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Frequency-Correlation Functions of Frequency Responses in Rooms

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A mathematical study of the random interference of sound waves in large rooms requires statistical methods. "Statistical wave acoustics" is based on the random interference of many simultaneously excited normal modes of a room. In general, the random interference takes place for frequencies above 2000 $(T_{60}/V)^{\frac{1}{2}}$, where T_{60} is the reverberation time (in sec) and V is the volume (in m^3) of the room. In the statistical theory, frequency responses between two points in a room are treated as random functions. The probability distributions, correlation functions, and "spectra" of these random functions are determined by physical parameters such as the distance between source and receiver, the volume and reverberation time of the room (or distribution of reverberation times), etc.

In this paper, correlation functions of frequency responses are derived for rooms with uniform reverberation time, and negligible direct-sound transmission between source and receiver. Analytic formulas for the following frequency-correlation functions are found: the autocorrelation functions of the real and imaginary parts, the modulus and the squared modulus of the frequency response, and the cross-correlation function between real and imaginary parts of the frequency response.

The significance of these correlation functions in room acoustics is discussed. Measurement of the autocorrelation function of the real (or imaginary) part of the frequency response allows a precise determination of the distribution of reverberation times. The autocorrelation function of the modulus (or squared modulus) determines the required frequency shift in public address systems to improve their stability. For measurement of electroacoustic transducers in reverberation chambers, optimum bandwidths of noise or warble tones are obtained.

INTRODUCTION

FREQUENCY responses in rooms between points with negligible direct-sound transmission are very irregular.[1] For large rooms, the average fluctuation between response maxima and minima is about 10 dB. Under certain conditions, the frequency responses of rooms can be considered random functions. Their probability distributions, frequency-correlation functions, and "spectra"[2] are determined by the geometrical and physical configurations such as distance between source and receiver, the volume of the room, the directivities of the source and receiver, the reverberation time (or distribution of reverberation times) of the room, etc. Frequency-correlation functions of frequency responses are particularly important, both from a theoretical and experimental point of view, because they can be more easily measured than probability distributions or "spectra." They also give a better insight into the characteristics of frequency responses than their "spectra." The "spectrum" of a frequency response (which itself is a kind of spectrum) is a rather unfamiliar concept to room acousticians.

The frequency-correlation functions are derived here under the following conditions:

(1) The normal modes of the room overlap each other at least 3:1; i.e., the average spacing of the resonance frequencies is smaller than one third of their bandwidth. This leads to the following inequalities between frequency, f (in cps), reverberation time T_{60}

[1] E. C. Wente, J. Acoust. Soc. Am. 7, 123 (1935).
[2] Quotation marks are used because "spectra" of frequency responses have time as the independent variable—not frequency.

(in sec), and room volume V (in m^3)[3]:

$$f>2000(T_{60}/V)^{\frac{1}{2}}, \tag{1}$$

or

$$V>4\times 10^6 T_{60}/f^2. \tag{1a}$$

Thus, for a reverberation time of 1 sec and frequencies above 100 cps, the results obtained here are applicable to rooms having volumes larger than 400 m^3, corresponding to an average linear dimension of about 7 m.

(2) The *directly* transmitted sound power between receiver and transmitter is much smaller (a third or less) than the power received via wall reflections. This leads to the following inequality for the distance r (in m) between source and receiver:

$$r>\frac{1}{4}\left(\frac{QS\bar{\alpha}}{1-\bar{\alpha}}\right)^{\frac{1}{2}}\approx\frac{1}{10}\left(\frac{QV}{T_{60}}\right)^{\frac{1}{2}}, \tag{2}$$

where Q is the directivity factor of the source (in the direction of the receiver), S is the room surface, and $\bar{\alpha}$ is the average energy-absorption coefficient. For $\bar{\alpha}\ll 1$, we may replace $1-\bar{\alpha}$ by 1 and express the total energy absorption $S\bar{\alpha}$ by the reverberation time T_{60} and the room volume V. With a sound velocity of 340 m/sec, we have

$$T_{60}\approx 0.16V/S\bar{\alpha},$$

and thus

$$r>0.1(QV/T_{60})^{\frac{1}{2}}. \tag{2a}$$

For a directivity factor $Q=1$, a reverberation time of 1 sec, and a room volume of 400 m^3, the distance between source and receiver must be larger than about 1.5 m.

