Distribution of Eigentones in a Rectangular Chamber at Low Frequency Range* †

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It has been shown by various authors[1] that the number of eigentones in a rectangular chamber with frequencies less than a certain limiting value ν is given by

$$N=4\pi V\nu^3/3c^3 \qquad (1)$$

where V is the volume of the chamber and c the wave velocity which, for sound waves, is 1125 feet per second at room temperature. This value given by (1) is correct only when the limiting wave-length is negligibly small compared to the dimensions of the chamber, as in case of light waves. When it is applied to problems in acoustics, as in the calculations of intensity distribution, growth and decay of sound energy in a room, etc., the results are far from satisfactory, because the wave-length now becomes comparable with the dimensions of the room. Mr. Bolt, by considering the density of representative points, arrived at a very satisfactory result, as shown in his diagrams.[2] From an alternative point of view, the following result

$$N=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}\right] \qquad (2)$$

may be obtained, in which S is the surface area, L the sum of the three dimensions, V the volume of the chamber, and c the sound velocity. Although this result seems quite different from that obtained by Bolt:[2]

$$N=\frac{4\pi V\nu^3}{3c^3}\left[\frac{2\nu V+cR^{\frac{1}{2}}}{2\nu V+\frac{1}{2}cR^{\frac{1}{2}}}\right]^3 \qquad (3)$$

(where R is the sum of the squares of three areas, $S_x^2+S_y^2+S_z^2$, and the other symbols have the same meaning as above), the two forms agree quite well. The values obtained from Eq. (2) are plotted as x's in the diagrams given by Bolt.[2]

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[1] See, e.g., Courant and Hilbert, *Methode der Mathematische Physik;* Morse, *Vibration and Sound.*

[2] See the preceding paper in this issue of the Journal: "Frequency Distribution of Eigentones in a Three-Dimensional Continuum," by R. H. Bolt.

DERIVATION

(A) The asymptotic formula (1)

Let the dimensions of the rectangular chamber be L_x, L_y, and L_z. If the surfaces of the chamber are all rigid, the sound wave in the chamber will be given by:

$$\nabla^2\phi-(1/c^2)\ddot{\phi}=0 \qquad (4)$$

under the boundary conditions:

$\partial\phi/\partial x=0$ at $x=0$ and $x=L_x$, $\partial\phi/\partial y=0$ at $y=0$ and $\cdots$;

where ϕ represents the velocity potential, $\partial\phi/\partial x$ the particle velocity along x-direction, etc. The solution will be of the form:

$$\phi=\epsilon^{-i2\pi\nu t}\cos\frac{p_x\pi x}{L_x}\cos\frac{p_y\pi y}{L_y}\cos\frac{p_z\pi z}{L_z}$$

with $4\pi^2\nu^2/c^2=p_x^2\pi^2/L_x^2+p_y^2\pi^2/L_y^2+p_z^2\pi^2/L_z^2$,

where p_x, p_y, and p_z are integers or zero. For the eigentones with frequency less than ν, one has

$$4\nu^2/c^2>p_x^2/L_x^2+p_y^2/L_y^2+p_z^2/L_z^2. \qquad (5)$$

Each set of positive integers (p_x, p_y, p_z) satisfying the relation (5) will give an eigentone of frequency less than ν. Therefore any point having positive integral coördinates in the (p_x, p_y, p_z) space will represent an eigentone and the number of eigentones having frequencies less than ν will be equal to the number of such representative points within the first octant of the ellipsoid

$$p_x^2/L_x^2+p_y^2/L_y^2+p_z^2/L_z^2=4\nu^2/c^2. \qquad (6)$$

It is easy to see that there is, on the average, one point having integral coördinates in every unit volume in space, and the number of such points within a certain volume is equal to the volume itself. Therefore, the number of eigentones having frequency less than ν is equal to one-eighth of the volume enclosed by the ellipsoid (6), or

$$N=\frac{1}{8}\frac{4\pi}{3}\left(\frac{2L_x\nu}{c}\right)\left(\frac{2L_y\nu}{c}\right)\left(\frac{2L_z\nu}{c}\right)=\frac{4\pi}{3}\frac{V\nu^3}{c^3}, \qquad (1)$$

where $V=L_xL_yL_z$ is the volume of the rectangular chamber.

