Exact effect of surface roughness on the reverberation time of a uniformly absorbing spherical enclosure[a)]

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We assess the validity of the Sabine, Eyring, and Kuttruff reverberation-time expressions, and of their underlying mean-path-length formula $\langle l \rangle = 4V/S$, by determining the exact reverberation time and asymptotic sound distribution for a uniformly absorbing spherical enclosure with any amount of air absorptivity and with a surface that can be continuously varied from nonabsorbing to completely absorbing and from specular to randomly reflecting. In disagreement with an extensive literature that presents the Eyring formula as a correction to that of Sabine, we find, on the one hand, that Sabine's formula and the applicability of $4V/S$ are vindicated in this application under Sabine's stated conditions of weak absorptivity and (any nonzero amount of) irregular reflection, while, on the other hand, we find that Eyring's and Kuttruff's expressions are less accurate than Sabine's unless the roughness and absorptivity of the surface exceed certain levels which are evaluated. Related concepts are discussed in some detail. Geometrical acoustics is assumed throughout, and its limitations are not considered here. The analysis is equally applicable to light in LEDs (light-emitting diodes) and to mechanical particles which stick or experience elastic reflection at the surface of their container.

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INTRODUCTION

Sabine's expression for the reverberation time of an enclosure is now widely criticized as illogical or fallacious.[1–7] In the limit of zero absorptivity each ray or sound particle (phonon) goes distances l_1, l_2, ... between successive wall reflections with a mean distance $\langle l \rangle$ given by

$$\langle l \rangle = \lim_{n\to\infty} \frac{1}{n}(l_1 + l_2 + \cdots + l_n) , \tag{1}$$

and the assertion of fallacy often centers on the validity of the evaluation

$$\langle l \rangle = 4V/S , \tag{2}$$

which underlies the Sabine expression. [In Eq. (2), V and S are the volume and surface area of the enclosure.]

The criticism is certainly valid for specularly reflecting spheres and rectangular parallelepipeds where Eq. (2) and Sabine's expression are wrong, but under Sabine's stated condition[8] of "irregular reflection" or, indeed, with purely specular reflection in enclosures of somewhat more complicated shape[9,10] we have defended[11] the validity of Eq. (2) for almost every initial ray direction and position. As an alternative and further vindication of Eq. (2), of Sabine's expression, and of the view that Sabine cannot be improved upon unless more information about the enclosure appears in the competing reverberation-time formula, we review the physically possible surface-reflection laws in Sec. II, and then in Sec. III we extend the integral-equation method of Kuttruff[12] to obtain the exact reverberation time for a uniformly absorbing spherical enclosure with a surface-reflection law that can be varied continuously from random to specular. In Sec. IV the exact result is interpreted as a confirmation of Sabine under his stated[8] condition of weak surface absorptivity α. For larger α in the range $\alpha \lesssim 0.6$ we find that, depending upon the degree of surface roughness, the exact reverberation time may be more or less than the Sabine value. Other expressions, such as those of Eyring[13–16] and Kuttruff[17] which always yield a shorter reverberation time than Sabine's, are therefore not always more accurate. A number of otherwise unmotivated pedagogical digressions are inserted in the following development because they bear on secondary points in the controversy.

I. SURFACE REFLECTIVITY

It is convenient to think of the energy which moves along the rays of geometrical acoustics as classical point particles of sound. As the term "electron" is used in both quantum and classical descriptions, so shall we employ the quantum mechanical term "phonon" for our classical sound particles. Justification for this particle model is given elsewhere.[18,19] Each phonon is assumed to follow a straight-line path between successive reflections, and the energy $E = \hbar\omega$ ($\omega/2\pi$ =frequency) and speed v of each phonon are conserved through repeated reflections until the phonon is absorbed on the surface or in the air.

