L = \boxed{1.33\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*d*) Express the total angular momentum of the coin:

$$L = L_{\text{orbit}} + L_{\text{spin}}$$

Find the orbital momentum of the coin:

$$L_{\text{orbit}} = \pm MvR$$
$$= \pm(0.015\,\text{kg})(0.05\,\text{m/s})(0.1\,\text{m})$$
$$= \pm 7.50\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}$$

where the ± is a consequence of the fact that the coin's direction is not specified.

Substitute to obtain:

$$L = \pm 7.50\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s} + 1.33\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}$$

The possible values for L are:

$$L = \boxed{8.83\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}}$$

or

$$L = \boxed{-6.17\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}}$$

48 ••

Picture the Problem Both the forces acting on the particles exert torques with respect to an axis perpendicular to the page and through point O and the net torque about this axis is their vector sum.

Express the net torque about an axis perpendicular to the page and through point O:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \sum_{\text{i}} \vec{\boldsymbol{\tau}}_{\text{i}} = \vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}_1 + \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2$$
$$= (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)\times \vec{\mathbf{F}}_1$$

because $\vec{F}_2 = -\vec{F}_1$

Because $\vec{r}_1 - \vec{r}_2$ points along $-\vec{F}_1$:

$$\boxed{(\vec{r}_1 - \vec{r}_2)\times \vec{F}_1 = 0}$$

Torque and Angular Momentum

49 •

Picture the Problem The angular momentum of the particle changes because a *net* torque acts on it. Because we know how the angular momentum depends on time, we can find the net torque acting on the particle by differentiating its angular momentum. We can use a constant-acceleration equation and Newton's 2nd law to relate the angular speed of the particle to its angular acceleration.

(*a*) Relate the magnitude of the torque acting on the particle to the rate at which its angular momentum changes:

$$\tau_{\text{net}} = \frac{dL}{dt} = \frac{d}{dt}\left[(4\,\text{N}\cdot\text{m})t\right]$$
$$= \boxed{4.00\,\text{N}\cdot\text{m}}$$

(*b*) Using a constant-acceleration equation, relate the angular speed of the particle to its acceleration and time-in-motion:

$$\omega = \omega_0 + \alpha t$$

where $\omega_0 = 0$

Use Newton's 2nd law to relate the angular acceleration of the particle to
the net torque acting on it:

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{\tau_{\text{net}}}{mr^2}$$

Substitute to obtain:

$$\omega = \frac{\tau_{\text{net}}}{mr^2}t$$

Substitute numerical values and evaluate ω:

$$\omega = \frac{(4\ \text{N}\cdot\text{m})t}{(1.8\,\text{kg})(3.4\,\text{m})^2} = \boxed{(0.192\,\text{rad/s}^2)t}$$

provided t is in seconds.

50 ••

Picture the Problem The angular momentum of the cylinder changes because a *net* torque acts on it. We can find the angular momentum at $t = 25$ s from its definition and the *net* torque acting on the cylinder from the rate at which the angular momentum is changing. The magnitude of the frictional force acting on the rim can be found using the definition of torque.

(*a*) Use its definition to express the angular momentum of the cylinder:

$$L = I\omega = \tfrac{1}{2}mr^2\omega$$

Substitute numerical values and evaluate L:

$$L = \tfrac{1}{2}(90\,\text{kg})(0.4\,\text{m})^2 \times \left(500\frac{\text{rev}}{\text{min}} \times \frac{2\pi\,\text{rad}}{\text{rev}} \times \frac{1\,\text{min}}{60\,\text{s}}\right) = \boxed{377\,\text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) Express and evaluate $\dfrac{dL}{dt}$:

$$\frac{dL}{dt} = \frac{(377\,\text{kg}\cdot\text{m}^2/\text{s})}{25\,\text{s}} = \boxed{15.1\,\text{kg}\cdot\text{m}^2/\text{s}^2}$$

(*c*) Because the torque acting on the uniform cylinder is constant, the rate

$$\tau = \frac{dL}{dt} = \boxed{15.1\,\text{kg}\cdot\text{m}^2/\text{s}^2}$$

of change of the angular momentum is constant and hence the instantaneous rate of change of the angular momentum at any instant is equal to the average rate of change over the time during which the torque acts:

(*d*) Using the definition of torque that relates the applied force to its lever arm, express the magnitude of the frictional force *f* acting on the rim:

$$f=\frac{\tau}{\ell}=\frac{15.1\,\text{kg}\cdot\text{m}^2/\text{s}^2}{0.4\,\text{m}}=\boxed{37.7\,\text{N}}$$

***51 ••**

Picture the Problem Let the system include the pulley, string, and the blocks and assume that the mass of the string is negligible. The angular momentum of this system changes because a *net* torque acts on it.

(*a*) Express the net torque about the center of mass of the pulley:

$$\tau_{\text{net}}=Rm_2g\sin\theta-Rm_1g=\boxed{Rg(m_2\sin\theta-m_1)}$$

where we have taken clockwise to be positive to be consistent with a positive upward velocity of the block whose mass is m_1 as indicated in the figure.

(*b*) Express the total angular momentum of the system about an axis through the center of the pulley:

$$L=I\omega+m_1vR+m_2vR=\boxed{vR\left(\frac{I}{R^2}+m_1+m_2\right)}$$

(*c*) Express τ as the time derivative of the angular momentum:

$$\tau=\frac{dL}{dt}=\frac{d}{dt}\left[vR\left(\frac{I}{R^2}+m_1+m_2\right)\right]=aR\left(\frac{I}{R^2}+m_1+m_2\right)$$

Equate this result to that of part (*a*) and solve for *a* to obtain:

$$a=\boxed{\frac{g(m_2\sin\theta-m_1)}{\frac{I}{R^2}+m_1+m_2}}$$

52 ••

Picture the Problem The forces resulting from the release of gas from the jets will exert a torque on the spaceship that will slow and eventually stop its rotation. We can relate this net torque to the angular momentum of the spaceship and to the time the jets must fire.

Relate the firing time of the jets to the desired change in angular momentum:

$$\Delta t = \frac{\Delta L}{\tau_{\text{net}}} = \frac{I\Delta\omega}{\tau_{\text{net}}}$$

Express the magnitude of the net torque exerted by the jets:

$$\tau_{\text{net}} = 2FR$$

Letting $\Delta m/\Delta t'$ represent the mass of gas per unit time exhausted from the jets, relate the force exerted by the gas on the spaceship to the rate at which the gas escapes:

$$F = \frac{\Delta m}{\Delta t'}v$$

Substitute and solve for Δt to obtain:

$$\Delta t = \frac{I\Delta\omega}{2\dfrac{\Delta m}{\Delta t'}vR}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{\left(4000\,\text{kg}\cdot\text{m}^2\right)\left(6\dfrac{\text{rev}}{\text{min}}\times\dfrac{2\pi\,\text{rad}}{\text{rev}}\times\dfrac{1\,\text{min}}{60\,\text{s}}\right)}{2\left(10^{-2}\,\text{kg/s}\right)\left(800\,\text{m/s}\right)\left(3\,\text{m}\right)} = \boxed{52.4\,\text{s}}$$

53 ••

Picture the Problem We can use constant-acceleration equations to express the projectile's position and velocity coordinates as functions of time. We can use these coordinates to express the particle's position and velocity vectors $\vec{r}$ and $\vec{v}$. Using its definition, we can express the projectile's angular momentum $\vec{L}$ as a function of time and then differentiate this expression to obtain $d\vec{L}/dt$. Finally, we can use the definition of the torque, relative to an origin located at the launch position, the gravitational force exerts on the projectile to express $\vec{\tau}$ and complete the demonstration that $d\vec{L}/dt = \vec{\tau}$.

Using its definition, express the angular momentum vector $\vec{L}$ of the projectile:

$$\vec{L} = \vec{r}\times m\vec{v} \qquad (1)$$

Using constant-acceleration

$$x = v_{0x}t = \left(V\cos\theta\right)t$$

equations, express the position coordinates of the projectile as a function of time:

and

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = (V\sin\theta)t - \tfrac{1}{2}gt^2$$

Express the projectile's position vector $\vec{r}$:

$$\vec{r} = [(V\cos\theta)t]\hat{i} + \left[(V\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j}$$

Using constant-acceleration equations, express the velocity of the projectile as a function of time:

$$v_x = v_{0x} = V\cos\theta$$

and

$$v_y = v_{0y} + a_y t = V\sin\theta - gt$$

Express the projectile's velocity vector $\vec{v}$:

$$\vec{v} = [V\cos\theta]\hat{i} + [V\sin\theta - gt]\hat{j}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{L} &= \left\{[(V\cos\theta)t]\hat{i} + \left[(V\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j}\right\} \\ &\quad \times m\left\{[V\cos\theta]\hat{i} + [V\sin\theta - gt]\hat{j}\right\} \\ &= \left(-\tfrac{1}{2}mgt^2V\cos\theta\right)\hat{k}\end{aligned}$$

Differentiate $\vec{L}$ with respect to t to obtain:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(-\tfrac{1}{2}mgt^2V\cos\theta\right)\hat{k} = (-mgtV\cos\theta)\hat{k} \qquad (2)$$

Using its definition, express the torque acting on the projectile:

$$\begin{aligned}\vec{\tau} &= \vec{r}\times(-mg)\hat{j} \\ &= [(V\cos\theta)t]\hat{i} + \left[(V\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j} \\ &\qquad \times(-mg)\hat{j}\end{aligned}$$

or

$$\vec{\tau} = (-mgtV\cos\theta)\hat{k} \qquad (3)$$

Comparing equations (2) and (3) we see that:

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Conservation of Angular Momentum

***54 •**

Picture the Problem Let m represent the mass of the planet and apply the definition of torque to find the torque produced by the gravitational force of attraction. We can use Newton's 2nd law of motion in the form $\vec{\tau} = d\vec{L}/dt$ to show that $\vec{L}$ is constant and apply conservation of angular momentum to the motion of the planet at points A and B.

(*a*) Express the torque produced by the gravitational force of attraction of the sun for the planet:

$\vec{\tau} = \vec{r} \times \vec{F} = \boxed{0}$ because $\vec{F}$ acts along the direction of $\vec{r}$.

(*b*) Because $\vec{\tau} = 0$:

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \vec{r} \times m\vec{v} = \text{constant}$$

Noting that at points *A* and *B* $|\vec{r} \times \vec{v}| = rv$, express the relationship between the distances from the sun and the speeds of the planets:

$$r_1 v_1 = r_2 v_2$$

or

$$\frac{v_1}{v_2} = \boxed{\frac{r_2}{r_1}}$$

55 ••

Picture the Problem Let the system consist of you, the extended weights, and the platform. Because the net external torque acting on this system is zero, its angular momentum remains constant during the pulling in of the weights.

(*a*) Using conservation of angular momentum, relate the initial and final angular speeds of the system to its initial and final moments of inertia:

$$I_i \omega_i = I_f \omega_f$$

Solve for ω_f:

$$\omega_f = \frac{I_i}{I_f}\omega_i$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \frac{6\,\text{kg}\cdot\text{m}^2}{1.8\,\text{kg}\cdot\text{m}^2}(1.5\,\text{rev/s}) = \boxed{5.00\,\text{rev/s}}$$

(*b*) Express the change in the kinetic energy of the system:

$$\Delta K = K_f - K_i = \tfrac{1}{2}I_f\omega_f^2 - \tfrac{1}{2}I_i\omega_i^2$$

Substitute numerical values and evaluate ΔK:

$$\begin{aligned}\Delta K &= \tfrac{1}{2}\left(1.8\,\text{kg}\cdot\text{m}^2\right)\left(5\frac{\text{rev}}{\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right)^2 \\ &\quad - \tfrac{1}{2}\left(6\,\text{kg}\cdot\text{m}^2\right)\left(1.5\frac{\text{rev}}{\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right)^2 \\ &= \boxed{622\,\text{J}}\end{aligned}$$

(*c*) $\boxed{\text{Because no external agent does work on the system, the energy comes from the internal energy of the man.}}$

***56 ••**

Picture the Problem Let the system consist of the blob of putty and the turntable. Because the net external torque acting on this system is zero, its angular momentum remains constant when the blob of putty falls onto the turntable.

(*a*) Using conservation of angular momentum, relate the initial and final angular speeds of the turntable to its initial and final moments of inertia and solve for ω_f:

$$I_0\omega_i = I_f\omega_f$$

and

$$\omega_f = \frac{I_0}{I_f}\omega_i$$

Express the final rotational inertia of the turntable-plus-blob:

$$I_f = I_0 + I_{blob} = I_0 + mR^2$$

Substitute and simplify to obtain:

$$\omega_f = \frac{I_0}{I_0 + mR^2}\omega_i = \boxed{\frac{1}{1+\frac{mR^2}{I_0}}\omega_i}$$

(*b*) If the blob flies off tangentially to the turntable, its angular momentum doesn't change (with respect to an axis through the center of turntable). Because there is no external torque acting on the blob-turntable system, the total angular momentum of the system will remain constant and the angular momentum of the turntable will not change. Because the moment of inertia of the table hasn't changed either, the turntable will continue to spin at $\boxed{\omega' = \omega_f}$.

