Case 1: Power gain factor for the velocity or pressure gradient microphone is

$$G_p=\frac{\cos^2\varphi\cdot 4\pi}{\int_0^{\pi}\cos^2\varphi\cdot 2\pi\sin\varphi d\varphi}=3\cos^2\varphi, \qquad \text{(a)}$$

where φ is the angle between the line joining the sound and the listener, and the polar axis of maximum sensitivity. Similarly, the value of G_p for the cardioid microphone is

$$G_p=\frac{3(1+\cos\varphi)^2}{4} \qquad \text{(b)}$$

for the condition of combining the pressure and velocity elements equally.

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Normal Frequency Spacing Statistics

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I. INTRODUCTION

CERTAIN problems in room acoustics cannot be analyzed adequately without consideration of *fluctuations from average functions*.* Thus the decay of sound in rooms is characterized by small scale modulations which influence hearing conditions in a manner not completely describable in terms of average decay rates.[1–6] The steady-state frequency response of a room is irregular in nature, and it has been shown experimentally that the "transmission irregularity" decreases monotonically with increasing absorption in the room.[7]

Consideration has been given[8,9] to the proportioning of rooms to minimize "piling up" of resonance frequencies and consequent increased roughness of response. For rooms basically rectangular in shape, the "$2^{1/3}$ rule" has been employed, by which room dimension ratios are chosen to be $2^{1/3}$ or powers of $2^{1/3}$ in order to spread normal frequencies with as little bunching as possible, and thereby minimize fluctuations.

Fluctuations are particularly prominent if the ratio of room dimensions to wave-length is so low that only a few normal modes of vibration are involved. In fact it may become quite unrealistic to deal only with smooth *average* expressions in such cases. Because of the unwieldiness of *exact* wave analyses, however, there arises a need for dealing with *average fluctuation properties*.

Results of a limited study on some of these problems are reported here. A generalized analysis is developed in terms of a dimensionless frequency parameter μ, and dimensionless expressions for normal frequency distributions in rectangular rooms. These quantities are chosen to be independent of room volume in order to facilitate the study of the influence of room proportions. Results are then applicable to rooms of any volume and/or to any frequency by means of conversion factors.

A *frequency spacing index* ψ is defined, and its possible utility as a measure of average fluctua-

* The word "fluctuations" is used broadly to include variations with time, distance, or frequency.
[1] R. L. Hanson, "Liveness of rooms," J. Acous. Soc. Am. **3**, 318 (1932).
[2] J. P. Maxfield and C. C. Potwin, "A modern concept of acoustical design," J. Acous. Soc. Am. **11**, 48 (1939).
[3] R. C. Jones, "Theory of fluctuations in decay of sound," J. Acous. Soc. Am. **11**, 324 (1940).
[4] R. B. Watson, "The modulations on sound decay curves," J. Acous. Soc. Am. **13**, 82A (1941). R. B. Watson, "Modulation of sound decay curves," J. Acous. Soc. Am. **18**, 119 (1946).
[5] D. Y. Maa, "Fluctuation phenomena in room acoustics," J. Acous. Soc. Am. **18**, 134 (1946).
[6] P. M. Morse and R. H. Bolt, "Sound waves in rooms," Rev. Mod. Phys. **16**, 69 (1944).
[7] E. C. Wente, "Characteristics of sound transmissions in rooms," J. Acous. Soc. Am. **7**, 123 (1935).
[8] J. E. Volkmann, "Polycylindrical diffusers in room acoustical design," J. Acous. Soc. Am. **13**, 234 (1942).
[9] C. P. Boner, "Performance of broadcast studios designed with convex surfaces of plywood," J. Acous. Soc. Am. **13**, 244 (1942).

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tion properties is explored. Values of ψ are calculated for certain general kinds of distribution. An approximate analytic expression for rectangular rooms is derived by use of number theory. Consideration is given to the application of ψ as a criterion for room proportions, requiring ψ to have as small a value as possible.[10] On this basis the $2^{\frac{1}{3}}$ rule appears to yield satisfactory, but not unique, results. The behavior of ψ is also examined in relation to sound diffusion.

More comprehensive statistical studies along the lines reported here and elsewhere[3, 5] are needed. That fluctuation phenomena influence, to a greater or lesser degree, the hearing qualities of rooms seems to be generally recognized.[1–9] The analyses of these problems are not fully developed, and more detailed experimental studies should be made to check theory and to establish the limits of significance of these effects in practical cases.

II. GENERALIZED FORMULATION FOR RECTANGULAR ROOMS

Using a dimensionless frequency parameter

$$\mu = V^{\frac{1}{3}}\nu/c, \tag{1}$$

the normal frequency of the Nth mode may be expressed as:

$$\mu_N = \tfrac{1}{2}(pq)^{\frac{1}{3}}[n_x^2+(n_y/p)^2+(n_z/q)^2]^{\frac{1}{2}} \tag{2}$$

in which the room dimensions are

$$\left.\begin{aligned} &L_x = L, \quad L_y = pL, \quad L_z = qL \\ &\text{and the room volume is} \\ &V = L_xL_yL_z = pqL^3. \end{aligned}\right\} \tag{3}$$

Similarly the smooth part of the normal frequency distribution equation[6, 11]

$$n(\nu) = \frac{4\pi V}{3c^3}\nu^3 + \frac{\pi S}{4c^2}\nu^2 + \frac{L_x+L_y+L_z}{2c}\nu \tag{4}$$

transforms into

$$n(\mu) = \frac{4}{3}\pi\mu^3 + \frac{\pi}{2}\frac{(p+pq+q)}{(pq)^{\frac{2}{3}}}\mu^2 + \frac{1}{2}\frac{(1+p+q)}{(pq)^{\frac{1}{3}}}\mu. \tag{5}$$

[10] R. H. Bolt, "Note on normal frequency statistics for rectangular rooms," J. Acous. Soc. Am. **18**, 130 (1946).
[11] D. Y. Maa, "The distribution of eigentones in a rectangular chamber at lower frequency ranges," J. Acous. Soc. Am. **10**, 258 (1939).