In order to characterize the sound at position $\mathbf{x}$, define the (*spectral*) *angular density* $\psi(t, \mathbf{x}, \theta, \phi)$ such that, at time t, $\psi dV d\Omega$ (dE) is the number of phonons (in energy range dE about E) in the volume dV enclosing $\mathbf{x}$ with direction of motion in the solid angle $d\Omega = \sin\theta d\theta d\phi$ pointing in the direction (θ, ϕ). Hereafter the term spectral is suppressed. An equivalent characterization is represented in terms of the *angular current density* $j = v\psi$ or the *angular power density* $Ej = Ev\psi$ where $j\cos\theta\, dS d\Omega$ is the number of phonons/s which reach a planar area element dS with direction of motion in the solid angle $d\Omega$ (Fig. 1). In a uniform isotropic medium, such as the air we consider here, a *random* sound distribution is homogeneous and isotropic, i.e., ψ and j are independent of position and direction. [In an inhomogeneous medium, such as an auditorium with a thermal gradient or a container of mechanical particles in a gravitational potential, the speed $v(\mathbf{x})$ varies with position, the paths are curved or bent, and a random or equilibrium distribu-

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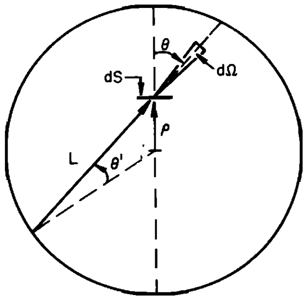

FIG. 1. Coordinate system for describing the phonon distribution.

tion is inhomogeneous. The extension to inhomogeneous media of randomness, Sabine's formula, the mean path length $\langle l \rangle = 4V/S$, and related expressions is given in Ref. 20.]

In Fig. 2 a phonon incident on the surface at position $\mathbf{x}$ with spherical direction coordinates (θ, ϕ) has nonnegative probability $P(\mathbf{x}, \theta, \phi, \theta', \phi')d\Omega'$ of being reflected into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$. (If one derives the results of this paper in terms of a continuum rather than a particle model, then $Pd\Omega'$ is the fraction of the power of an incident beam which is reflected into $d\Omega'$, etc.) Thus, if $j_s(t, \mathbf{x}, \theta, \phi)$ is the current incident on the surface at time t, the reflected current at the surface j_s' is given by

$$j_s'(t, \mathbf{x}, \theta', \phi')\cos\theta' = \int\int P j_s(t, \mathbf{x}, \theta, \phi)\cos\theta d\Omega \,, \tag{3}$$

or, equivalently, in terms of the *reflection matrix* $R(\mathbf{x}, \theta, \phi, \theta', \phi')$, defined as $P/\cos\theta'$, by

$$j_s'(t, \mathbf{x}, \theta', \phi') = \int\int R j_s(t, \mathbf{x}, \theta, \phi)\cos\theta d\Omega \,, \tag{4}$$

where $\cos\theta d\Omega$ is conveniently thought of as a projected solid angle $d\Omega_p$, and unspecified integration ranges are always over the hemisphere $0 \le \phi < 2\pi$, $0 \le \theta \le \pi/2$.

In our universe the reflection matrix R satisfies three previously derived[11] conditions imposed by the first law of thermodynamics, i.e.,

$$1 \ge \int\int R\cos\theta' d\Omega' \,, \tag{5}$$

by the second law of thermodynamics, i.e.,

$$1 \ge \int\int R\cos\theta d\Omega \,, \tag{6}$$

and, in the case of a reciprocal surface, by the principle of detailed balance,[21] i.e.,

$$R(\mathbf{x}, \theta, \phi, \theta', \phi') = R(\mathbf{x}, \theta', \phi', \theta, \phi) \,. \tag{7}$$

Equation (7) is a stronger condition in the sense that it and Eq. (5) imply Eq. (6). Equation (6), but not Eq. (7), is necessary to the validity of $\langle l \rangle = 4V/S$ and the Sabine reverberation-time expression in the limit $\alpha \to 0$ [cf. Eq. (38)]. That is, Sabine's equation is also often valid in universes that violate detailed balance. When there is no absorption Eqs. (5) and (6) become equalities.

