A simple iteration scheme for the computation of decay constants in enclosures with diffusely reflecting boundaries

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This paper describes a procedure for the exact determination of the decay constant and hence of the reverberation time of enclosures with diffusely reflecting boundaries. It is based on the integral equation which describes the energy propagation in such rooms under the condition of exponential decay. In contrast to Gilbert's well-known method, the decay constant is determined by an iterative solution merely of that integral equation. Some typical results for rectangular rooms are presented. Depending on the distribution of sound absorbing areas, considerable differences between reverberation times after Sabine's equation, on the one hand, and the present method, on the other, are observed. Furthermore, the question to what extent the applicability of the procedure is restricted by the assumption of diffuse wall reflections is discussed. © *1995 Acoustical Society of America.*

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INTRODUCTION

Since W. C. Sabine's pioneering work the reverberation time is generally accepted as the most important and meaningful acoustical figure of merit for any kind of halls. According to his famous formula the reverberation time T of an enclosure with the volume V and the boundary surface S is

$$T=0.163\,\frac{V}{Sa}, \tag{1}$$

where all quantities are expressed in mks units. According to L. Cremer, the quantity a may be referred to as "absorption exponent." In Sabine's original formula, a is just $\langle\alpha\rangle$, the absorption coefficient averaged over the whole boundary. If $\langle\alpha\rangle$ is not very small compared to unity, it is often replaced with $-\ln(1-\langle\alpha\rangle)$. With this modification, Eq. (1) is mostly referred to as "Eyring's formula."

Equation (1) is used by most practitioners; in fact, it predicts the reverberation time of a hall in many cases quite correctly, especially if the absorption exponent is small. However, its derivation is based upon the assumption that each wall element of the enclosure is hit by the same energy amount per second. This is the case if the sound field in the room is "diffuse" or "homogeneous." In many practical situations, however, this condition is not fulfilled, either because of the particular room shape or, what is more probable in practice, because of a highly nonuniform distribution of absorption. If the assumption of a diffuse sound field is dropped, both the steady-state distribution of sound energy and its decay can be described by an integral equation.[1–3] Even then, the final sound decay follows an exponential law with the same decay rate at each location as has been shown by Miles.[4] It is the goal of this paper to discuss methods of finding the decay rate for this part.

I. INTEGRAL EQUATION

In its time-dependent form, the above-mentioned integral equation reads

$$B(\mathbf{r},t)=\int_S K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')B(\mathbf{r}',t-R/c)\,dS'+B_0(\mathbf{r},t). \tag{2}$$

This integral equation is the basis for what is nowadays sometimes referred to as the "radiosity method."[5] In this equation, B denotes the "irradiation density" of the boundary, i.e., the sound energy arriving at the wall per second and unit area. B_0 characterizes the contribution of the direct sound. $R=|\mathbf{r}-\mathbf{r}'|$ is the distance between two wall points with coordinates $\mathbf{r}$ and $\mathbf{r}'$, c is the sound velocity, and dS' is an area element at $\mathbf{r}'$. The integration is extended over the whole boundary of the enclosure. By $\rho=1-\alpha$ the "reflection coefficient" of the boundary element dS' is introduced which is assumed as independent of the angle of sound incidence. The function K is given by

$$K(\mathbf{r},\mathbf{r}')=\frac{\cos\vartheta\,\cos\vartheta'}{\pi R^2}. \tag{3}$$

For the meaning of ϑ and ϑ' see Fig. 1. Equation (2) holds for enclosures with "rough" boundaries which scatter the reflected sound energy in an ideally diffuse way. This condition (which must not mistakenly be assumed to result in a diffuse sound field) implies that the intensity of the reflected or rather scattered sound a certain distance from the considered wall element is proportional to the cosine of the angle between the wall normal and the scattering direction (Lambert's law). Although there exists no real wall which strictly meets this condition, it is probably a better approximation to reality than the assumption of specular sound reflections, especially with regard to decaying sound fields. (This is probably the reason why Sabine's equation proved so useful in practical applications.) We shall come back to this point at the end of this paper.

