A MODIFIED FORMULA FOR REVERBERATION

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I. GENERAL INTRODUCTION

A paper has been published by Eyring[1] in which he gives as a modification of Sabine's well-known reverberation formula namely:

$$T = \frac{0.05V}{\alpha S} \tag{1}$$

the following formula:

$$T = -\frac{0.05V}{S \log_e (1 - \alpha)} \tag{2}$$

where $\alpha = \Sigma\alpha_1 S_1/S$, ft-sec. units being used throughout.

This formula is designed to deal with the case of a heavily damped room where $\alpha \rightarrow 1$, for which Sabine's formula obviously breaks down as it makes T approximate to 0.05 V/S instead of to zero. Sabine's formula is seen to be an approximation to Eyring's formula for small values of α.

In some recent work on reverberation measurements on the absorption coefficients of highly absorbing materials anomalous results have been obtained, Eyring's formula giving coefficients greater than unity for materials for which values of about 0.7 have been obtained by other workers. This suggested that under the conditions used the formula is not accurate as the anomalies were too large to be attributed to experimental error. The theory of reverberation has therefore been developed afresh from first principles, and it appears that under certain conditions the formula takes a modified form.

II. STATEMENT OF THE MODIFIED FORMULA

Eyring's formula can be worked out on the assumption that at each reflection the sound energy in the room is reduced to $(1-\alpha)$ times its previous value where α is the average absorption coefficient. This will be true if at each reflection the energy reflected from each of the areas S_1, S_2, etc., spreads by scattering or by the divergence of the wave so that at the next reflection it is distributed over the remaining areas. If, how-

[1] Eyring, Bell System Tech. Jour., April (1930).

ever, the areas are large enough to act as plane reflectors, any incident narrow cone of rays will be reflected as a confined cone and the energy will not be distributed over the remaining surfaces, but will be incident on some particular one. It will therefore suffer the absorption associated with that surface. Instead of the energy being absorbed the same amount at each reflection it will be absorbed the amount corresponding to the surface involved.

It is shown in section VI that this line of reasoning leads to a new formula which may be stated as follows:

$$T = -\frac{0.05V}{\sum S_1 \log_e (1 - \alpha_1)}. \tag{3}$$

This corresponds to the ideal case when the areas S_1, S_2, etc. have dimensions several times the wave-length of the sound, and are moreover large enough to make edge effects and effects due to the spreading by divergence of the waves negligible. Also each of the ratios S_1/S, S_2/S, etc., must be large enough for the statistical methods used to apply.

In practical cases it is difficult to assess the weight which should be given to the various factors involved, but the formula corresponds to the extreme case towards which conditions may approximate. In a typical studio the walls are rectangular and of large area, and large areas of damping are used in the form of curtains, carpets, and panels. Thus in spite of obstacles present such as chairs and instruments the Eyring formula should need some modification in the direction indicated. In a reverberation room such as the one in use, the modification may need to be almost complete.

III. Application to Reverberation Rooms in General

Assuming that the modified formula can be applied to a reverberation room, the analysis may be stated as follows:

Let T be the reverberation time in the bare room, then

$$\sum S_1 \log_e (1 - \alpha_1) = -\frac{0.05V}{T}. \tag{4}$$

If the total area is S and the absorption is uniform and has a coefficient α this gives

$$S \log_e (1 - \alpha) = -\frac{0.05V}{T}. \tag{5}$$

Suppose that an area S_p of damping coefficient α_p is hung on the wall or placed on the floor where the coefficient is α_r and that T_p is the new reverberation time, we now have:

$$\sum S_1 \log_e (1-\alpha_1) + S_p[\log_e (1-\alpha_p) - \log_e (1-\alpha_r)] = -\frac{0.05V}{T_p}\cdot \tag{6}$$

If α_r is known we can solve (4) and (6) for α_p. If it is small compared with α_p and we assume that it is approximately equal to α then

$$\sum S_1 \log_e (1-\alpha_1) + S_p[\log_e (1-\alpha_p) - \log_e (1-\alpha)] = -\frac{0.05V}{T_p} \tag{7}$$

and therefore from (4) and (7)

$$S_p[\log_e (1-\alpha_p) - \log_e (1-\alpha)] = -\frac{0.05V}{T_p} + \frac{0.05V}{T}$$

which may be written

$$\log_e\left[1-\left(\frac{\alpha_p-\alpha}{1-\alpha}\right)\right] = -\frac{0.05V}{S_p}\left[\frac{1}{T_p}-\frac{1}{T}\right]. \tag{8}$$

We may notice from this formula that however small T_p may be $(\alpha_p-\alpha)/(1-\alpha)$ is >0 and <1 and therefore α_p is <1, whereas Eyring's formula can give values greater than unity. The case where T_p is $>T$ corresponds as it should do to $\alpha_p<\alpha$.