(B) The correction

When the whole space was divided into eight octants, all the points on the coördinate planes were, effectively, cut into two halves and all the points on the coördinate axes were, effectively, cut into four quarters. So, in Eq. (1) only one-half of the points on the coördinate planes and only one-fourth of the points on the coördinate axes have been counted. The correction will be the addition of the missing points. Evidently there is, on the average, one point with integral coördinates in every unit area on the coördinate planes. The total number of representative points on the planes within the ellipsoid (6) is equal to the sum of the areas of the first quadrants of the ellipses

$$\frac{p_y^2}{(2L_y\nu/c)^2}+\frac{p_z^2}{(2L_z\nu/c)^2}=1,\cdots,$$

that is

$$\frac{1}{4}\left[\pi\frac{2L_y\nu}{c}\frac{2L_z\nu}{c}+\pi\frac{2L_z\nu}{c}\frac{2L_x\nu}{c}+\pi\frac{2L_x\nu}{c}\frac{2L_y\nu}{c}\right]=\frac{\pi\nu^2S}{2c},$$

where $S=2(L_yL_z+L_zL_x+L_xL_y)$ is the total surface area of the chamber. One-half of this number had been missed. So we have to add another half, i.e.,

$$\pi S\nu^2/4c^2 \tag{7}$$

to the value given by Eq. (1). As for the points on the axes, one-fourth were counted in Eq. (1) and another half in Eq. (7). One-fourth of the points are still uncounted. It is easily seen that the number of points on the axes is equal to the sum of lengths of the three semi-axes of the ellipsoid (6), i.e.,

$$2L_x\nu/c+2L_y\nu/c+2L_z\nu/c=2L\nu/c,$$

where $L=(L_x+L_y+L_z)$ is the sum of three dimensions of the chamber; and one-fourth of this number is

$$L\nu/2c. \tag{8}$$

Therefore, the total number of representative points, those within the positive octant of the ellipsoid, (6), or the number of eigentones of frequencies less than ν is given by

$$N=\frac{4\pi}{3}\frac{V\nu^3}{c^3}+\frac{\pi S\nu^2}{4c^2}+\frac{L\nu}{2c}$$

$$=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}\right]. \tag{2}$$

Or in terms of wave-length, the number of eigentones having wave-lengths greater than λ is

$$N=\frac{4\pi V}{3\lambda^3}\left[1+\frac{3S\lambda}{16V}+\frac{3L\lambda^2}{8\pi V}\right]. \tag{9}$$

Discussion

It will be seen that the correction terms depend only on the ratio of wave-length to the dimension of the chamber. In all practical cases, the second correction term is always very much smaller than one, except for the first few eigentones. For example, in a room $10'\times15'\times30'$ (discussed by Bolt)

$3L\lambda^2/8\pi V=0.19$ at 100 cycles per second

and 0.02 at 300 cycles per second.

So, in general, it is negligible. But the first correction term must be considered throughout most of the audiofrequency range. For example, in the same room, $10'\times15'\times30'$,

$3S\lambda/16V=1.13$ at 100 c.p.s.
0.38 at 300 c.p.s.

and 0.04 at 3000 c.p.s.

