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On Frequency Response Curves in Rooms. Comparison of Experimental, Theoretical, and Monte Carlo Results for the Average Frequency Spacing between Maxima

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The average frequency spacing $\langle\Delta f_{\max}\rangle$ between adjacent maxima of the frequency response curve between two points in a room is determined by experiment, theory, and Monte Carlo computation. In earlier papers it had been shown that, above a certain critical frequency, $\langle\Delta f_{\max}\rangle$ is reciprocally related to reverberation time and not dependent on other room characteristics—disproving a belief that $\langle\Delta f_{\max}\rangle$ is a useful measure of the acoustical quality of rooms. Theory predicts $\langle\Delta f_{\max}\rangle=3.91/T_{60}$, where T_{60} is the reverberation time. Monte Carlo computation gives $3.90/T_{60}$. Measurements in two very different rooms using a vacuum tube voltmeter for reading the sound pressure are in good agreement with these predictions. Measurements with a logarithmic level recorder give, as in earlier investigations, much larger values. The discrepancy cannot be explained by the level recorder's 0.5-db quantization. This paper shows that: (1) The Monte Carlo method is a useful tool for solving complex problems in room acoustics. (2) Level recorder measurements of frequency (or space) irregularities of rooms must be taken with several grains of salt.

I. INTRODUCTION

SINCE Wente,[1] in 1935, directed the attention of room acousticians to stationary frequency response curves of rooms, a number of papers have been devoted to this topic. A certain completion of these investigations was reached in 1954 in papers by Schroeder[2] and Kuttruff and Thiele,[3] who were able to show both theoretically and experimentally that above a certain critical frequency the statistical parameters of frequency response curves for all rooms are either identical or depend at most on reverberation time. Specifically, the average frequency spacing between adjacent maxima was found to be

$$\langle f_{\max}\rangle=6.7/T_{60}, \tag{1}$$

where T_{60} is the reverberation time for a 60-db decay. The above-mentioned critical frequency is given by[4]

$$f_c=2000(T_{60}/V)^{\frac{1}{2}}, \tag{2}$$

where V is the volume of the room in cubic meters.

Recently, a new theory for $\langle\Delta f_{\max}\rangle$ has been developed which is considerably simpler and uses fewer assumptions than the earlier theory.[2] Its main result which is derived in Sec. III of this paper is

$$\langle\Delta f_{\max}\rangle\cdot T_{60}=3.91\pm0.04. \tag{3}$$

If a frequency response curve is measured with a logarithmic level recorder, the output is usually quantized in steps of about 0.5 db and a certain number of the "smaller" maxima are not registered. For a quantization of 0.5 db, a Monte Carlo computation (see Sec. IV) gives the following result:

$$\langle\Delta f_{\max}\rangle_{0.5\text{ db}}\cdot T_{60}=4.4. \tag{4}$$

II. EXPERIMENTAL RESULTS

These predictions were compared with experimental values obtained in two very different rooms. Room I is an auditorium with a volume of 2500 m³ (Arnold Auditorium at Bell Telephone Laboratories, Murray Hill, New Jersey). Room II is a highly sound-absorbent laboratory with a volume of approximately 200 m³. The reverberation times of both rooms are plotted in Fig. 1 as a function of frequency. The critical frequency according to Eq. (2) is 45 cps in the auditorium and 80 cps in the laboratory.

