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Measurement of Sound Diffusion in Reverberation Chambers*

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This paper describes a variety of methods for the measurement of the diffusion of sound fields in reverberation chambers. Diffusion is defined on the basis of the angular distribution of sound energy flux, in accordance with the definition that has found its visual expression in the "sound hedgehog" of Meyer and Thiele. The theoretical foundations of the methods proposed here are: normal mode expansion, the sampling theorem (both in time and two-dimensional space), and either Fourier or correlation analysis. The quantities to be measured are sound pressures and, in some cases, sound pressure gradients at a number of sampling points on the measuring wall. Results of these measurements are suitably transformed to give the sound energy fluxes for all possible angles of incidence. The accuracy of measurement is determined by the Q (frequency times reverberation time) of the chamber and is typically of the order of 1°. This extraordinary directivity is achieved without substantial perturbation of the sound field. Methods applicable to both single frequencies and finite frequency bands are described.

I. INTRODUCTION

IN two earlier papers[1,2] it was shown that a number of room acoustical quantities, as for instance the distribution of eigenfrequencies and the "frequency irregularity," are *not* related to the sound diffusion in *large* rooms. In the present paper, the attempt of a *constructive* contribution to the measurement of sound diffusion is made. As a first step, a specific definition of diffusion is introduced which is based on the angular distribution of sound energy flux. This procedure seems most logical and is in accordance with the definition of diffusion which has found its visible expression in the well-known "sound hedgehog" of Meyer and Thiele.[3] The methods described in this paper emphasize those aspects of sound diffusion which are important for the measurements in reverberation chambers of the "statistical" absorption coefficient of acoustic materials.

In order to measure in a reverberation chamber the sound absorption averaged over all angles of incidence one requires, at the location of the absorber, a "diffuse" sound field. At low frequencies, where the wavelength is comparable to the linear dimensions of the room, the realization of completely diffuse sound fields is difficult if not impossible.[4] Special provisions have to be made in order to assure a sufficient approximation to the state of complete diffusion for measuring purposes.[5]

In this paper the question of how the diffusion in a reverberation chamber can be improved is not discussed.

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[1] M. R. Schroeder, Acustica 4, Beih. 1, 456 (1954).

[2] M. R. Schroeder, Acustica 4, Beih. 2, 594 (1954).

[3] E. Meyer and R. Thiele, Acustica 6, Beih. 2, 425 (1956).

[4] P. M. Morse and R. H. Bolt, Revs. Modern Phys. 16, 137 (1944).

[5] E. Meyer and H. Kuttruff, "Akust. Modellversuche zum Aufbau eines Hallraumes," Göttinger Nachrichten, No. 6, p. 97 (1958).

The emphasis is on methods for the *measurement* of sound diffusion.

Such measurements are useful for a number of purposes. They can be used to monitor the diffusion in a given reverberation chamber. They allow one to study the sound diffusing capability of different kinds of diffusers and the effectiveness of different locations of suitable diffusing elements. Finally, they enable one to correct the absorption coefficients, measured in a room lacking complete diffusion, for the case of complete diffusion.

The methods described here use simple sound pressure or pressure gradient microphones. Parabolic reflectors or other bulky instruments are not required. The perturbation of the sound field is correspondingly small. The necessary directivity is achieved by a 2-dimensional Fourier or correlation analysis of the sound field at the measuring wall. The accuracy of this method is extremely high. The angle of incidence can be measured with a precision of a fraction of a degree of arc even at the lowest frequencies.

The measurements can be made in the empty chamber or with the absorber on the measuring wall. It is possible to measure at a single frequency or with finite frequency bands, for instance, octave or third-octave bands. In the latter cases, periodic pulses, warble tones or noise bands are suitable to excite the room.

II. DEFINITION OF DIFFUSION OF A SOUND FIELD

We define the "diffusion of a sound field at a point" as the angular distribution of sound energy flux in the plane wave expansion of the sound field at that point. If the distribution over the solid angle is uniform, we shall call the sound field at this point "completely diffuse."

In reverberation chambers for the measurement of *isotropic* and *homogeneous* absorbers, one is interested in the distribution of the *angle of incidence* of the sound energy flux *averaged over the measuring wall.* Therefore, we shall call the sound field at the measuring wall in a reverberation chamber completely diffuse, if the angle of incidence averaged over the measuring wall is distributed like the angle of incidence at a point of complete diffusion. One should note that, according to this definition, the sound field at a measuring wall of a reverberation chamber can be completely diffuse without the sound field at any single point being completely diffuse. In that case, the complete diffusion at the wall is the result of the averaging over all azimuth angles and all points of the measuring wall.