TABLE I. Errors in value of $n(\mu)$ as given by Eq. (6) relative to Eq. (5).

p, q values	μ values	error
$q=p^2$	all values	zero
$p, q>0.3$	$\mu>0.5$	$<1\%$
$p, q\geqslant0.2$	$\mu>0.5$	$<3\%$
$p, q\geqslant0.2$	$\mu>2$	$<1\%$
$p, q\geqslant0.1$	$\mu>0.5$	$<7\%$
$p, q\geqslant0.1$	$\mu>2$	$<3\%$
$p, q\geqslant0.1$	$\mu>4$	$<1\%$

The (p, q) functions in the second and third terms of Eq. (5) are of the same order of magnitude for all realistic values of p, q; they are approximately equal over a considerable range of p, q values, being identically equal for $q=p^2$. Furthermore, the third term is smaller than the second (except for $\mu<\frac{1}{2}$ which is meaningless for statistical consideration) and decreases relatively with increasing μ. It is, therefore, a good approximation to use the (p, q) function of the second term in the third term also, yielding:

$$n(\mu) \simeq \frac{4\pi}{3}\mu^3 + \frac{\pi}{2}P\mu^2 + \tfrac{1}{2}P\mu, \tag{6}$$

$$P = \frac{(p+pq+q)}{(pq)^{\frac{2}{3}}}; \tag{7}$$

Eq. (6) thus gives (approximately) the number of normal frequencies up to μ in a rectangular room specified by the *room proportion factor P*. Volume, area, and lengths do not enter explicitly.

The percentage error in the value of $n(\mu)$ introduced by the approximation of using P in the third term is indicated for several cases in Table I. Equations (6) and (7) have been presented graphically in Fig. 1 of reference 10; $P=f(p, q)$ is given in the upper left corner, and $n(\mu)$ is plotted against μ in a series of curves with P as a parameter.

We are also interested in the average spacing between adjacent normal frequencies, $\bar{\delta}$, obtained by differentiating Eq. (6) and letting $dn=1$:

$$\bar{\delta} = (4\pi\mu^2+\pi P\mu+P/2)^{-1}. \tag{8}$$

Values of $\bar{\delta}=f(\mu)$, with P as parameter, are also plotted in Fig. 1 of reference 10, as well as curves of $\nu=f(\mu)$ with V as parameter, computed from

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Eq. (1) for $c=1127$. It is thus possible to read from this chart the average number of normal frequencies up to μ or ν, or the average frequency spacing, for a rectangular room of arbitrary proportion and volume, up to frequencies for which $n(\mu)$ is about 250. This procedure has saved considerable time in calculations for the present study.

III. BASIC STATISTICAL RELATIONS

We next develop the basic statistical treatment leading to a method for expressing the fluctuation properties of normal frequency distributions. The following terminology and definitions are needed.

$\mu_1, \mu_2, \cdots, \mu_N$	Normal frequencies (dimensionless generalization)
$\delta_N=\mu_{N+1}-\mu_N$	Actual space at Nth normal frequency
$\bar{\delta}$	Expected average space at μ or μ_N.
$\Delta_a{}^b=\mu_b-\mu_a$	Interval containing several normal frequencies.
$\Delta/\bar{\delta}$	Expected number of normal frequencies in Δ.
$\rho=\delta_N/\bar{\delta}$	Ratio of actual to average space at μ_N.
M_ρ	Probability of finding spaces with ratios between ρ and $\rho+d\rho$ per unit space $\bar{\delta}$.
$M_\rho d\rho$	Number of spaces with ratios between ρ and $\rho+d\rho$ per unit *average* space $\bar{\delta}$ (or per unit n).
$M_\rho d\rho\delta_N/\bar{\delta}$	Number of spaces with ratios between ρ and $\rho+d\rho$ in the *actual* space δ_N.

In the previous section our problem was generalized through the introduction of a dimensionless frequency parameter μ. This makes it possible to study normal frequencies for a general room without regard to its size. In effect, we are using the n scale as a base line against which to measure the normal frequencies and their spacing properties.

We adopt the unit interval on the n scale, which has a space width $\bar{\delta}$, as a basic unit for measuring frequency spaces. In terms of this unit, a given space δ_N may be specified by ρ_N, the ratio of the actual space to the average space at the particular frequency in question. This choice of scale factor aids considerably in simplifying our problem, as will become apparent when we discuss certain types of distribution.*

* The author is indebted to P. M. Morse for valuable suggestions and aid on the statistical formulation.

Using the above definitions, we write:

$$\begin{aligned}\psi^{(0)}&\equiv\int_0^\infty M_\rho d\rho=1,\\ \psi^{(1)}&\equiv\int_0^\infty \rho M_\rho d\rho=1,\\ \psi^{(m)}&\equiv\int_0^\infty \rho^m M_\rho d\rho\geqslant 1.\end{aligned}\tag{9}$$

The quantity $\psi^{(m)}$ is the mth moment of the probability function against the spacing ratio ρ. *The zeroth moment $\psi^{(0)}$ is always equal to unity,* for any kind of distribution function, because of the normalization condition implied in the above definitions. *The first moment $\psi^{(1)}$ is always equal to unity,* for any distribution function, because of our choice of scale factor. The meaning of these conditions is more readily apparent in the equivalent summation forms of the above integrals:

$$\begin{aligned}\psi^{(0)}&=\frac{1}{\Delta_a{}^b}\sum_a^b\bar{\delta}=1,\\ \psi^{(1)}&=\frac{1}{\Delta_a{}^b}\sum_a^b\delta_N=1,\\ \psi^{(m)}&=\frac{\delta}{\Delta_a{}^b}\sum_a^b\left(\frac{\delta_N}{\bar{\delta}}\right)^m\geqslant 1.\end{aligned}\tag{10}$$

The zeroth moment is equal to unity because the sum of the *average* spaces in a given interval is equal to the width of that interval. The first moment is equal to unity because the sum of the *actual* spaces in a given interval is equal to the width of that interval.

The higher probability moments are never less than unity, and are equal to unity only for the special distribution in which every actual space is equal to the average space at the corresponding frequency. This we call "uniform" distribution (uniform on the n scale) and we make the (reasonable) postulate that uniform distribution will result in the smoothest possible frequency response (for a given set of conditions as to damping, room size, etc.). It is also reasonable to expect that increasing departure from uniform

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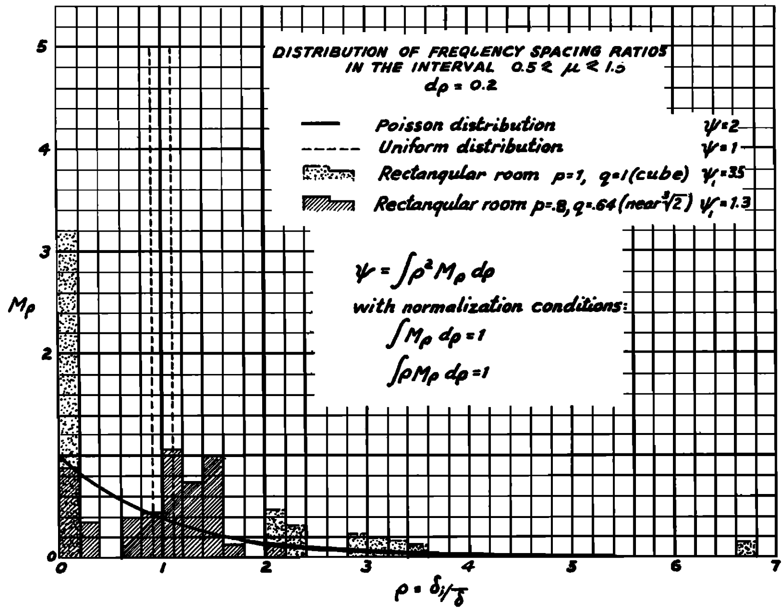

FIG. 1.

distribution will result in increasing fluctuation or irregularity of sound response.**

The above reasoning leads us to adopt the probability moments as a measure of spacing fluctuation. In principle, if all moments (from $m=0$ to $m=\infty$) are specified, the complete frequency distribution is thereby determined. Calculation of all of these moments, however, would hardly be a simplification over the procedure of determining all normal frequencies directly to study their distribution in detail! On the basis of exploratory study of several actual distributions, we surmise that the second moment alone should express adequately the major fluctuation characteristics of frequency spacing in rooms. The possible need for added complication remains to be determined by further theory and experimental application.