57 ••

Picture the Problem Because the net external torque acting on the Lazy Susan-cockroach system is zero, the net angular momentum of the system is constant (equal to zero because the Lazy Susan is initially at rest) and we can use conservation of angular momentum to find the angular velocity ω of the Lazy Susan. The speed of the cockroach relative to the floor v_f is the difference between its speed with respect to the Lazy Susan and the speed of the Lazy Susan at the location of the cockroach with respect to the floor.

Relate the speed of the cockroach with respect to the floor v_f to the speed of the Lazy Susan at the location of the cockroach:

$$v_f = v - \omega r \qquad (1)$$

Use conservation of angular momentum to obtain:

$$L_{LS} - L_C = 0$$

Express the angular momentum of the Lazy Susan:

$$L_{LS} = I_{LS}\omega = \tfrac{1}{2}MR^2\omega$$

Express the angular momentum of the cockroach:

$$L_C = I_C\omega_C = mr^2\left(\frac{v}{r} - \omega\right)$$

Substitute to obtain:

$$\tfrac{1}{2}MR^2\omega - mr^2\left(\frac{v}{r} - \omega\right) = 0$$

Solve for ω to obtain:

$$\omega = \frac{2mrv}{MR^2 + 2mr^2}$$

Substitute in equation (1):

$$v_f = v - \frac{2mr^2 v}{MR^2 + 2mr^2}$$

Substitute numerical values and evaluate v_f:

$$v_f = 0.01\,\text{m/s} - \frac{2(0.015\,\text{kg})(0.08\,\text{m})^2(0.01\,\text{m/s})}{(0.25\,\text{m})(0.15\,\text{m})^2 + 2(0.015\,\text{kg})(0.08\,\text{m})^2} = \boxed{9.67\,\text{mm/s}}$$

***58 ••**

Picture the Problem The net external torque acting on this system is zero and so we know that angular momentum is conserved as these disks are brought together. Let the numeral 1 refer to the disk to the left and the numeral 2 to the disk to the right. Let the angular momentum of the disk with the larger radius be positive.

Using conservation of angular momentum, relate the initial angular speeds of the disks to their common final speed and to their moments of inertia:

$$I_i\omega_i = I_f\omega_f$$

or

$$I_1\omega_0 - I_2\omega_0 = (I_1 + I_2)\omega_f$$

Solve for ω_f:

$$\omega_f = \frac{I_1 - I_2}{I_1 + I_2}\omega_0$$

Express I_1 and I_2:

$$I_1 = \tfrac{1}{2}m(2r)^2 = 2mr^2$$

and

$$I_2 = \tfrac{1}{2}mr^2$$

Substitute and simplify to obtain:

$$\omega_f = \frac{2mr^2 - \tfrac{1}{2}mr^2}{2mr^2 + \tfrac{1}{2}mr^2}\omega_0 = \boxed{\tfrac{3}{5}\omega_0}$$

59 ••

Picture the Problem We can express the angular momentum and kinetic energy of the block directly from their definitions. The tension in the string provides the centripetal force required for the uniform circular motion and can be expressed using Newton's 2nd law. Finally, we can use the work-kinetic energy theorem to express the work required to reduce the radius of the circle by a factor of two.

(*a*) Express the initial angular momentum of the block:

$$L_0 = \boxed{r_0 m v_0}$$

(*b*) Express the initial kinetic energy of the block:

$$K_0 = \boxed{\tfrac{1}{2} m v_0^2}$$

(*c*) Using Newton's 2nd law, relate the tension in the string to the centripetal force required for the circular motion:

$$T = F_c = \boxed{m\frac{v_0^2}{r_0}}$$

Use the work-kinetic energy theorem to relate the required work to the change in the kinetic energy of the block:

$$\begin{aligned} W = \Delta K = K_f - K_0 &= \frac{L_f^2}{2I_f} - \frac{L_0^2}{2I_0} \\ &= \frac{L_0^2}{2I_f} - \frac{L_0^2}{2I_0} = \frac{L_0^2}{2}\left(\frac{1}{I_f - I_0}\right) \\ &= \frac{L_0^2}{2}\left(\frac{1}{m\left(\frac{1}{2}r_0\right)^2 - mr_0^2}\right) = -\frac{2}{3}\frac{L_0^2}{mr_0^2} \end{aligned}$$

Substitute the result from part (*a*) and simplify to obtain:

$$W = \boxed{-\tfrac{2}{3} m v_0^2}$$

***60** ••

Picture the Problem Because the force exerted by the rubber band is parallel to the position vector of the point mass, the net external torque acting on it is zero and we can use the conservation of angular momentum to determine the speeds of the ball at points *B* and *C*. We'll use mechanical energy conservation to find *b* by relating the kinetic and elastic potential energies at *A* and *B*.

(*a*) Use conservation of momentum to relate the angular momenta at points *A*, *B* and *C*:

$$L_A = L_B = L_C$$

or

$$m v_A r_A = m v_B r_B = m v_C r_C$$

Solve for v_B in terms of v_A:

$$v_B = v_A \frac{r_A}{r_B}$$

Substitute numerical values and evaluate v_B:

$$v_B = (4\,\text{m/s})\frac{0.6\,\text{m}}{1\,\text{m}} = \boxed{2.40\,\text{m/s}}$$

Solve for v_C in terms of v_A:

$$v_C = v_A \frac{r_A}{r_C}$$

Substitute numerical values and evaluate v_C:

$$v_C = (4\,\text{m/s})\frac{0.6\,\text{m}}{0.6\,\text{m}} = \boxed{4.00\,\text{m/s}}$$

(*b*) Use conservation of mechanical energy between points *A* and *B* to relate the kinetic energy of the point mass and the energy stored in the stretched rubber band:

$$E_A = E_B$$

or

$$\tfrac{1}{2}mv_A^2 + \tfrac{1}{2}br_A^2 = \tfrac{1}{2}mv_B^2 + \tfrac{1}{2}br_B^2$$

Solve for *b*:

$$b = \frac{m\left(v_B^2 - v_A^2\right)}{r_A^2 - r_B^2}$$

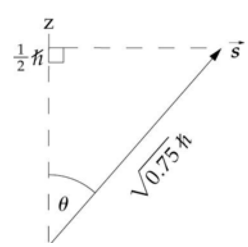
Substitute numerical values and evaluate *b*:

$$b = \frac{(0.2\,\text{kg})\left[(2.4\,\text{m/s})^2 - (4\,\text{m/s})^2\right]}{(0.6\,\text{m})^2 - (1\,\text{m})^2} = \boxed{3.20\,\text{N/m}}$$

Quantization of Angular Momentum

***61 •**

Picture the Problem The electron's spin angular momentum vector is related to its *z* component as shown in the diagram.



Using trigonometry, relate the magnitude of $\vec{s}$ to its z component:

$$\theta = \cos^{-1}\frac{\frac{1}{2}\hbar}{\sqrt{0.75}\hbar} = \boxed{54.7°}$$

62 ••

Picture the Problem Equation 10-27*a* describes the quantization of rotational energy. We can show that the energy difference between a given state and the next higher state is proportional to $\ell + 1$ by using Equation 10-27*a* to express the energy difference.

From Equation 10-27*a* we have:

$$K_\ell = \ell(\ell+1)E_{0\mathrm{r}}$$

Using this equation, express the difference between one rotational state and the next higher state:

$$\Delta E = (\ell+1)(\ell+2)E_{0\mathrm{r}} - \ell(\ell+1)E_{0\mathrm{r}} = \boxed{2(\ell+1)E_{0\mathrm{r}}}$$

63 ••

Picture the Problem The rotational energies of HBr molecule are related to ℓ and $E_{0\mathrm{r}}$ according to $K_\ell = \ell(\ell+1)E_{0\mathrm{r}}$ where $E_{0\mathrm{r}} = \hbar^2/2I$.

(*a*) Express and evaluate the moment of inertia of the H atom:

$$I = m_{\mathrm{p}}r^2 = \left(1.67\times10^{-27}\ \mathrm{kg}\right)\left(0.144\times10^{-9}\ \mathrm{m}\right)^2 = \boxed{3.46\times10^{-47}\ \mathrm{kg}\cdot\mathrm{m}^2}$$

(*b*) Relate the rotational energies to ℓ and $E_{0\mathrm{r}}$:

$$K_\ell = \ell(\ell+1)E_{0\mathrm{r}}$$

Evaluate $E_{0\mathrm{r}}$:

$$E_{0\mathrm{r}} = \frac{\hbar^2}{2I} = \frac{\left(1.05\times10^{-34}\ \mathrm{J\cdot s}\right)^2}{2\left(3.46\times10^{-47}\ \mathrm{kg}\cdot\mathrm{m}^2\right)} = 1.59\times10^{-22}\ \mathrm{J}\times\frac{1\,\mathrm{eV}}{1.60\times10^{-19}\ \mathrm{J}} = 0.996\,\mathrm{meV}$$

Evaluate E for $\ell = 1$:

$$E_1 = (1+1)(0.996\,\mathrm{meV}) = \boxed{1.99\,\mathrm{meV}}$$

Evaluate E for $\ell = 2$:

$$E_2 = 2(2+1)(0.996\,\mathrm{meV}) = \boxed{5.98\,\mathrm{meV}}$$

Evaluate E for $\ell = 3$:

$$E_3 = 3(3+1)(0.996\,\mathrm{meV}) = \boxed{12.0\,\mathrm{meV}}$$

64 ••

Picture the Problem We can use the definition of the moment of inertia of point particles to calculate the rotational inertia of the nitrogen molecule. The rotational energies of nitrogen molecule are related to ℓ and $E_{0\mathrm{r}}$ according to $K_\ell = \ell(\ell+1)E_{0\mathrm{r}}$ where $E_{0\mathrm{r}} = \hbar^2/2I$.

(*a*) Using a rigid dumbbell model, express and evaluate the moment of inertia of the nitrogen molecule about its center of mass:

$$I = \sum_i m_i r_i^2 = m_N r^2 + m_N r^2$$
$$= 2m_N r^2$$

Substitute numerical values and evaluate I:

$$I = 2(14)\left(1.66\times10^{-27}\text{ kg}\right)\left(5.5\times10^{-11}\text{ m}\right)^2$$
$$= \boxed{1.41\times10^{-46}\text{ kg}\cdot\text{m}^2}$$

(*b*) Relate the rotational energies to ℓ and E_{0r}:

$$E_\ell = \ell(\ell+1)E_{0r}$$

Evaluate E_{0r}:

$$E_{0r} = \frac{\hbar^2}{2I} = \frac{\left(1.05\times10^{-34}\text{ J}\cdot\text{s}\right)^2}{2\left(1.41\times10^{-46}\text{ kg}\cdot\text{m}^2\right)}$$
$$= 3.91\times10^{-23}\text{ J}\times\frac{1\text{eV}}{1.60\times10^{-19}\text{ J}}$$
$$= 0.244\text{ meV}$$

Substitute to obtain:

$$E_\ell = \boxed{0.244\ell(\ell+1)\text{ meV}}$$

***65 ••**

Picture the Problem We can obtain an expression for the speed of the nitrogen molecule by equating its translational and rotational kinetic energies and solving for v. Because this expression includes the moment of inertia I of the nitrogen molecule, we can use the definition of the moment of inertia to express I for a dumbbell model of the nitrogen molecule. The rotational energies of a nitrogen molecule depend on the quantum number ℓ according to $E_\ell = L^2/2I = \ell(\ell+1)\hbar^2/2I$.

Equate the rotational kinetic energy of the nitrogen molecule in its $\ell = 1$ quantum state and its translational kinetic energy:

$$E_1 = \tfrac{1}{2}m_N v^2 \qquad (1)$$

Express the rotational energy levels of the nitrogen molecule:

$$E_\ell = \frac{L^2}{2I} = \frac{\ell(\ell+1)\hbar^2}{2I}$$

For $\ell = 1$:

$$E_1 = \frac{1(1+1)\hbar^2}{2I} = \frac{\hbar^2}{I}$$

Substitute in equation (1):

$$\frac{\hbar^2}{I} = \tfrac{1}{2} m_N v^2$$

Solve for v to obtain:

$$v = \sqrt{\frac{2\hbar^2}{m_N I}} \qquad (2)$$

Using a rigid dumbbell model, express the moment of inertia of the nitrogen molecule about its center of mass:

$$I = \sum_i m_i r_i^2 = m_N r^2 + m_N r^2 = 2m_N r^2$$

and

$$m_N I = 2m_N^2 r^2$$

Substitute in equation (2):

$$v = \sqrt{\frac{2\hbar^2}{2m_N^2 r^2}} = \frac{\hbar}{m_N r}$$

Substitute numerical values and evaluate v:

$$v = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{14\left(1.66\times10^{-27}\ \text{kg}\right)\left(5.5\times10^{-11}\ \text{m}\right)} = \boxed{82.5\ \text{m/s}}$$

Collision Problems

66 ••

Picture the Problem Let the zero of gravitational potential energy be at the elevation of the rod. Because the net external torque acting on this system is zero, we know that angular momentum is conserved in the collision. We'll use the definition of angular momentum to express the angular momentum just after the collision and conservation of mechanical energy to determine the speed of the ball just before it makes its perfectly inelastic collision with the rod.