How is the angle of incidence distributed for complete diffusion? According to our definition, the sound energy flux $E(\Omega)$ does not depend on the solid angle Ω for complete diffusion. For the half-space, the normalization constant is $1/2\pi$. Thus, for complete diffusion at a point,

$$E(\Omega)=1/2\pi.$$

If we call the angle of incidence γ and the azimuth angle φ, we obtain (because of $d\Omega=\sin\gamma d\varphi$)

$$E(\gamma,\varphi)=1/2\pi\ \sin\gamma;\quad 0\leqq\gamma\leqq\pi/2,\quad 0\leqq\varphi\leqq 2\pi.$$

By integrating over the azimuth angle, φ, which is of no interest in the case of isotropic absorbers, one obtains

$$E(\gamma)=\sin\gamma. \tag{1}$$

Transforming to the *cosine* of the angle of incidence results in an even simpler law. Because of $d\gamma=1/\sin\gamma \times d(\cos\gamma)$, one has instead of Eq. (1)

$$E(\cos\gamma)=1;\quad 0\leqq\cos\gamma\leqq 1. \tag{2}$$

Equation (2) is a very simple law which is excellently suited for checking the state of diffusion experimentally. Any deviation of the measured distribution of the incident cosine of the sound energy flux from Eq. (2), i.e., from a constant, signals a deviation from complete diffusion as defined above.

III. REPRESENTATION OF THE SOUND FIELD AT THE MEASURING WALL

As a matter of course, one can expand the sound field at a surface into the eigenfunctions of the surface and its boundary conditions. To simplify, we assume a rectangular wall with dimensions $0\leqq x\leqq a$ and $0\leqq y\leqq b$ and an infinite impedance at the boundaries. Thus, the eigenfunctions are the well-known circular functions.

$$\psi_{lm}(x,y)=\cos[(\pi l/a)x]\cdot\cos[(\pi m/b)y].$$

With a sinusoidal excitation of radian frequency ω, the sound pressure at the wall can be expressed as follows:

$$p(x,y,t)=\sum_{lm} a_{lm}\cos(\omega t+\varphi_{lm})\cos[(\pi l/a)x]\cdot\cos[(\pi m/b)y]. \tag{3}$$

With the wave equation for a homogeneous medium having a sound velocity c,

$$\Delta p+(\omega^2/c^2)p=0,$$

the dependence of the sound pressure in the z direction (perpendicular to the wall), at least in its immediate vicinity, becomes

$$p(x,y,z,t)=\sum_{lm}[a_{lm}^{+}\cos(\omega t+\varphi_{lm}^{+}-k_{lm}z)+a_{lm}^{-}\cos(\omega t+\varphi_{lm}^{-}+k_{lm}z)]\cdot\cos[(\pi l/a)x]\cdot\cos[(\pi m/b)y], \tag{4}$$

where

$$k_{lm}=+[(\omega/c)^2-(\pi l/a)^2-(\pi m/b)^2]^{\frac{1}{2}}. \tag{5}$$

Equation (4) makes it evident that the sound field at the wall can be considered as resulting from the superposition of incoming and outgoing plane waves.

The cosines corresponding to the *incident* waves follow from Eq. (5).

$$\cos\gamma_{lm}=(c/\omega)k_{lm}=+[1-(l\lambda/2a)^2-(m\lambda/2b)^2]^{\frac{1}{2}}, \quad (6)$$

where

$$\lambda=2\pi c/\omega.$$

At this juncture, we must focus our attention on a point which is important for the further discussion. According to Eq. (6), it is quite possible that some of the $\cos\gamma_{lm}$ are imaginary. This can happen if in the expansion, Eq. (3), sufficiently large l or m occur with nonzero amplitude. The existence of imaginary $\cos\gamma_{lm}$ signals the presence of exponentially attenuated waves in space.

In a perfectly rectangular room in which not even the excitation constitutes a perturbation, such waves do not exist. They are a consequence of not completely attenuated diffraction phenomena. In a reverberation chamber with oblique or "zig-zag" walls, propellers, prisms, or other diffusing elements such spatially attenuated waves do appear at the measuring wall, if these perturbations are located in close proximity to that wall. In addition, these "forbidden" waves can be caused by the manner and location of the excitation or by partial covering of the measuring wall with the absorber. In any case, they result in faulty measurements and should not occur in good reverberation chambers. In Sec. XI a method will be described which indicates the presence of such "forbidden" waves.

In the following, we shall assume that no diffraction phenomena reach the measuring wall and, consequently, that expansion (3) contains only terms for which the incident cosine is real. With this assumption, we can introduce finite limits into (3).

$$p(x,y,t)=\sum_{l=-[2a/\lambda]}^{[2a/\lambda]}\sum_{m=-[2b/\lambda]}^{[2b/\lambda]} a_{lm}\cos(\omega t+\varphi_{lm}) \cdot\cos[(\pi l/a)x]\cdot\cos[(\pi m/b)y]$$

$$\cdot a_{lm}=a_{\pm l,\pm m};\ \varphi_{lm}=\varphi_{\pm l,\pm m}. \quad (7)$$

Here $[2a/\lambda]$ is the largest integer which does not exceed $2a/\lambda$. In the following, we shall simply omit the limits of the summations and use only one summation sign for each two indices. Thus,

$$\sum_{lm}\equiv\sum_{l=-[2a/\lambda]}^{[2a/\lambda]}\sum_{m=-[2b/\lambda]}^{[2b/\lambda]}.$$

The summation over positive and negative indices is a mathematical trick. It permits one to treat oblique, tangential and axial waves in a unified manner.