A very important point arises when $\alpha_p=1$ since even when S_p is very small the formula gives $T_p=0$. Of course this cannot be so in practice, and the theory breaks down simply because the statistical methods used cannot be applied to the cases of very short decays and of very small surfaces, even assuming the latter to remain plane reflectors. But we can interpret the result by saying that a small area of heavy absorbent can reduce the reverberation time of a room to a very much smaller value than is generally recognised.

For the bare room Eyring's formula also gives Eq. (5), but Eq. (7) is replaced by one of the form:

$$S \log_e (1-\alpha') = -\frac{0.05V}{T_p}$$

where α' is a new average coefficient given by

$$\alpha' = \frac{\alpha_p S_p + (S-S_p)\alpha}{S}$$

and these equations with Eq. (5) give

$$\log_e\left[1-\frac{S_p}{S}\frac{(\alpha_p-\alpha)}{(1-\alpha)}\right] = -\frac{0.05V}{S}\left[\frac{1}{T_p}-\frac{1}{T}\right] \tag{9}$$

for comparison with (8), and converting to common logarithms we have instead of (8)

$$\log_{10}\left[1-\left\{\frac{\alpha_p-\alpha}{1-\alpha}\right\}\right]=-\frac{0.0217V}{S_p}\left[\frac{1}{T_p}-\frac{1}{T}\right] \tag{10}$$

and instead of (9)

$$\log_{10}\left[1-\frac{S_p}{S}\left\{\frac{\alpha_p-\alpha}{1-\alpha}\right\}\right]=-\frac{0.0217V}{S}\left[\frac{1}{T_p}-\frac{1}{T}\right]. \tag{11}$$

IV. APPLICATION TO THE REVERBERATION ROOM IN USE

A series of measurements was made upon a sample of highly absorbent material in a room having the following constants:

Volume $V=1933$ cu. ft. Total surface $S=932$ sq. ft.

The average absorption α is of the order of 0.02 to 0.03. Some of this absorption takes place at cracks round the door and window, but in most cases it is accurate enough to assume that α_r, the coefficient of the wall or floor to be covered by damping, is equal to α. This error may be serious when materials with very small coefficients are tested, but obviously special precautions must be taken for such cases to use a very reverberant room or to make α_r small in comparison with α_p e.g., by using a rigid metal surface over which to place the damping. In a particular experiment an area $S=27.8$ sq. ft. of absorbent was used, and the reverberation time at 1000 cycles per second was $T_p=1.80$ sec. The corresponding bare room time was $T=3.77$ sec.

The coefficient α may be obtained to within experimental error by Sabine's formula, i.e.,

$$\alpha=\frac{0.05V}{ST}=\frac{0.05\times1933}{932\times3.77}=0.027$$

whence $1-\alpha=0.973$.

Eyring's formula (11) gives:

$$\log_{10}\left[1-\frac{27.8}{932}\frac{(\alpha_p-\alpha)}{(1-\alpha)}\right]=-\frac{0.0217\times1933}{932}\left[\frac{1}{1.80}-\frac{1}{3.77}\right]$$

whence

$$1-\frac{27.8}{932}\frac{(\alpha_p-\alpha)}{(1-\alpha)}=0.9702$$

$$\therefore\quad\frac{\alpha_p-\alpha}{1-\alpha}=0.0298\times\frac{932}{27.8}=1.00$$

whence

$$\alpha_{p\text{Eyring}} = 1.00.$$

The modified formula (10) gives

$$\log_{10}\left[1 - \frac{(\alpha_p - \alpha)}{(1 - \alpha)}\right] = -\frac{0.0217 \times 1933}{27.8}\left[\frac{1}{1.80} - \frac{1}{3.77}\right]$$

$$\therefore \quad 1 - \frac{(\alpha_p - \alpha)}{(1 - \alpha)} = 0.363$$

$$\therefore \quad \alpha_{p\text{Modified}} = 0.65.$$

It is thus clear that whereas the value of absorption coefficient calculated according to Eyring is impossibly high, that calculated by the modified formula is reasonable. It is also in close accord with that which was expected for the material in question.