The loudspeaker used for the excitation of the room was supplied with a sinusoidal voltage whose frequency was changed slowly enough to ensure stationary conditions.[5] The microphone voltage was amplified and

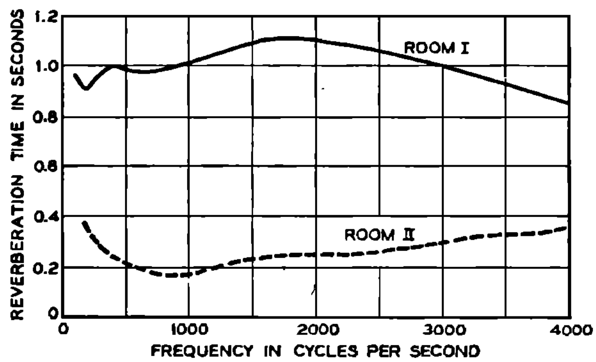

FIG. 1. Reverberation time T_{60} as a function of frequency for room I (auditorium, volume 2500 m³) and room II (laboratory, volume 200 m³). Measurements were performed with third-octave noise bands.

* Work performed as a consultant to the Acoustics Research Department at Bell Telephone Laboratories, Inc.

[1] E. C. Wente, J. Acoust. Soc. Am. 7, 123 (1935).

[2] M. R. Schroeder, Acustica 4, 594 (1954).

[3] K. H. Kuttruff and R. Thiele, Acustica 4, 614 (1954).

[4] The critical frequency given in reference 2 is $4000(T_{60}/V)^{\frac{1}{2}}$, corresponding to 10 overlapping normal modes. Measurements by various authors have shown that the theory is actually valid for frequencies as low as $2000(T_{60}/V)^{\frac{1}{2}}$.

[5] R. H. Bolt and R. W. Roop, J. Acoust. Soc. Am. 22, 280 (1950).

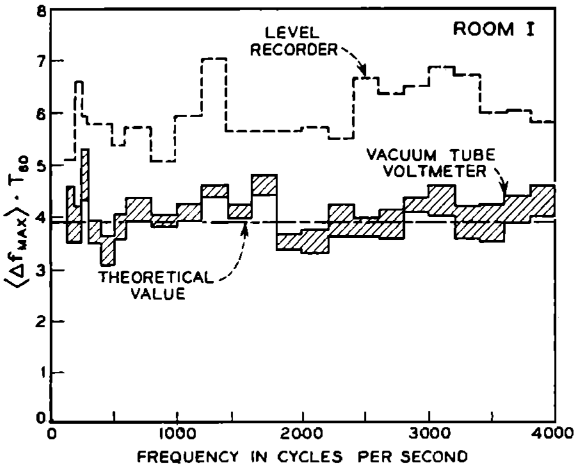

FIG. 2. Average spacing between adjacent maxima of frequency response curve in room I multiplied by the reverberation time. Upper curve: results obtained with level recorder; average over entire frequency range $\langle \Delta f_{\max} \rangle \cdot T_{60} = 6.00$. Crosshatched regions: results obtained with vacuum-tube voltmeter. The upper boundary of each region is the value of $\langle \Delta f_{\max} \rangle \cdot T_{60}$ when only actually observed maxima are counted; the lower boundary obtains when apparent "saddle points" are included. Average over entire frequency range $\langle \Delta f_{\max} \rangle \cdot T_{60} = 4.00$. The theoretical value, $\langle \Delta f_{\max} \rangle \cdot T_{60} = 3.90$, is indicated by a horizontal dashed line.

recorded in the usual manner by a logarithmic level recorder. Simultaneously, the sound pressure was measured by a vacuum-tube voltmeter, so that the response maxima could be observed by eye and counted as they occurred. In this manner, small maxima which were not registered by the level recorder could be counted.

Immediately following the recording of the frequency response curve, the reverberation time was measured with third-octave bands of noise—without changing any of the conditions in the room.

The observation of the vacuum-tube voltmeter resulted in occasional uncertainties whether or not a maximum was present. These doubtful cases were counted separately and are called subsequently "saddle points." (These points are, of course, not saddle points in the mathematical sense because the probability for the occurrence of a true saddle point is zero in any finite frequency interval.)

For the evaluation of the results, the entire frequency range from 150 to 4000 cps was divided into smaller intervals having bandwidths W between 50 and 200 cps. For each of these intervals the quantity $T_{60} \cdot W/n = T_{60} \cdot \langle \Delta f_{\max} \rangle$ was formed, where n is the number of maxima observed in W and T_{60} is the reverberation time averaged over the interval in question.