Use conservation of angular momentum to relate the angular momentum before the collision to the angular momentum just after the perfectly inelastic collision:

$$L_f = L_i = mvr$$

Use conservation of mechanical energy to relate the kinetic energy of the ball just before impact to its initial potential energy:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = U_f = 0$,

$$K_f - U_i = 0$$

Letting h represent the distance the

$$v = \sqrt{2gh}$$

ball falls, substitute for K_f and U_i and solve for v to obtain:

Substitute for v to obtain:

$$L_f = mr\sqrt{2gh}$$

Substitute numerical values and evaluate L_f:

$$L_f = (3.2\,\text{kg})(0.9\,\text{m})\sqrt{2(9.81\,\text{m/s}^2)(1.2\,\text{m})} = \boxed{14.0\,\text{J}\cdot\text{s}}$$

***67 ••**

Picture the Problem Because there are no external forces or torques acting on the system defined in the problem statement, both linear and angular momentum are conserved in the collision and the velocity of the center of mass after the collision is the same as before the collision. Let the direction the blob of putty is moving initially be the positive x direction and toward the top of the page in the figure be the positive y direction.

Using its definition, express the location of the center of mass relative to the center of the bar:

$$y_{cm} = \frac{md}{M+m} \text{ below the center of the bar.}$$

Using its definition, express the velocity of the center of mass:

$$v_{cm} = \boxed{\frac{mv}{M+m}}$$

Using the definition of L in terms of I and ω, express ω:

$$\omega = \frac{L_{cm}}{I_{cm}} \qquad (1)$$

Express the angular momentum about the center of mass:

$$L_{cm} = mv(d - y_{cm}) = mv\left(d - \frac{md}{M+m}\right) = \frac{mMvd}{M+m}$$

Using the parallel axis theorem, express the moment of inertia of the system relative to its center of mass:

$$I_{cm} = \tfrac{1}{12}ML^2 + My_{cm}^2 + m(d - y_{cm})^2$$

Substitute for y_{cm} and simplify to obtain:

$$I_{\text{cm}} = \tfrac{1}{12}ML^2 + M\left(\frac{md}{M+m}\right)^2 + m\left(d - \frac{md}{M+m}\right)^2$$

$$= \tfrac{1}{12}ML^2 + \frac{Mm^2d^2}{(M+m)^2} + m\left(\frac{d(M+m)-md}{M+m}\right)^2$$

$$= \tfrac{1}{12}ML^2 + \frac{Mm^2d^2}{(M+m)^2} + \frac{mM^2d^2}{(M+m)^2} = \tfrac{1}{12}ML^2 + \frac{(M+m)mMd^2}{(M+m)^2}$$

$$= \tfrac{1}{12}ML^2 + \frac{mMd^2}{M+m}$$

Substitute for I_{cm} and L_{cm} in equation (1) and simplify to obtain:

$$\omega = \boxed{\frac{mMvd}{\tfrac{1}{12}ML^2(M+m) + Mmd^2}}$$

Remarks: You can verify the expression for I_{cm} by letting $m \to 0$ to obtain $I_{cm} = \tfrac{1}{12}ML^2$ and letting $M \to 0$ to obtain $I_{\text{cm}} = 0$.

68 ••

Picture the Problem Because there are no external forces or torques acting on the system defined in the statement of Problem 67, both linear and angular momentum are conserved in the collision and the velocity of the center of mass after the collision is the same as before the collision. Kinetic energy is also conserved as the collision of the hard sphere with the bar is elastic. Let the direction the sphere is moving initially be the positive x direction and toward the top of the page in the figure be the positive y direction and v' and V' be the final velocities of the objects whose masses are m and M, respectively.

Apply conservation of linear momentum to obtain:

$$p_{\text{i}} = p_{\text{f}}$$

or

$$mv = mv' + MV' \qquad (1)$$

Apply conservation of angular momentum to obtain:

$$L_{\text{i}} = L_{\text{f}}$$

or

$$mvd = mv'd + \tfrac{1}{12}ML^2\omega \qquad (2)$$

Set $v' = 0$ in equation (1) and solve for V':

$$V' = \frac{mv}{M} \qquad (3)$$

Use conservation of mechanical energy to relate the kinetic energies of translation and rotation before

$$K_{\text{i}} = K_{\text{f}}$$

or

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}MV'^2 + \tfrac{1}{2}\left(\tfrac{1}{12}ML^2\right)\omega^2 \qquad (4)$$

and after the elastic collision:

Substitute (2) and (3) in (4) and simplify to obtain:

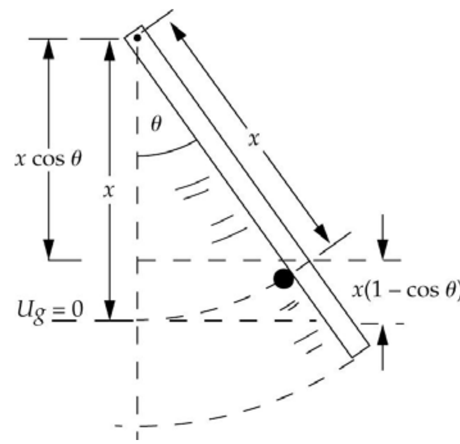
$$1=\frac{m}{M}+\frac{12m}{M}\left(\frac{d^2}{L^2}\right)$$

Solve for d:

$$d=\boxed{L\sqrt{\frac{M-m}{12m}}}$$

69 ••

Picture the Problem Let the zero of gravitational potential energy be a distance x below the pivot as shown in the diagram. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. We can also use conservation of mechanical energy to relate the initial kinetic energy of the system after the collision to its potential energy at the top of its swing.



Using conservation of mechanical energy, relate the rotational kinetic energy of the system just after the collision to its gravitational potential energy when it has swung through an angle θ:

$$\Delta K+\Delta U=0$$

or, because $K_f = U_i = 0$,

$$-K_i+U_f=0$$

and

$$\tfrac{1}{2}I\omega^2=\left(Mg\frac{d}{2}+mgx\right)(1-\cos\theta) \quad (1)$$

Apply conservation of momentum to the collision:

$$L_i=L_f$$

or

$$0.8dmv=I\omega=\left[\tfrac{1}{3}Md^2+(0.8d)^2m\right]\omega$$

Solve for ω to obtain:

$$\omega=\frac{0.8dmv}{\frac{1}{3}Md^2+0.64md^2} \quad (2)$$

Express the moment of inertia of the system about the pivot:

$$I=m(0.8d)^2+\tfrac{1}{3}Md^2=0.64md^2+\tfrac{1}{3}Md^2 \quad (3)$$

Substitute equations (2) and (3) in equation (1) and simplify to obtain:

$$\left(Mg\frac{d}{2}+mgd\right)(1-\cos\theta)=\frac{0.32(dmv)^2}{\frac{1}{3}Md^2+0.64md^2}$$

Solve for v:

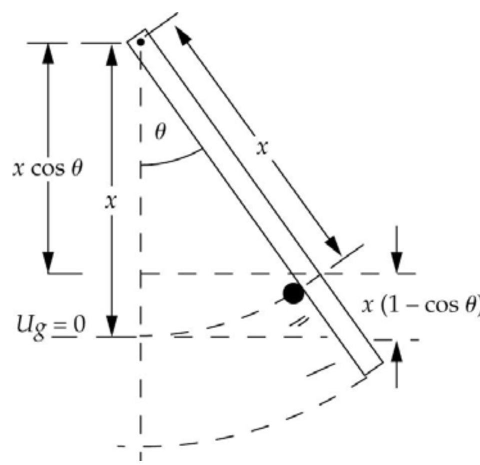
$$v=\sqrt{\frac{(0.5M+0.8m)\left(\frac{1}{3}Md^2+0.64md^2\right)g(1-\cos\theta)}{0.32dm^2}}$$

Evaluate v for $\theta = 90°$ to obtain:

$$\boxed{v=\sqrt{\frac{(0.5M+0.8m)\left(\frac{1}{3}Md^2+0.64md^2\right)g}{0.32dm^2}}}$$

70 ••

Picture the Problem Let the zero of gravitational potential energy be a distance x below the pivot as shown in the diagram. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. We can also use conservation of mechanical energy to relate the initial kinetic energy of the system after the collision to its potential energy at the top of its swing.



Using conservation of mechanical energy, relate the rotational kinetic energy of the system just after the collision to its gravitational potential energy when it has swung through an angle θ:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

and

$$\tfrac{1}{2}I\omega^2=\left(Mg\frac{d}{2}+mgx\right)(1-\cos\theta) \quad (1)$$

Apply conservation of momentum to the collision:

$$L_i = L_f$$

or

$$0.8dmv = I\omega = \left[\tfrac{1}{3}Md^2+(0.8d)^2m\right]\omega$$

Solve for ω to obtain:

$$\omega = \frac{0.8dmv}{\frac{1}{2}Md^2 + 0.64md^2} \qquad (2)$$

Express the moment of inertia of the system about the pivot:

$$\begin{aligned} I &= m(0.8d)^2 + \tfrac{1}{3}Md^2 \\ &= \left(0.64m + \tfrac{1}{3}M\right)d^2 \\ &= \left[0.64(0.3\,\text{kg}) + \tfrac{1}{3}(0.8\,\text{kg})\right](1.2\,\text{m})^2 \\ &= 0.660\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Substitute equation (2) in equation (1) and simplify to obtain:

$$\left(Mg\frac{d}{2} + 0.8dmg\right)(1-\cos\theta) = \frac{0.32(dmv)^2}{I}$$

Solve for v:

$$v = \sqrt{\frac{g(0.5M + 0.8m)(1-\cos\theta)I}{0.32dm^2}}$$

Substitute numerical values and evaluate v for $\theta = 60°$ to obtain:

$$v = \sqrt{\frac{(9.81\,\text{m/s}^2)\left[0.5(0.8\,\text{kg}) + 0.8(0.3\,\text{kg})\right](0.5)(0.660\,\text{kg}\cdot\text{m}^2)}{0.32(1.2\,\text{m})(0.3\,\text{kg})^2}} = \boxed{7.74\,\text{m/s}}$$

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Picture the Problem Let the length of the uniform stick be ℓ. We can use the impulse-change in momentum theorem to express the velocity of the center of mass of the stick. By expressing the velocity V of the end of the stick in terms of the velocity of the center of mass and applying the angular impulse-change in angular momentum theorem we can find the angular velocity of the stick and, hence, the velocity of the end of the stick.

(*a*) Apply the impulse-change in momentum theorem to obtain:

$$K = \Delta p = p - p_0 = p$$

or, because $p_0 = 0$ and $p = Mv_{cm}$,

$$K = Mv_{cm}$$

Solve for v_{cm} to obtain:

$$v_{cm} = \boxed{\frac{K}{M}}$$

(*b*) Relate the velocity V of the end of the stick to the velocity of the center of mass v_{cm}:

$$V = v_{cm} + v_{\text{rel to c of m}} = v_{cm} + \omega\left(\tfrac{1}{2}\ell\right) \qquad (1)$$

Relate the angular impulse to the change in the angular momentum of the stick:

$$K\left(\tfrac{1}{2}\ell\right) = \Delta L = L - L_0 = I_{cm}\omega$$

or, because $L_0 = 0$,

$$K\left(\tfrac{1}{2}\ell\right) = I_{cm}\omega$$

Refer to Table 9-1 to find the moment of inertia of the stick with respect to its center of mass:

$$I_{cm} = \tfrac{1}{12}M\ell^2$$

Substitute to obtain:

$$K\left(\tfrac{1}{2}\ell\right) = \tfrac{1}{12}M\ell^2\omega$$

Solve for ω:

$$\omega = \frac{6K}{M\ell}$$

Substitute in equation (1) to obtain:

$$V = \frac{K}{M} + \left(\frac{6K}{M\ell}\right)\frac{\ell}{2} = \boxed{\frac{4K}{M}}$$

(*c*) Relate the velocity V' of the other end of the stick to the velocity of the center of mass v_{cm}:

$$V = v_{cm} - v_{\text{rel to c of m}} = v_{cm} - \omega\left(\tfrac{1}{2}\ell\right)$$
$$= \frac{K}{M} - \left(\frac{6K}{M\ell}\right)\frac{\ell}{2} = \boxed{-\frac{2K}{M}}$$

(*d*) Letting x be the distance from the center of mass toward the end not struck, express the condition that the point at x is at rest:

$$v_{cm} - \omega x = 0$$

Solve for x to obtain:

$$\frac{K}{M} - \frac{6K}{M\ell}x = 0$$

Solve for x to obtain:

$$x = \frac{\frac{K}{M}}{\frac{6K}{M\ell}} = \boxed{\tfrac{1}{6}\ell}$$

Note that for a meter stick struck at the 100-cm mark, the stationary point would be at the 33.3-cm mark.

Remarks: You can easily check this result by placing a meterstick on the floor and giving it a sharp blow at the 100-cm mark.

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Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision.