The *absorptivity* $\alpha(\mathbf{x}, \theta, \phi)$ and *reflectivity* $r(\mathbf{x}, \theta, \phi) = 1 - \alpha$ for a phonon incident from direction (θ, ϕ) are given by

$$r = 1 - \alpha = \int\int P d\Omega' = \int\int R\cos\theta' d\Omega' \,, \tag{8}$$

and Eq. (5) merely requires that r not exceed unity.

Examples of reflection matrices that satisfy Eqs. (5)–(7) include the case of *specular reflection*,

$$R = 2r(\mathbf{x}, \theta, \phi)\delta(\sin^2\theta - \sin^2\theta')\delta(\phi - \phi' \pm \pi) \,, \tag{9}$$

and the case of *random reflection*,

$$R = r(\mathbf{x})/\pi \,. \tag{10}$$

In Eq. (9) the delta functions, interpreted in an obvious way for the azimuthal ϕ angle, force θ to equal θ' and ϕ' to equal $\phi \pm \pi$. A randomly reflecting surface is one which appears equally bright, to use an optical term, regardless of the angle that the observer's viewing direction makes with the surface normal; that is, R, and hence j_s', does not depend upon the reflected direction (θ', ϕ'). Note that if R does not depend upon the reflected direction (θ', ϕ'), then Eq. (7) prevents r from depending upon the incidence direction (θ, ϕ) in Eq. (10).

Equation (6) keeps the surface from acting as a Maxwell demon which could combine low-power beams incident from various directions into a single higher-power/area-solid angle reflected beam. For example, a commonly encountered reflection law which violates Eq. (6) is the "uniform" distribution over the hemisphere of reflected solid angles, i.e.,

$$P d\Omega' = R\cos\theta' d\Omega' = (r/2\pi)d\Omega' . \tag{11}$$

For the main example of this paper we take a particular linear combination of specular and random reflection. Specifically, for every surface position $\mathbf{x}$, we assume that a fraction s of the reflected phonons are specularly reflected and the remainder, $1 - s$, are randomly directed; i.e., for $0 \le r \le 1$ and $0 \le s < 1$,

$$R = (1 - s)(r/\pi) + 2sr\delta(\sin^2\theta - \sin^2\theta')\delta(\phi - \phi' \pm \pi). \tag{12}$$

For simplicity the r multiplying the delta function is taken to be independent of the incidence angle, and thus r and s are constants in the main example.

It should be emphasized that Eq. (12) is a linear combination of random and specular scattering chosen for mathematical convenience, not physical necessity. As an example of an alternative possibility, consider the special case that R is the product of functions of the incidence and reflected directions; e.g.,

$$R = (1/\pi)\sin^\eta\theta \sin^\eta\theta' , \quad 0 \le \eta \le \infty \,. \tag{13}$$

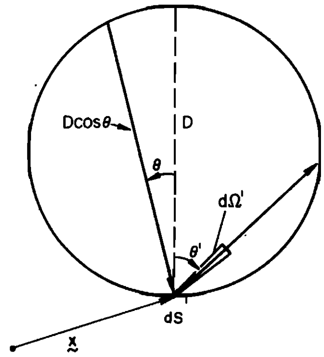

FIG. 2. Nonspecular reflection of a phonon into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$.

Using Eq. (8) we can write Eq. (13) in the more suggestive form

$$R = r(\theta)[(\eta+2)/2\pi]\sin^{\eta}\theta' , \tag{14}$$

with

$$r(\theta) = 1 - \alpha(\theta) = [2/(\eta+2)]\sin^{\eta}\theta . \tag{15}$$

In Eq. (15), $\alpha(\theta)$ is the absorption probability for a phonon incident at angle θ, and $r(\theta) = 1 - \alpha(\theta)$ is the probability that it be reflected (in any direction). For $\eta = 0$, R is independent of the reflected direction (random scattering), and there is no absorption. For $\eta > 0$, absorption is introduced and the reflected sound intensity becomes nonrandom—peaked away from the normal by the factor $\sin^{\eta}\theta'$. Equation (7) forces the inclusion of the factor $\sin^{\eta}\theta$ in $r(\theta)$; that is, if the scattered current has a preferred direction, then the probability of absorption α must depend upon the incidence direction θ. For $\eta = \infty$, there is complete absorption. Replacement of each sin in Eq. (13) with, for example, cos or $1 - \sin$ peaks the scattered sound intensity toward the normal.