IV. MEASUREMENT OF $\langle\cos^2\gamma\rangle$

Before developing the more difficult methods for the measurement of the angular distribution of the sound energy flux, we shall describe a method which is simple, both conceptually and in application, for the measurement of the mean square of the incident cosine.

First, we have to answer the question what the value of $\langle\cos^2\gamma\rangle$ is in the case of complete diffusion. This is easily done. With Eq. (2), we have

$$\langle\cos^2\gamma\rangle=\int_0^1\cos^2\gamma E(\cos\gamma)\,d\cos\gamma=\int_0^1\cos^2\gamma\,d\cos\gamma,$$

or

$$\langle\cos^2\gamma\rangle=\tfrac{1}{3}. \quad (8)$$

For a discrete distribution, we have with Eq. (3)

$$\langle\cos^2\gamma\rangle=\frac{\sum_{lm}\cos^2\gamma_{lm}a_{lm}^2}{\sum_{lm}a_{lm}^2},$$

or with Eq. (6)

$$\langle\cos^2\gamma\rangle=\frac{\sum_{lm}[1-(l\lambda/2a)^2-(m\lambda/2b)^2]a_{lm}^2}{\sum_{lm}a_{lm}^2}. \quad (9)$$

Instead of the amplitudes a_{lm} we should have used the amplitudes of the incident waves a_{lm}^+ which appear in Eq. (4). For infinite or zero terminal impedance or in general, if the wall impedance does not depend on the angle of incidence, the difference between a_{lm} and a_{lm}^+ is a constant factor and, therefore, insignificant. For measurements with the unknown absorber at the measuring wall, the identification of a_{lm} and a_{lm}^+ leads to errors which, however, in most practical applications should be negligible.

The $\langle\cos^2\gamma\rangle$, according to Eq. (9), can be determined by measuring the intensity and the pressure gradients on the measuring wall. Defining the sound intensity at the point (x,y)

$$u(x,y)\equiv\frac{\omega}{2\pi}\int_0^{2\pi/\omega}p^2(x,y,t)dt, \quad (10)$$

one has with Eq. (7)

$$u(x,y)=\tfrac{1}{2}\sum_{lm}\sum_{l'm'}a_{lm}a_{l'm'}\cos(\varphi_{lm}-\varphi_{l'm'}) \cdot\cos[(\pi l/a)x]\cos[(\pi l'/a)x] \cdot\cos[(\pi m/b)y]\cos[(\pi m'/b)y].$$

Averaging over the wall results in

$$\langle u\rangle\equiv\frac{1}{ab}\int_0^a dx\int_0^b dy\,u(x,y)=\tfrac{1}{2}\sum_{lm}a_{lm}^2. \quad (11)$$

With Eq. (3) the gradient in the x direction becomes

$$[\partial p(x,y,t)/\partial x]=-\sum_{lm}(\pi l/a)a_{lm}\cos(\omega t+\varphi_{lm}) \cdot\sin[(\pi l/a)x]\cos[(\pi m/b)y]. \quad (12)$$

The mean square gradient in the direction of x at the point (x,y) is

$$u_x(x,y)=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[\frac{\partial p(x,y,t)}{\partial x}\right]^2 dt. \qquad (13)$$

Averaging over the walls gives

$$\langle u_x\rangle\equiv\frac{1}{ab}\int_0^a dx\int_0^b dy\, u_x(x,y)=\tfrac{1}{2}\sum_{lm}\left(\frac{\pi l}{a}\right)^2 a_{lm}^2. \qquad (14)$$

Analogously, one obtains for the mean square gradient in the direction of y, averaged over the wall,

$$\langle u_y\rangle=\tfrac{1}{2}\sum_{lm}(\pi m/b)^2 a_{lm}^2. \qquad (15)$$

By substituting Eqs. (11), (14), and (15) into Eq. (9) for $\langle\cos^2\gamma\rangle$, one obtains the final result

$$\langle\cos^2\gamma\rangle=1-(\lambda/2\pi)^2(\langle u_x\rangle+\langle u_y\rangle/\langle u\rangle). \qquad (16)$$

Before proceeding to the next section, a word of caution against the indiscriminate use of Eq. (16) is in order. As we saw above, $\langle\cos^2\gamma\rangle$ equals $\frac{1}{3}$ for complete diffusion. However, the converse is not true: It does not follow from $\langle\cos^2\gamma\rangle=\frac{1}{3}$ that the sound field at the wall is completely diffuse. For instance, a single angle of incidence, namely, $\gamma=55°$, can result in the value $\langle\cos^2\gamma\rangle=\frac{1}{3}$.