After developing a few general expressions for the probability moments, we shall restrict the rest of the present study to the second moment which we arbitrarily define as the *frequency spacing index:*

$$\psi \equiv \psi^{(2)} = \int_0^\infty \rho^2 M_\rho d\rho \qquad (11)$$

** The quantitative relationship between frequency spacing and response irregularity involves damping, position of source and pick-up, room dimensions, etc.; this problem is postponed for future study.

or

$$\psi \equiv \frac{1}{\mu_b - \mu_a} \sum_a^b \left(\frac{\delta_N{}^2}{\bar{\delta}} \right). \qquad (12)$$

Defined in this way, ψ is the mean squared ratio of actual to average normal frequency spaces.

IV. THREE IDEALIZED DISTRIBUTIONS

It is helpful to investigate the behavior of probability moments for certain idealized types of distribution, as background for studying actual distributions in practical cases. We shall consider: (a) *random* (Poisson), (b) *constant ratio,* and (c) *constant difference* distributions, defined and evaluated as follows:

(a) *Random distribution* signifies that there is equal probability of occurrence of a normal frequency at any point on the frequency scale, consistent with the average distribution law which specifies the total expected number of normal frequencies in a given band. It can be shown (Appendix I) that the probability function in this case is

$$M_\rho = e^{-\rho}. \qquad (13)$$

This function is plotted in Fig. 1, together with other distribution functions discussed below. The simplicity of the probability function in this case is in part a consequence of our choice of normal-

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ization and scale factor discussed above. Now

$$\left.\begin{aligned}&\psi^{(m)}=\int_0^\infty \rho^m e^{-\rho}d\rho=m!\\&\psi^{(0)}=\psi^{(1)}=1,\quad\text{as required and}\\&\quad\psi\equiv\psi^{(2)}=2.\end{aligned}\right\}\qquad(14)$$

We thus see that if the normal frequencies are randomly distributed, the frequency spacing index (as well as every higher moment) is *independent of frequency*. Also, the mean squared actual space is twice the average space for random distribution.

(b) *Constant ratio distribution* means that every actual, non-zero space δ_N is a constant multiple $k(\geqslant 1)$ of the average space $\bar{\delta}$ so that $\rho_N=\delta_N/\bar{\delta}$ $=$constant. If $\rho_N=1$, all actual spaces equal the corresponding average space, and we have "uniform" distribution. For constant ratio distribution:

$$\left.\begin{aligned}&M_\rho=\begin{cases}1, & \rho=k\\0, & \rho\neq k,\end{cases}\quad\text{so that:}\\&\psi^{(m)}=k^{m-1},\qquad m=1,2,3,\cdots\\&\quad\psi\equiv\psi^{(2)}=k.\end{aligned}\right\}\qquad(15)$$

Again, the frequency spacing index is independent of frequency.

We should note that a certain number of degeneracies are required whenever $k>1$, for otherwise the sum of the actual spaces in Δ would be greater than the allowed width of Δ. Thus if $k=2$, every actual space is twice as wide as the average space; then there can be only half as many distinct normal frequencies, so that half of the modes must be doubly degenerate.

(c) *Constant difference distribution* is here defined by the condition that μ_N^2 takes on successive values which differ by a constant amount. For example, this would be the case with:

$$4\mu_N^2=n_x^2+n_y^2+n_z^2=Z_N$$
$$n_x,n_y,n_z=0,1,2,3,\cdots\qquad(16)$$

if the sum of the n^2's could take on all integral values, for then $4\mu_{N+1}^2-4\mu_N^2=1$ always, or $\mu_{N+1}^2-\mu_N^2=\frac{1}{4}$, a constant difference. Actually, we shall find that the above sum of squares *cannot* take on all integral values and we shall investigate missing values of Z in our study of actual distributions in rectangular rooms.

Letting u be the value of the constant difference on the Z scale, we obtain an approximate relation for ψ as follows:

$$\frac{Z_{N+1}-Z_N}{4}=\frac{u}{4}=\mu_{N+1}^2-\mu_N^2\simeq 2\mu_N\delta_N$$

and

$$\delta\simeq\frac{1}{4\pi\mu^2}\ \text{asymptotically (Eq. (8)).}$$

Then $\rho\simeq\delta_N/\bar{\delta}=(\pi u/2)\mu=Ku\mu$. From Eq. (15) we then obtain

$$\psi^{(m)}\simeq(Ku\mu)^{m-1}\quad\text{and}\quad\psi\equiv\psi^{(2)}=Ku\mu.\qquad(17)$$

More accurate expressions for ψ in special cases involving constant difference distribution will be developed below; but this approximate relation exhibits some significant properties which will now be discussed.

In the first place, *ψ is proportional to frequency* in a constant difference distribution, in striking contrast to Poisson and constant ratio distributions which were both shown to have probability moments independent of frequency. This result indicates that a constant difference distribution as defined above becomes progressively less random as frequency increases. The physical meaning of this will be discussed later.

In the second place, ψ will increase with frequency irrespective of the magnitude of u, even though we make the size of the constant steps u as small as we choose.

If the value of Z cannot take on all integral values but has some missing values, then some of the Z steps will be larger than u. The expression for ψ can be modified to include this possibility (see Eq. (28)) and it turns out that the effect of missing values is always to increase the value of ψ. We deduce generally that any distribution of normal frequencies which has some minimum possible value for steps on the Z scale can never become truly random and, in fact, becomes "less random" with increasing frequency.

V. THE CUBICAL ROOM

We shall find it instructive to study the cube in some detail before proceeding to rectangular

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rooms of less simple proportions. The cube is strongly degenerate so the influence of degeneracies (several modes at one frequency) on spacing irregularity may be readily demonstrated. Also, the symmetry of the normal frequency expression for this case leads very simply to an approximate analytic expression for ψ.