(*a*) Use its definition to express the total angular momentum of the disk and projectile just before impact:

$$L_0 = \boxed{m_p v_0 b}$$

(*b*) Use conservation of angular momentum to relate the angular momenta just before and just after the collision:

$$L_0 = L = I\omega \text{ and } \omega = \frac{L_0}{I}$$

Express the moment of inertia of the disk + projectile:

$$I = \tfrac{1}{2}MR^2 + m_p b^2$$

Substitute for I in the expression for ω to obtain:

$$\omega = \boxed{\frac{2m_p v_0 b}{MR^2 + 2m_p b^2}}$$

(*c*) Express the kinetic energy of the system after impact in terms of its angular momentum:

$$K_f = \frac{L^2}{2I} = \frac{(m_p v_0 b)^2}{2\left(\frac{1}{2}MR^2 + m_p b^2\right)}$$

$$= \boxed{\frac{(m_p v_0 b)^2}{MR^2 + 2m_p b^2}}$$

(*d*) Express the difference between the initial and final kinetic energies, substitute, and simplify to obtain:

$$\Delta E = K_i - K_f$$

$$= \tfrac{1}{2}m_p v_0^2 - \frac{(m_p v_0 b)^2}{MR^2 + 2m_p b^2}$$

$$= \boxed{\tfrac{1}{2}m_p v_0^2\left(1 - \frac{m_p b^2}{MR^2 + 2m_p b^2}\right)}$$

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Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the perfectly inelastic collision with the particle and the rotational kinetic of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the masses M and m.

Use conservation of energy to relate the initial and final potential energy of the rod to its rotational kinetic energy just before it collides with the particle:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solve for ω:

$$\omega = \sqrt{\frac{3g}{L_1}}$$

Letting ω' represent the angular speed of the rod-and-particle system just after impact, use conservation of angular momentum to relate the angular momenta before and after the collision:

$$L_i = L_f$$

or

$$\left(\tfrac{1}{3}ML_1^2\right)\omega = \left(\tfrac{1}{3}ML_1^2 + mL_2^2\right)\omega'$$

Solve for ω':

$$\omega' = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\omega$$

Use conservation of energy to relate the rotational kinetic energy of the rod-plus-particle just after their collision to their potential energy when they have swung through an angle θ_{max}:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-\tfrac{1}{2}I\omega'^2 + Mg\left(\tfrac{1}{2}L_1\right)\left(1-\cos\theta_{max}\right) + mgL_2\left(1-\cos\theta_{max}\right) = 0 \quad (1)$$

Express the moment of inertia of the system with respect to the pivot:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for θ_{max}, I and ω' in equation (1):

$$\frac{3\dfrac{g}{L_1}\left(\tfrac{1}{3}ML_1^2\right)^2}{\tfrac{1}{3}ML_1^2 + mL_2^2} = Mg\left(\tfrac{1}{2}L_1\right) + mgL_2$$

Simplify to obtain:

$$L_1^3 = 2\frac{m}{M}L_1^2L_2 + 3L_2^2L_1 + 6\frac{m}{M}L_2^3 \quad (2)$$

Simplify equation (2) by letting $\alpha = m/M$ and $\beta = L_2/L_1$ to obtain:

$$6\alpha^2\beta^3 + 3\beta^2 + 2\alpha\beta - 1 = 0$$

Substitute for α and simplify to obtain the cubic equation in β:

$$12\beta^3 + 9\beta^2 + 4\beta - 3 = 0$$

Use the solver function* of your calculator to find the only real value

$$\beta = \boxed{0.349}$$

of β:

***Remarks**: **Most graphing calculators have a "solver" feature. One can solve the cubic equation using either the "graph" and "trace" capabilities or the "solver" feature. The root given above was found using SOLVER on a TI-85.**

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Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the perfectly inelastic collision with the particle and the rotational kinetic energy of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the mass M to m.

(*a*) Use conservation of energy to relate the initial and final potential energy of the rod to its rotational kinetic energy just before it collides with the particle:

$$K_\mathrm{f} - K_\mathrm{i} + U_\mathrm{f} - U_\mathrm{i} = 0$$

or, because $K_\mathrm{i} = 0$,

$$K_\mathrm{f} + U_\mathrm{f} - U_\mathrm{i} = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solve for ω:

$$\omega = \sqrt{\frac{3g}{L_1}}$$

Letting ω' represent the angular speed of the system after impact, use conservation of angular momentum to relate the angular momenta before and after the collision:

$$L_\mathrm{i} = L_\mathrm{f}$$

or

$$\left(\tfrac{1}{3}ML_1^2\right)\omega = \left(\tfrac{1}{3}ML_1^2 + mL_2^2\right)\omega' \qquad (1)$$

Solve for ω':

$$\omega' = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\,\omega$$

$$= \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\sqrt{\frac{3g}{L_1}}$$

Substitute numerical values to obtain:

$$\omega' = \frac{\frac{1}{3}(2\,\text{kg})(1.2\,\text{m})^2}{\frac{1}{3}(2\,\text{kg})(1.2\,\text{m})^2 + m(0.8\,\text{m})^2} \times \sqrt{\frac{3(9.81\,\text{m/s}^2)}{1.2\,\text{m}}}$$

$$= \frac{4.75\,\text{kg}\cdot\text{m}^2/\text{s}}{0.960\,\text{kg}\cdot\text{m}^2 + (0.64\,\text{m}^2)m}$$

$$= \frac{4.75\,\text{kg/s}}{0.960\,\text{kg} + 0.64m}$$

Use conservation of energy to relate the rotational kinetic energy of the rod-plus-particle just after their collision to their potential energy when they have swung through an angle θ_{max}:

$$K_\text{f} - K_\text{i} + U_\text{f} - U_\text{i} = 0$$

or, because $K_\text{f} = 0$,

$$-K_\text{i} + U_\text{f} - U_\text{i} = 0$$

Substitute for K_i, U_f, and U_i to obtain:

$$-\tfrac{1}{2}I\omega'^2 + Mg\left(\tfrac{1}{2}L_1\right)(1-\cos\theta_{\text{max}}) + mgL_2(1-\cos\theta_{\text{max}}) = 0$$

Express the moment of inertia of the system with respect to the pivot:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for θ_{max}, I and ω' in equation (1) and simplify to obtain:

$$\frac{\frac{1}{2}(4.75\,\text{kg/s})^2}{0.960\,\text{kg} + 0.64m} = 0.2g(ML_1 + mL_2)$$

Substitute for M, L_1 and L_2 and simplify to obtain:

$$m^2 + 3.00m - 8.901 = 0$$

Solve the quadratic equation for its positive root:

$$m = \boxed{1.84\,\text{kg}}$$

(*b*) The energy dissipated in the inelastic collision is:

$$\Delta E = U_\text{i} - U_\text{f} \qquad (2)$$

Express U_i:

$$U_\text{i} = Mg\frac{L_1}{2}$$

Express U_f:

$$U_\text{f} = (1-\cos\theta_{\text{max}})g\left(M\frac{L_1}{2} + mL_2\right)$$

Substitute in equation (2) to obtain:

$$\Delta E = Mg\frac{L_1}{2} - (1-\cos\theta_{\max})g\left(M\frac{L_1}{2} + mL_2\right)$$

Substitute numerical values and evaluate ΔE:

$$\begin{aligned} U_f &= \frac{(2\,\text{kg})(9.81\,\text{m/s}^2)(1.2\,\text{m})}{2} \\ &\quad -(1-\cos 37°)(9.81\,\text{m/s}^2)\left(\frac{(2\,\text{kg})(1.2\,\text{m})}{2} + (1.85\,\text{kg})(0.8\,\text{m})\right) \\ &= \boxed{6.51\,\text{J}} \end{aligned}$$

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Picture the Problem Let ω_i and ω_f be the angular velocities of the rod immediately before and immediately after the inelastic collision with the mass m. Let ω_0 be the initial angular velocity of the rod. Choose the zero of gravitational potential energy be at a distance L_1 below the pivot. We apply energy conservation to determine ω_f and conservation of angular momentum to determine ω_i. We'll apply energy conservation to determine ω_0. Finally, we'll find the energies of the system immediately before and after the collision and the energy dissipated.

Express the energy dissipated in the inelastic collision:

$$\Delta E = U_i - U_f \qquad (1)$$

Use energy conservation to relate the kinetic energy of the system immediately after the collision to its potential energy after a 180° rotation:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = K_{top} = 0$ and $K_i = K_{bottom}$,

$$-K_{bottom} + U_{top} - U_{bottom} = 0$$

Substitute for K_{bottom}, U_{top}, and U_{bottom} to obtain:

$$-\tfrac{1}{2}I\omega_f^2 + \tfrac{3}{2}MgL_1 + mg(L_1 + L_2) - \tfrac{1}{2}MgL_1 - mg(L_1 - L_2) = 0$$

Simplify to obtain:

$$-\tfrac{1}{2}I\omega_f^2 + MgL_1 + 2mgL_2 = 0 \qquad (2)$$

Express I:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for I in equation (2) and solve for ω_f to obtain:

$$\omega_f = \sqrt{\frac{2g(ML_1 + 2mL_2)}{\tfrac{1}{3}ML_1^2 + mL_2^2}}$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \sqrt{\frac{2(9.81\,\text{m/s}^2)[(0.75\,\text{kg})(1.2\,\text{m}) + 2(0.4\,\text{kg})(0.8\,\text{m})]}{\frac{1}{3}(0.75\,\text{kg})(1.2\,\text{m})^2 + (0.4\,\text{kg})(0.8\,\text{m})^2}} = 7.00\,\text{rad/s}$$

Use conservation of angular momentum to relate the angular momentum of the system just before the collision to its angular momentum just after the collision:

$$L_i = L_f$$

or

$$I_i\omega_i = I_f\omega_f$$

Substitute for I_i and I_f and solve for ω_i:

$$\left(\tfrac{1}{3}ML_1^2\right)\omega_i = \left(\tfrac{1}{3}ML_1^2 + mL_2^2\right)\omega_f$$

and

$$\omega_i = \left[1 + \frac{3m}{M}\left(\frac{L_2}{L_1}\right)^2\right]\omega_f$$

Substitute numerical values and evaluate ω_i:

$$\omega_i = \left[1 + \frac{3(0.4\,\text{kg})}{0.75\,\text{kg}}\left(\frac{0.8\,\text{m}}{1.2\,\text{m}}\right)^2\right](7.00\,\text{rad/s})$$
$$= 12.0\,\text{rad/s}$$

Apply conservation of mechanical energy to relate the initial rotational kinetic energy of the rod to its rotational kinetic energy just before its collision with the particle:

$$K_f - K_i + U_f - U_i = 0$$

Substitute to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega_i^2 - \tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega_0^2 + Mg\frac{L_1}{2}$$
$$- MgL_1 = 0$$

Solve for ω_0:

$$\omega_0 = \sqrt{\omega_i^2 - \frac{3g}{L_1}}$$

Substitute numerical values and evaluate ω_0:

$$\omega_0 = \sqrt{(12\,\text{rad/s})^2 - \frac{3(9.81\,\text{m/s}^2)}{1.2\,\text{m}}}$$
$$= \boxed{10.9\,\text{rad/s}}$$

Substitute in equation (1) to express the energy dissipated in the collision:

$$\Delta E = \tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega_i^2 - MgL_1 + 2mgL_2$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = \tfrac{1}{6}(0.75\,\text{kg})(1.2\,\text{m})^2(12\,\text{rad/s})^2 - \left(9.81\,\text{m/s}^2\right)[(0.75\,\text{kg})(1.2\,\text{m}) + 2(0.4\,\text{kg})(0.8\,\text{m})]$$
$$= \boxed{10.8\,\text{J}}$$

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Picture the Problem Let v be the speed of the particle immediately after the collision and ω_i and ω_f be the angular velocities of the rod immediately before and immediately after the elastic collision with the mass m. Choose the zero of gravitational potential energy be at a distance L_1 below the pivot. Because the net external torque acting on the system is zero, angular momentum is conserved in this elastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the elastic collision with the particle and the kinetic energy of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the mass M to m.