Since the mean free path of sound waves in the room is given by

$$\Lambda=4V/S,$$

the first correction term may also be put in the form $3\lambda/4\Lambda$, and Eq. (2) will become

$$N=(4\pi V/3\lambda^3)[1+3\lambda/4\Lambda+3L\lambda^2/8\pi V]. \tag{10}$$

From Eq. (2) many other useful relations can be derived; the following are a few examples:

(I) The number of eigentones having frequencies between ν and $\nu+\delta\nu$ is

$$\delta N=\frac{4\pi V}{3c^3}[(\nu+\delta\nu)^3-\nu^3]+\frac{\pi S}{4c^2}[(\nu+\delta\nu)^2-\nu^2]$$

$$+\frac{L}{2c}[(\nu+\delta\nu)-\nu],$$

or

$$\delta N=\frac{4\pi V\nu^2}{c^3}\left(1+\frac{Sc}{8V}\frac{1}{\nu}+\frac{Lc^2}{8\pi V}\frac{1}{\nu^2}\right)\delta\nu$$

$$+\frac{4\pi V\nu}{c^3}\left(1+\frac{Sc}{16V}\frac{1}{\nu}\right)\delta\nu^2+\frac{4\pi V}{3c^3}\delta\nu^3. \quad (11)$$

For very small values of $\delta\nu$, i.e., when $\delta\nu \ll \nu$, the higher order terms of $\delta\nu$ may be neglected, and we have

$$\delta N=\frac{4\pi V\nu^2}{c^3}\left(1+\frac{Sc}{8V}\frac{1}{\nu}+\frac{Lc^2}{8\pi V}\frac{1}{\nu^2}\right)\delta\nu. \quad (12)$$

The first correction term is just

$$\lambda/2\Lambda.$$

In the room $10'\times 15'\times 30'$, $\Lambda=10'$. Except when $\lambda \ll 20'$, or $\nu \gg 56$ c.p.s., the old form

$$\delta N=(4\pi V/c^3)\nu^2\delta\nu$$

will not be a good approximation. The second correction is still very small except when the frequency is very low.

(II) If the walls of the chamber are not totally reflective, there will be damping for any sound wave in the chamber and the above derivation will not be exact because the relation (5) is only an approximation. But when the absorption of the walls is small, no further correction will be required.

If, in such a room, a source of strength $Q_0\epsilon^{-i\omega t}$ is turned on, the final average energy density will be, according to Morse,[1]

$$W=\frac{4\rho c^2Q_0^2}{V^2}\sum_{(n)}\frac{\sigma_n\psi_n^2(S)}{(2\omega_nk_n/\omega)^2+(\omega-\omega_n^2/\omega)^2}, \quad (13)$$

where ρ is the density of air, k_n is the damping factor for the nth eigentone ω_n, S is the position of source,

$$\psi(x,y,z)=\cos\frac{p_x\pi x}{L_x}\cos\frac{p_y\pi y}{L_y}\cos\frac{p_z\pi z}{L_z}$$

and $\sigma_n=1$, $\frac{1}{2}$, $\frac{1}{4}$, according to whether none, one, or two of the p's are zero. If ν is high enough so that the eigentones around it are very close to one another, the average energy density will be

$$W=\frac{\pi\rho Q_0^2\nu^2}{4Vk}\left(1+\frac{Sc}{8V}\frac{2}{\nu}+\frac{Lc^2}{8\pi V}\frac{1}{\nu^2}\right) \quad (14)$$

by following Morse's method.

(III) The present result (2) is readily applicable to chambers of other shapes. It is suggested that when the formula is applied to a chamber of conventional type, V and S will still be taken as the total volume and surface area, respectively, and L will be the sum of the height and one-half of the perimeter of the chamber.

(IV) In the above derivation, no consideration has been given to the eigentone (0,0,0) which, in general, is not counted. Effectively, one-eighth of the eigentone (0,0,0) has been counted in the first term of N, three-eighths in the second and three-eighths in the third. If in the counting of the eigentones, the mode (0,0,0), which is not vibratory at all, is excluded, the number attributed by it, i.e.,

$$\tfrac{1}{8}+\tfrac{3}{8}+\tfrac{3}{8}=\tfrac{7}{8},$$

should be subtracted from the value of N obtained above and Eq. (2) will become

$$N=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}\right]-\frac{7}{8}$$

or

$$N=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}+\frac{21c^3}{32\pi V}\frac{1}{\nu^3}\right]. \quad (15)$$

The value of δN will be still given by Eq. (11) or (12). When the value of N is much larger than one, the last correction may be neglected. But if N is not so large, all the correction terms should be used.