The results are plotted in Fig. 2 (for room I) and Fig. 3 (for room II). For the vacuum-tube voltmeter, the results both with and without "saddle points" are plotted. The resulting uncertainty ranges are indicated by crosshatching. The true values can lie anywhere within these uncertainty ranges.

In Table I the observed number of maxima and "saddle points" for the entire frequency range from 150 to 4000 cps are given. Also shown is $\langle \Delta f_{\max} \rangle \langle T_{60} \rangle$, where $\langle T_{60} \rangle$ is the reverberation time averaged linearly over the entire frequency range.

TABLE I. Average frequency spacing of maxima for two different rooms measured with a vacuum-tube voltmeter and a level recorder.

		Room I		Room II	
		Vacuum-tube voltmeter	Level recorder	Vacuum-tube voltmeter	Level recorder
Total number of maxima		938	662	217	146
Total number of "saddle points"		109		42	
$\langle \Delta f_{\max} \rangle \langle T_{60} \rangle$	Without "saddle points"	4.23	6.00	4.67	6.95
	With "saddle points"	3.79		3.92	

As can be seen, the results obtained with the vacuum tube voltmeter are compatible with the theoretically predicted value, $\langle \Delta f_{\max} \rangle \cdot T_{60} = 3.91$. On the other hand, the results obtained with the level recorder are in fairly good agreement with the result $\langle \Delta f_{\max} \rangle \cdot T_{60} = 6.7$ of the earlier experiments[3]—provided that only the actually observed maxima are taken into account, in accordance with the manner of evaluation of the previous experiments.

However, the level recorder results deviate considerably from the value 4.4 predicted by the Monte Carlo method for a recording quantized in steps of 0.5 db. Therefore, we conclude that frequency response curves when plotted by logarithmic level recorders exhibit systematically fewer maxima than could be accounted for if their only source of error was a 0.5-db quantization. This result may be of importance for many measurements made with logarithmic level recorders.

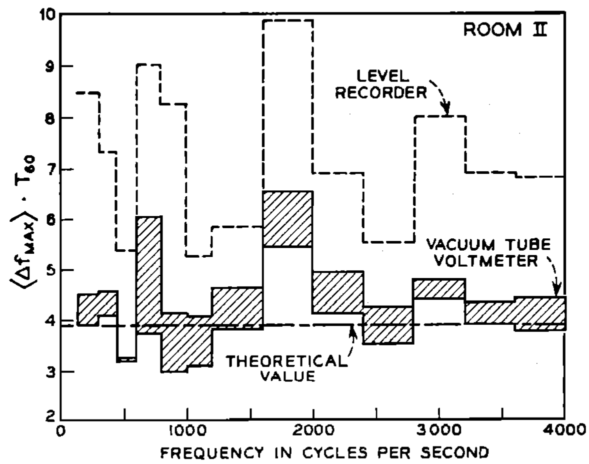

FIG. 3. Same as Fig. 2, but for room II. Average over entire frequency range for level recorder $\langle \Delta f_{\max} \rangle \cdot T_{60} = 6.95$; for vacuum-tube voltmeter $\langle \Delta f_{\max} \rangle \cdot T_{60} = 4.25$.

III. THEORY

Here we are interested in deriving a *theoretical* formula for the average frequency spacing between adjacent maxima of $|p(f)|$ or, equivalently, of $|p(f)|^2$, where $p(f)$ is the complex sound pressure transmission function between two points in a room. We shall consider $|p(f)|^2$ our dependent variable and denote it by $y(f)$:

$$y(f) \equiv |p(f)|^2. \tag{5}$$

According to a formula by Rice,[6] if both the first and second derivatives y' and y'' are Gaussian, the average spacing between adjacent maxima of $y(f)$ is

$$\langle \Delta f_{\max} \rangle = \left[\int_0^\infty t^2 \langle Y^2(t) \rangle dt \bigg/ \int_0^\infty t^4 \langle Y^2(t) \rangle dt \right]^{\frac{1}{2}}, \tag{6}$$

where $\langle Y^2(t) \rangle$ is the mean square Fourier transform of $y(f)$.