Use energy conservation to relate the energies of the system immediately before and after the elastic collision:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}mv^2 + Mg\frac{L_1}{2}\left(1 - \cos\theta_{max}\right) - Mg\frac{L_1}{2} = 0$$

Solve for mv^2:

$$mv^2 = MgL_1\cos\theta_{max} \qquad (1)$$

Apply conservation of energy to express the angular speed of the rod just before the collision:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega_i^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solve for ω_i:

$$\omega_i = \sqrt{\frac{3g}{L_1}}$$

Apply conservation of energy to the rod after the collision:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega_f^2 - Mg\frac{L_1}{2}\left(1-\cos\theta_{max}\right)=0$$

Solve for ω_f:

$$\omega_f=\sqrt{\frac{0.6g}{L_1}}$$

Apply conservation of angular momentum to the collision:

$$L_i=L_f$$

or

$$\left(\tfrac{1}{3}ML_1^2\right)\omega_i=\left(\tfrac{1}{3}ML_1^2\right)\omega_f+mvL_2$$

Solve for mv:

$$mv=\frac{\tfrac{1}{3}ML_1^2\left(\omega_i-\omega_f\right)}{L_2}$$

Substitute for ω_f and ω_i to obtain:

$$mv=\frac{ML_1^2\left(\sqrt{\frac{3g}{L_1}}-\sqrt{\frac{0.6g}{L_1}}\right)}{3L_2}\qquad(2)$$

Divide equation (1) by equation (2) to eliminate m and solve for v:

$$v=\frac{MgL_1\cos\theta_{max}}{\dfrac{ML_1^2\left(\sqrt{\frac{3g}{L_1}}-\sqrt{\frac{0.6g}{L_1}}\right)}{3L_2}}=\frac{3gL_2\cos\theta_{max}}{\sqrt{3gL_1}-\sqrt{0.6gL_1}}$$

Substitute numerical values and evaluate v:

$$v=\frac{3\left(9.81\,\text{m/s}^2\right)\left(0.8\,\text{m}\right)\cos37^\circ}{\sqrt{3\left(9.81\,\text{m/s}^2\right)\left(1.2\,\text{m}\right)}-\sqrt{0.6\left(9.81\,\text{m/s}^2\right)\left(1.2\,\text{m}\right)}}=5.72\,\text{m/s}$$

Solve equation (1) for m:

$$m=\frac{MgL_1\cos\theta_{max}}{v^2}$$

Substitute for v in the expression for mv and solve for m:

$$m=\frac{\left(2\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right)\left(1.2\,\text{m}\right)\cos37^\circ}{\left(5.72\,\text{m/s}\right)^2}=\boxed{0.575\,\text{kg}}$$

Because the collision was elastic:

$$\Delta E=\boxed{0}$$

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Picture the Problem We can determine the angular momentum of the wheel and the angular velocity of its precession from their definitions. The period of the precessional motion can be found from its angular velocity and the angular momentum associated with the motion of the center of mass from its definition.

(*a*) Using the definition of angular momentum, express the angular momentum of the spinning wheel:

$$L = I\omega = MR^2\omega = \frac{w}{g}R^2\omega$$

Substitute numerical values and evaluate *L*:

$$L = \left(\frac{30\,\text{N}}{9.81\,\text{m/s}^2}\right)(0.28\,\text{m})^2 \times \left(12\frac{\text{rev}}{\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right) = \boxed{18.1\,\text{J}\cdot\text{s}}$$

(*b*) Using its definition, express the angular velocity of precession:

$$\omega_\text{p} = \frac{d\phi}{dt} = \frac{MgD}{L}$$

Substitute numerical values and evaluate ω_p:

$$\omega_\text{p} = \frac{(30\,\text{N})(0.25\,\text{m})}{18.1\,\text{J}\cdot\text{s}} = \boxed{0.414\,\text{rad/s}}$$

(*c*) Express the period of the precessional motion as a function of the angular velocity of precession:

$$T = \frac{2\pi}{\omega_\text{p}} = \frac{2\pi}{0.414\,\text{rad/s}} = \boxed{15.2\,\text{s}}$$

(*d*) Express the angular momentum of the center of mass due to the precession:

$$L_\text{p} = I_\text{cm}\omega_\text{p} = MD^2\omega_\text{p}$$

Substitute numerical values and evaluate L_p:

$$L_\text{p} = \left(\frac{30\,\text{N}}{9.81\,\text{m/s}^2}\right)(0.25\,\text{m})^2(0.414\,\text{rad/s}) = \boxed{0.0791\,\text{J}\cdot\text{s}}$$

The direction of L_p is either up or down, depending on the direction of *L*.

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Picture the Problem The angular velocity of precession can be found from its definition. Both the speed and acceleration of the center of mass during precession are related to the angular velocity of precession. We can use Newton's 2nd law to find the vertical and horizontal components of the force exerted by the pivot.

(a) Using its definition, express the angular velocity of precession:

$$\omega_{\mathrm{p}}=\frac{d\phi}{dt}=\frac{MgD}{I_{\mathrm{s}}\omega_{\mathrm{s}}}=\frac{MgD}{\frac{1}{2}MR^2\omega_{\mathrm{s}}}=\frac{2gD}{R^2\omega_{\mathrm{s}}}$$

Substitute numerical values and evaluate ω_{p}:

$$\omega_{\mathrm{p}}=\frac{2\left(9.81\,\mathrm{m/s^2}\right)(0.05\,\mathrm{m})}{(0.064\,\mathrm{m})^2\left(700\frac{\mathrm{rev}}{\mathrm{min}}\times\frac{2\pi\,\mathrm{rad}}{\mathrm{rev}}\times\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right)}=\boxed{3.27\,\mathrm{rad/s}}$$

(b) Express the speed of the center of mass in terms of its angular velocity of precession:

$$v_{\mathrm{cm}}=D\omega_{\mathrm{p}}=(0.05\,\mathrm{m})(3.27\,\mathrm{rad/s})=\boxed{0.164\,\mathrm{m/s}}$$

(c) Relate the acceleration of the center of mass to its angular velocity of precession:

$$a_{\mathrm{cm}}=D\omega_{\mathrm{p}}^2=(0.05\,\mathrm{m})(3.27\,\mathrm{rad/s})^2=\boxed{0.535\,\mathrm{m/s^2}}$$

(d) Use Newton's 2nd law to relate the vertical component of the force exerted by the pivot to the weight of the disk:

$$F_{\mathrm{v}}=Mg=(2.5\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right)=\boxed{24.5\,\mathrm{N}}$$

Relate the horizontal component of the force exerted by the pivot to the acceleration of the center of mass:

$$F_{\mathrm{v}}=Ma_{\mathrm{cm}}=(2.5\,\mathrm{kg})\left(0.535\,\mathrm{m/s^2}\right)=\boxed{1.34\,\mathrm{N}}$$

General Problems

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Picture the Problem While the 3-kg particle is moving in a straight line, it has angular momentum given by $\vec{L}=\vec{r}\times\vec{p}$ where $\vec{r}$ is its position vector and $\vec{p}$ is its linear momentum. The torque due to the applied force is given by $\vec{\tau}=\vec{r}\times\vec{F}$.

(a) Express the angular momentum of the particle:

$$\vec{L}=\vec{r}\times\vec{p}$$

Express the vectors $\vec{r}$ and $\vec{p}$:

$$\vec{r}=(12\,\mathrm{m})\hat{i}+(5.3\,\mathrm{m})\hat{j}$$

and

$$\vec{p} = mv\hat{i} = (3\,\text{kg})(3\,\text{m/s})\hat{i}$$
$$= (9\,\text{kg}\cdot\text{m/s})\hat{i}$$

Substitute and simplify to find $\vec{L}$:

$$\vec{L} = \left[(12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}\right] \times (9\,\text{kg}\cdot\text{m/s})\hat{i}$$
$$= (47.7\,\text{kg}\cdot\text{m}^2/\text{s})(\hat{j}\times\hat{i})$$
$$= \boxed{-(47.7\,\text{kg}\cdot\text{m}^2/\text{s})\hat{k}}$$

(*b*) Using its definition, express the torque due to the force:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Substitute and simplify to find $\vec{\tau}$:

$$\vec{\tau} = \left[(12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}\right] \times (-3\,\text{N})\hat{i}$$
$$= -(15.9\,\text{N}\cdot\text{m})(\hat{j}\times\hat{i})$$
$$= \boxed{(15.9\,\text{N}\cdot\text{m})\hat{k}}$$

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Picture the Problem The angular momentum of the particle is given by $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is its position vector and $\vec{p}$ is its linear momentum. The torque acting on the particle is given by $\vec{\tau} = d\vec{L}/dt$.

Express the angular momentum of the particle:

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$
$$= m\vec{r} \times \frac{d\vec{r}}{dt}$$

Evaluate $\dfrac{d\vec{r}}{dt}$:

$$\frac{d\vec{r}}{dt} = 6t\hat{j}$$

Substitute and simplify to find $\vec{L}$:

$$\vec{L} = \left[(3\,\text{kg})\left\{(4\,\text{m})\hat{i} + (3t^2\,\text{m/s}^2)\hat{j}\right\}\right]$$
$$\times (6t\,\text{m/s})\hat{j}$$
$$= \boxed{(72.0t\,\text{J}\cdot\text{s})\hat{k}}$$

Find the torque due to the force:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left[(72.0t\,\text{J}\cdot\text{s})\hat{k}\right]$$
$$= \boxed{(72.0\,\text{N}\cdot\text{m})\hat{k}}$$

81 ••

Picture the Problem The ice skaters rotate about their center of mass; a point we can locate using its definition. Knowing the location of the center of mass we can determine their moment of inertia with respect to an axis through this point. The angular momentum of the system is then given by $L = I_{cm}\omega$ and its kinetic energy can be found from $K = L^2/2I_{cm}$.

(*a*) Express the angular momentum of the system about the center of mass of the skaters:

$$L = I_{cm}\omega$$

Using its definition, locate the center of mass, relative to the 85-kg skater, of the system:

$$x_{cm} = \frac{(55\,\text{kg})(1.7\,\text{m}) + (85\,\text{kg})(0)}{55\,\text{kg} + 85\,\text{kg}} = 0.668\,\text{m}$$

Calculate I_{cm}:

$$I_{cm} = (55\,\text{kg})(1.7\,\text{m} - 0.668\,\text{m})^2 + (85\,\text{kg})(0.668\,\text{m})^2 = 96.5\,\text{kg}\cdot\text{m}^2$$

Substitute to determine L:

$$L = \left(96.5\,\text{kg}\cdot\text{m}^2\right)\left(\frac{1\,\text{rev}}{2.5\,\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right) = \boxed{243\,\text{J}\cdot\text{s}}$$

(*b*) Relate the total kinetic energy of the system to its angular momentum and evaluate K:

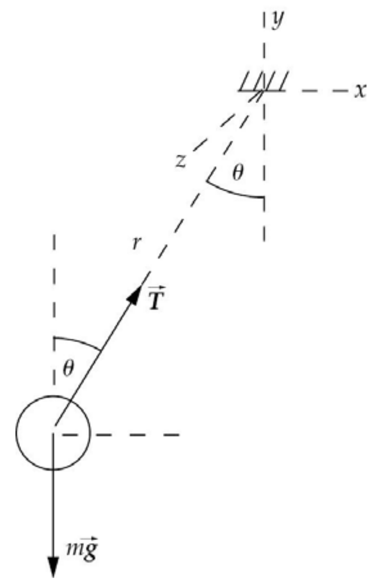
$$K = \frac{L^2}{2I_{cm}}$$

Substitute numerical values and evaluate K:

$$K = \frac{(243\,\text{J}\cdot\text{s})^2}{2\left(96.5\,\text{kg}\cdot\text{m}^2\right)} = \boxed{306\,\text{J}}$$

***82 ••**

Picture the Problem Let the origin of the coordinate system be at the pivot (point P). The diagram shows the forces acting on the ball. We'll apply Newton's 2nd law to the ball to determine its speed. We'll then use the derivative of its position vector to express its velocity and the definition of angular momentum to show that $\vec{L}$ has both horizontal and vertical components. We can use the derivative of $\vec{L}$ with respect to time to show that the rate at which the angular momentum of the ball changes is equal to the torque, relative to the pivot point, acting on it.



(*a*) Express the angular momentum of the ball about the point of support:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \quad (1)$$

Apply Newton's 2nd law to the ball:

$$\sum F_x = T\sin\theta = m\frac{v^2}{r\sin\theta}$$

and

$$\sum F_z = T\cos\theta - mg = 0$$

Eliminate T between these equations and solve for v:

$$v = \sqrt{rg\sin\theta\tan\theta}$$

Substitute numerical values and evaluate v:

$$v = \sqrt{(1.5\,\text{m})(9.81\,\text{m/s}^2)\sin30°\tan30°} = 2.06\,\text{m/s}$$

Express the position vector of the ball:

$$\vec{r} = (1.5\,\text{m})\sin30°\left(\cos\omega t\,\hat{\boldsymbol{i}} + \sin\omega t\hat{\boldsymbol{j}}\right) - (1.5\,\text{m})\cos30°\hat{\boldsymbol{k}}$$

where $\omega = \omega\hat{\boldsymbol{k}}$.

Find the velocity of the ball:

$$\vec{v} = \frac{d\vec{r}}{dt} = (0.75\omega\,\text{m/s})\left(-\sin\omega t\,\hat{\boldsymbol{i}} + \cos\omega t\,\hat{\boldsymbol{j}}\right)$$

Evaluate ω:

$$\omega = \frac{2.06\,\text{m/s}}{(1.5\,\text{m})\sin30°} = 2.75\,\text{rad/s}$$

Substitute for ω to obtain: $\vec{v} = (2.06\,\text{m/s})(-\sin\omega t\,\hat{i} + \cos\omega t\hat{j})$

Substitute in equation (1) and evaluate $\vec{L}$:

$$\begin{aligned}\vec{L} &= (2\,\text{kg})\left[(1.5\,\text{m})\sin 30°(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) - (1.5\,\text{m})\cos 30°\hat{k}\right] \\ &\quad\times\left[(2.06\,\text{m/s})(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})\right] \\ &= \left[5.36(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) + 3.09\,\hat{k}\right]\text{J}\cdot\text{s}\end{aligned}$$

The horizontal component of $\vec{L}$ is: $\boxed{5.36(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})\,\text{J}\cdot\text{s}}$

The vertical component of $\vec{L}$ is: $\boxed{3.09\,\hat{k}\,\text{J}\cdot\text{s}}$

(*b*) Evaluate $\dfrac{d\vec{L}}{dt}$: $\dfrac{d\vec{L}}{dt} = \left[5.36\omega(-\sin\omega t\,\hat{i} + \cos\omega t\hat{j})\right]\text{J}$

Evaluate the magnitude of $\dfrac{d\vec{L}}{dt}$:

$$\left|\frac{d\vec{L}}{dt}\right| = (5.36\,\text{N}\cdot\text{m}\cdot\text{s})(2.75\,\text{rad/s}) = \boxed{14.7\,\text{N}\cdot\text{m}}$$

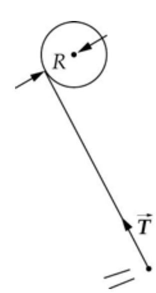
Express the magnitude of the torque exerted by gravity about the point of support: $\tau = mgr\sin\theta$

Substitute numerical values and evaluate τ:

$$\tau = (2\,\text{kg})(9.81\,\text{m/s}^2)(1.5\,\text{m})\sin 30° = \boxed{14.7\,\text{N}\cdot\text{m}}$$

83 ••

Picture the Problem In part (*a*) we need to decide whether a net torque acts on the object. In part (*b*) the issue is whether any external forces act on the object. In part (*c*) we can apply the definition of kinetic energy to find the speed of the object when the unwrapped length has shortened to $r/2$.