(3) The decay rates of all normal modes in the frequency range of interest are identical. The reverberation time may, however, be *slowly* varying with frequency, provided that the correlation analyses are performed over frequency ranges for which the reverberation time is relatively constant.

Condition (1) is basic to statistical wave acoustics and cannot be removed without leaving the realm of probabilistic treatment.

Conditions (2) and (3) may be dropped, if desired, leading to more general (but also more complicated) formulas.

FREQUENCY-AUTOCORRELATION FUNCTIONS OF THE REAL AND IMAGINARY PARTS OF THE FREQUENCY RESPONSE

Let $p(f)$ be the complex steady-state frequency response between two points in a room:

$$p(f)=p_r(f)+ip_i(f), \tag{3}$$

where $p_r(f)$ and $p_i(f)$ are its real and imaginary parts, respectively. The inverse Fourier transform of $p(f)$ is the *impulse response* $P(t)$ between the two points. The inverse Fourier transform of the real part $p_r(f)$ is $P(|t|)/2$, the "power spectrum" is $P^2(|t|)/4$.

In statistical wave acoustics, the frequency responses in a room are considered a random process. The statistical ensemble consists of different frequency responses measured for many source and receiver positions. The ensemble average of $P^2(|t|)$ is

$$\langle P^2(|t|)\rangle = P_0^2 e^{-|t|/\tau}, \tag{4}$$

where τ is the time for which the sound energy in the room decays to $1/e$ of its initial value after impulsive excitation ($\tau=T_{60}/13.8$).

Since the autocorrelation function is the Fourier transform of its power spectrum,[4] we obtain[5] for the (normalized) frequency-autocorrelation function of $p_r(f)$

$$\rho_r(\Delta f)=1/[1+(2\pi\tau\Delta f)^2], \tag{5}$$

where Δf is the "frequency interval" (corresponding to the delay in autocorrelation functions of time waves). The bandwidth between the two points for which $\rho_r=\frac{1}{2}$ is

$$\Delta f_{\frac{1}{2}}=1/\pi\tau=4.4/T_{60}. \tag{6}$$

This is twice as wide as the -6-dB bandwidth of a single room mode ($\Delta f=2.2/T_{60}$). In fact, $\rho_r(\Delta f)$ has the same *shape* as the response (in terms of power) of a single room mode.

$\rho_r(\Delta f)$ can also be obtained by expanding $p_r(f)$ into contributions from the individual normal modes. The averaging process over different source and receiver locations cancels all but the autocorrelation of each normal-mode response with itself because the modes have random phases for different locations. Since the autocorrelation function of a single resonance has the same shape, but is twice as wide as its response (in terms of power), the above derived result for $\rho_r(\Delta f)$ is

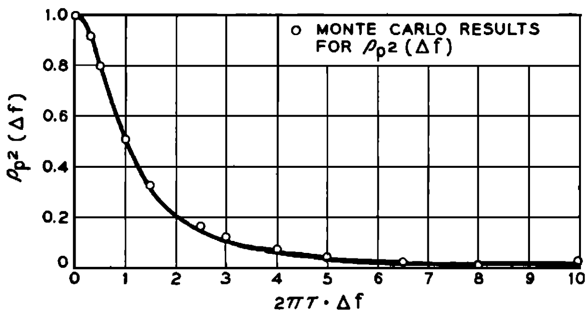

FIG. 1. Normalized frequency-autocorrelation function of the real and imaginary parts and squared modulus of the frequency response of a room with exponential reverberation. τ is the time interval in which the energy of the modes decays by a factor $1/e$ in a free decay. $\tau=T_{60}/13.8$. The circle indicates results of a Monte Carlo computation (see reference 6).

[3] M. R. Schroeder, Acustica 4, 594 (1954). The frequency given in this reference is 4000 $(T_{60}/V)^{\frac{1}{2}}$ corresponding to 10 overlapping normal modes. Measurements by various authors have shown that the theory is actually valid for frequencies as low as $2000(T_{60}/V)^{\frac{1}{2}}$.

[4] N. Wiener, *Extrapolation, Interpolation and Smoothing of Stationary Time Series* (John Wiley & Sons, Inc., New York, 1949), p. 42.

[5] *Tables of Integral Transforms,* edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1954), p. 14, formula 1.4.(1).

obtained. This derivation (only sketched here) affords a better intuitive understanding of why $\rho_r(\Delta f)$ is shaped like a single room resonance. $\rho_r(\Delta f)$ for positive Δf is shown in Fig. 1.