Figure 2 shows a few decay curves obtained by numerically solving this integral equation for a rectangular room with relative dimensions 1:2:3. The "floor" (i.e., the wall with dimensions 2:3) was assumed to be totally absorbent ($\rho=0$), whereas the remaining walls are free of absorption

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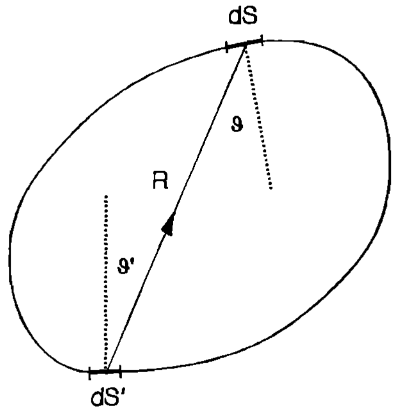

FIG. 1. Notations in Eq. (3).

($\rho=1$). It underlines the above statement that the initial fluctuations of the decay curves gradually fade out, leaving a straight decay curve in a logarithmic scale. Accordingly the momentary energy in this phase of decay is proportional to $\exp(-\lambda t)$ with a constant decay constant λ throughout the whole room.

By introducing this exponential time dependence into Eq. (2) one obtains, with $B_0=0$,

$$B(\mathbf{r})=\int_S K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')B(\mathbf{r}')\exp(\lambda R/c)dS'. \tag{4}$$

(Joyce[3] has published a somewhat more general version of this integral equation which allows for angle-dependent reflectivity and also for partially diffuse, partially specular wall reflections.) Our further goal is to find the decay constant λ from which we can obtain the "absorption exponent" a to be inserted into Eq. (1) using

$$a=\frac{\lambda}{c}\langle R\rangle, \tag{5a}$$

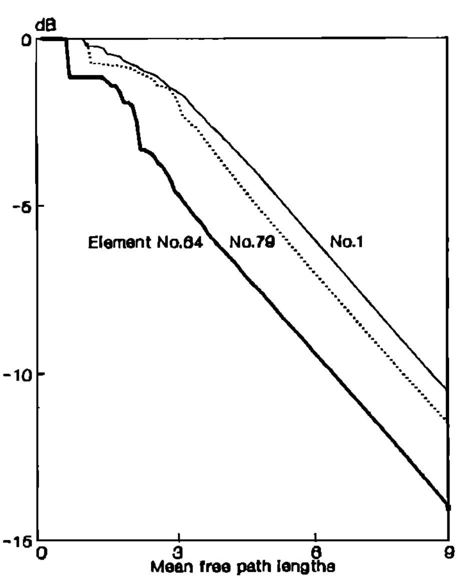

FIG. 2. Decay curves in a rectangular room with relative dimensions 1:2:3 with totally absorbing and reflecting side walls and ceiling. The receiver positions are indicated in Fig. 3. The unit at the abscissa is the mean-free-path length according to Eq. (5b).

where $\langle R\rangle$ denotes the classical mean-free-path length

$$\langle R\rangle=\frac{4V}{S}. \tag{5b}$$

II. ITERATIVE DETERMINATION OF THE DECAY CONSTANT

The decay constant λ which we are looking for must be determined by solving the integral equation (4) in some way. Unfortunately, this equation is not of standard (Fredholm) type, since its parameter (eigenvalue) is not just a constant factor in front of the integral; instead it is hidden in the exponential and is multiplied with a function of the integration variables. Probably the first attempt to overcome this problem is due to Gerlach and Mellert[6] who replaced the actual path lengths R with their average $\langle R\rangle$. Then the integral equation obtains the usual form with the parameter $\exp(\lambda\langle R\rangle/c)$ and can be solved by standard procedures. However, the decay constants calculated in this way turn out to deviate from the correct values as determined, for instance, by a Monte Carlo method. [Originally, Gerlach and Mellert considered the discretized version of Eq. (5), i.e., a set of linear equations for the unknown variables B_i. They interpreted the coefficients K_{ij} which correspond to the function $K(\mathbf{r},\mathbf{r}')$ as transition probabilities and the whole decay as a Markov process.]

To obtain the correct result, Gilbert[7] proposed an iterative process which consists of the simultaneous solution of Eq. (4) and of a supplementary integral equation for λ. In our present notation and for angle-independent absorption coefficients, this latter equation reads

$$\lambda=\frac{\int_S[1-\rho(\mathbf{r})]B(\mathbf{r})dS}{\int\int_{S,S'}\rho(\mathbf{r})B(\mathbf{r})\{[\exp(\lambda R/c)-1]/\lambda\}K(\mathbf{r},\mathbf{r}')dS\,dS'}. \tag{6}$$

The iteration starts by inserting tentative values of B and λ into the right-hand side of Eq. (6). A more correct value of λ obtained by evaluating this expression is used to compute a new function B from Eq. (4) which again is inserted into Eq. (6), etc. This process converges after a few steps. Gilbert applied it to hemispherical auditoria, whereas Schroeder and Hackman[8] used it to find the reverberation time of some two-dimensional enclosures.