The (inverse) Fourier transform of the complex transmission function $p(f)$ is the impulse response $P(t)$ between the two points:

$$P(t) = \int_{-\infty}^{+\infty} p(f) e^{i2\pi f t} df. \tag{7}$$

By Wiener's famous theorem[7] the Fourier transform $Y(t)$ of $y(f) = |p(f)|^2$ is the autocorrelation function of $P(t)$:

$$Y(t) = \int_{-\infty}^{+\infty} P(t') P(t'+t) dt'. \tag{8}$$

To evaluate this integral, we expand the impulse response into decaying normal modes:

$$P(t) = \sum_{k=-\infty}^{\infty} a_k \exp(-t/2\tau_k + i\omega_k t), \quad \text{for} \quad t>0, \tag{9}$$

$$P(t) = 0 \qquad \text{for} \quad t<0.$$

Here a_k is the complex amplitude of the kth normal mode for $t \to 0$, τ_k is the time in which its *energy* decays to $1/e$ of its initial value, and ω_k its eigenfrequency. In the expansion (9), every normal mode is represented by two conjugate complex terms: $a_{-k} = a_k^*$, $\tau_{-k} = \tau_k$, and $\omega_{-k} = -\omega_k$. In the following we shall assume a uniform decay time τ for all normal modes—as realized in the case of a diffuse sound field.

[6] S. O. Rice, in *Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, New York, 1954), p. 211. In Rice's paper the independent variable is time and the results are applied to functions of *time*. Here we are interested in functions of *frequency* whose Fourier transforms (spectra) are functions of time. Because a spectrum is usually thought of as occurring in the frequency domain, the use of the term has been avoided in this paper. For "average power spectrum" the term "mean square Fourier transform" is used.

[7] N. Wiener, *Extrapolation, Interpolation and Smoothing of Stationary Time Series* (John Wiley & Sons, Inc., New York, 1949), p. 42.

Inserting (9) into (8) yields

$$Y(t) = e^{-|t|/2\tau} \sum_{k,l} a_k a_l \frac{\exp(i\omega_k t)}{(-1/2\tau) + i(\omega_k + \omega_l)}. \tag{10}$$

This equation holds for a particular room with eigenfrequencies ω_k and for a particular pair of loudspeaker and microphone positions with excitation coefficients a_k. When forming $\langle Y^2(t) \rangle$, we average over all possible values of the eigenfrequencies ω_k and excitation coefficients a_k and obtain

$$\langle Y^2(t) \rangle = \text{const} \cdot \exp(-|t|/\tau). \tag{11}$$

Here the constant depends on the assumptions made in the averaging process but, and this is the important thing, it does not depend on time for the distributions of excitation coefficients and eigenfrequencies encountered in real rooms.[8]

With $\langle Y^2(t) \rangle$ from (11) inserted into (6), we obtain

$$\langle \Delta f_{\max} \rangle = [\tau^3 2! / \tau^5 4!]^{\frac{1}{2}} = \tfrac{1}{2}\sqrt{3}\,\tau. \tag{12}$$

Substituting the reverberation time $T_{60} = 6 \cdot \ln 10 \cdot \tau = 13.8 \cdot \tau$ yields

$$\langle \Delta f_{\max} \rangle = (\sqrt{3} \cdot \ln 10)/T_{60} = 3.99/T_{60}. \tag{13}$$

This result would be exact if both y' and y'' were Gaussian distributed. For frequencies above $2000(T_{60}/V)^{\frac{1}{2}}$, both the real and imaginary parts of $p(f)$ approach Gaussian distributions.[2] From this it follows that y' is also Gaussian distributed,[9] but not necessarily y''. However, an examination of Rice's derivation of Eq. (6) shows that only the distribution of y' is critical. If y'' is nonGaussian, the error is usually very small. In two similar cases, for which exact results are known, the error in the spacing between maxima is $+1.5\%$ and $+3\%$, respectively.