The inverse Fourier transform of the imaginary part $p_i(f)$ is $\operatorname{sgn}(t)\times P(|t|)$, where $\operatorname{sgn}(t)$ is the algebraic sign of t (+ for positive time, − for negative time). Thus, the "power spectrum" of $p_i(f)$ and its autocorrelation function $\rho_i(\Delta f)$ are the same as those for $p_r(f)$:

$$\rho_i(\Delta f)=\rho_r(\Delta f)=1/[1+(2\pi\tau\Delta f)^2]. \qquad (7)$$

FREQUENCY-AUTOCORRELATION FUNCTION OF THE SQUARED MODULUS OF THE FREQUENCY RESPONSE

The "power spectrum" of $|p(f)|^2$ has been derived in an earlier paper.[6] Except for a constant factor, the "power spectrum" of $|p(f)|^2-\bar{p}^2$ is identical to that of $p_r(f)$ given in Eq. (4). Here $\bar{p}^2$ is the mean-square modulus of the frequency response (averaged over frequency). Thus, the (normalized) frequency-correlation function $\rho_{p^2}(\Delta f)$ of $|p(f)|^2-\bar{p}^2$ is given by Eq. (5):

$$\rho_{p^2}(\Delta f)=1/[1+(2\pi\tau\Delta f)^2]. \qquad (8)$$

FREQUENCY CROSS-CORRELATION FUNCTIONS BETWEEN REAL AND IMAGINARY PARTS OF THE FREQUENCY RESPONSE

The imaginary part of the frequency response is the "Hilbert transform"[7] of its real part, a property shared by all passive linear filters.[8] Thus,

$$p_i(f)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{p_r(x)}{f-x}dx, \qquad (9)$$

where the Cauchy principal value of the integral is to be taken.

Multiplying Eq. (9) by $p_r(f-\Delta f)$ and averaging over frequency, we obtain the desired cross-correlation function

$$C_{ri}(\Delta f)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho_r(x-f+\Delta f)}{f-x}dx. \qquad (9a)$$

Substituting ρ_r from Eq. (5) into (9a) yields[9]

$$C_{ri}(\Delta f)=2\pi\tau\Delta f/[1+(2\pi\tau\Delta f)^2]. \qquad (10)$$

$C_{ri}(\Delta f)$ is shown in Fig. 2. It is antisymmetric around the origin ($\Delta f=0$), reaches a peak value of $\frac{1}{2}$ at $\Delta f=1/2\pi\tau=2.2/T_{60}$, and, for large Δf, approaches zero asymptotically as $1/(2\pi\tau\Delta f)$.

[6] M. R. Schroeder and H. Kuttruff, J. Acoust. Soc. Am. 34, 76 (1962).

[7] S. Seshu and N. Balabanian, *Linear Network Analysis* (John Wiley & Sons, Inc., New York, 1959), p. 265.

[8] Rooms are not usually thought of as "filters." If the transducers are linear, we can include even them and consider the input terminals of the loudspeaker the input to the filter, and the microphone-output terminals the output of the filter.

[9] H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (The Macmillan Company, New York, 1957), formula 136.

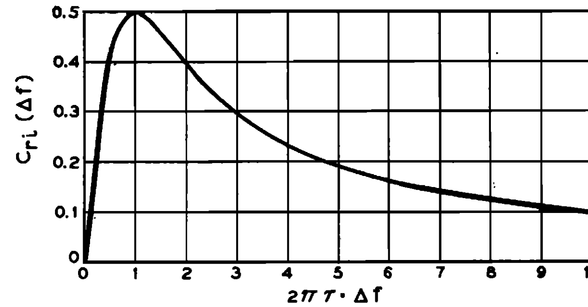

FIG. 2. Normalized frequency cross-correlation function between real and imaginary parts of the frequency response of a room with exponential reverberation. The cross correlation vanishes for zero-frequency interval and reaches a maximum at $\Delta f=1/2\pi\tau=2.2/T_{60}$, or about 2 cps for rooms with 1-sec reverberation time. For large frequency intervals the cross correlation is proportional to the reciprocal of the frequency interval.

FREQUENCY-AUTOCORRELATION FUNCTION OF THE MODULUS OF THE FREQUENCY RESPONSE

The autocorrelation function of the *modulus* of the frequency response is more difficult to derive than that of the *squared modulus*.