In an earlier example,[11] in the case of *no absorption* and a scattered angular distribution with no memory of the incidence direction, it was pointed out that the only scattering law permitted by Eqs. (5) and (6) is the random form $R = 1/\pi$ [the $r = 1$ case of Eq. (10), or the $s = 0$, $r = 1$ case of Eq. (12), or the $\eta = 0$ case of Eq. (13)]. There is, however, no unique extrapolation of this example to absorbing surfaces. One possible extrapolation is $R = r/\pi$, as used in Eq. (12). Another of the nondenumerably many physical possibilities is Eq. (14). In Eq. (14) the scattered distribution is not random for $0 < \eta < \infty$. However, apart from the absorption probability $\alpha(\theta)$, the scattered angular distribution does *not* depend upon the incidence direction. That is, in the presence of absorption, memoryless scattering does not imply random reflectivity nor isotropic absorptivity.

For a two-dimensional enclosure the probability of reflection into $d\theta'$ is $Pd\theta' = R\cos\theta' d\theta'$. One then makes the substitution $d\Omega \to d\theta$ in Eqs. (5) and (6), and the single integration is over the range $-\pi/2$ to $\pi/2$. Analogs of Eqs. (7), (12), and (13) are, respectively, $R(\mathbf{x}, \theta, \theta') = R(\mathbf{x}, \theta', \theta)$, $R(\theta, \theta') = (1-s)r/2 + sr\delta(\sin\theta + \sin\theta')$, and $R = \frac{1}{2}\sin^{\eta}\theta \sin^{\eta}\theta'$.

II. REVERBERATION TIME

If $N(t)$ is the total number of phonons or total sound energy (in a narrow phonon energy range ΔE) at a time t seconds after the cessation of sound generation, then we take for the definition of the *decay rate* A or the *reverberation time* T, if T exists and has a finite value greater than zero,

$$A = 1/T = \lim_{t\to\infty}(N^{-1}dN/dt) . \tag{16}$$

Thus T is related to the usual 60-dB-attenuation reverberation time T_{60} by

$$T_{60} = 6T\ln 10 \approx 13.82T . \tag{17}$$

When finite nonzero T exists, the asymptotic sound distribution is, apart from an arbitrary numerical factor, given by

$$v\psi_a(\mathbf{x}, \theta, \phi) = j_a(\mathbf{x}, \theta, \phi) = \lim_{t\to\infty}[e^{At}j(t, \mathbf{x}, \theta, \phi)] . \tag{18}$$

We consider only those enclosures where, for almost every initial sound distribution, the limit $t\to\infty$ in Eqs. (16) and (18) causes memory of the initial distribution to fade away and yields a value for A and an asymptotic distribution that are unique intrinsic properties of the enclosure, independent of the initial distribution. Thus, for example, the asymptotic decay rate of the sound's angular energy density $E\psi$ is the same almost everywhere in the enclosure.

In contrast, as an example of an enclosure with a nonunique decay rate, note that a specularly reflecting isotropically absorbing sphere of diameter D causes a phonon to have the same incidence angle θ at every reflection. Then, for initial conditions where a reverberation time exists, Eq. (16) yields

$$T = -D\cos\theta_0/v\ln r . \tag{19}$$

In Eq. (19), θ_0 is the smallest incidence angle present in the initial distribution; i.e., any reverberation time shorter than $-D/v\ln r$ is possible if one begins with sound of sufficiently grazing incidence. The case $s = 1$ is therefore excluded from Eq. (12).