V. MEASUREMENT OF $\langle\cos^2\gamma\rangle$ FOR FINITE FREQUENCY BANDS

Quite often the approximation to complete diffusion is not very good at a single frequency. Since the absorption of most absorbers does not change much in relatively narrow frequency bands, one can use finite frequency bands in order to obtain a better approximation to complete diffusion. In the following we shall always assume, for mathematical convenience, that the exciting power spectrum consists of a number of equally strong harmonic components at frequencies

$$\omega_n=n\omega_1;\quad n=N,\quad N+1,\ \cdots\ N+K. \qquad (17)$$

This corresponds to the case of excitation with periodic impulses of period $2\pi/\omega_1$, band-pass filtered for frequencies between ω_N and ω_{N+K}. Warble tones with sawtooth frequency modulation have a similar spectrum. Finally, the assumption (17) is also valid for band-pass filtered noise.[6] However, one has to average the observation over a sufficient number of "periods" of lengths $2\pi/\omega_1$. This averaging is done, at least in part, by the inertia of the measuring instruments. The remaining fluctuations can be eliminated by visual averaging by the observer.

When working with finite frequency bands, Eq. (3) is amplified by an additional summation over the frequencies of the band;

$$p(x,y,t)=\sum_{lm}\sum_{n=N}^{N+K}a_{lmn}\cos(\omega_n t+\varphi_{lmn})\cos\left(\frac{\pi l}{a}x\right)\cdot\cos\left(\frac{\pi m}{b}y\right),\quad\text{with }\omega_n=n\omega_1. \qquad (18)$$

Equation (6) for the incident cosine takes on the following form

$$\cos^2\gamma_{lmn}=1-(\pi cl/a\omega_n)^2-(\pi cm/b\omega_n)^2. \qquad (19)$$

From this we obtain for the $\langle\cos^2\gamma\rangle$:

$$\langle\cos^2\gamma\rangle=\frac{\sum_{lmn}[1-(\pi cl/a\omega_n)^2-(\pi cm/b\omega_n)^2]a_{lmn}^2}{\sum_{lmn}a_{lmn}^2}. \qquad (20)$$

The definition of the sound intensity is exactly as in (10) except that the radian frequency ω is replaced by ω_1. Because of the orthogonality of the frequency components $n\omega_1$ in the interval $2\pi/\omega_1$, we have, in complete analogy to (11),

$$\langle u\rangle=\tfrac{1}{2}\sum_{lmn}a_{lmn}^2. \qquad (21)$$

For the mean squared gradient we shall introduce a definition which deviates from Eq. (13) by replacing the sound pressure by its *integral* over time, $p_i(x,y,t)$. With (18) we have

$$p_i(x,y,t)=\frac{1}{\tau}\sum_{lmn}(a_{lmn}/\omega_n)\sin(\omega_n t+\varphi_{lmn})\cdot\cos[(\pi l/a)x]\cos[(\pi m/b)y]. \qquad (22)$$

This means that one should use an amplitude spectrum which is proportional to $1/\omega$ instead of a flat spectrum. Such a spectrum can be obtained by integrating with a simple *RC* section. If we call the (sufficiently large) time constant of the *RC* section τ, the amplitude spectrum is multiplied by $1/\tau\omega$.

It is interesting to note that for the integrated signal the power spectrum as a function of the wavelength, λ, is flat.

Designating all quantities which are measured with a $(1/\omega)$ spectrum by an additional index "i", one has, after averaging over the wall, in analogy to (14) and (15)

$$\langle u_{xi}\rangle=(1/2\tau^2)\sum_{lmn}(\pi l/\omega_n a)^2 a_{lmn}^2, \qquad (23)$$

and

$$\langle u_{yi}\rangle=(1/2\tau^2)\sum_{lmn}(\pi m/\omega_n b)^2 a_{lmn}^2. \qquad (24)$$

Substituting (21), (23), and (24) into Eq. (20) results in a formula for $\langle\cos^2\gamma\rangle$ for *arbitrarily* wide frequency bands.

$$\langle\cos^2\gamma\rangle=1-(\tau c)^2[(\langle u_{xi}\rangle+\langle u_{yi}\rangle)]/\langle u\rangle]. \qquad (25)$$

[6] S. O. Rice, Bell System Tech. J. **23**, 306 (1944). See also *Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954), p. 157.