In the following, a considerably simpler iteration scheme will be described which merely employs the integral equation (4). The kernel of this equation may be symmetrized by substitution of

$$\beta(\mathbf{r})-B(\mathbf{r})\sqrt{\rho(\mathbf{r})}\quad\text{and}\quad\kappa(\mathbf{r},\mathbf{r}')=\sqrt{\rho(\mathbf{r})\rho(\mathbf{r}')}K(\mathbf{r},\mathbf{r}').$$

The modified integral equation is

$$\beta(\mathbf{r})=\int_S\kappa(\mathbf{r},\mathbf{r}')\beta(\mathbf{r}')\exp(\lambda R/c)dS'. \tag{7}$$

Because of its homogenity we may suppose without loss of generality that the function β is normalized, i.e., that $\int\beta^2\,dS=1$.

TABLE I. Absorption exponent $a=(4V/cS)\lambda$ of a cube.

Method	Uniform absorption ($\rho=5/6$)	Absorption concentrated on one or two sides with $\rho=0$ (all other sides with $\rho=1$): one side	two sides, parallel	two sides, perpendicular
Sabine ($a=1-\langle\rho\rangle$)	0.1667	0.1667	0.3333	0.3333
Eyring ($a=-\ln\langle\rho\rangle$)	0.1823	0.1823	0.4055	0.4055
Gerlach and Mellert	0.1822	0.2008	0.4970	0.4352
Present method (or Gilbert's)	0.1733	0.1982	0.4224	0.4540

The iteration process starts by inserting constant $\beta^{(0)}=1/\sqrt{S}$ into Eq. (7); as an initial value of the decay constant its Eyring value $\lambda_0=-c\ \ln\langle\rho\rangle/\langle R\rangle$ or even $\lambda_0=0$ may be used [$\langle R\rangle$ is the mean-free-path length given by Eq. (5b)]. The result of the numerical integration is some function β'. Since this function is not the exact solution of the integral equation but at best an approximation to it, β' will not fulfill the condition $\int\beta'^2\ dS=1$ and hence must be multiplied with the constant

$$\mu=\left(\int\beta'^2\ dS\right)^{-1/2}.$$

Since $\exp(\lambda_0 R/c)$ is a monotonic function of λ_0, $\mu<1$ indicates that λ_0 is larger than the correct decay constant λ and vice versa. Accordingly, changing λ by some increment $\Delta\lambda$ changes the normalization constant μ by a factor $\exp(R_0\Delta\lambda/c)$ with R_0 denoting a "typical" value of the path lengths R. To find a proper increment we equate μ to $\exp(\langle R\rangle\Delta\lambda/c)$. This yields

$$\Delta\lambda=\frac{c}{\langle R\rangle}\ln(\mu)=-\frac{c}{2\langle R\rangle}\ln\left(\int\beta'^2\ dS\right). \tag{8}$$

For the next iteration step, the function $\beta^{(1)}(\mathbf{r})=\mu\beta'(\mathbf{r})$ and the improved eigenvalue $\lambda_1=\lambda_0+\Delta\lambda$ are inserted into Eq. (7) etc. As Gilbert's method, this procedure converges relatively fast toward the correct decay constant λ and the correct values of β. From the latter, the "irradiation strength" B can also be obtained—so to speak, as a by-product (In case of totally absorbing wall elements it is useful to assign a very small but finite reflection coefficient to them).

This procedure is somewhat reminiscent of Gerlach and Mellert's with the difference, however, that the replacement of R with $\langle R\rangle$ is not applied to λ but only to the correction $\Delta\lambda$ which becomes smaller with each iteration step.