Assuming that the error of (13) lies between $+1\%$ and $+3\%$, we may write

$$\langle \Delta f_{\max} \rangle = (3.91 \pm 0.04)/T_{60}. \tag{14}$$

The result obtained in the earlier theoretical paper[2] was $\langle \Delta f_{\max} \rangle \approx 7/T_{60}$. In spite of the good agreement with the experimental results which became known shortly after the theory was completed, it was felt that the coincidence was somewhat fortuitous because of the rather drastic simplifying assumptions made in the early theory. This feeling provided the impetus to seek a more precise theory—the one presented in this Section.

IV. MONTE CARLO COMPUTATION OF $\langle \Delta f_{\max} \rangle$

Another means of computing $\langle \Delta f_{\max} \rangle$ is afforded by the Monte Carlo[10] method in which a statistical process

[8] M. R. Schroeder, Acustica 4, 456 (1954).

[9] S. O. Rice, Bell System Tech. J. 27, 109 (1948).

[10] For a definition and history of the Monte Carlo method, see the Preface and Introductory Note in *Symposium on Monte Carlo Methods* (John Wiley & Sons, Inc., New York, 1956), edited by H. A. Meyer.

identical to the one under consideration is generated (on a digital computer for instance) and evaluated much like experimental data.

In our computation a large digital computer (IBM 7090) which has a subroutine for generating independent random numbers with a Gaussian distribution was used. This white Gaussian noise $n(f)$ is "filtered" in the computer to give it the same mean square Fourier transform as the real part $p_r(f)$ of the complex transmission function, $p(f)=p_r(f)+ip_i(f)$, between two points in a room.

It has been shown in Sec. III that the mean square Fourier transform of $|p(f)|^2$ is proportional to $\exp(-|t|/\tau)$. By a similar derivation, the same can be shown to be true for $p_r(f)$. The appropriate filter function to convert the flat mean square Fourier transform of the white Gaussian noise into one proportional to $\exp(-|t|/\tau)$ is[11] $1/[1+(4\pi\tau f)^2]$. For a reverberation time of $T_{60}=1$ sec, corresponding to $\tau=1/13.8$ sec, the filter function becomes $1/[1+(f/1.1)^2]$. Mathematically, the filtering is accomplished by a convolution of the white noise with this filter function. Hence,

$$p_r(f)=\int_{-\infty}^{\infty}\frac{n(f-f')}{1+(f/1.1)^2}df'. \tag{15}$$

Here f' is the integration variable which is represented in the computer in discrete steps corresponding to frequency intervals of 0.22 cps. The integration is truncated when the denominator in the integrand becomes larger than 20. The error resulting from these approximations in the computer is estimated to be less than 1%.

Next, we compute the imaginary part $p_i(f)$ which is the Hilbert transform of $p_r(f)$[12]:

$$p_i(f)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{p_r(f')}{f-f'}df'. \tag{16}$$

Both $p_r(f)$ and $p_i(f)$ computed in this manner have the same characteristics as the real and imaginary parts of the complex transmission functions in a room with 1 sec reverberation time: They are both Gaussian (because they are derived from a Gaussian process by linear operations), they have the appropriate mean square Fourier transforms, and they have the proper interdependence (ensured by the Hilbert transform). Thus, apart from an unimportant scale factor, $p_r(f)$ and $p_i(f)$ are completely determined.