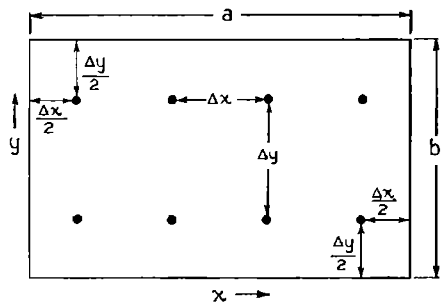

FIG. 1. Distribution of sample points on the measuring wall of a reverberation chamber.

This formula is very simple indeed. The experimental effort is quite small, too, if one remembers that the sound pressure and the pressure gradient do not have to be measured at each point of the wall but rather at a lattice of sample points with a spacing of approximately $\lambda/2$ (see Fig. 1). If one chooses the sample points as described in Sec. XI, all integrals over the measuring wall can be replaced by summations over the sample points without introducing any error.

Next we shall describe two methods for the measurement of the angular distribution of the sound energy flux.

VI. MEASUREMENT OF ANGULAR DISTRIBUTION BY FOURIER ANALYSIS OF SOUND FIELD AT THE WALL FOR EXCITATION WITH A SINGLE FREQUENCY

By Fourier inversion of Eq. (3), at the time t_0, one obtains

$$a_{lm}\cos(\omega t_0+\varphi_{lm})=\frac{1}{ab}\int_0^a dx\int_0^b dy\, p(x,y,t_0)\cdot\cos\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right). \quad (26)$$

By squaring and averaging over time, one gets the intensity of the lm wave, a_{lm}^2.Th is would require infinitely many Fourier inversions. However, by applying the "sampling theorem" in the time domain we need to invert only for two discrete instants of time which are separated by a quarter period. Thus,

$$a_{lm}\cos\left(\omega t_0+\frac{\pi}{2}+\varphi_{lm}\right)=\frac{1}{ab}\int_0^a dx\int_0^b dy\, p\left(x,y,t_0+\frac{\pi}{2\omega}\right)\cdot\cos\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right). \quad (27)$$

By squaring and adding (26) and (27), we obtain the intensity of the lm wave

$$a_{lm}^2=\sum_{k=0}^{1}\left[\frac{1}{ab}\int_0^a dx\int_0^b dy\, p\left(x,y,t_0+\frac{\pi}{2\omega}k\right)\cdot\cos\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right)\right]^2. \quad (28)$$

The angle of incidence belonging to this intensity is given by Eq. (6).

VII. MEASUREMENT OF ANGULAR DISTRIBUTION BY FOURIER ANALYSIS FOR FINITE FREQUENCY BANDS

The analysis is based on the same assumption made at the beginning of Sec. V. In particular, we assume again a discrete form of the spectrum,

$$\omega_N=N\omega_1,\ \omega_{N+1}=(N+1)\omega_1,\ \cdots,\ \omega_{N+K}=(N+K)\omega_1. \quad (29)$$

For the sound pressure, we assume again the form shown in Eq. (18). By two-dimensional Fourier inversion of (18), we obtain

$$\sum_{n=N}^{N+K} a_{lmn}\cos(\omega_n t_0+\varphi_{lmn})=\frac{1}{ab}\int_0^a dx\int_0^b dy\, p(x,y,t_0)\cdot\cos\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right). \quad (30)$$

The intensity of the lm waves in the frequency band $[\omega_N, \omega_{N+K}]$, $\sum_n a_{lmn}^2$, is obtained from (30) by squaring and averaging over a time interval of length $2\pi/\omega_1$. With the sampling theorem for low-pass signals[7] (see also Courant[8] for a historically earlier development) we can limit ourselves in the averaging process to a finite number of discrete times t_k. The interval between two neighboring sampling times is somewhat smaller than half the period of the highest frequency. Quantitatively,

$$t_{k+1}-t_k=\{1/[2(N+K)+1]\}(2\pi/\omega_1). \quad (31)$$

Thus, the desired intensity becomes

$$\sum_{n=N}^{N+K} a_{lmn}^2=\frac{1}{2(N+K)+1}\cdot\sum_{k=0}^{2(N+K)}\left[\frac{1}{ab}\int_0^a dx\int_0^b dy\, p(x,y,t_k)\cdot\cos\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right)\right]^2,$$

[7] C. E. Shannon, Proc. Inst. Radio Engrs. 37, 11 (1949).
[8] R. Courant, *Vorlesungen über Differential- und Integralrechnung, Erster Band* (Springer-Verlag, Berlin, 1955), third edition, p. 417.

where

$$t_k = t_0 + \{k/[2(N+K)+1]\}(2\pi/\omega_1). \quad (32)$$

The number of Fourier transformations implicit in (32) equals $2(N+K)+1$. For relatively narrow frequency bands it is recommended to use the sampling theorem for band-pass signals.[9] This reduces the number of Fourier transformations to $2(K+1)$. In addition to the sound pressure one needs to know its Hilbert transform[10] which is also known as the "quadrature signal" among communication engineers. It differs from the signal itself by a constant phase difference of $\pi/2$. Simple LC filters to generate pairs of Hilbert transforms, i.e., signals with a constant phase difference of $\pi/2$, in a finite frequency band have been described in detail by Darlington.[11]