TABLE I.

Freq. cycles	T_p secs.	$\log_{10}(1-\alpha_p)$	$1-\alpha_p$	α_p	α_pEyring
120	0.73	$-0.260=\bar{1}.740$	0.550	0.45	0.49
240	0.49	$-0.388=\bar{1}.612$	0.409	0.59	0.65
500	0.38	$-0.500=\bar{1}.500$	0.316	0.68	0.77
1000	0.34	$-0.559=\bar{1}.441$	0.276	0.72	0.82
2000	0.34	$-0.559=\bar{1}.441$	0.276	0.72	0.82
4000	0.32	$-0.594=\bar{1}.406$	0.255	0.74	0.86

V. RECALCULATION OF EYRING'S RESULTS

It is interesting in the light of these results to recalculate on the modified formula some figures given in Eyring's paper.

The following data are given:

Volume of room	= 73,475 cu. ft.	V
Total area of the walls	= 11,553 sq. ft.	S
Area of insulating material under test	= 8401 sq. ft.	S_p

The bare walls made of concrete, glass, etc. were considered to be perfectly reflecting, i.e., T was taken as infinite.

The results calculated on the modified formula are given in Table I with Eyring's values for comparison. The coefficient was also found for the insulating material with Monk's cloth hung in front of it, and an area of 1670 sq. ft. was used with 6887 sq. ft. uncovered.

TABLE II.

Freq. (cycles)	T_p (secs.)	$S_1 \log_{10}(1-\alpha_1)$ $+S_p \log_{10}(1-\alpha_p)$	$S_1 \log_{10}(1-\alpha_1)$	$S_p \log_{10}(1-\alpha_p)$	$\log_{10}(1-\alpha_p)$	$1-\alpha_p$	α_p	$\alpha_{p\text{Eyring}}$
120	0.52	−3060	−1790	−1270	$-0.761=\bar{1}.239$	0.173	0.83	1.10
240	0.40	−3980	−2680	−1300	$-0.779=\bar{1}.221$	0.166	0.83	1.03
500	0.32	−4980	−3440	−1540	$-0.923=\bar{1}.077$	0.120	0.88	1.00
1000	0.31	−5140	−3850	−1290	$-0.773=\bar{1}.227$	0.169	0.83	1.01
2000	0.28	−5690	−3850	−1840	$-1.110=\bar{2}.890$	0.0776	0.92	1.13
4000	0.26	−6130	−4090	−2040	$-1.220=\bar{2}.780$	0.0603	0.94	1.27

If S_1 represents this uncovered area, and α_1 the coefficient as found from Table I, and if S_p is the area covered, and α_p the coefficient of the composite damping, the value of $S_1 \log_{10}(1-\alpha_1)+S_p \log_{10}(1-\alpha_p)$ is found by direct application of the modified formula, and $S_1 \log_{10}(1-\alpha_1)$ is obtained from the third column of the table. The results of Eyring are again given for comparison in Table II.

Eyring attributes the fact that his values are greater than unity to large experimental errors in measuring very short times. It seems, however, more rational to explain them as due to the inadequacy of his formula, especially as it follows from first principles that the modified formula is more applicable the larger the actual areas of damping used and the greater the ratio of the areas to the total area of the walls. Both of these conditions are much more complied with in Eyring's case than in our own where we have seen that his formula is definitely inadequate.