III. REMARKS ON THE NUMERICAL EVALUATION OF EQ. (7)

To evaluate Eq. (7), it has to be discretized, i.e., the boundary of the enclosure is subdivided into N surface elements which are small enough to justify the assumption that β can be considered constant within each wall element and that the variation of $\exp(\lambda R/c)$ for any pair of wall elements is small. Then Eq. (7) reads

$$\beta_i=\sum_{k=1}^{N}\kappa_{ik}\exp(\lambda R_{ik}/c)\beta_k, \tag{9}$$

with R_{ik} denoting the distance between the centres of the ith and the kth surface element, and with

$$\kappa_{ik}=\kappa_{ki}=\sqrt{\rho_i\rho_k}K_{ik} \tag{10}$$

and

$$K_{ik}=\frac{1}{\pi}\iint_{\Delta S,\Delta S'}\frac{\cos\vartheta\cos\vartheta'}{R^2}\ dS\ dS'. \tag{11}$$

Analytical expressions for the above integral are available for a few cases only, for instance, for pairs of rectangular, perpendicular elements. In general, however, it must to be solved numerically. Since usually both cosines show relatively large variations, the approximation

$$K_{ik}\approx\frac{\cos\vartheta_i\cos\vartheta_k}{\pi R_{ik}^2}\Delta S_i\Delta S_k \tag{12}$$

is admissible only for very small surface elements. An approximation which seems to be a compromise between Eqs. (11) and (12) has been given by Lewers.[5]

For the examples described in the following section (and also for the decay curves of Fig. 2), the coefficients K_{ik} for mutually perpendicular wall elements have been obtained analytically, while all other coefficients have been computed by numerical evaluation of the double (or rather quadruple) integral in Eq. (11).

IV. SOME RESULTS

As a first example we consider a cubical enclosure the boundary of which was subdivided into 96 equal square elements. Table I shows absorption exponents calculated with different methods for various distributions of the wall absorption. If a constant absorption coefficient of 1/6 is assumed for all walls (first column) the exact absorption exponent turns out to be about 5% lower than the Eyring value which in turn agrees with the absorption exponent after Gerlach and Mellert. In all other cases, both the values after Gerlach and Mellert and those by the present method differ in an unsystematic manner from each other; however, both are noticeably higher than the Sabine or Eyring absorption exponents. Note that the total absorption within the first and the second pair of absorption distributions is equal.

Some more results have been computed for a rectangular room with dimensions 1:2:3. To apply the present iteration scheme, the boundary of this enclosure has been subdivided into 88 equal square elements. If the absorption coefficient is

TABLE II. Absorption exponent $a-(4V/cS)\lambda$ of a rectangular room 1:2:3.

Method	Uniform absorption (ρ=8/11)	Absorption concentrated on one wall 3:2 ("floor")	3:1 ("long sidewall")	2:1 ("short sidewall")
Sabine ($a=1-\langle\rho\rangle$)	0.2727	0.2727	0.1364	0.0909
Eyring ($a=-\ln\langle\rho\rangle$)	0.3185	0.3185	0.1466	0.0953
Gerlach and Mellert	0.3181	0.4329	0.1457	0.0754
Present method (or Gilbert's)	0.2884	0.3643	0.1432	0.0739

constant over the whole boundary (first column of Table II) the results are similar as for the corresponding case in the cube. The most interesting case, however, is with the absorption concentrated onto one of the large walls ("floor," 2:3) since it is typical for an occupied hall with a flat floor. To deal with an extreme situation, this wall was assumed as totally absorbent (ρ=0) whereas the remaining ones are free of absorption (ρ=1) (Decay curves for this case are shown in Fig. 2). In this case (see column 3:2 of Table II) the correct absorption exponent lies about between that according to Gerlach and Mellert and that predicted by the Eyring formula. If the absorbing wall, however, is one of the long sidewalls (1:3), the differences between the results of different methods are not significant. And finally, if the absorption is concentrated on one of the short sidewalls (last column, 1:2), the Gerlach–Mellert result is close to that of the present iteration method which is smaller than the Eyring value by about 24%. It should be added that the correctness of results has been checked by comparing them to the outcome of the exact computation of decay curves by solving Eq. (2) and to Monte Carlo results.

To give an idea on the convergence of the iteration process it may be mentioned that computing the absorption exponents with an accuracy of 1% required between seven and ten iteration steps; about the same number of steps is needed with Gilbert's iteration method. For better illustration, Table III presents the absorption exponent for the rectangular room with an absorbing floor as it evolves in the iteration process. Of course, the accuracy of the result is not only determined by the number of iterations but also by the way in which the boundary is subdivided into surface elements. To check this influence, some calculations have been carried out for the rectangular enclosure with only 22 quadratic wall elements instead of 88. The results are listed in Table IV. The increase of accuracy by using smaller elements is evident, but it is not drastic. From this we may conclude that further increasing the number of elements would not improve the result sensibly and that therefore a number of about 100 elements should be sufficient to treat rooms of such simple geometry.