For the case considered here, the real and imaginary parts of the frequency response are approximately stationary Gaussian processes.[3] The modulus of a complex Gaussian process, where real and imaginary parts are related by a Hilbert transformation (as they are in our case), corresponds to the envelope of a real Gaussian process. Rice[10] has derived a formula for the joint distribution of two values of the *envelope* of a real stationary Gaussian process taken at a nonzero interval. If the spectral moments appearing in Rice's formula are computed for the exponential power spectrum [Eq. (4)], one obtains the following joint distribution:

$$P(p,p_\Delta)=\frac{4p\,p_\Delta}{\langle p^2\rangle^2}\left(1+\frac{1}{x^2}\right)I_0\left[\frac{2p\,p_\Delta}{\langle p^2\rangle}\frac{1}{x}\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}\right]$$
$$\times\exp\left[-(p^2+p_\Delta^2)\left(1+\frac{1}{x^2}\right)/\langle p^2\rangle\right], \qquad (11)$$

where $x=2\pi\tau\Delta f$ and $p=|p(f)|$ and $p_\Delta=|p(f+\Delta f)|$. The brackets $\langle\ \rangle$ denote an ensemble average. I_0 is the Bessel function of the first kind for imaginary argument.

If we assume the random process to be ergodic, we can make the usual substitution of ensemble averages for frequency averages. The ensemble average $p\,p_\Delta$ is defined by the following integral:

$$\langle p\,p_\Delta\rangle=\int_0^\infty\int_0^\infty p\,p_\Delta P(p,p_\Delta)dp\,dp_\Delta. \qquad (12)$$

The integral can be evaluated by expanding I_0 into a

[10] S. O. Rice, "Mathematical Analysis of Random Noise" in *Noise and Stocastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954), p. 216, formula 3.7-13.

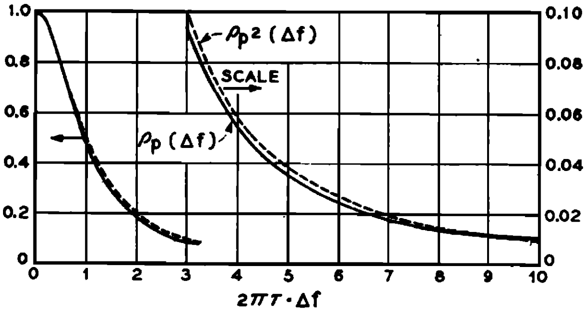

FIG. 3. Normalized frequency-autocorrelation function of the modulus of a frequency response in a room with exponential reverberation. For abscissa values greater than 3.0, the ordinate is magnified 10 times. For comparison the autocorrelation function of the squared modulus has been included in this figure (dashed curve). The close resemblance of the two autocorrelation functions is noteworthy.

power series[11] and integrating term by term. The result is

$$\langle p\,p_\Delta\rangle=\frac{\pi}{4}\langle p^2\rangle\left(\frac{x^2}{1+x^2}\right)^2\sum_{m=0}^{\infty}\frac{1}{(1+x^2)^m}\frac{[(2m+1)!]^2}{2^{4m}[m!]^4}. \tag{13}$$

Now we substitute $1/(1+x^2)=z^2$ and write

$$\frac{(2m+1)!}{2^{2m}m!}=\tfrac{3}{2}\,\tfrac{5}{2}\cdots\frac{2m+1}{2}=(\tfrac{3}{2})_m \quad\text{and}\quad m!=(1)_m.$$

With this notation we have

$$\langle p\,p_\Delta\rangle=\frac{\pi}{4}\langle p^2\rangle(1-z^2)^2\sum_{m=0}^{\infty}\frac{z^{2m}}{(1)_m}\frac{(\frac{3}{2})_m(\frac{3}{2})_m}{(1)_m},$$

where the sum is a hypergeometric function:

$$\langle p\,p_\Delta\rangle=\tfrac{1}{4}\pi\langle p^2\rangle(1-z^2)^2{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};1;z^2).$$

With the transformation formula[12]

$${}_2F_1(\alpha,\beta;\gamma;z^2)=(1-z^2)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\beta,\gamma-\alpha;1;z^2),$$

we have

$$\langle p\,p_\Delta\rangle=\tfrac{1}{4}\pi\langle p^2\rangle{}_2F_1(-\tfrac{1}{2},-\tfrac{1}{2};1;z^2). \tag{14}$$