VI. THE PROOF OF THE MODIFIED FORMULA

As above we consider the walls (including the floor and roof) to have a total area S made up of a number of areas $S_1, S_2 \cdots S_n$ with corresponding absorption coefficients $\alpha_1, \alpha_2 \cdots \alpha_n$. For an average shape of room there will be a mean free path p which is given by the relation $p=4V/S$, and considering any particular ray it will make vt/p reflections in a time t secs. The simple ray theory can be applied provided each of the areas S_1, S_2 etc., is large enough to act as a reflector of the sound and edge effects are negligible. Provided that none of the ratios $S_1/S, S_2/S$, etc., is very small, then of the vt/p reflections a fraction S_1/S will take place on the surface S_1. If S_1/S is equal to $1/n$, then n must be sufficiently small in comparison with the number of reflections vt/p for the probability that S_1/S of them will take place on S_1 to be high enough for statistical methods to be applied. Similarly a fraction S_2/S of the reflections will take place on the surface S_2 and so on for S_3, etc. At each reflection from the surface S_1 the energy reflected is $(1-\alpha_1)$ times the incident energy, and is $(1-\alpha_2)$ for the surface S_2 and so on. If the initial energy is E_0 then the energy remaining at the time t sec. later after vt/p reflections is

$$E_t = E_0(1-\alpha_1)^{(S_1/S)(vt/p)}(1-\alpha_2)^{(S_2/S)(vt/p)} \cdots (1-\alpha_n)^{(S_n/S)(vt/p)}$$

and it does not matter in which order the reflections have occurred. The expression for E_t may be written in exponential form thus:

$$\begin{aligned} E_t &= E_0 e^{(S_1/S)(vt/p)\log_e(1-\alpha_1)} e^{(S_2/S)(vt/p)\log_e(1-\alpha_2)} \cdots e^{(S_n/S)(vt/p)\log_e(1-\alpha_n)} \\ &= E_0 e^{(S_1/S)(vt/p)\log_e(1-\alpha_1)+(S_2/S)(vt/p)\log_e(1-\alpha_2)+\cdots+S_n/S(vt/p)\log_e(1-\alpha_n)} \\ &= E_0 e^{-kt} \end{aligned}$$

where

$$\begin{aligned} k &= -\frac{v}{pS}\sum S_1 \log_e(1-\alpha_1) \\ &= -\frac{v}{4V}\sum S_1 \log_e(1-\alpha_1) \text{ since } p = \frac{4V}{S}\cdot \end{aligned}$$

If T is the reverberation time

$$\begin{aligned} e^{kT} &= 10^6 \\ \therefore \quad kT &= 6\log_e 10 = 13.82 \end{aligned}$$

and

$$\begin{aligned} T &= \frac{13.82}{k} \\ &= -\frac{55.2V}{v\sum S_1 \log_e(1-\alpha_1)} \end{aligned}$$

putting $v = 1100$ ft. per sec. gives

$$T = -\frac{0.05V}{\sum S_1 \log_e(1-\alpha_1)}$$

which is the modified formula which has been given above.

Eyring's formula may be derived by putting $\alpha_1=\alpha_2=\cdots=\alpha$ all through the argument, but this would assume that all the surfaces had the average coefficient α and would moreover give no justification for assuming that $\alpha=\Sigma\alpha_1 S_1/S$ rather than a value worked out on any other system of averaging. A better method of deriving the formula and of showing the true nature of the assumptions made is as follows: Suppose that at any moment during the decay the energy in the room is E and that it is distributed evenly so that at the next reflection an amount S_1E/S is incident upon the surface S_1 and so on, the amount of energy reflected from S_1 will be $(S_1/S)E(1-\alpha_1)$ and from S_2 will be $(S_2/S)E(1-\alpha_2)$ etc.

The total remaining energy will be given by

$$\frac{S_1}{S}E(1-\alpha_1)+\frac{S_2}{S}E(1-\alpha_2)+\cdots+\frac{S_n}{S}E(1-\alpha_n)$$

$$=\frac{E}{S}[S_1+S_2+\cdots+S_n-(\alpha_1S_1+\alpha_2S_2+\cdots+\alpha_nS_n)]$$

$$=\frac{E}{S}[S-\alpha S]=E(1-\alpha)$$

where α is defined as $\Sigma\alpha_1S_1/S$. Now if this remaining energy is also evenly distributed, after the next reflection the energy remaining will be $E(1-\alpha)^2$ and thus in a time t the energy becomes $E(1-\alpha)^{vt/p}$ or Ee^{-kt} where

$$k=-\frac{v}{p}\log_e(1-\alpha)=-\frac{vS}{4V}\cdot\log_e(1-\alpha)$$

whence

$$T=-\frac{0.05V}{S\log_e(1-\alpha)}$$

which is Eyring's formula. This method of proving it shows that account is taken of the particular coefficient corresponding to the surface on which the energy happens to be incident; the fundamental difference between the two formulae is that the modified formula considers the energy reflected in a series of confined cones and traces the history of any particular cone of energy, whereas the Eyring formula considers the energy as uniformly spread out after each reflection. The truth will in general lie somewhere between these two points of view, and the situation has been stated in section II after the formal statement of the modified formula.