In the theory of nonabsorbing enclosures[9,10] a *mixing* enclosure is defined by the property that it randomize almost every initial distribution. That is, if all of the phonons initially fill uniformly any area-solid angle $\Delta V'\Delta\Omega'$, then as $t\to\infty$ the fraction of the phonons which fill any area-solid angle $\Delta V\Delta\Omega$ approaches $(\Delta V/V) \times (\Delta\Omega/4\pi)$. In the case of absorption we define a *mixing* or *randomizing enclosure* as one having a unique asymptotic distribution for almost every[22] initial distribution. Restriction to the forms of R permitted by Eqs. (5) and (6) presumably ensures the equivalence of these definitions in the limit of zero absorption.

Even among randomizing enclosures we consider only those that forget the initial conditions fast enough to yield a finite value for T. As a counter example note that $T = \infty$ results from Eq. (16) in the case of a rectangular room with two opposing nonabsorbing specular walls and two rough partially absorbing walls. (For nonrandomizing enclosures T may fail to exist not only because T may be infinite, but alternately for many initial conditions because the sound remains bunched up and the energy decrease is not exponential but rather stepped each time the bunch hits the surface.[11])

If the surface is perfectly absorbing ($\alpha = 1$), then T is undefined by Eq. (16) because both N and dN/dt are identically zero after a finite length of time. For $\alpha = 1$, we arbitrarily define the reverberation time by interchanging the limiting operations, i.e.,

$$T(\alpha = 1) = \lim_{\alpha\to 1} T(\alpha) . \tag{20}$$

Following Kuttruff's[12] idea we now derive an integral-equation with T as the eigenvalue. The reflection matrix R is, however, not necessarily of the form of Eq. (10), and the enclosure is such that ψ_a and finite nonzero T exist and are unique.

As an extension to time-varying beams of the steady-

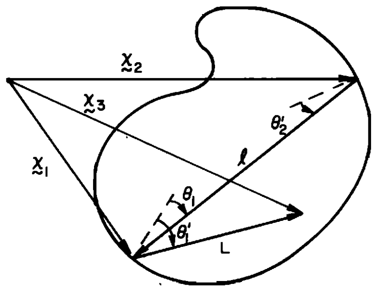

FIG. 3. Coordinate system for the reverberation-time integral equation, Eq. (22).

state brightness-conservation theorem[23] it can be shown[18] that $j(t, \mathbf{x}, \theta, \phi)$ is a conserved quantity along the path of natural reversible motion in uniform media. Thus, apart from the irreversible air-absorption factor e^{-ml} ($m=$ air absorptivity), the incident surface current $j_s(t, \mathbf{x}_1, \theta_1, \phi_1)$ at the lower end of the chord of length l in Fig. 3 is the reflected current $j_s'(t-\Delta t, \mathbf{x}_2, \theta_2', \phi_2')$ which left the upper end a time $\Delta t = l/v$ earlier, i.e.,

$$j_s(t, \mathbf{x}_1, \theta_1, \phi_1) = j_s'(t - l/v, \mathbf{x}_2, \theta_2', \phi_2')e^{-ml} . \tag{21}$$

Substituting Eq. (21) into Eq. (4) and trying a solution of the form

$$j_s'(t, \mathbf{x}, \theta', \phi') = j_{s,a}'(\mathbf{x}, \theta', \phi')\exp(-At)$$

yield as the desired extension of Kuttruff's integral equation[12] for $j_{s,a}'$

$$j_{s,a}'(\mathbf{x}_1, \theta_1', \phi_1') = \int\int R(\mathbf{x}_1, \theta_1, \phi_1, \theta_1'\phi_1') \times j_{s,a}'(\mathbf{x}_2, \theta_2', \phi_2')e^{(A/v-m)l}\cos\theta_1 d\Omega_1 , \tag{22}$$

where $\mathbf{x}_2, \theta_2', \phi_2'$, and l are regarded as functions of $\mathbf{x}_1$, θ_1 and ϕ_1, and the solution $j_{s,a}'$ of physical significance does not change sign.