(*a*) Consider the overhead view of the cylindrical post and the object shown in the adjoining figure. The object rotates counterclockwise. The torque about the center of the cylinder is clockwise and of magnitude RT, where R is the radius of the cylinder and T is the tension. So

L must decrease.

$\boxed{\text{No, } L \text{ decreases.}}$

(*b*) Because, in this frictionless environment, no net external forces act on the object:

$\boxed{\text{Its kinetic energy is constant.}}$

(*c*) Express the kinetic energy of the object as it spirals inward:

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(mr^2\right)\frac{v^2}{r^2} = \tfrac{1}{2}mv^2$$

$\boxed{v_0\text{. (The kinetic energy remains constant.)}}$

84 ••

Picture the Problem Because the net torque acting on the system is zero; we can use conservation of angular momentum to relate the initial and final angular velocities of the system.

Using conservation of angular momentum, relate the initial and final angular velocities to the initial and final moments of inertia:

$$L_i = L_f$$

or

$$I_i\omega_i = I_f\omega_f$$

Solve for ω_f:

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{I_i}{I_f}\omega$$

Express I_i:

$$I_i = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)$$

Express I_f:

$$I_f = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right)$$

Substitute to express ω_f in terms of ω:

$$\omega_f = \frac{\tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)}{\tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right)}\omega = \boxed{\frac{M + 5m\dfrac{\ell^2}{L^2}}{M + 5m}\omega}$$

Express the initial kinetic energy of the system:

$$K_i = \tfrac{1}{2}I_i\omega^2 = \tfrac{1}{2}\left[\tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)\right]\omega^2 = \boxed{\tfrac{1}{20}\left(ML^2 + 5m\ell^2\right)\omega^2}$$

Express the final kinetic energy of the system and simplify to obtain:

$$K_f = \tfrac{1}{2} I_f \omega_f^2 = \tfrac{1}{2}\left[\tfrac{1}{10} ML^2 + 2\left(\tfrac{1}{4} mL^2\right)\right]\omega_f^2 = \tfrac{1}{20}\left(ML^2 + 5mL^2\right)\omega_f^2$$

$$= \tfrac{1}{20}\left(ML^2 + 5mL^2\right)\left(\frac{M + 5m\dfrac{\ell^2}{L^2}}{M + 5m}\omega\right)^2 = \tfrac{1}{20}\left[\frac{\left(ML + 5m\dfrac{\ell^2}{L}\right)^2}{M + 5m}\right]\omega^2$$

$$= \boxed{\tfrac{1}{20}\left[\frac{\left(ML^2 + 5m\ell^2\right)^2}{ML^2 + 5mL^2}\right]\omega^2}$$

85 ••

Determine the Concept Yes. The net external torque is zero and angular momentum is conserved as the system evolves from its initial to its final state. Because the disks come to the same final position, the initial and final configurations are the same as in Problem 84. Therefore, the answers are the same as for Problem 84.

86 ••

Picture the Problem Because the net torque acting on the system is zero; we can use conservation of angular momentum to relate the initial and final angular velocities of the system.

Using conservation of angular momentum, relate the initial and final angular velocities to the initial and final moments of inertia:

$$L_i = L_f$$

or

$$I_i\omega_i = I_f\omega_f$$

Solve for ω_f:

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{I_i}{I_f}\omega \qquad (1)$$

Relate the tension in the string to the angular speed of the system and solve for and evaluate ω:

$$T = mr\omega^2 = m\frac{\ell}{2}\omega^2$$

and

$$\omega = \sqrt{\frac{2T}{m\ell}} = \sqrt{\frac{2(108\,\text{N})}{(0.4\,\text{kg})(0.6\,\text{m})}}$$

$$= \boxed{30.0\,\text{rad/s}}$$

Express and evaluate I_i:

$$\begin{aligned} I_i &= \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right) \\ &= \tfrac{1}{10}(0.8\,\text{kg})(2\,\text{m})^2 + \tfrac{1}{2}(0.4\,\text{kg})(0.6\,\text{m})^2 \\ &= 0.392\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Express and evaluate I_f:

$$\begin{aligned} I_f &= \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right) \\ &= \tfrac{1}{10}(0.8\,\text{kg})(2\,\text{m})^2 + \tfrac{1}{2}(0.4\,\text{kg})(2\,\text{m})^2 \\ &= 1.12\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Substitute in equation (1) and solve for ω_f:

$$\begin{aligned} \omega_f &= \frac{I_i}{I_f}\omega = \frac{0.392\,\text{kg}\cdot\text{m}^2}{1.12\,\text{kg}\cdot\text{m}^2}(30.0\,\text{rad/s}) \\ &= \boxed{10.5\,\text{rad/s}} \end{aligned}$$

Express and evaluate the initial kinetic energy of the system:

$$\begin{aligned} K_i &= \tfrac{1}{2}I_i\omega^2 \\ &= \tfrac{1}{2}\left(0.392\,\text{kg}\cdot\text{m}^2\right)(30.0\,\text{rad/s})^2 \\ &= \boxed{176\,\text{J}} \end{aligned}$$

Express and evaluate the final kinetic energy of the system:

$$\begin{aligned} K_f &= \tfrac{1}{2}I_f\omega_f^2 \\ &= \tfrac{1}{2}\left(1.12\,\text{kg}\cdot\text{m}^2\right)(10.5\,\text{rad/s})^2 \\ &= \boxed{61.7\,\text{J}} \end{aligned}$$

87 ••

Picture the Problem Until the inelastic collision of the cylindrical objects at the ends of the cylinder, both angular momentum and energy are conserved. Let K' represent the kinetic energy of the system just before the disks reach the end of the cylinder and use conservation of energy to relate the initial and final kinetic energies to the final radial velocity.

Using conservation of mechanical energy, relate the initial and final kinetic energies of the disks:

$$K_i = K'$$

or

$$\tfrac{1}{2}I_i\omega^2 = \tfrac{1}{2}I_f\omega_f^2 + \tfrac{1}{2}\left(2mv_r^2\right)$$

Solve for v_r:

$$v_r = \sqrt{\frac{I_i\omega^2 - I_f\omega_f^2}{2m}} \qquad (1)$$

Using conservation of angular momentum, relate the initial and final angular velocities to the initial

$$L_i = L_f$$

or

and final moments of inertia: $I_{\mathrm{i}}\omega_{\mathrm{i}} = I_{\mathrm{f}}\omega_{\mathrm{f}}$

Solve for ω_{f}:

$$\omega_{\mathrm{f}} = \frac{I_{\mathrm{i}}}{I_{\mathrm{f}}}\omega = \frac{I_{\mathrm{i}}}{I_{\mathrm{f}}}\omega$$

Express I_{i}:

$$I_{\mathrm{i}} = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)$$

Express I_{f}:

$$I_{\mathrm{f}} = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right)$$

Substitute to obtain ω_{f} in terms of ω:

$$\omega_{\mathrm{f}} = \frac{\tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)}{\tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right)}\omega = \frac{ML^2 + 5m\ell^2}{ML^2 + 5mL^2}\omega$$

Substitute in equation (1) and simplify to obtain:

$$v_{\mathrm{r}} = \boxed{\frac{\ell\omega}{2L}\sqrt{\left(L^2 - \ell^2\right)}}$$

88 ••

Picture the Problem Because the net torque acting on the system is zero, we can use conservation of angular momentum to relate the initial and final angular velocities and the initial and final kinetic energy of the system.

Using conservation of angular momentum, relate the initial and final angular velocities to the initial and final moments of inertia:

$$L_{\mathrm{i}} = L_{\mathrm{f}}$$

or

$$I_{\mathrm{i}}\omega_{\mathrm{i}} = I_{\mathrm{f}}\omega_{\mathrm{f}}$$

Solve for ω_{f}:

$$\omega_{\mathrm{f}} = \frac{I_{\mathrm{i}}}{I_{\mathrm{f}}}\omega_{\mathrm{i}} = \frac{I_{\mathrm{i}}}{I_{\mathrm{f}}}\omega \qquad (1)$$

Relate the tension in the string to the angular speed of the system:

$$T = mr\omega^2 = m\frac{\ell}{2}\omega^2$$

Solve for ω:

$$\omega = \sqrt{\frac{2T}{m\ell}}$$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{2(108\,\mathrm{N})}{(0.4\,\mathrm{kg})(0.6\,\mathrm{m})}} = \boxed{30.0\,\mathrm{rad/s}}$$

Express and evaluate I_i:

$$I_i = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right) = \tfrac{1}{10}(0.8\,\text{kg})(2\,\text{m})^2 + \tfrac{1}{2}(0.4\,\text{kg})(0.6\,\text{m})^2 = 0.392\,\text{kg}\cdot\text{m}^2$$

Letting L' represent the final separation of the disks, express and evaluate I_f:

$$I_f = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL'^2\right) = \tfrac{1}{10}(0.8\,\text{kg})(2\,\text{m})^2 + \tfrac{1}{2}(0.4\,\text{kg})(1.6\,\text{m})^2 = 0.832\,\text{kg}\cdot\text{m}^2$$

Substitute in equation (1) and solve for ω_f :

$$\omega_f = \frac{I_i}{I_f}\omega = \frac{0.392\,\text{kg}\cdot\text{m}^2}{0.832\,\text{kg}\cdot\text{m}^2}(30.0\,\text{rad/s}) = 14.1\,\text{rad/s}$$

Express and evaluate the initial kinetic energy of the system:

$$K_i = \tfrac{1}{2}I_i\omega^2 = \tfrac{1}{2}\left(0.392\,\text{kg}\cdot\text{m}^2\right)(30.0\,\text{rad/s})^2 = \boxed{176\,\text{J}}$$

Express and evaluate the final kinetic energy of the system:

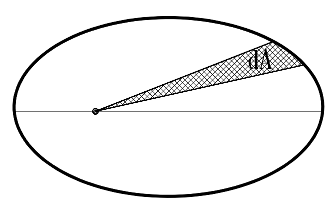
$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}\left(0.832\,\text{kg}\cdot\text{m}^2\right)(14.1\,\text{rad/s})^2 = \boxed{82.7\,\text{J}}$$

The energy dissipated in friction is:

$$\Delta E = K_i - K_f = 176\,\text{J} - 82.7\,\text{J} = \boxed{93.3\,\text{J}}$$

***89 ••**

Picture the Problem The drawing shows an elliptical orbit. The triangular element of the area is $dA = \tfrac{1}{2}r(rd\theta) = \tfrac{1}{2}r^2d\theta.$



Differentiate dA with respect to t to obtain:

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}r^2\omega$$

Because the gravitational force acts along the line joining the two objects, $\tau = 0$ and:

$$L = mr^2\omega = \text{constant}$$

Eliminate $r^2\omega$ between the two equations to obtain:

$$\frac{dA}{dt} = \boxed{\frac{L}{2m} = \text{constant}}$$

90 ••

Picture the Problem Let x be the radial distance each disk moves outward. Because the net torque acting on the system is zero, we can use conservation of angular momentum to relate the initial and final angular velocities to the initial and final moments of inertia. We'll assume that the disks are thin enough so that we can ignore their lengths in expressing their moments of inertia.