Next, the absolute square of $p(f)$ is formed according to the relation

$$|p(f)|^2=p_r^2(f)+p_i^2(f), \tag{17}$$

and the number of its maxima over intervals corresponding to bandwidths of 1000 cps are counted. The reciprocals of these numbers correspond to $\langle\Delta f_{\max}\rangle$. Table II lists the results for 40 frequency intervals. The average distance for all 40 intervals is 3.90 cps, its statistical error ±0.02 cps. Hence, the Monte Carlo method yields

TABLE II. Monte Carlo results for $\langle\Delta f_{\max}\rangle\cdot T_{60}$ for 40 frequency intervals $W=1000$ cps. Average value 3.90±0.02.

4.18	3.91	4.02	3.90	3.78	3.88	4.04	3.97
3.82	3.92	3.87	3.97	4.02	4.01	3.75	3.85
3.70	3.88	3.82	3.80	3.70	4.00	3.96	3.92
3.98	3.66	3.86	3.85	4.09	3.91	3.94	3.85
4.06	3.90	3.86	3.91	3.82	3.77	3.77	4.06

$$\langle\Delta f_{\max}\rangle=(3.90\pm0.02)/T_{60}, \tag{18}$$

which is in good agreement with the theoretical result (3.91±0.04).

The Monte Carlo method is particularly useful—and often the only method—in obtaining numerical results for problems which are theoretically completely intractable or very complex. The average spacing between maxima registered by a device which quantizes sound pressure in equal logarithmic steps is such an intractable problem. (It is indeed related to the unsolved—except for very few special cases—problem of the zero-crossing distribution of Gaussian noise.)

For the Monte Carlo computation of this room-plus-level recorder problem, we take the logarithm of $|p(f)|^2$ and round it off to the nearest integer multiple of the chosen step size before counting the maxima. Figure 4 shows the results for $\langle\Delta f_{\max}\rangle\cdot T_{60}$ as a function of the quantization. As can be seen, $\langle\Delta f_{\max}\rangle$ increases with increasing step size, reflecting the fact that more and more maxima are missed as the step size is increased. For the usual step size of 0.5 db, the Monte Carlo result is

$$\langle\Delta f_{\max}\rangle_{0.5\text{ db}}\doteq4.4/T_{60}. \tag{19}$$

In addition to the computation of $\langle\Delta f_{\max}\rangle$ as a function of the quantization, many other problems related

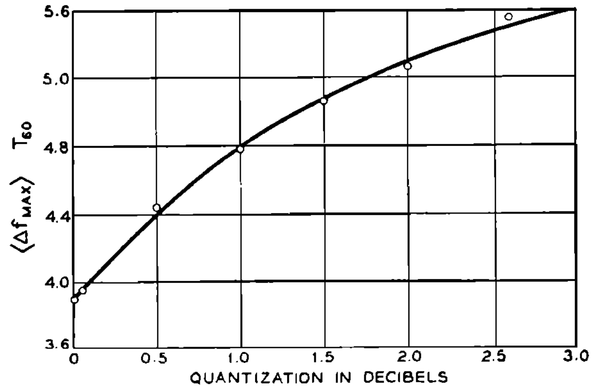

FIG. 4. Results of Monte Carlo computation of average spacing between maxima as a function of step size when the frequency response curve is quantized in equal-decibel steps. For 0.5-db quantum steps (the usual step size of level recorders), $\langle\Delta f_{\max}\rangle\cdot T_{60}=4.4$.

[11] See, for example, I. M. Ryzhik and I. S. Gradstein, *Tables of Series, Products, and Integrals* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1957), p. 250.

[12] S. Shesku and N. Balabanian, *Linear Network Analysis* (John Wiley & Sons, Inc., New York, 1959), p. 265.

to frequency response curves have been solved by the Monte Carlo method. Among these are feedback stability of public address systems operating in reverberant rooms, frequency irregularity[5] of room response curves as a function of the ratio of reverberant-to-direct sound intensity, frequency irregularity as a function of the bandwidth of the signal with which the room is excited, space irregularity for different degrees of diffusion of the sound field, the correlation of sound pressure in frequency and space, and various other interesting problems. While some of these problems were amenable to theoretical treatment, others were not, and the Monte Carlo method proved to be the only one to furnish useful results.