The frequency distribution is of the modified constant difference type. The values of μ for a cube are obtained by setting p, $q=1$ in Eq. (2), resulting in Eq. (16). It is obvious that Z_n can take only integral values, since n_x, n_y, n_z are integers. Some values of Z_n can be obtained in more than one way (e.g., $0^2+0^2+3^2=9$, $2^2+2^2+1^2=9$, etc.), and some values do not exist (e.g., 7, 15, $\cdots$). All possible values of Z_n up to $256(\mu=8, n(\mu)=2459)$ have been computed to determine the number of ways in which each value can be obtained, i.e., the order of the degeneracy, designated O_n. In Fig. 2 is plotted the highest order of degeneracy which occurs up to the frequency μ in a cube. Thus at $\mu=7$ there appears a 33-fold degeneracy: 33 different normal modes having the same normal frequency.

Missing values of Z_n have also been determined up to $Z_n=256$, and are 7, 15, 23, 28; 31, 39, 47, 55, 60; etc. (see Eq. (22)). The differences in Z-value between successive "zeros" follow recurrence patterns: 8, 8, 5, 3; 8, 8, 8, 5, 3; etc. which we call "a" patterns, and 8, 8, 1, 7, 5, 3; designated "b" patterns. The complete pattern, up to $Z_n=256$, is then a_0, a, a, b, a, a, a, b. (a_0 indicating a missing "8," which would be present if the $Z_n=-1$ mode were a zero and the $Z_n=0$ mode existed!) There is no assurance here that these same patterns continue beyond $Z_n=256$, but this more general question is covered in the next section.

As a first approximation, consider the pattern "a" as being typical of the distribution. This encompasses an interval $Z_b-Z_a=32$, which contains 22 "actual" spaces of width unity and 5 spaces of width 2. The spaces of zero width do not enter in ψ (i.e., are given zero weight in Eq. (11) or (17)), and we are neglecting the spaces of width 3 which occur seldom (just once in each "b" pattern).

$$Z_B-Z_A=4(\mu_B^2-\mu_A^2)=32=4(\mu_B+\mu_A)\Delta_A^B \quad (18)$$

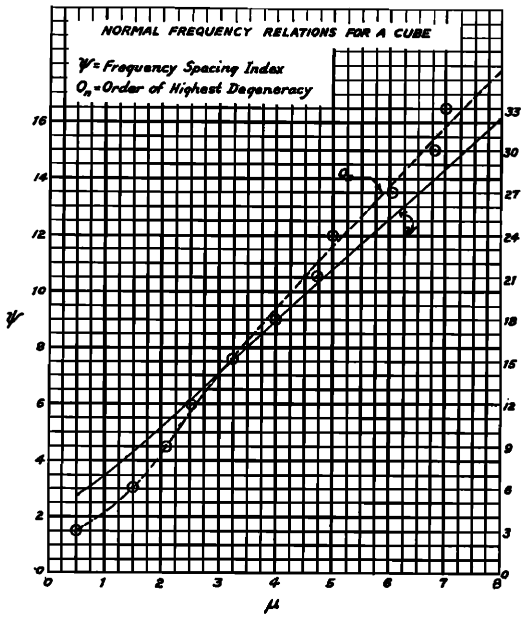

FIG. 2.

and from the above counting: (see Eq. (17))

$$u=8\mu_N\delta_N=\begin{cases}1, & 22 \text{ times in } 32\\ 2, & 5 \text{ times in } 32.\end{cases} \quad (19)$$

We now insert Δ_A^B from Eq. (18), δ_N from Eq. (19) and $\bar{\delta}$ from Eq. (8) into Eq. (12) and obtain

$$\psi\simeq\frac{8\mu}{32}\left[\frac{22(1/8\mu)^2+5(2/8\mu)^2}{(4\pi\mu^2+3\pi\mu+3/2)^{-1}}\right]$$
$$=2.05\mu+1.54+0.24/\mu$$

or, approximately

$$\psi\simeq 2\mu+1.5. \quad (20)$$

This equation gives a good approximation to the exact value ψ_A^B computed from Eq. (12) if the interval Δ_A^B covers about 32 on the Z scale. For smaller samples the actual value will fluctuate more widely about the average value given by this equation.

The cube illustrates an average spacing ratio (as expressed by ψ) which increases indefinitely with frequency. To throw further light on this situation let us re-examine the orders of degeneracies. Equation (20) is plotted in Fig. 2, with the scale arbitrarily adjusted so as to make the ψ curve more or less coincide with O_n values

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discussed above. This gives an illuminating, though purely qualitative, comparison.

As frequency increases, more and more modes appear with identical normal frequencies. Therefore, the spacing is continually increasing relative to average spacing (it is decreasing in absolute value, but by the order of $1/\mu$ instead of $1/\mu^2$ as is $\bar{\delta}$). In a 20′ cube $\psi \simeq 5.5$ at 100 c.p.s., 13.5 at 300 c.p.s., and 41.5 at 1000 c.p.s. Thus at 1000 c.p.s. the index ψ, which we are taking to be a measure of fluctuation, is 20 times the Poisson value in this room. Actually the inevitable irregularities of construction of the room will alter the situation; at some frequency, depending on the amount of perturbation, the value of ψ will level off or start decreasing, owing basically to the splitting up of degeneracies.

VI. THEORY FOR ψ IN RECTANGULAR ROOMS OF CERTAIN PROPORTIONS

Consider an idealized rectangular parallelopiped, for which frequency space is a perfectly regular lattice specified by Eq. (2). We first allow proportion coefficients which are not necessarily integral, but are rational (as for example $(1/p^2)=1.5$, $(1/q^2)=2.7$). It is then possible for Z steps to be less than unity, but there will be some minimum value (such as 0.1 in the above example), because all Z values are got by multiplications and additions involving the integral numbers n_x, n_y, n_z, and the proportion coefficients. Subsequently, we shall evaluate ψ explicitly for certain cases in which $(1/p^2)$ and $(1/q^2)$ are integers, so that the smallest possible finite space on the Z scale is unity.

We define u as the least common divisor of the widths of spaces on the Z scale. Obviously there is a minimum value of u wherever $(1/p^2)$ and $(1/q^2)$ are rational, and this value depends on the number of decimal places required to specify $(1/p^2)$ and $(1/q^2)$. To this value of u there corresponds a least value of δ designated δ_u and given by

$$\delta_u = \frac{(pq)^{\frac{2}{3}}u}{8\mu}$$

and all possible δ values are given by $n\delta_u$, $n=1, 2, 3, \cdots$. We define h_n as the number spaces of width $n\delta_u$ in an arbitrary interval Δ. The sum of the actual spaces equals the total interval: $\delta_u(h_1+2h_2+3h_3+\cdots)=\sum nh_n\delta_u=\Delta$. The frequency spacing index ψ then is

$$\psi = \frac{h_1\delta_u^2+h_2(2\delta_u)^2+h_3(3\delta_u)^2+\cdots}{\Delta\bar{\delta}}$$
$$= \frac{\sum n^2h_n\delta_u}{\sum nh_n\bar{\delta}} = \left[\frac{(pq)^{\frac{1}{3}}\pi\mu}{2}\right] \times\left[1+\frac{P}{4\mu}+\frac{P}{8\pi\mu^2}\right]\left[u\sum n^2h_n/\sum nh_n\right]. \quad (21)$$

This general formula reduces to Eq. (28) for the special cases treated below, in which number theory is used to evaluate the weighting function in the last bracket. The first bracket times u is equivalent to Eq. (17) for constant difference distribution.