Use conservation of angular momentum to relate the initial and final angular velocities of the disks:

$$L_{\rm i} = L_{\rm f}$$

or

$$I_{\rm i}\omega_{\rm i} = I_{\rm f}\omega_{\rm f}$$

Solve for $\omega_{\rm f}$:

$$\omega_{\rm f} = \frac{I_{\rm i}}{I_{\rm f}}\omega_{\rm i} \qquad (1)$$

Express the initial moment of inertia of the system:

$$I_{\rm i} = I_{\rm cyl} + 2I_{\rm disk}$$

Express the moment of inertia of the cylinder:

$$\begin{aligned} I_{\rm cyl} &= \tfrac{1}{12}ML^2 + \tfrac{1}{2}MR^2 \\ &= \tfrac{1}{12}M\left(L^2 + 6R^2\right) \\ &= \tfrac{1}{12}(0.8\,\text{kg})\left[(1.8\,\text{m})^2 + 6(0.2\,\text{m})^2\right] \\ &= 0.232\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Letting ℓ represent the distance of the clamped disks from the center of rotation and ignoring the thickness of each disk (we're told they are thin), use the parallel-axis theorem to express the moment of inertia of each disk:

$$\begin{aligned} I_{\rm disk} &= \tfrac{1}{4}mr^2 + m\ell^2 \\ &= \tfrac{1}{4}m\left(r^2 + 4\ell^2\right) \\ &= \tfrac{1}{4}(0.2\,\text{kg})\left[(0.2\,\text{m})^2 + 4(0.4\,\text{m})^2\right] \\ &= 0.0340\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

With the disks clamped:

$$\begin{aligned} I_{\rm i} &= I_{\rm cyl} + 2I_{\rm disk} \\ &= 0.232\,\text{kg}\cdot\text{m}^2 + 2\left(0.0340\,\text{kg}\cdot\text{m}^2\right) \\ &= 0.300\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

With the disks unclamped, $\ell = 0.6$ m and:

$$\begin{aligned} I_{\text{disk}} &= \tfrac{1}{4}m\left(r^2 + 4\ell^2\right) \\ &= \tfrac{1}{4}(0.2\,\text{kg})\left[(0.2\,\text{m})^2 + 4(0.6\,\text{m})^2\right] \\ &= 0.0740\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Express and evaluate the final moment of inertia of the system:

$$\begin{aligned} I_{\text{f}} &= I_{\text{cyl}} + 2I_{\text{disk}} \\ &= 0.232\,\text{kg}\cdot\text{m}^2 + 2\left(0.0740\,\text{kg}\cdot\text{m}^2\right) \\ &= 0.380\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Substitute in equation (1) to determine ω_{f}:

$$\begin{aligned} \omega_{\text{f}} &= \frac{0.300\,\text{kg}\cdot\text{m}^2}{0.380\,\text{kg}\cdot\text{m}^2}(8\,\text{rad/s}) \\ &= \boxed{6.32\,\text{rad/s}} \end{aligned}$$

Express the energy dissipated in friction:

$$\begin{aligned} \Delta E &= E_{\text{i}} - E_{\text{f}} \\ &= \tfrac{1}{2}I_{\text{i}}\omega_{\text{i}}^2 - \left(\tfrac{1}{2}I_{\text{f}}\omega_{\text{f}}^2 + \tfrac{1}{2}kx^2\right) \end{aligned}$$

Apply Newton's 2nd law to each disk when they are in their final positions:

$$\sum F_{\text{radial}} = kx = mr\omega^2$$

Solve for k:

$$k = \frac{mr\omega^2}{x}$$

Substitute numerical values and evaluate k:

$$\begin{aligned} k &= \frac{(0.2\,\text{kg})(0.6\,\text{m})(6.32\,\text{rad/s})^2}{0.2\,\text{m}} \\ &= 24.0\,\text{N/m} \end{aligned}$$

Express the energy dissipated in friction:

$$\begin{aligned} W_{\text{fr}} &= E_{\text{i}} - E_{\text{f}} \\ &= \tfrac{1}{2}I_{\text{i}}\omega_{\text{i}}^2 - \left(\tfrac{1}{2}I_{\text{f}}\omega_{\text{f}}^2 + \tfrac{1}{2}kx^2\right) \end{aligned}$$

Substitute numerical values and evaluate W_{fr}:

$$\begin{aligned} W_{\text{fr}} &= \tfrac{1}{2}\left(0.300\,\text{kg}\cdot\text{m}^2\right)(8\,\text{rad/s})^2 - \tfrac{1}{2}\left(0.380\,\text{kg}\cdot\text{m}^2\right)(6.32\,\text{rad/s})^2 - \tfrac{1}{2}(24\,\text{N/m})(0.2\,\text{m})^2 \\ &= \boxed{1.53\,\text{J}} \end{aligned}$$

91 ••

Picture the Problem Let the letters d, m, and r denote the disk and the letters t, M, and R the turntable. We can use conservation of angular momentum to relate the final angular speed of the turntable to the initial angular speed of the Euler disk and the moments of inertia of the turntable and the disk. In part (b) we'll need to use the parallel-axis theorem to express the moment of inertia of the disk with respect to the rotational axis of the turntable. You can find the moments of inertia of the disk in its two orientations and that of the turntable in Table 9-1.

(*a*) Use conservation of angular momentum to relate the initial and final angular momenta of the system:

$$I_{\text{di}}\omega_{\text{di}} = I_{\text{df}}\omega_{\text{df}} + I_{\text{tf}}\omega_{\text{tf}}$$

Because $\omega_{\text{tf}} = \omega_{\text{df}}$:

$$I_{\text{di}}\omega_{\text{di}} = I_{\text{df}}\omega_{\text{tf}} + I_{\text{tf}}\omega_{\text{tf}}$$

Solve for ω_{tf}:

$$\omega_{\text{tf}} = \frac{I_{\text{di}}}{I_{\text{df}} + I_{\text{tf}}}\omega_{\text{di}} \qquad (1)$$

Ignoring the negligible thickness of the disk, express its initial moment of inertia:

$$I_{\text{di}} = \tfrac{1}{4}mr^2$$

Express the final moment of inertia of the disk:

$$I_{\text{df}} = \tfrac{1}{2}mr^2$$

Express the final moment of inertia of the turntable:

$$I_{\text{tf}} = \tfrac{1}{2}MR^2$$

Substitute in equation (1) to obtain:

$$\omega_{\text{tf}} = \frac{\tfrac{1}{4}mr^2}{\tfrac{1}{2}mr^2 + \tfrac{1}{2}MR^2}\omega_{\text{di}} = \frac{1}{2 + 2\dfrac{MR^2}{mr^2}}\omega_{\text{di}} \qquad (2)$$

Express ω_{di} in rad/s:

$$\omega_{\text{di}} = 30\frac{\text{rev}}{\text{min}} \times \frac{2\pi\,\text{rad}}{\text{rev}} \times \frac{1\,\text{min}}{60\,\text{s}} = \pi\,\text{rad/s}$$

Substitute numerical values in equation (2) and evaluate ω_{tf}:

$$\omega_{\text{tf}} = \frac{\pi\,\text{rad/s}}{2 + 2\dfrac{(0.735\,\text{kg})(0.25\,\text{m})^2}{(0.5\,\text{kg})(0.125\,\text{m})^2}} = \boxed{0.228\,\text{rad/s}}$$

(*b*) Use the parallel-axis theorem to express the final moment of inertia of the disk when it is a distance L from the center of the turntable:

$$I_{\text{df}} = \tfrac{1}{2}mr^2 + mL^2 = m\left(\tfrac{1}{2}r^2 + L^2\right)$$

Substitute in equation (1) to obtain:

$$\omega_{\text{tf}} = \frac{\frac{1}{4}mr^2}{m\left(\frac{1}{2}r^2 + L^2\right) + \frac{1}{2}MR^2}\omega_{\text{di}} = \frac{1}{2 + 4\dfrac{L^2}{r^2} + 2\dfrac{MR^2}{mr^2}}\omega_{\text{di}}$$

Substitute numerical values and evaluate ω_{tf}:

$$\omega_{\text{tf}} = \frac{\pi\text{ rad/s}}{2 + 4\dfrac{(0.1\text{ m})^2}{(0.125\text{ m})^2} + 2\dfrac{(0.735\text{ kg})(0.25\text{ m})^2}{(0.5\text{ kg})(0.125\text{ m})^2}} = \boxed{0.192\text{ rad/s}}$$

92 ••

Picture the Problem We can express the period of the earth's rotation in terms of its angular velocity of rotation and relate its angular velocity to its angular momentum and moment of inertia with respect to an axis through its center. We can differentiate this expression with respect to *T* and then use differentials to approximate the changes in *r* and *T*.

(*a*) Express the period of the earth's rotation in terms of its angular velocity of rotation:

$$T = \frac{2\pi}{\omega}$$

Relate the earth's angular velocity of rotation to its angular momentum and moment of inertia:

$$\omega = \frac{L}{I} = \frac{L}{\frac{2}{5}mr^2}$$

Substitute and simplify to obtain:

$$T = \frac{2\pi\left(\frac{2}{5}mr^2\right)}{L} = \boxed{\frac{4\pi m}{5L}r^2}$$

(*b*) Find dT/dr:

$$\frac{dT}{dr} = 2\left(\frac{4\pi m}{5L}\right)r = 2\left(\frac{T}{r^2}\right)r = \frac{2T}{r}$$

Solve for dT/T:

$$\frac{dT}{T} = 2\frac{dr}{r} \text{ or } \boxed{\frac{\Delta T}{T} \approx 2\frac{\Delta r}{r}}$$

(*c*) Using the equation we just derived, substitute for the change in the period of the earth:

$$\frac{\Delta T}{T} = \frac{\frac{1}{4}\text{d}}{\text{y}} \times \frac{1\text{ y}}{365.24\text{ d}} = \frac{1}{1460} = 2\frac{\Delta r}{r}$$

Solve for and evaluate Δr:

$$\Delta r = \frac{r}{2(1460)} = \frac{6.37\times10^{3}\ \text{km}}{2(1460)} = \boxed{2.18\,\text{km}}$$

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Picture the Problem Let ω_P be the angular velocity of precession of the earth-as-gyroscope, ω_s its angular velocity about its spin axis, and I its moment of inertia with respect to an axis through its poles, and relate ω_P to ω_s and I using its definition.

Use its definition to express the precession rate of the earth as a giant gyroscope:

$$\omega_P = \frac{\tau}{L}$$

Substitute for I and solve for τ:

$$\tau = L\omega_P = I\omega\omega_P$$

Express the angular velocity ω_s of the earth about its spin axis:

$\omega = \dfrac{2\pi}{T}$ where T is the period of rotation of the earth.

Substitute to obtain:

$$\tau = \frac{2\pi I\omega_P}{T}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{2\pi\left(8.03\times10^{37}\ \text{kg}\cdot\text{m}^2\right)\left(7.66\times10^{-12}\ \text{s}^{-1}\right)}{1\,\text{d}\times\dfrac{24\,\text{h}}{\text{d}}\times\dfrac{3600\,\text{s}}{\text{h}}} = \boxed{4.47\times10^{22}\ \text{N}\cdot\text{m}}$$

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Picture the Problem The applied torque accelerates the system and increases the tension in the string until it breaks. The work done before the string breaks is the change in the kinetic energy of the system. We can use Newton's 2nd law to relate the breaking tension to the angular velocity of the system at the instant the string breaks. Once the applied torque is removed, angular momentum is conserved.

Express the work done before the string breaks:

$$W = \Delta K = K_f = \tfrac{1}{2}I_f\omega_f^2 \qquad (1)$$

Express the moment of inertia of the system (see Table 9-1):

$$\begin{aligned} I &= I_{cyl} + 2I_m = I(x) = \tfrac{1}{12}M_{cyl}L_{cyl}^2 + 2mx^2 \\ &= \tfrac{1}{12}(1.2\,\text{kg})(1.6\,\text{m})^2 + 2(0.4\,\text{kg})x^2 \\ &= 0.256\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})x^2 \end{aligned}$$

Evaluate $I_f = I(0.4\text{ m})$:

$$\begin{aligned} I_f &= I(0.4\,\text{m}) \\ &= 0.256\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})(0.4\,\text{m})^2 \\ &= 0.384\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Using Newton's 2nd law, relate the forces acting on a disk to its angular velocity:

$$\sum F_{\text{radial}} = T = mr\omega_f^2$$

where T is the tension in the string at which it breaks.

Solve for ω_f:

$$\omega_f = \sqrt{\frac{T}{mr}}$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \sqrt{\frac{100\,\text{N}}{(0.4\,\text{kg})(0.4\,\text{m})}} = 25.0\,\text{rad/s}$$

Substitute in equation (1) to express the work done before the string breaks:

$$W = \tfrac{1}{2} I_f \omega_f^2$$

Substitute numerical values and evaluate W:

$$\begin{aligned} W &= \tfrac{1}{2}(0.384\,\text{kg}\cdot\text{m}^2)(25\,\text{rad/s})^2 \\ &= \boxed{120\,\text{J}} \end{aligned}$$

With the applied torque removed, angular momentum is conserved and we can express the angular momentum as a function of x:

$$\begin{aligned} L &= I_f\omega_f \\ &= I(x)\omega(x) \end{aligned}$$

Solve for $\omega(x)$:

$$\omega(x) = \frac{I_f\omega_f}{I(x)}$$

Substitute numerical values to obtain:

$$\begin{aligned} \omega(x) &= \frac{(0.384\,\text{kg}\cdot\text{m}^2)(25\,\text{rad/s})}{0.256\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})x^2} \\ &= \boxed{\frac{9.60\,\text{J}\cdot\text{s}}{0.256\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})x^2}} \end{aligned}$$

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Picture the Problem The applied torque accelerates the system and increases the tension in the string until it breaks. The work done before the string breaks is the change in the kinetic energy of the system. We can use Newton's 2nd law to relate the breaking tension to the angular velocity of the system at the instant the string breaks. Once the applied torque is removed, angular momentum is conserved.