If we designate the Hilbert transform by a "$\wedge$", we have

$$\sum_{n=N}^{N+K} a_{lmn}^2 = \frac{1}{K+1}\sum_{k=0}^{K}\left\{\left[\frac{1}{ab}\int_0^a dx\int_0^b dy\, p(x,y,t_k)\cdot\cos\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right)\right]^2 + \left[\frac{1}{ab}\int_0^a dx\int_0^b dy\, \hat{p}(x,y,t_k)\cos\left(\frac{\pi l}{b}y\right)\right]^2\right\}$$

$$t_k = t_0 + \frac{k}{K+1}\cdot\frac{2\pi}{\omega_1}. \quad (33)$$

This looks somewhat more "dangerous" than Eq. (32) but requires a smaller experimental and computational effort.

Returning to the computation of the mean cosine for the intensity given by Eq. (33), we have with (19)

$$\langle\cos^2\gamma_{lm}\rangle = 1 - \left[\left(\frac{\pi c l}{a}\right)^2 + \left(\frac{\pi c m}{b}\right)^2\right]\frac{\sum_n (a_{lmn}/\omega_n^2)}{\sum_n a_{lmn}^2}. \quad (34)$$

We have just shown how the sum in the denominator can be measured. The sum in the numerator is obtained in exactly the same manner, using a $(1/\omega)$ spectrum instead of a flat spectrum for the excitation. Calling the time constant of the *RC* section again τ and designating all quantities obtained with the $(1/\omega)$ spectrum by an additional index "i", one obtains

$$\sum_{n=N}^{N+K}\frac{a_{lmn}^2}{\omega_n^2} = \frac{2\tau^2}{2(N+K)+1}\cdot\sum_{k=0}^{2(N+K)}\left[\frac{1}{ab}\int_0^a dx\int_0^b dy\, p_i(x,y,t_k)\cdot\cos\left(\frac{\pi l}{a}x\right)\cdot\cos\left(\frac{\pi m}{b}y\right)\right]^2. \quad (35)$$

For narrow frequency bands, one can write instead of (35) in good approximation

$$\sum_{n=N}^{N+K}\frac{a_{lmn}^2}{\omega_n^2} = \frac{1}{\omega_N\omega_{N+K}}\sum_{n=N}^{N+K} a_{lmn}^2. \quad (36)$$

If in addition

$$(\pi c l/a\omega_N)^2 + (\pi c m/b\omega_N)^2 < 1, \quad (37)$$

we can approximate Eq. (34) as follows:

$$\langle\cos^2\gamma_{lm}\rangle = 1 - \lambda_N\lambda_{N+K}[(l/2a)^2 + (m/2b)^2]. \quad (38)$$

Here λ_N and λ_{N+K} are the wavelengths corresponding to the lowest and highest frequencies of the band.

VIII. MEASUREMENT OF ANGULAR DISTRIBUTION BY CORRELATION ANALYSIS OF THE SOUND FIELD AT THE WALL FOR A SINGLE FREQUENCY

First, we define the two-dimensional correlation function

$$\Phi(\xi,\eta) = \frac{1}{ab}\int_0^a dx\int_0^b dy\frac{\omega}{2\pi}\int_0^{2\pi/\omega} p(x,y,t)\cdot p(x+\xi,\, y+\eta,\, t)dt. \quad (39)$$

Here $p(x+\xi, y+\eta, t)$ must be replaced by $p(x+\xi-a, y+\eta, t)$ if $x+\xi>a$. A corresponding replacement has to be made if $y+\eta>b$.

With $p(x,y,t)$ from (3), we obtain

$$\Phi(\xi,\eta) = \tfrac{1}{2}\sum_{lm} a_{lm}^2\cos[(\pi l/a)\xi]\cos[(\pi m/b)\eta]. \quad (40)$$

Fourier-inversion of (40) gives the desired intensity

$$a_{lmn}^2 = \frac{2}{ab}\int_0^a d\xi\int_0^b d\eta\Phi(\xi,\eta)\cdot\cos\left(\frac{\pi l}{a}\xi\right)\cos\left(\frac{\pi m}{b}\eta\right). \quad (41)$$

The corresponding cosine is given by Eq. (6).