An approximate equation for ψ for rectangular rooms of certain proportions can be derived with the aid of Waring's problem in number theory.[12,*] As was shown in Section V, not all integers can be obtained from the sums of squares of three integers. One aspect of Waring's problem yields formulas for computing these missing numbers. Thus, the missing values of

$$Z=n_x^2+n_y^2+n_z^2 \quad n_x, n_y, n_z, 0, 1, 2, \cdots$$

are given by

$$O_z=4^n(8m+7) \quad n, m=0, 1, 2, 3, \cdots. \quad (22)$$

More generally, equations of the form

$$\left.\begin{array}{ll} O_z=A^n(Bm+C), & \text{and in some cases one or more additional equations} \\ O_z=D_jn+E_j & n,\ m=0,\ 1,\ 2,\ 3,\ \cdots \\ & j=1, 2, 3 \text{ (see Table II)} \end{array}\right\} \quad (23)$$

are given for computing the missing values of

$$Z=\alpha n_x^2+\beta n_y^2+\gamma n_z^2. \quad (24)$$

Equations (23) have been evaluated[12] for 105 sets of values of α, β, γ in Eq. (24). These results can be used to compute missing values of $Z_n=4\mu_n^2$ for various proportions of rectangular rooms by

[12] L. E. Dickson, *Modern Elementary Theory of Numbers* (University of Chicago Press, Chicago, 1939), p. 111.

* The author is grateful to L. J. Savage for indicating this approach and to L. L. Foldy for many cogent suggestions on its application.

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putting Eq. (24) in the form

$$Z'=n_x^2+\left(\frac{n_y}{p}\right)^2+\left(\frac{n_z}{q}\right)^2 \quad \text{(takes only integral values by choice of } p,\, q\text{).} \qquad (25)$$

Fifty cases of p, q values are listed in Table II, with the corresponding values of A, B, C, and D_j of Eqs. (23).

To utilize these results in a formula for ψ we must compute the average number of missing Z values in a given frequency band $d\mu$. We first find the number of missing values $N^0(Z')$ up to Z', given by the number of $n-m$ intersections up to the curve $Z'=A^n(Bm+C)$. Along this curve, $m=((Z'/A^n)-C)/B$; and for $m=0$, $n=\log_A(Z'/C)$. These values of m and n are then used as upper summation limits

$$N_0(Z')=\sum_{n=0}^{\log_A Z'/C}\ \sum_{m=0}^{(Z'/A^n-C)/B}(1),$$

$$=\sum_{n=0}^{\log_A Z'/C}\left[\frac{Z'/B}{A^n}-\frac{C}{B}+1\right]=\frac{Z'}{B}\sum\frac{1}{A^n}+\sum\left(1-\frac{C}{B}\right),$$

$$=\frac{Z'}{B}\left[\frac{1-\left(\frac{1}{A}\right)^{(\log_A Z'/C+1)}}{1-\frac{1}{A}}\right]+\left(1-\frac{C}{B}\right)\left(\log_A\frac{Z'}{C}+1\right).$$

Further:

$$\left(\frac{1}{A}\right)^{(\log_A Z'/C+1)}=\left(\frac{1}{A}\right)\left(\frac{1}{A}\right)^{(\log_A Z'/C)}=\frac{1}{A}(A)^{(\log_A C/Z')}=\frac{C}{AZ'}.$$

Similarly the second type of Eq. (23) yields

$$\sum_{n=0}^{(Z'-E)/D_j}(1)=\frac{Z'}{D_j}-\frac{E}{D_j}+1.$$

The complete equation for the number of missing Z' values then is

$$N_0(Z')=\frac{1}{B}\left[\frac{AZ'-C}{A-1}\right]+\left(1-\frac{C}{B}\right)\left(\frac{\ln Z'/C}{\ln A}+1\right)+\sum_j\left[\frac{Z'}{D_j}-\frac{E_j}{D_j}+1\right]. \qquad (26)$$

TABLE II. Statistical coefficients for normal frequency spacing in rectangular rooms of various proportions.