VII. GENERAL CONCLUSIONS FROM THE THEORY

Before proceeding to the general conclusions to be deduced from the theory it is convenient to prove two theorems.

Theorem I.—If we write $\Sigma S_1 \log_e (1-\alpha_1) = -A$ and $S \log_e (1-\alpha) = -B$ where A and B are both positive, then A is always greater than B. For we have:

$$\begin{aligned} A - B &= - \sum S_1 \log_e (1-\alpha_1) + S \log_e (1-\alpha) \\ &= - \sum S_1 \log_e (1-\alpha_1) + \sum S_1 \log_e (1-\alpha) \\ &= - \sum S_1 [\log_e (1-\alpha_1) - \log_e (1-\alpha)] \\ &= - \sum S_1 \log_e \frac{1-\alpha_1}{1-\alpha} \\ &= - \sum S_1 \log_e \left(1 + \frac{\alpha-\alpha_1}{1-\alpha}\right) \end{aligned}$$

Now if $\alpha_1 < \alpha$ $(\alpha-\alpha_1)/(1-\alpha)$ is positive, and $\log_e (1+(\alpha-\alpha_1)/(1-\alpha))$ is positive but $<(\alpha-\alpha_1)/(1-\alpha)$ and may be written $(\alpha-\alpha_1)/(1-\alpha)-\epsilon_1$ where ϵ_1 is positive. But if $\alpha_1 > \alpha$ $(\alpha-\alpha_1)/(1-\alpha)$ is negative, and $\log_e (1+(\alpha-\alpha_1)/(1-\alpha))$ may be written $(\alpha-\alpha_1)/(1-\alpha)-\epsilon_1'$ where ϵ_1' is positive.

These results follow from the fact that if x is positive the numerical value of $\log_e (1+x)$ is $<x$ while the numerical value of $\log_e (1-x)$ is $>x$.

Thus in both cases we may write

$$\log_e \left(1 + \frac{\alpha-\alpha_1}{1-\alpha}\right) = \frac{\alpha-\alpha_1}{1-\alpha} - \epsilon_1, \text{ where } \epsilon_1 \text{ is positive.}$$

Thus

$$\begin{aligned} A - B &= - \sum S_1 \left[\frac{\alpha-\alpha_1}{1-\alpha} - \epsilon_1\right] \\ &= \sum \left[\frac{\alpha_1 S_1 - \alpha S_1}{1-\alpha} + \epsilon_1 S_1\right] \\ &= \frac{\sum \alpha_1 S_1 - \alpha \sum S_1}{1-\alpha} + \sum \epsilon_1 S_1 \end{aligned}$$

But $\alpha \Sigma S_1 = \alpha S = \Sigma \alpha_1 S_1$

$\therefore A - B = \Sigma \epsilon_1 S_1$ which is positive and under all conditions $A > B$ except when $\alpha_1 = \alpha_2 = \cdots = \alpha$ and $A = B$.

Theorem II. If in the formulae (10) and (11) of section III the coefficient of absorption of a sample is α_M when worked out on the modified formula, and is α_E when worked out on Eyring's formula then: α_E is approximately equal to α_M when S_p/S is nearly unity and α_E is approximately equal to $\alpha_M+\alpha_M{}^2/2+\alpha_M{}^3/3+$—— when S_p/S is very small compared with unity.