The reason for the disagreement between the predictions according to Sabine's and Eyring's formula, on the one hand, and the correct absorption exponents, on the other, can be explained by considering the illumination of the boundary during the exponential decay process, represented by the B values over the boundary. The following figures are to give an idea of the illumination. In the case of a totally absorbing floor (3:2), this area receives more energy per second and unit area than the remaining walls (Fig. 3); in particular, the energy flow toward the opposite side (the "ceiling") is much smaller. As a consequence, the energy absorption by the floor is much higher than it would be in a diffuse sound field. If, on the contrary, all absorption is concentrated on one of the short sidewalls (Fig. 4 and last column of Table II), this wall causes a depletion of energy in the right half of the enclosure; most of the energy is contained in the left half. Accordingly, the absorbing wall receives and absorbs less energy per second that it would in a diffuse sound field.

V. CONCLUSIONS

It has been shown that the described iteration scheme is an effective mean to exactly determine the decay constant of an enclosure. It should be added that it yields at the same time the illumination B of the boundary from which the sound intensity $\mathbf{I}$ in any point $\mathbf{r}''$ within the room can be computed by

$$\mathbf{I}(\mathbf{r}'')=\frac{1}{\pi}\int_S B(\mathbf{r})\rho(\mathbf{r})\exp\left(\frac{\lambda}{c}R'\right)\frac{\mathbf{R}'\cos\vartheta}{R'^3}\,dS. \quad (13)$$

Likewise, the energy density in $\mathbf{r}''$ is given by

TABLE III. Absorption exponent $a=(4V/cS)\lambda$ of a rectangular room 1:2:3 with totally absorbing floor, as a function of the number of iteration steps (88 wall elements).

Number of iterations	Absorption exponent
5	0.35791
6	0.36634
7	0.36369
8	0.36450
9	0.36425
10	0.36433
11	0.36430
12	0.36431

TABLE IV. Absorption exponent $a=(4V/cS)\lambda$ of a rectangular room 1:2:3 computed with 88 and 22 wall elements.

Location of totally absorbing surface	Absorption exponent computed with 88 elements	22 elements
3:2 (floor)	0.3643	0.3735
3:1 (long sidewall)	0.1432	0.1438
2:1 (short sidewall)	0.0739	0.0755

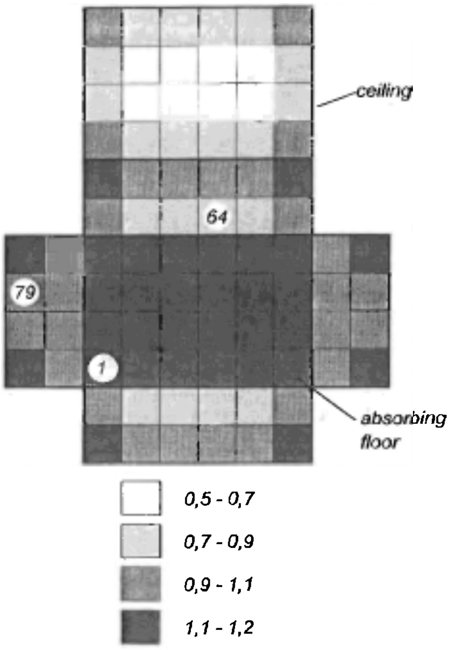

FIG. 3. Illumination of wall elements during the exponential sound decay in a rectangular enclosure (1:2:3) with totally absorbing floor, the remaining walls are free of absorption. The ranges of numbers refer to the quantity B normalized to unit average. (The numbers in circles indicate the receiving wall elements in Fig. 2.)

$$u(\mathbf{r}'')=\frac{1}{\pi c}\int_S B(\mathbf{r})\rho(\mathbf{r})\exp\left(\frac{\lambda}{c}R'\right)\frac{\cos\vartheta}{R'^2}\,dS. \tag{14}$$

In these expressions, $\mathbf{R}'=\mathbf{r}''-\mathbf{r}$ is the vector pointing from the reflecting boundary element toward the point $\mathbf{r}''$ and $R'=|\mathbf{R}'|$ is its magnitude; as before, ϑ is the angle between the wall normal $\mathbf{n}$ and the vector $\mathbf{R}'$.