With another transformation formula,[13] this may be written as

$$\langle p\,p_\Delta\rangle=\tfrac{1}{4}\pi\langle p^2\rangle(1+z){}_2F_1[-\tfrac{1}{2},\tfrac{1}{2};1;4z/(1+z)^2]. \tag{15}$$

Now the hypergeometric function is in the form of a complete elliptic integral of the second kind. With the relation[14]

$$E(k)=\tfrac{1}{4}\pi F(-\tfrac{1}{2},\tfrac{1}{2};1;k^2),$$

we have

$$\langle p\,p_\Delta\rangle=\tfrac{1}{2}\langle p^2\rangle(1+z)E[2z^{\frac{1}{2}}/(1+z)]. \tag{16}$$

The *normalized* autocorrelation function of $p-\langle p\rangle$, where $\langle p\rangle$ is the average modulus of the frequency response, is given by

$$\rho_p(\Delta f)=\frac{\langle p\,p_\Delta\rangle-\langle p\rangle^2}{\langle p^2\rangle-\langle p\rangle^2}. \tag{17}$$

Because of the Rayleigh distribution[15] of p, we have

$$\langle p\rangle^2=\tfrac{1}{4}\pi\langle p^2\rangle. \tag{18}$$

With this value for $\langle p\rangle^2$ substituted into Eq. (17), we obtain

$$\rho_p(\Delta f)=\frac{4}{4-\pi}\left[\frac{\langle p\,p_\Delta\rangle}{\langle p^2\rangle}-\frac{\pi}{4}\right]. \tag{19}$$

With $\langle p\,p_\Delta\rangle$ from Eq. (16), we have the final result:

$$\rho_p(\Delta f)=\frac{1}{4-\pi}\left[2(1+z)E\left(\frac{2z^{\frac{1}{2}}}{1+z}\right)-\pi\right], \tag{20}$$

where $z=[1+(2\pi\tau\Delta f)^2]^{-\frac{1}{2}}$.

$\rho_p(\Delta f)$ is plotted in Fig. 3 as a function of $2\pi\tau\Delta f$ (solid curve). For $2\pi\tau\Delta f>3$, the ordinate has been magnified ten times so as to show ρ_p for large Δf in greater detail. For comparison, the autocorrelation function of the squared modulus has also been plotted in Fig. 3 (dashed curve). The near identity of ρ_p and ρ_{p^2} means that the normalized autocorrelation function does not change much when one goes from the modulus to the squared modulus of the frequency response. This implies, at least in the present case, that the autocorrelation function is a measure of correlation along the frequency axis largely independent of whether modulus, squared modulus (and perhaps other monotonic functions of the modulus) are considered.

DISCUSSION AND APPLICATIONS

Autocorrelation and cross-correlation functions are among the most important tools in the realm of random processes. The random responses obtained in rooms, as a result of the statistical interference of many sound waves (or many normal modes), are no exception. Apart from specific applications, frequency autocorrelation and cross-correlation functions provide a deeper insight and better understanding of the random-wave interference phenomena occurring in rooms. In the following, some specific applications of frequency-autocorrelation functions in room acoustics are discussed. The examples given by no means exhaust the possible uses.

Figures 1 and 3 show that the normalized autocorrelation functions of the modulus and the squared modulus of the frequency response decrease to about 0.1 for frequency intervals $\Delta f=3/2\pi\tau=6.6/T_{60}$. This means that for rooms with a reverberation time of 1 sec, the frequency response has substantially independent values at two frequencies separated by more than 6 cps.

[11] I. M. Ryzhik and I. S. Gradstein, *Tables of Series, Products, and Integrals* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1959), p. 320, formula 6.457.1.
[12] Reference 11, p. 386, formula 7.231.1.
[13] Reference 11, p. 387, formula 7.234.3.
[14] Reference 11, p. 271, formula 6.114.1.
[15] P. E. Doak, Acustica 9, 1 (1959).

Suppose, for example, that frequency-response measurements are made on a loudspeaker in a reverberation chamber and these measurements are made at discrete frequencies. Then, the above analysis indicates that a frequency spacing of about $6/T_{60}$ should be used for maximum measuring efficiency. A smaller frequency spacing does not give significantly more information. A larger frequency spacing gives fewer measuring points and a reduced accuracy.