(V) From the solution

$$\phi=\epsilon^{-i\omega t}\cos\frac{p_x\pi}{L_x}x\cos\frac{p_y\pi}{L_y}y\cos\frac{p_z\pi}{L_z}z$$

of the wave equation, it is evident that $(2L_x/p_x)$, $(2L_y/p_y)$, $(2L_z/p_z)$ are the "components" of the wave-length along the axes (here the "component" of a wave-length along a certain direction means the line segment along that direction cut by two successive wave fronts at a wave-length apart), hence the direction cosines of the wave normal of this particular eigentone are equal to, but may differ in signs from

$$\frac{\lambda}{2L_x/p_x}, \quad \frac{\lambda}{2L_y/p_y}, \quad \frac{\lambda}{2L_z/p_z}$$

or

$$\frac{p_x}{2L_x}, \frac{p_y}{2L_y}, \frac{p_z}{2L_z} \Big/ \left[\left(\frac{p_x}{2L_x}\right)^2 + \left(\frac{p_y}{2L_y}\right)^2 + \left(\frac{p_z}{2L_z}\right)^2\right]^{\frac{1}{2}},$$

the number, N_θ, of the eigentones with frequencies less than ν and with wave normals making angles less than θ with the x axis will be the number of representative points in the p space within the first octant of the ellipsoid (6) and also within the conical surface

$$\frac{p_x}{2L_x} \Big/ \left[\left(\frac{p_x}{2L_x}\right)^2 + \left(\frac{p_y}{2L_y}\right)^2 + \left(\frac{p_z}{2L_z}\right)^2\right]^{\frac{1}{2}} = \cos\theta. \tag{16}$$

Without the correction for the points on the coördinate planes and the axes, this number may be found by integration as

$$(4\pi V\nu^3/3c^3)(1-\cos\theta).$$

By the same method as before, the correction for the points on the coördinate planes is

$$\theta S'\nu^2/c^2$$

where $\qquad S' = L_xL_y + L_xL_z,$

for those on the p_x axis

$$2L_x\nu/c$$

and for the origin

$$\tfrac{1}{8}\cos\theta - \theta/2\pi - \tfrac{1}{4}.$$

Therefore

$$N_\theta = \frac{4\pi V\nu^3}{3c^3}(1-\cos\theta) + \frac{S'\nu^2}{c^2}\theta + \frac{2L\nu}{c} + \left(\tfrac{1}{8}\cos\theta - \frac{\theta}{2\pi} - \tfrac{1}{4}\right). \tag{17}$$

In general, the last term is neglected. For the eigentones in the directions between θ and $\theta+\delta\theta$ the number is

$$\delta N_\theta = \left[\frac{4\pi V\nu^3}{3c^3}\sin\theta + \frac{S'\nu^2}{c^2} - \tfrac{1}{8}\sin\theta - \frac{1}{2\pi}\right]\delta\theta. \tag{18}$$

If only the first term is taken, or what amounts to the same thing, if the eigentone with wave normal parallel to any of the walls be given a weight $\frac{1}{2}$ and the eigentone with wave normal perpendicular to any of them be given a weight $\frac{1}{4}$, Eq. (17) will give

$$N_\theta = (4\pi V\nu^3/3c^3)(1-\cos\theta). \tag{19}$$

This suggests a random distribution of the eigentones so far as the direction is concerned. Thus the usual way of assuming diffused waves in a room is justified provided the frequency is so high that the secondary terms are negligible or particular weights are given to the waves parallel to the walls or the edges, according to the above manner. In such a case the number of eigentones with frequencies between ν and $\nu+\delta\nu$ and in direction between θ and $\theta+\delta\theta$ will be given by

$$\delta^2 N = (4\pi V\nu^2/c^3)\sin\theta\,\delta\nu\,\delta\theta. \tag{20}$$