From the successful application of the correlation method to the measurement of the distribution of the *incident cosine*, one must not conclude that the correla-

[9] S. Goldman, *Information Theory* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1953), p. 75.
[10] D. Hilbert, Göttinger Nachrichten, p. 213 (1904). See also Hilbert's book *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (B. G. Teubner, Leipzig, 1924), first edition, p. 75.
[11] S. Darlington, Bell System Tech. J. 29, 94 (1950).

tion method is applicable to the measurement of the distribution over the *solid angle*. As a matter of fact, the correlation method gives ambiguous results for this case. Dämmig[12] has shown that a spatial correlation function identical to that obtained for complete diffusion does not necessarily mean that the sound field is actually completely diffuse. In particular, he pointed out that for a uniform distribution of the sound energy flux over all angles of one *octant* (instead of the entire sphere) the spatial correlation is the same as in the case of a completely diffuse sound field.

IX. MEASUREMENT OF ANGULAR DISTRIBUTION BY CORRELATION ANALYSIS OF THE SOUND FIELD AT THE WALL FOR FINITE FREQUENCY BANDS

The correlation method can be easily generalized to excitation with finite frequency bands. In the definition of the correlation function (39) we simply have to replace the averaging interval $(2\pi/\omega)$ by the fundamental period $(2\pi/\omega_1)$. In the case of excitation with noise bands, we must average over sufficiently long times (compared to the reciprocal band widths) in order to eliminate random fluctuations.

With $p(x,y,t)$ from (18), we obtain by Fourier inversion of (40)

$$\sum_{n=N}^{N+K} a_{lmn}^2=\frac{2}{ab}\int_0^a d\xi\int_0^b d\eta\Phi(\xi,\eta)\cdot\cos\left(\frac{\pi l}{a}\xi\right)\cos\left(\frac{\pi m}{b}\eta\right). \quad (42)$$

The corresponding cosine is given by equation (36) or (38). When using (36) we need to know $\sum_n a_{lmn}^2/\omega_n^2$. This quantity can be obtained by replacing the flat spectrum by a $(1/\omega)$ spectrum. Thus,

$$\sum_{n=N}^{N+K}\frac{a_{lmn}}{\omega_n^2}=\frac{2\tau^2}{ab}\int_0^a d\xi\int_0^b d\eta\Phi_i(\xi,\eta)\cdot\cos\left(\frac{\pi l}{a}\xi\right)\cos\left(\frac{\pi m}{b}\eta\right). \quad (43)$$

Here the index "i" indicates that the correlation function was obtained with a $(1/\omega)$ spectrum. τ is the time constant of the *RC* section which was used to obtain the $(1/\omega)$ spectrum.

X. CORRECTION OF ABSORPTION MEASUREMENTS OBTAINED WITH NOT COMPLETELY DIFFUSE SOUND FIELDS

With the wall impedance, W, and the characteristic impedance of air, Z, the absorption as a function of the incident cosine is[13]

$$s(\cos\gamma)=1-\left|\frac{W\cos\gamma-Z}{W\cos\gamma+Z}\right|^2. \quad (44)$$

For complete diffusion, the absorption averaged over all angles of incidence is

$$s_{\text{diffuse}}=1-\int_0^1\left|\frac{W\cos\gamma-Z}{W\cos\gamma+Z}\right|^2\cos\gamma d(\cos\gamma). \quad (45)$$

This is the desired "diffuse absorption." Actually, one measures

$$s_{\text{measured}}=1-\int_0^1 E(\cos\gamma)\left|\frac{W\cos\gamma-Z}{W\cos\gamma+Z}\right|^2\cos\gamma d\cos\gamma, \quad (46)$$

where the intensity

$$E(\cos\gamma)=\frac{\sum_{lm} a_{lm}^2\delta(\cos\gamma-\cos\gamma_{lm})}{\sum_{lm} a_{lm}^2} \quad (47)$$

is defined with the help of the well-known delta function. With (46) we have

$$s_{\text{diffuse}}=s_{\text{measured}}+\int_0^1[1-E(\cos\gamma)]\cdot\left|\frac{W\cos\gamma-Z}{W\cos\gamma+Z}\right|^2\cos\gamma d\cos\gamma. \quad (48)$$

In this formula the integral is the correction. $E(\cos\gamma)$ is known from the measurement. For the wall impedance W as a function of the incident cosine, one uses an approximation taking into account the measured (uncorrected) absorption and other known properties of the absorber.

XI. SAMPLING THEOREM FOR A 2-DIMENSIONAL CONTINUUM

In this paper we have encountered a number of integrals over the sound pressure, the pressure gradient, their squares and other quantities of the sound field which necessitate the knowledge of the integrand at each point of the region of integration, i.e., the measuring wall. Experimentally, this is unrealistic. In a good reverberation chamber, according to the definition given in Sec. III, the Fourier expansion of the sound pressure is finite [see Eq. (7)]. Hence, we can convert all integrals into sums by means of the sampling theorem (for further references, see reference 7). The summation has to be carried out over a lattice of sampling points as illustrated in Fig. 1. The distances between the sampling points, Δx and Δy, are subject to two conditions according to the sampling theorem:

[12] P. Dämmig, Acustica **7**, 387 (1957).