If warble tones or noise bands are to be used in order to smooth out the irregularities of the chamber's frequency response, the autocorrelation functions can be used to predict the necessary bandwidth of these signals. For substantial smoothing, noise bands or warble tones with a bandwidth of approximately $20/T_{60}$ are a good compromise for optimum smoothing without losing too much spectral resolution.

In an earlier paper[16] on improvement of acoustic feedback stability of public address systems in rooms, a frequency shift of $\Delta f=4/T_{60}$ (corresponding to $2\pi\tau\Delta f=1.8$) was recommended as sufficient for a substantial increase in stability. Figures 1 and 3 show that the autocorrelation function for such a frequency shift is approximately $\frac{1}{5}$ of the value for no shift. Thus, the individual frequency components of the audio signal encounter substantially new values of the frequency response for every trip around the feedback loop. Figures 1 and 3 also show that for frequency shifts $\Delta f>\frac{3}{2}\pi\tau=6.6/T_{60}$ the autocorrelation function decreases rather slowly. Therefore, only moderate increases of feedback stability can be expected for the frequency shifts exceeding 6.6 cps for rooms with 1-sec reverberation time. This agrees well with measured feedback stabilities for various frequency shifts.[16]

If different normal modes have different "reverberation times" τ_k, it can be shown that the autocorrelation function of $p_r(f)$ is as follows:

$$\rho_r(\Delta f)=\sum_{k=1}^{K}\frac{a_k}{[1+(2\pi\tau_k\Delta f)^2]}, \tag{21}$$

where a_k is the relative contribution of the modes with "reverberation time" τ_k to the entire reverberation process. The number of terms K equals the number of distinct reverberation times.

Equation (21) can be used to measure the different "reverberation times" τ_k and their relative weights a_k. The τ_k and a_k are determined by least-square fitting of Eq. (21) to the measured autocorrelation function $\rho_r(\Delta f)$.

This method may yield more accurate results than the fitting of straight lines to a logarithmic decay curve, especially if the autocorrelation and the least-square approximation are performed by a digital computer. With the increasing availability of digital computers and analog-to-digital converters, computer processing of room acoustical data may be within the reach of many room acousticians in the future.

The measurement of correlation functions by analog equipment presents some difficulty because four-quadrant multipliers are required. (To avoid this difficulty, the real and imaginary parts of the frequency response may be fed into a digital computer for processing, especially if high accuracy and further computations are required.) However, an interesting relation[17] between the normalized autocorrelation function of a Gaussian process and that of its algebraic sign allows one to compute the autocorrelation function from measurements on the "infinitely clipped" process. This relation is:

$$\rho(\Delta f)=\sin[\tfrac{1}{2}\pi\hat{\rho}(\Delta f)], \tag{22}$$

where $\rho(\Delta f)$ is the autocorrelation function of the Gaussian process and $\hat{\rho}(\Delta f)$ that of its sign. Since $p_r(f)$ and $p_i(f)$ are Gaussian processes, Eq. (22) may be applied to determine their autocorrelation functions by measuring that of their signs. To obtain $p_r(f)$, the microphone signal is "chopped" by the loudspeaker signal and low-pass filtered. Then $p_r(f)$ is clipped, and multiplied by the clipped $p_r(f+\Delta f)$. The product is averaged over the frequency interval of interest. From the resulting $\hat{\rho}(\Delta f)$, $\rho(\Delta f)$ is computed according to Eq. (22). The chopping, clipping, and multiplying operations can be performed by simple transistor circuits.

In this paper, the discussion is limited to *frequency* correlation functions. *Space* correlation functions of sound fields in rooms are equally, if not more, important because they give insight into the spatial diffusion of sound fields and allow accurate measurement of the diffusion.

Further important applications lie outside acoustics in fields in which random interference of waves occurs. A well-known case is multipath propagation of electromagnetic waves. A remedy against multipath distortion is "frequency diversity," i.e., the use of several simultaneous frequency bands for the transmission of intelligence. The frequency correlation of the prevailing transmission function enables one to select an optimum set of frequency bands.

In general, wherever waves of any nature (acoustic, electromagnetic, quantum-mechanical) interfere randomly, the method of treating their frequency and space responses as random functions should prove useful.

ACKNOWLEDGMENTS

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[16] M. R. Schroeder, in *Proceedings of the 3rd International Congress on Acoustics*, edited by L. Cremer (Elsevier Publishing Company, Amsterdam, 1961), Vol. II, p. 771.

[17] J. A. McFadden, IRE Trans. Inform. Theory **IT-2**, 146 (1956).