CONCLUSIONS

The work reported in the present paper began with the derivation of a new theory for the average spacing $\langle\Delta f_{\max}\rangle$ between adjacent maxima of frequency response curves in rooms. In a previous theoretical treatment of this problem, many simplifying assumptions had been made and it was felt that the good agreement with experimental data available at that time was rather fortuitous. The new theory presented here (Sec. III) is much more compact than the older one and involves only one minor simplification. Its result is $\langle\Delta f_{\max}\rangle = 3.91/T_{60}$. To further corroborate this result, which differs by a factor of 1.8 from the earlier theory, a Monte Carlo computation on a digital computer (IBM 7090) has been undertaken (Sec. IV). Its outcome, $\langle\Delta f_{\max}\rangle = 3.90/T_{60}$, is in good agreement with the theoretical result.

Existing experimental values of $\langle\Delta f_{\max}\rangle$, obtained with commercially available logarithmic sound-level recorders, differed widely from those predicted by the new theory. We therefore decided to repeat the measurement of $\langle\Delta f_{\max}\rangle$ in two very different rooms using both a level recorder and a vacuum-tube voltmeter to observe the sound-pressure level (Sec. II). While the results obtained with the vacuum-tube voltmeter are compatible with the new theory, the values of $\langle\Delta f_{\max}\rangle$ obtained with the level recorder are more than 50% too large (as were the earlier level-recorder measurements). This error indicates that more than one-third of the maxima are not registered. At first, this discrepancy was attributed to the quantization of sound-pressure level which takes place in level recorders. However, a Monte Carlo computation showed that only a fraction of the error could be accounted for by quantization. The source of the remaining discrepancy is possibly due to hysteresis in the servomechanism driving the potentiometer and recording stylus. But this question requires further study.

Note added in proof. The measurements reported here were performed with a Brüel and Kjaer level recorder model No. 2304 using a 50-db logarithmic potentiometer with a quantization of 0.5 db. Since then, an improved level recorder, Brüel and Kjaer model No. 2305 with a 50-db potentiometer and 0.25-db quantization has become available. Measurements in room I (Arnold Auditorium) were repeated using this improved level recorder. A total of 940 maxima were observed in the frequency range from 150 to 4000 cps corresponding to $\langle\Delta f_{\max}\rangle\cdot\langle T_{60}\rangle = 4.22$ (instead of 6.00 with the old model as in Table I). This result is in perfect agreement with the Monte Carlo prediction for a 0.25-db quantization (see Fig. 4). Thus, it appears that whatever shortcoming in the older model prevented faithful registration (within 0.5 db) has been overcome.

It is important to realize that our critique of level recorders applies exclusively to their use in measuring frequency and space responses in reverberant rooms. In these cases, presently available level recorders do not register a large proportion of actually present maxima, presumably those of small amplitude. There is no reason to doubt that level recorders perform satisfactorily if one is not interested in the *detailed* structure of room responses.

Another conclusion of our work is that complex problems, such as the number of maxima registered by a device which quantizes the sound pressure in equal-decibel steps, can be solved by means of Monte Carlo computation on high-speed digital computers. For this reason, a rather detailed description of the Monte Carlo computation has been given in Sec. IV.

The theoretical treatment of the problem given in Sec. III is likewise intended as an example of how analytical methods can be applied to the solution of problems in room acoustics which have hitherto appeared rather difficult.

In both sections the fundamental distributions and Fourier transforms governing sound transmission in rooms have been given. The authors hope that this information may prove stimulating and useful to others faced with complex problems resulting from the random interference of sound waves in rooms.

ACKNOWLEDGMENT

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