[13] L. Cremer, *Die wissenschaftlichen Grundlagen der Raumakustik. Band III, Wellentheoretische Raumakustik* (S. Hirzel, Leipzig, 1950), p. 125.

1. They must be integral fractions of the wall dimensions.
2. They must be smaller than the smallest half-wavelength that is contained in the exciting spectrum. Thus,

$$\Delta x = a/L < \lambda/2; \quad y = b/M < \lambda/2 \qquad (L \text{ and } M \text{ integers}). \tag{49}$$

In order to keep the number of sampling points $L \cdot M$ as small as possible, we require

$$a/(L-1) \geqq \lambda/2; \quad b/(M-1) \geqq \lambda/2.$$

Inequalities (49) and (50) combined give

$$L = 1 + (2a/\lambda); \quad M = 1 + (2a/\lambda).$$

Substituting this into Eq. (49) results in

$$\Delta x = \frac{a}{1+(2a/\lambda)}; \quad \Delta y = \frac{b}{1+(2b/\lambda)}. \tag{51}$$

The sampling lattice defined by these two equations is valid for all integrals over the measuring wall $a \cdot b$ which appear in this paper.

However, if one wants to check whether the Fourier expansion of the sound pressure at the wall actually breaks off where it ought to, a smaller sampling interval is required. In order to detect the presence of undesirable diffraction phenomena or "forbidden" waves, it is usually sufficient to make Δx and Δy half as large as given by Eq. (51).

XII. MEASURING ACCURACY FOR THE ANGLE OF INCIDENCE AS A FUNCTION OF REVERBERATION TIME

The measuring accuracy of the angle of incidence is limited due to the fact that the wavelength appearing in Eq. (6) is not necessarily the wavelength of the exciting oscillation for finite reverberation times, T. For example, let us consider a cubical room with uniformly absorbing walls which is oscillating in the (1,1,1) mode. From simple symmetry considerations it follows that the incident cosine equals $(\frac{1}{3})^{\frac{1}{2}}$ independent of the exciting frequency. Thus, the wavelengths in (6) is not that of the exciting oscillation but rather the resonant wavelengths of the (1,1,1)-mode.

In general, the wavelength appearing in Eq. (6) is not known exactly when the resonances of the room have finite widths. The only assertion one can make is that it probably does not differ from the exciting wavelength by more than the band width. We shall now compute what result the incomplete knowledge of the wavelength has on the measurement of the angle of incidence.

By solving (6), we obtain

$$\gamma_{lm} = \arcsin \frac{\lambda}{2}\left(\frac{l}{a} + \frac{m}{b}\right), \tag{52}$$

and by differentiating

$$d\gamma_{lm} = \frac{(l/a)+(m/b)}{2\cos\gamma_{lm}} d\lambda.$$

By putting $d\lambda = \Delta\lambda$, the half-power width of the resonance, and expressing the latter by the reverberation time, T,

$$\Delta\lambda = (\lambda/\omega)(13.8/T),$$

one obtains

$$\Delta\gamma = (2.2 \tan\gamma)/(f \cdot T). \tag{53}$$

Example: $T = 7$ sec; $f = 100$ cps; $\gamma = \pi/4$.

$\Delta\gamma = 3 \times 10^{-3} = 0.2$ degree of arc (!)

This is an extraodrinary accuracy indeed. It raises the question "where, in the reverberation chamber, did we hide the directional microphone which has such an enormous directivity?"

The answer to this question is not difficult. As we saw in Sec. XI, the sound pressures are measured at a lattice of sampling points and combined with appropriate weighting factors in order to determine the intensity for a particular possible angle of incidence. In this manner we have created, without explicitly wanting to do so, a 2-dimensional directive antenna. The important point is that the dimensions of this antenna are by no means that of the wall, *ab*. The effective dimensions of the directive antenna are considerably increased by repeated reflections at the sidewalls of the reverberation chamber. The fact that the directivity of such a resonator surface antenna increases with the Q of the resonator was already pointed out by W. Güth.[14]

XIII. ACKNOWLEDGMENT

The expansion of the sound field into orthogonal modes employed in this paper is quite analogous to that used for electromagnetic waves in metallic wave guides. This analogy includes the existence of a "cutoff frequency" which in this paper is reflected by the finite number of modes that can be excited with a given frequency.

It is therefore with particular pleasure that the author recalls the fact that he was able to experiment with guided waves for the first time in Professor Erwin Meyer's III. Physical Institute at the University of Göttingen. At that time he (the author) was skeptical because the wave guides used to be round and filled with Bessel functions. However, once it became evident that knowledge of the numerical values 1.84 and 2.4 was quite sufficient for the first twelve months, the initial skepticism vanished. Today the author is very grateful to Professor Meyer for initial guidance in this interesting and important field.

[14] W. Güth, Z. angew. Phys. 8, 368 (1956).