Relative dimensions	Proportion ratios p	q	Number theory coefficients A	B	C	D_1	D_2	D_3	Spacing index correction factors F	G
$1:1:1$	1	1	4	8	7				1.33	0.05
$1:2^{1/2}:2^{1/2}$	1	.71	4	16	14				1.17	.045
$1:3^{1/2}:3^{1/2}$	1	.58	9	9	6				1.25	.076
$1:2:2$	1	.5	4	8	7.	8			1.58	.045
$1:5^{1/2}:5^{1/2}$	1	.45	4	8	3				1.33	.225
$1:6^{1/2}:6^{1/2}$	1	.41	9	9	3				1.25	.152
$1:2\cdot 2^{1/2}:2\cdot 2^{1/2}$	1	.35	4	16	14	4	16		1.80	.045
$1:3:3$	1	.33	4	8	7	9/2			1.78	.045
$1:2\cdot 3^{1/2}:2\cdot 3^{1/2}$	1	.29	9	9	6	4			1.75	.076
$1:4:4$	1	.25	4	8	7	4	8	32	2.15	.045
$1:2\cdot 6^{1/2}:2\cdot 6^{1/2}$	1	.20	9	9	3	4	8		2.00	.152
$1:1:2$	.71	.71	4	8	7				1.33	0.05
$1:(3/2)^{1/2}:3^{1/2}$	.71	.58	4	16	10				1.17	.135
$1:2^{1/2}:2$	.71	.5	4	16	14				1.17	.045
$1:3^{1/2}:6^{1/2}$	.71	.41	4	16	14	3			1.83	.045
$1:2:2\cdot 2^{1/2}$	.71	.35	4	8	7	8			1.58	.045
$1:2\cdot 2^{1/2}:4$	.71	.25	4	16	14	8/2	16		1.79	.045
$1:1:3^{1/2}$	.58	.58	9	3	2				1.75	0.08
$1:2\cdot 3^{1/2}:2$	.58	.5	9	9	6	4			1.75	.076
$1:2^{1/2}:6^{1/2}$	.58	.41	4	16	14	3			1.83	.045
$1:3^{1/2}:3$	.58	.33	9	9	6	3			1.92	.076
$1:2:2\cdot 3^{1/2}$	.58	.29	9	3	2	4			2.25	.076
$1:6^{1/2}:3\cdot 2^{1/2}$	.58	.24	4	16	10	3	9		2.06	.135
$1:1:2$	.5	.5	4	8	7	4/3			2.83	0.05
$1:(3/2)^{1/2}:6^{1/2}$	.5	.41	9	9	3	16			1.38	.152
$1:2^{1/2}:2\cdot 2^{1/2}$	.5	.35	4	16	14	4/3			2.67	.045
$1:3^{1/2}:2\cdot 3^{1/2}$	.5	.29	9	9	6	4/3			2.75	.076
$1:2:4$	.5	.25	4	8	7	4/3	16		2.96	.045
$1:6^{1/2}:2\cdot 6^{1/2}$	.5	.20	9	9	3	4/3			2.75	.152
$1:3:6$	.5	.17	4	8	7	4/3	9/2		3.28	.045
$1:1:5^{1/2}$	.45	.45	4	8	7	5/2			2.13	0.05
$1:5^{1/2}:5$	.45	.20	4	8	3	5/2	25/2		2.29	.225
$1:1:6^{1/2}$	.41	.41	9	3	2	8			2.00	0.08
$1:(3/2)^{1/2}:3$	.41	.33	9	9	3	3			1.92	.152
$1:2(2/3)^{1/2}:4$	.41	.25	9	9	3	8/2	16/2	64	2.03	.152
$1:3^{1/2}:3\cdot 2^{1/2}$	.41	.24	4	8	5	3	9		2.22	.135
$1:2:2\cdot 6^{1/2}$	.41	.20	9	3	2	8/2	32		2.31	.076
$1:1:2\cdot 2^{1/2}$	.35	.35	4	8	7	4/3	8		3.08	0.05
$1:2^{1/2}:4$	.35	.25	4	16	14	4/3	8		2.92	.045
$1:3^{1/2}:2\cdot 6^{1/2}$	.35	.20	4	8	5	4/3			2.83	.135
$1:1:3$	.33	.33	4	8	7	3	9/2		2.44	0.05
$1:2\cdot 3^{1/2}:2\cdot 3^{1/2}$	.33	.29	9	9	6	3	4		2.42	.076
$1:2/3\cdot 6^{1/2}:2\cdot 6^{1/2}$	.33	.20	9	9	3	3	4	8	2.67	.152
$1:1:2\cdot 3^{1/2}$	.29	.29	9	3	2	4/3			3.25	0.08
$1:3^{1/2}:6$	.29	.17	9	9	6	3	4/3		3.42	.076
$1:1:4$	.25	.25	4	8	7	4/3	8	16/2	3.33	0.05
$1:(3/2)^{1/2}:2\cdot 6^{1/2}$	.25	.20	9	9	3	4/3	8	64	3.03	.152
$1:3^{1/2}:4\cdot 3^{1/2}$	.25	.14	9	9	6	4/3	8	16/2	3.25	.076
$1:1:2\cdot 6^{1/2}$	.20	.20	9	3	2	4/3	8	32	3.56	0.08
$1:3^{1/2}:6\cdot 2^{1/2}$	.20	.12	4	8	5	3	4/3	9	3.72	.135

The number of missing values in an interval $d\mu$ on the μ scale is obtained by substituting $Z'=4\mu^2/(pq)^{\frac{2}{3}}$ in Eq. (26) and differentiating

$$\frac{dN_0(\mu)}{d\mu}=\left[\frac{A}{B(A-1)}+\sum_j\frac{1}{D_j}\right]\frac{8\mu}{(pq)^{\frac{2}{3}}}+\frac{2\left(1-\frac{C}{B}\right)}{\mu\ln A}. \qquad (27)$$

This result is now used to weight the counting in ψ (last bracket of Eq. (21)). From Eqs. (2), (9), and (24) we have

$$(Z_{N+1}-Z_N)=8\mu_N\delta_N=(pq)^{\frac{2}{3}}(Z_{N+1}'-Z_N').$$

The spaces $(Z_{N+1}'-Z_N')$ must always equal zero

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or 1 or 2 or 3 etc., since all cases in Table II are so chosen that Z' takes on only integral values ($1/p^2$ and $1/q^2$ are integers). Zero values of δ_N do not contribute to ψ. Unit values of $(Z_{N+1}'-Z_N')$ occur between all non-missing values of Z' ($u=1$ in Eq. (21)); if one Z' is missing, there is a double space and the number of these in Δ is given by $[dN_0(\mu)/d\mu]\Delta=h_2$. Two or more adjacent missing values, creating Z' spaces of 3 or more in width, occur relatively infrequently in the cases considered, and are therefore neglected.

The value of δ_N for a unit Z_N'-space is

$$\delta_u=\frac{(pq)^{\frac{2}{3}}}{8\mu_N}.$$

Since Δ/δ_u is the total number of "available" unit spaces, there will be $(\Delta/\delta_u-2h_2)=h_1$ actual single spaces in Δ. Inserting these values of h_1, h_2, $u=1$, $n=1$, 2 in Eq. (21) we obtain directly

$$\left.\begin{aligned}\psi&=\left[\frac{(pq)^{\frac{2}{3}}\pi\mu}{2}\right]\left[1+\frac{P}{4\mu}+\frac{P}{8\pi\mu^2}\right]\left[F+\frac{G}{\mu^2}\right]\\ &\text{in which (see Table II)}\\ F&=1+\frac{2A}{B(A-1)}+\sum_j\frac{2}{D_j},\\ G&=\left(1-\frac{C}{B}\right)(pq)^{\frac{2}{3}}/2\ln A.\end{aligned}\right\}\qquad(28)$$

This equation illustrates explicitly several interesting characteristics of the frequency spacing index for rectangular rooms. We recall that the frequency spacing index is defined as the second moment of the distribution function $M_\rho=f(\rho)$ taken about the axis $\rho=0$. Equations similar to the one above could be derived for higher moments, but these would not illustrate any essential properties not contained in the present formula.

The first bracketed factor gives the asymptotic value of ψ for the hypothetical case in which all integral values of Z exist (*constant difference* of distribution). The second bracket introduces low frequency correction terms which arise from frequency lattice points lying in the coordinate planes. The influence of shape is introduced through the room proportion factor P.

The third factor introduces a correction owing to missing values of Z. As seen from Table II, the value of this factor is always greater than unity; in fact, it would equal unity only if there were no missing values of Z. Since the first term, F, is independent of frequency, ψ is always increased by missing values, even at very high frequencies.

The above equation is applicable only to those cases for which the number theory coefficients are available. Many additional cases are calculable, at least in principle, by Eq. (21), the only restriction being that the proportion coefficients $1/p^2$ and $1/q^2$ are rational. We see at present no way of formulating a rigorous theory for irrational coefficients, but we can infer the behavior of ψ for these cases by the following reasoning.