From formulae (10) and (11) it follows that:

$$S_p \log_{10}\left(1-\frac{\alpha_M-\alpha}{1-\alpha}\right)=S\log_{10}\left(1-\frac{S_p}{S}\frac{(\alpha_E-\alpha)}{(1-\alpha)}\right)$$

$$\therefore \quad \left[1-\frac{\alpha_M-\alpha}{1-\alpha}\right]^{S_p/S}=1-\frac{S_p}{S}\cdot\frac{\alpha_E-\alpha}{1-\alpha}.$$

Writing

$$\frac{\alpha_M-\alpha}{1-\alpha}=\alpha_M' \text{ and } \frac{\alpha_E-\alpha}{1-\alpha}=\alpha_E',$$

i.e., α_M' and α_E' are the uncorrected values neglecting the absorption of the bare room, we have:

$$(1-\alpha_M')^{S_p/S}=1-\frac{S_p}{S}\alpha_E'$$

$$\therefore \quad 1-\frac{S_p}{S}\alpha_M'=1-\frac{S_p}{S}\alpha_M'+\frac{S_p}{S}\left(\frac{S_p}{S}-1\right)\frac{\alpha_M'^2}{2}-\text{etc.}$$

$$\therefore \qquad \alpha_E'=\alpha_M'+\left(1-\frac{S_p}{S}\right)\frac{\alpha_M'^2}{2}+\left(1-\frac{S_p}{S}\right)\left(2-\frac{S_p}{S}\right)\frac{\alpha_M'^3}{3}+\cdots$$

Now if S_p/S is nearly unity $\alpha_E'=\alpha_M'$ and if $S_p/S<<1$ $\alpha_E'=\alpha_M'+\alpha_M'^2/2+\alpha_M'^3/3+$———. Since in practice the absorption coefficient of the bare room is very small compared with that of the sample, the results also hold if we replace α_M' by α_M and α_E' by α_E and the theorem follows.

The following conclusions from the theory may now be stated:

(i) If in a given room the true values of the coefficients are known, e.g., from tube measurements, the reverberation worked out on the modified formula is always less than the time worked out on Eyring's formula. For if we call the two times T_M and T_E respectively then $T_M=0.05V/A$ and $T_E=0.05V/B$ and since $A>B$ by theorem I it follows that T_M is less than T_E.

(ii) In the converse process in which a reverberation room is used to find the absorption coefficient of a given sample, the coefficient obtained by using Eyring's formula is always greater than that obtained by using the modified formula. This follows immediately from theorem II. Moreover the difference between them gets larger as the ratio S_p/S gets smaller and the material more absorbent.

(iii) The modified formula is more direct for using in reverberation work, and the errors in the value of α for given errors in the measurement of the time T_p are much smaller than with Eyring's formula when S_p/S is small. This can be seen by differentiating Eq. (8) and (9). Writing the respective values of α_p as α_M and α_E as above we have

$$\frac{-\dfrac{d\alpha_M}{1-\alpha}}{1-\dfrac{\alpha_M-\alpha}{1-\alpha}}=\frac{0.05V}{S_p}\,\frac{dT_p}{T_p{}^2}$$

and

$$\frac{-\dfrac{S_p}{S}\,\dfrac{d\alpha_E}{1-\alpha}}{1-\dfrac{S_p}{S}\,\dfrac{\alpha_E-\alpha}{1-\alpha}}=\frac{0.05V}{S}\,\frac{dT_p}{T_p{}^2}$$

and for a given error dT_p

$$\frac{d\alpha_M}{d\alpha_E}=\frac{1-\dfrac{\alpha_M-\alpha}{1-\alpha}}{1-\dfrac{S_p}{S}\,\dfrac{\alpha_E-\alpha}{1-\alpha}}$$

and the general results stated follow.

VIII. RECTANGULAR ROOM

The special case of a rectangular room with a source of sound which has marked directional properties is of special importance in considering the usual type of reverberation rooms. Suppose that the edges of the room define a set of rectangular axes OX, OY, and OZ, the dimensions being a, b and c along these directions respectively. Consider a ray travelling in a direction determined by cosines $l\ m\ n$, then by symmetry we need only to consider rays starting off in the positive octant. We can divide the walls up into three pairs, the "x" pair parallel to the YOZ plane and separated by the distance 'a' and the 'y' and 'z' pairs corresponding to the 'b' and 'c' distances. If the total area of the 'x' walls is S_x and is made up of areas S_{x1} S_{x2} etc. with coefficients α_{x1} α_{x2} etc., and similarly for the 'y' walls areas S_{y1} S_{y2} etc., with coefficients α_{y1} α_{y2} etc., and for the 'z' walls S_{z1} S_{z2} etc., with coefficients α_{z1} α_{z2} etc., then the number of reflections at the 'x' walls in t seconds will be lvt/a and similarly mvt/b and nvt/c for the 'y' and 'z' walls respectively, and it follows that the value of the decay constant k is given by:

$$k = -v\left[\frac{l}{a}\,\frac{\sum S_{x1}\log_e(1-\alpha_{x1})}{S_x} + \frac{m}{b}\,\frac{\sum S_{y1}\log_e(1-\alpha_{y1})}{S_y} + \frac{n}{c}\,\frac{\sum S_{z1}\log_e(1-\alpha_{z1})}{S_z}\right]$$

$$= -\frac{v}{2V}\left[l\sum S_{x1}\log_e(1-\alpha_{x1}) + m\sum S_{y1}\log_e(1-\alpha_{y1}) + n\sum S_{z1}\log_e(1-\alpha_{z1})\right] \qquad (12)$$

since $aS_x = bS_y = cS_z = 2V$. This is the fundamental expression for a plane wave travelling in the direction $(l\ m\ n)$.

In the particular case in which the sound is directed at an equal inclination to all three principal axes, $l = m = n = 3^{-\frac{1}{2}}$ then:

$$k = -\frac{v}{2(3)^{\frac{1}{2}}V}\sum S_1\log_e(1-\alpha_1) \qquad (13)$$

where the summation term is the sum of the three summation terms of Eq. (12). This expression is equivalent to the general formula with $2(3)^{1/2}$ instead of 4 in the denominator.

To obtain the expression for a diffuse state in a rectangular room, the result must be averaged over all values of $(l\ m\ n)$ in the octant, and this is equivalent to replacing l m and n by $\bar{l}$ $\bar{m}$ and $\bar{n}$ where

$$\bar{l} = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\phi\cos\theta\, d\phi\, d\theta$$

$$\bar{m} = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\phi\sin\theta\, d\phi\, d\theta$$

$$\bar{n} = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\cos\phi\, d\phi\, d\theta$$

whence

$$\bar{l} = \bar{m} = \bar{n} = \tfrac{1}{2}$$

giving

$$k = -\frac{v}{4V}\sum S_1 \log_e (1 - \alpha_1)$$

i.e., the general formula, which is equivalent to saying that in a rectangular room the mean free path is the statistical value $4V/S$.

The general expression given in Eq. (12) can be used to determine the reverberation time under any given conditions in which the sound field is produced by an ordered set of waves. It expresses for instance the obvious fact that if the waves travel to and fro between one pair of walls (i.e., parallel to a principal direction, e.g., $l=1$ $m=n=0$) then the area and absorbing nature of the other walls do not enter into the argument. It must be remembered, however, that under certain conditions the statistical theory cannot be applied in detail. For instance in a room in which one pair of opposite walls is left bare and the others heavily damped, there will exist in the room even with a spherical sound field a 'rattle' effect. This rattle which is due to multiple reflections between the bare walls is only obvious because the rest of the walls are damped and is only noticeable when a short impulsive sound such as a hand clap is used. The effective rate of decay of the rattle can be worked out analytically by the method of images. If the reverberation is measured in the usual way by first allowing the steady state to be reached, only a small proportion of the sound is responsible for the rattle decay as received at any given point in the room, and the sound does in effect die away exponentially in accordance with the theory.

IX. CONCLUSION

In conclusion it may be noted that the modified formula has a marked resemblance to Sabine's original formula, since it can be obtained from it merely by replacing α_1, α_2, etc., by $-\log_e(1-\alpha)$ $-\log_e(1-\alpha_2)$ etc. We can therefore work out problems by the old Sabine method provided we remember that the quantity α which we work out by it is not really an absorption coefficient in the accepted sense (as used by Sabine in deriving his formula) but a logarithmic function. Moreover, if values of α are worked out on Sabine's formula and then used to calculate the reverberation of another room we should expect to get agreement between theory and experiment. This possibly explains part of the success which Sabine had in this direction, although he seldom dealt with rooms dead enough for us to expect a marked discrepancy between any of the formulae.

The foregoing investigation was carried out in the Research Laboratory of the Columbia Graphophone Co. Ltd., under the general direction of Mr. P. W. Willans to whom I am indebted for much stimulating criticism and advice in connection with this paper and the experimental work incidental to it, as also to Mr. G. Baillie for carrying out much of the practical work involved.

January 5, 1932