Express the work done before the string breaks:

$$W = \Delta K = K_\mathrm{f} = \tfrac{1}{2} I_\mathrm{f} \omega_\mathrm{f}^2 \qquad (1)$$

Express the moment of inertia of the system (see Table 9-1):

$$I = I_\mathrm{cyl} + 2I_\mathrm{m} = I(x) = \tfrac{1}{12} M_\mathrm{cyl} L_\mathrm{cyl}^2 + 2mx^2$$

Substitute numerical values to obtain:

$$\begin{aligned} I &= \tfrac{1}{12}(1.2\,\mathrm{kg})(1.6\,\mathrm{m})^2 + 2(0.4\,\mathrm{kg})x^2 \\ &= 0.256\,\mathrm{kg\cdot m^2} + (0.8\,\mathrm{kg})x^2 \end{aligned}$$

Evaluate $I_\mathrm{f} = I(0.4\text{ m})$:

$$\begin{aligned} I_\mathrm{f} &= I(0.4\,\mathrm{m}) \\ &= 0.256\,\mathrm{kg\cdot m^2} + (0.8\,\mathrm{kg})(0.4\,\mathrm{m})^2 \\ &= 0.384\,\mathrm{kg\cdot m^2} \end{aligned}$$

Using Newton's 2nd law, relate the forces acting on a disk to its angular velocity:

$$\sum F_\mathrm{rad} = T = mr\omega_\mathrm{f}^2$$

where T is the tension in the string at which it breaks.

Solve for ω_f:

$$\omega_\mathrm{f} = \sqrt{\frac{T}{mr}}$$

Substitute numerical values and evaluate ω_f:

$$\omega_\mathrm{f} = \sqrt{\frac{100\,\mathrm{N}}{(0.4\,\mathrm{kg})(0.4\,\mathrm{m})}} = 25.0\,\mathrm{rad/s}$$

With the applied torque removed, angular momentum is conserved and we can express the angular momentum as a function of x:

$$\begin{aligned} L &= I_\mathrm{f}\omega_\mathrm{f} \\ &= I(x)\omega(x) \end{aligned}$$

Solve for $\omega(x)$:

$$\omega(x) = \frac{I_\mathrm{f}\omega_\mathrm{f}}{I(x)}$$

Substitute numerical values and simplify to obtain:

$$\omega(x) = \frac{(0.384\,\mathrm{kg\cdot m^2})(25\,\mathrm{rad/s})}{0.256\,\mathrm{kg\cdot m^2} + (0.8\,\mathrm{kg})x^2} = \frac{9.60\,\mathrm{J\cdot s}}{0.256\,\mathrm{kg\cdot m^2} + (0.8\,\mathrm{kg})x^2}$$

Evaluate $\omega(0.8\,\mathrm{m})$:

$$\omega(0.8\,\text{m}) = \frac{9.60\,\text{J}\cdot\text{s}}{0.256\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})(0.8\,\text{m})^2} = \boxed{12.5\,\text{rad/s}}$$

Remarks: Note that this is the angular velocity in both instances. Because the disks leave the cylinder with a tangential velocity of $\frac{1}{2}L\omega$, the angular momentum of the system remains constant.

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Picture the Problem The applied torque accelerates the system and increases the tension in the string until it breaks. The work done before the string breaks is the change in the kinetic energy of the system. We can use Newton's 2nd law to relate the breaking tension to the angular velocity of the system at the instant the string breaks. Once the applied torque is removed, angular momentum is conserved.

Express the work done before the string breaks:

$$W = \Delta K = K_\text{f} = \tfrac{1}{2} I_\text{f}\omega_\text{f}^2 \qquad (1)$$

Using the parallel axis theorem and treating the disks as thin disks, express the moment of inertia of the system (see Table 9-1):

$$\begin{aligned} I(x) &= I_\text{cyl} + 2I_\text{m} \\ &= \tfrac{1}{12}ML^2 + \tfrac{1}{2}MR^2 + 2\left(\tfrac{1}{4}mR^2 + mx^2\right) \\ &= \tfrac{1}{12}M\left(L^2 + 6R^2\right) + 2m\left(\tfrac{1}{4}R^2 + x^2\right) \end{aligned}$$

Substitute numerical values to obtain:

$$\begin{aligned} I(x) &= \tfrac{1}{12}(1.2\,\text{kg})\left[(1.6\,\text{m})^2 + 6(0.4\,\text{m})^2\right] \\ &\quad + 2(0.4\,\text{kg})\left[\tfrac{1}{4}(0.4\,\text{m})^2 + x^2\right] \\ &= 0.384\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})x^2 \end{aligned}$$

Evaluate $I_\text{f} = I(0.4\text{ m})$:

$$\begin{aligned} I_\text{f} &= I(0.4\,\text{m}) \\ &= 0.384\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})(0.4\,\text{m})^2 \\ &= 0.512\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

Using Newton's 2nd law, relate the forces acting on a disk to its angular velocity:

$$\sum F_\text{rad} = T = mr\omega_\text{f}^2$$

where T is the tension in the string at which it breaks.

Solve for ω_f:

$$\omega_\text{f} = \sqrt{\frac{T}{mr}}$$

Substitute numerical values and evaluate ω_f:

$$\omega_\text{f} = \sqrt{\frac{100\,\text{N}}{(0.4\,\text{kg})(0.4\,\text{m})}} = 25.0\,\text{rad/s}$$

Substitute in equation (1) to express the work done before the string breaks:

$$W = \tfrac{1}{2} I_f \omega_f^2$$

Substitute numerical values and evaluate W:

$$W = \tfrac{1}{2}\left(0.512\,\text{kg}\cdot\text{m}^2\right)\left(25\,\text{rad/s}\right)^2 = \boxed{160\,\text{J}}$$

With the applied torque removed, angular momentum is conserved and we can express the angular momentum as a function of x:

$$L = I_f \omega_f = I(x)\omega(x)$$

Solve for $\omega(x)$:

$$\omega(x) = \frac{I_f \omega_f}{I(x)}$$

Substitute numerical values to obtain:

$$\omega(x) = \frac{\left(0.512\,\text{kg}\cdot\text{m}^2\right)\left(25\,\text{rad/s}\right)}{0.384\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})x^2} = \boxed{\frac{12.8\,\text{J}\cdot\text{s}}{0.384\,\text{kg}\cdot\text{m}^2 + (0.8\,\text{kg})x^2}}$$

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Picture the Problem Let the origin of the coordinate system be at the center of the pulley with the upward direction positive. Let λ be the linear density (mass per unit length) of the rope and L_1 and L_2 the lengths of the hanging parts of the rope. We can use conservation of mechanical energy to find the angular velocity of the pulley when the difference in height between the two ends of the rope is 7.2 m.

(*a*) Apply conservation of energy to relate the final kinetic energy of the system to the change in potential energy:

$$\Delta K + \Delta U = 0$$

or, because $K_i = 0$,

$$K + \Delta U = 0 \qquad (1)$$

Express the change in potential energy of the system:

$$\begin{aligned}\Delta U &= U_f - U_i \\ &= -\tfrac{1}{2}L_{1f}\left(L_{1f}\lambda\right)g - \tfrac{1}{2}L_{2f}\left(L_{2f}\lambda\right)g \\ &\quad -\left[-\tfrac{1}{2}L_{1i}\left(L_{1i}\lambda\right)g - \tfrac{1}{2}L_{2i}\left(L_{2i}\lambda\right)g\right] \\ &= -\tfrac{1}{2}\left(L_{1f}^2 + L_{2f}^2\right)\lambda g + \tfrac{1}{2}\left(L_{1i}^2 + L_{2i}^2\right)\lambda g \\ &= -\tfrac{1}{2}\lambda g\left[\left(L_{1f}^2 + L_{2f}^2\right) - \left(L_{1i}^2 + L_{2i}^2\right)\right]\end{aligned}$$

Because $L_1 + L_2 = 7.4$ m, $L_{2i} - L_{1i} = 0.6$ m, and $L_{2f} - L_{1f} = 7.2$ m, we obtain:

$L_{1i} = 3.4$ m, $L_{2i} = 4.0$ m, $L_{1f} = 0.1$ m, and $L_{2f} = 7.3$ m.

Substitute numerical values and evaluate ΔU:

$$\begin{aligned}\Delta U &= -\tfrac{1}{2}(0.6\,\mathrm{kg/m})(9.81\,\mathrm{m/s^2})\\ &\quad\times\left[(0.1\,\mathrm{m})^2 + (7.3\,\mathrm{m})^2 - (3.4\,\mathrm{m})^2 - (4\,\mathrm{m})^2\right]\\ &= -75.75\,\mathrm{J}\end{aligned}$$

Express the kinetic energy of the system when the difference in height between the two ends of the rope is 7.2 m:

$$\begin{aligned}K &= \tfrac{1}{2}I_\mathrm{p}\omega^2 + \tfrac{1}{2}Mv^2\\ &= \tfrac{1}{2}\left(\tfrac{1}{2}M_\mathrm{p}R^2\right)\omega^2 + \tfrac{1}{2}MR^2\omega^2\\ &= \tfrac{1}{2}\left(\tfrac{1}{2}M_\mathrm{p} + M\right)R^2\omega^2\end{aligned}$$

Substitute numerical values and simplify:

$$\begin{aligned}K &= \tfrac{1}{2}\left[\tfrac{1}{2}(2.2\,\mathrm{kg}) + 4.8\,\mathrm{kg}\right]\left(\frac{1.2\,\mathrm{m}}{2\pi}\right)^2\omega^2\\ &= (0.1076\,\mathrm{kg\cdot m^2})\omega^2\end{aligned}$$

Substitute in equation (1) and solve for ω:

$$(0.1076\,\mathrm{kg\cdot m^2})\omega^2 - 75.75\,\mathrm{J} = 0$$

and

$$\omega = \sqrt{\frac{75.75\,\mathrm{J}}{0.1076\,\mathrm{kg\cdot m^2}}} = \boxed{26.5\,\mathrm{rad/s}}$$

(*b*) Noting that the moment arm of each portion of the rope is the same, express the total angular momentum of the system:

$$\begin{aligned}L &= L_\mathrm{p} + L_\mathrm{r} = I_\mathrm{p}\omega + M_\mathrm{r}R^2\omega\\ &= \left(\tfrac{1}{2}M_\mathrm{p}R^2 + M_\mathrm{r}R^2\right)\omega \qquad (2)\\ &= \left(\tfrac{1}{2}M_\mathrm{p} + M_\mathrm{r}\right)R^2\omega\end{aligned}$$

Letting θ be the angle through which the pulley has turned, express $U(\theta)$:

$$U(\theta) = -\tfrac{1}{2}\left[(L_{1i} - R\theta)^2 + (L_{2i} + R\theta)^2\right]\lambda g$$

Express ΔU and simplify to obtain:

$$\begin{aligned}\Delta U &= U_\mathrm{f} - U_\mathrm{i} = U(\theta) - U(0)\\ &= -\tfrac{1}{2}\left[(L_{1i} - R\theta)^2 + (L_{2i} + R\theta)^2\right]\lambda g\\ &\quad + \tfrac{1}{2}\left(L_{1i}^2 + L_{2i}^2\right)\lambda g\\ &= -R^2\theta^2\lambda g + (L_{1i} - L_{2i})R\theta\lambda g\end{aligned}$$

Assuming that, at $t = 0$, $L_{1i} \approx L_{2i}$:

$$\Delta U \approx -R^2\theta^2\lambda g$$

Substitute for K and ΔU in equation (1) to obtain:

$$\left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2 - R^2\theta^2\lambda g = 0$$

Solve for ω:

$$\omega = \sqrt{\frac{R^2\theta^2\lambda g}{0.1076\,\text{kg}\cdot\text{m}^2}}$$

Substitute numerical values to obtain:

$$\omega = \sqrt{\frac{\left(\dfrac{1.2\,\text{m}}{2\pi}\right)^2\left(0.6\,\text{kg/m}\right)\left(9.81\,\text{m/s}^2\right)}{0.1076\,\text{kg}\cdot\text{m}^2}}\,\theta = \left(1.41\,\text{s}^{-1}\right)\theta$$

Express ω as the rate of change of θ:

$$\frac{d\theta}{dt} = \left(1.41\,\text{s}^{-1}\right)\theta \Rightarrow \frac{d\theta}{\theta} = \left(1.41\,\text{s}^{-1}\right)dt$$

Integrate θ from 0 to θ to obtain:

$$\ln\theta = \left(1.41\,\text{s}^{-1}\right)t$$

Transform from logarithmic to exponential form to obtain:

$$\theta(t) = e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Differentiate to express ω as a function of time:

$$\omega(t) = \frac{d\theta}{dt} = \left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Substitute for ω in equation (2) to obtain:

$$L = \left(\tfrac{1}{2}M_\text{p} + M_\text{r}\right)R^2\left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Substitute numerical values and evaluate L:

$$L = \left[\tfrac{1}{2}\left(2.2\,\text{kg}\right) + \left(4.8\,\text{kg}\right)\right]\left(\frac{1.2\,\text{m}}{2\pi}\right)^2\left[\left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}\right] = \boxed{\left(0.303\,\text{kg}\cdot\text{m}^2/\text{s}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}}$$