As room proportions or shape are gradually altered, holding volume constant, the normal frequencies are gradually shifted without discontinuity in value.[13] Therefore, the value of ψ must also change smoothly, without discontinuity. Now an irrational number can be approximated to any desired accuracy by a rational number (by dropping all figures beyond a specified decimal place). Then we can bracket any set of irrational proportion coefficients by two sets of rational ones, each of which has a ψ which increases with frequency (applying Eq. (21)). Thus (e, π) can be bracketed by (2.7, 3.1) and (2.8, 3.2).

The above theory for ψ in *rectangular* rooms is no longer valid above some limiting μ at which inevitable irregularities in room construction become comparable in size to the wave-length. In fact, this limiting accuracy in construction imposes a limiting accuracy with which $1/p^2$ and $1/q^2$ can be specified with significance: so they are, perforce, rational for practical purposes.

Thus we conclude that ψ always tends to increase with frequency, up to any practical limit we choose. And we conjecture (without mathematical proof) that this behavior is generally characteristic of any perfectly rectangular parallelopiped irrespective of the dimension ratios.

Perhaps this conclusion is not so surprising: as long as the room is completely regular (perfectly smooth and exactly perpendicular walls) the corresponding frequency space must also be a regular rectangular lattice. This imposes a certain

[13] R. H. Bolt, H. Feshbach, and A. M. Clogston, "Perturbation of sound waves in irregular rooms," J. Acous. Soc. Am. 13, 65 (1942).

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degree of order on the totality of frequency points, and this lack of randomness causes the average ratio of actual-to-average spacing to increase with frequency. The absolute width of the actual space will decrease with frequency since the average space is decreasing as $1/\mu^2$. But in terms of a frequency *ratio* distribution function the system becomes less random with increasing frequency.

This situation has been pointed out in connection with "ergodic" distribution of energy in a room.[6] From a study of the wave functions it was shown that the distribution of sound in a regular rectangular room will never become thoroughly diffuse, and in fact becomes less diffuse, as frequency is increased.

In actual cases this situation is modified by inevitable irregularities in the room. When the perturbation due to these irregularities produces normal frequency shifts of the order of magnitude of the average spacing (and it has been shown that these shifts in frequency may be either positive or negative)[13] the regularity of the normal frequency lattice breaks down and the above theory leading to Eq. (28) is no longer valid.

From this picture we may make the following very general comment about normal frequency distributions. It is incorrect to state categorically that the normal frequencies in any enclosure will, at some high frequency, become randomly spaced simply because "there are so many frequencies present." Rather, one must say that normal frequencies will be distributed at random only if the system is perturbed, either in boundary shape or in boundary value, to a "sufficient" degree.

Furthermore, *the frequency spacing index may be used as a measure of diffusion.* As diffusion increases, ψ becomes smaller and less frequency dependent, approaching the constant value $\psi=2$ for complete randomness.*

VII. SOME FORMULAS FOR DIRECT CALCULATION

The frequency spacing index may be calculated directly for any case in which the normal frequencies ν_j are known (either by calculation or measurement), from the formula:

$$\psi_a{}^b=\frac{\pi}{c^2(\nu_b-\nu_a)}\sum_{j=a}^{j=b}(\nu_{j+1}-\nu_j)^2\left(\frac{4V}{c}\nu_j{}^2+\frac{S}{2}\nu_j\right) \quad (29)$$

in which V is the volume and S is the surface area of the room. To minimize statistical deviations in ψ, the interval $a-b$ should contain a "reasonable" number of normal frequencies; apparently some 20 or 30 are sufficient.

For rectangular rooms having proportions listed in Table II, the index at any frequency is given approximately by

$$\psi\simeq\frac{\pi}{L_{\max}{}^2}\left[\frac{V\nu}{2c}+\frac{S}{16}\right]F \quad (30)$$

in which $L_{\max}$ is the longest dimension of the room, and F is a number theory correction factor given in Table II. This formula must be used with caution: it is valid only as long as the room is perfectly rectangular (negligible perturbation with respect to the wave-lengths).

The frequency spacing index ψ is in effect a *mean square* spacing ratio. Also, $\bar{\delta}$ of Eq. (8) is the *linear* average spacing between (dimensionless) normal frequencies. Therefore, the *root mean square* average spacing is given by

$$\delta_{\text{r.m.s.}}=\psi^{\frac{1}{2}}\bar{\delta}.$$

By use of Eqs. (1), (3), and (28) this becomes

$$\langle d\nu\rangle_{\text{r.m.s.}}\simeq\frac{c^{5/2}F^{1/2}}{10V^{1/2}\nu^{3/2}L_{\max}}, \quad (31)$$

$$\langle d\nu\rangle\equiv d\nu/dn \text{ for } dn=1 \text{ (see Eq. 4)},$$

which gives the r.m.s. average spacing between normal frequencies, subject to the caution noted for Eq. (30).

It is instructive to compute the r.m.s. spacing at a standard low frequency position on the normal frequency scale, as an indication of the low frequency response irregularity of a room. Let us choose $\mu=1$, below which there are about 10 normal frequencies. Substituting $\mu=V^{\frac{1}{3}}\nu/c=1$ in Eq. (31) we obtain:

$$\langle d\nu_1\rangle_{\text{r.m.s.}}\simeq\frac{cF^{\frac{1}{2}}}{10L_{\max}} \quad \text{at} \quad \nu_1=c/V^{\frac{1}{3}}. \quad (32)$$

* This rule is not strictly applicable to the lowest range of normal frequencies (below about $\mu=3$). The small number of frequencies here may be more nearly *uniform* ($\psi=1$) than *random* ($\psi=2$) in distribution, depending in detail on the room proportions. Even here, however, ψ is a measure of room response smoothness, as indicated in Figs. 2 and 3 of reference 10.

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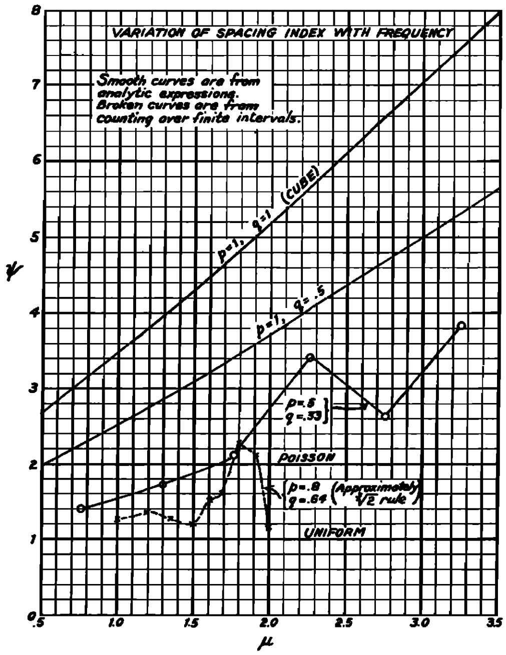

FIG. 3.