In Sec. I we assumed that the sound energy reflected from the boundary is dispersed over all directions according

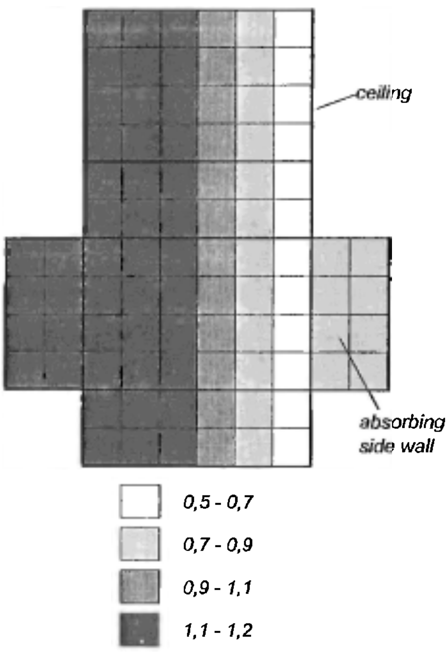

FIG. 4. As Fig. 3 but the absorbing wall is the short sidewall at right hand.

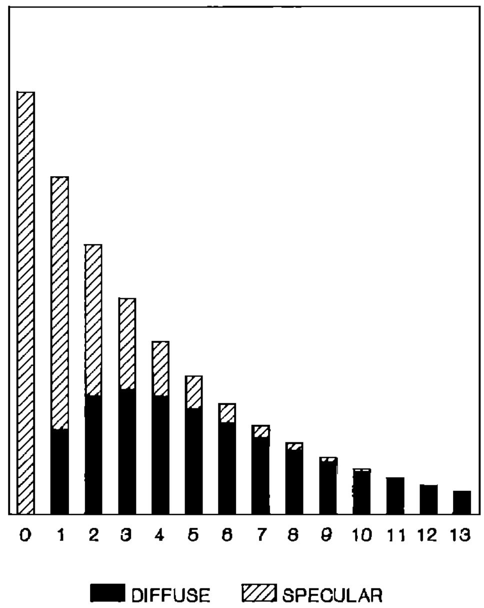

FIG. 5. Conversion of specularly into diffusely reflected sound energy, illustrated by an example. Hatched areas. specular energy; full areas. diffuse energy. The numbers denote the order of reflection.

to Lambert's cosine law. In practical situations, however, this will almost never occur. Instead, only a certain fraction d will be reflected in a diffuse manner while the remaining fraction $1-d$ will be specularly reflected. Therefore the initial part of decay curves calculated under the assumption of diffuse reflections is not too realistic. The result could be improved by employing a mixed specular–diffuse scattering model. Such a model has already been applied to a spherical room by Joyce.[3] For rectangular rooms, Baines[9] proposed to treat the diffusely reflected energy by solving the time-dependent integral equation (2), but to account for the specular component by applying the well-known image source model.

The exponential part of a decay curve, however, should not be very sensitive to the fact that usually only a part of the reflected energy is scattered into all directions. This is because this part is made up mainly of higher-order reflections, and because the conversion of specular sound energy into diffuse sound is irreversible. This means that in each reflection some of the energy of an incident sound ray is converted into nonspecular sound, but the reverse will never occur, i.e., the reflection of nonspecular energy will never result in the formation of a single sound ray. The conversion of geometrical into diffuse energy during subsequent reflections in equal time intervals is illustrated in Fig. 5, assuming that during one reflection 25% of the energy is scattered whereas 75% of it is specularly reflected; the boundary is supposed to have an uniform absorption coefficient of 0.2. It is evident that after a few reflections nearly all energy has undergone at least one diffuse reflection process.

From this we may conclude that our computation of decay constants is valid even if only a part of the reflected sound energy is dispersed into nonspecular directions. This conclusion is supported by earlier Monte Carlo simulations of sound decay, applied to polyhedral, in particular to rectangular rooms[10] according to which a relatively small amount of diffusion is sufficient to bring the decay curve close to that obtained with perfectly diffuse wall reflections. Similar results are reported in Joyce's paper mentioned above, although his example was a very special room shape which has a strong tendency to concentrate the specularly reflected energy into certain regions. On the other hand, there seems to be strong evidence that even untreated room walls produce diffuse reflections.[11] Therefore, it is believed that our present method is applicable to nearly all realistic situations.

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