This surprisingly simple formula is applicable to at least the 50 cases in Table II. If applied to two rectangular rooms of equal volume but different proportions, $\langle d\nu_1\rangle_{\text{r.m.s.}}$ is directly a measure of the relative "peakiness" or response irregularity of the two rooms at the specified frequency ν_1.

This formula clearly points up two factors* which influence frequency response in rectangular rooms. Increasing the maximum dimension $L_{\max}$, while holding volume constant, decreases the effective spacing because the fundamental (lowest) normal frequency of the room becomes lower and additional normal frequencies are introduced into the "low frequency" band in question. Altering the relative dimensions influences the symmetry of the normal frequency lattice, and modifies the value of F which expresses, though indirectly, the degree of degenerateness.

Actually, these two factors work together in a rather complicated way. Thus, consider the cases 1:1:1 and 1:5:5 in Table II. Reduced to equal volume, the latter room has $L_{\max}$ about 1.3 times $L_{\max}$ of the former, yet $F=1.33$ for both cases. Here the reduced symmetry yields a lower $\langle d\nu_1\rangle_{\text{r.m.s.}}$ by F *not* increasing to counteract the increasing $L_{\max}$. Again consider 1:1:3 and 1:3/2:3. Reduced to equal volumes, these two rooms have almost the same value of $L_{\max}$ (actually 7 percent difference). But here the value of F drops from 1.75 to 1.17 (33 percent) to reflect the decreased symmetry.

* Only these two factors enter in the limited study of the present paper. The influence of other important factors, such as damping and position in the room, are postponed for future study.

CONCLUSIONS

1. *Fluctuation* properties of normal frequency spacing have been investigated by statistical methods, as an extension of the normal frequency distribution theory[11] which yields Eqs. (5) and (8) for *smooth* average properties.

2. The study is generalized by use of dimensionless parameters (Eqs. (1)–(3)) to yield statistical quantities which are independent of room volume but dependent explicitly on room shape.

3. A *frequency spacing index* ψ is defined (Eq. (11)), and evaluated for certain cases, notably: random (Poisson) distribution ($\psi=2$), uniform distribution ($\psi=1$), cubical room (Eq. (20)), and a class of rectangular rooms (Eq. (28)) for which weighting factors can be evaluated from number theory (Table II). For other cases ψ can be calculated by Eqs. (10) or (29) if normal frequency values are available. Typical results for $\psi=f(\mu)$ are shown in Fig. 3; ψ is independent of frequency for uniform and random distributions, increases steeply with frequency for a cube, and increases less steeply for other room proportions.

4. In a room for which frequency space is a regular lattice, ψ generally increases with frequency, indicating increasing departure from a random state. True randomness (ergodic motion, Poisson distribution) requires perturbation of the system in boundary shape or condition.

5. The index ψ is a relative measure of sound diffusion or response irregularity. Values of ψ at $\mu=1$ for a wide range of rectangular room proportions have been plotted (reference 10),** yielding a criterion for relative smoothness of

** For this plot, values of ψ were determined at about 70(p, q) values. Some of these were obtained from Eq. (28), and many were calculated by Eq. (12), over a band of about 25 normal frequencies, to check Eq. (28) or to fill in cases not included in Table II. The author is indebted to J. R. Engstrom, K. C. Morrical, and C. M. Harris for supplying many of the normal frequency calculations, and for giving helpful suggestions on this study.

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response. The $2^{\frac{1}{3}}$ rule and the 2:3:5 proportions employed in practice satisfy this criterion, but the present result suggests that a relatively broad range of proportions may be equally satisfactory. A quantitative evaluation of response irregularity, involving damping and other factors neglected here, remains for further study.

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APPENDIX I

Probability Function for Random (Poisson) Distribution

Consider a random distribution in which every normal frequency μ_N in an interval Δ has an equal probability of occurring at any frequency in that interval. This is an example of Poisson distribution, for which it can be shown that

$$P_m = \frac{N^m e^{-N}}{m!}$$

is the probability that there are m normal frequencies in an interval in which $N = \Delta/\bar{\delta}$ is the expected (average) number.

The probability of finding *no* normal frequencies ($m=0$) in the interval Δ is

$$P_0 = e^{-N} = \exp(-\Delta/\bar{\delta}).$$

But if there are no normal frequencies in Δ then there is a space of width Δ. Therefore, $\exp(-\Delta/\bar{\delta})$ *is the probability of finding a space of width* Δ *where the average expected space width is* $\bar{\delta}$.

The probability of occurrence of an actual space δ is then $\exp(-\delta/\bar{\delta}) = e^{-\rho}$; the probability per unit average space is $(1/\bar{\delta})e^{-\rho}$; and the number per unit $\bar{\delta}$ in the range between δ and $\delta + d\delta$ is $(1/\bar{\delta})e^{-\rho}d\delta$. But $\rho = \delta/\bar{\delta}$ and $d\rho = d\delta/\bar{\delta}$, so $(1/\bar{\delta})e^{-\rho}d\delta = e^{-\rho}d\rho = M_\rho d\rho$, where: $M_\rho = e^{-\rho}$ is the probability of finding spacing ratios between ρ and $\rho + d\rho$, per unit average space $\bar{\delta}$, for Poisson distribution.

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Factors Governing the Intelligibility of Speech Sounds

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The characteristics of speech, hearing, and noise are discussed in relation to the recognition of speech sounds by the ear. It is shown that the intelligibility of these sounds is related to a quantity called articulation index which can be computed from the intensities of speech and unwanted sounds received by the ear, both as a function of frequency. Relationships developed for this purpose are presented. Results calculated from these relations are compared with the results of tests of the subjective effects on intelligibility of varying the intensity of the received speech, altering its normal intensity-frequency relations and adding noise.

1. INTRODUCTION

THIS paper discusses the factors which govern the intelligibility of speech sounds and presents relationships for expressing quantitatively, in terms of the fundamental characteristics of speech and hearing, the capability of the ear in recognizing these sounds. The relationships are based on studies of speech and hearing which have been carried on at Bell Telephone Laboratories over a number of years. The results of these studies have in large measure already been published. The formulation of the results into relationships for expressing speech intelligibility, which has also been in progress for a number of years, has not been previously published. The purpose of this paper is to bring the relationships and basic data together into one report.

Speech consists of a succession of sounds varying rapidly from instant to instant in intensity and frequency. Assuming that the various components are received by the ear in their initial order and spacing in time, the success of the listener in recognizing and interpreting these sounds depends upon their intensity in his ear and the intensity of unwanted sounds that may

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