For a two-dimensional enclosure one makes the substitution $d\Omega_1 \to d\theta_1$ in Eq. (22), and the single integration is over $-\pi/2 \le \theta_1 \le \pi/2$.

Once Eq. (22) is solved for $j_{a,s}'(\mathbf{x}, \theta', \phi')$, the idea of Eq. (21) can be used again to find the asymptotic current j_a, angular power Ej_a, and phonon density ψ_a at any interior point and direction, i.e., in the coordinates of Fig. 3 at the end of the vector of length L

$$v\psi_a(\mathbf{x}_3, \theta_1', \phi_1') = j_a(\mathbf{x}_3, \theta_1', \phi_1') = j_{a,s}'(\mathbf{x}_1, \theta_1', \phi_1')e^{(A/v-m)L} . \tag{23}$$

In Eqs. (22) and (23) the effect of air absorption is in accord with Knudsen's rule,[24]

$$A = vm + A(m=0); \tag{24}$$

so ordinarily one simply neglects m during calculations and then corrects the result with Eq. (24).

Any connected enclosure with a reflection matrix given by Eq. (12) is assumed to be randomizing for $0 \le s < 1$. In the case of a sphere of diameter D there is no $\mathbf{x}$ or ϕ dependence, and Eq. (22) reduces to

$$j_{s,a}'(\theta')[1 - sre^{\mu\cos\theta'}] = (1-s)r\int_0^{\pi/2} j_{s,a}'(\theta)e^{\mu\cos\theta}\sin 2\theta d\theta = g \tag{25}$$

with

$$\mu = (A/v - m)D \ge 0 . \tag{26}$$

Because θ' does not appear after the first equality sign in Eq. (25), the integral defines a function g which does not depend upon θ'. Thus the left side of Eq. (25) shows that the θ' dependence of $j_{s,a}'(\theta')$ must be of the form

$$j_{s,a}'(\theta') = [1 - sre^{\mu\cos\theta'}]^{-1}g . \tag{27}$$

Using Eq. (27) to eliminate $j_{s,a}'(\theta)$ under the integral of Eq. (25) reduces the last equality of Eq. (25) to

$$\frac{1}{2}\frac{s}{(1-s)} = \frac{1}{2}\int_0^{\pi/2}[(sr)^{-1}e^{-\mu\cos\theta} - 1]^{-1}\sin 2\theta d\theta$$

$$= \mu^{-2}(\ln sr)\ln\frac{1-sre^{\mu}}{1-sr} + \mu^{-2}\int_{sr}^{sre^{\mu}}\frac{\ln y}{1-y}dy$$

$$= \mu^{-2}(\ln sr)\ln\frac{1-sre^{\mu}}{1-sr} + \mu^{-2}f(sre^{\mu}) - \mu^{-2}f(sr), \tag{28}$$

where $0 \le s < 1$, and $f(x)$, defined by

$$f(x) = \int_1^x \frac{\ln y}{1-y}dy , \tag{29}$$

is the dilogarithm function.[25,26] Equation (28) is the eigenvalue equation for $A = 1/T$; that is, the right-hand side can be evaluated for any values of $\mu = (A/v - m)D$ and the sr product in the physical range $0 \le \mu < \infty$, $0 \le sr < e^{-\mu}$. Then s and $r = 1 - \alpha = (sr)/s$ are found separately from the left side and the assumed value of sr. (For fixed μ, both α and s increase with increasing sr.)

For the two-dimensional circular enclosure one makes the substitution $d\theta \to d\theta/2\sin\theta$ in the first form of Eq. (28).

For random reflection only ($s=0$), Eq. (28) integrates to the result first obtained by Carroll and Chien[27] who used Kuttruff's integral equation[12]

$$\alpha = 1 - r = 1 - \tfrac{1}{2}\mu[\mu^{-1} + e^{\mu}(1-\mu^{-1})]^{-1} , \quad s = 0 . \tag{30}$$

Equation (30) is plotted as a solid line in Fig. 4 assuming no air absorption ($\mu = AD/v$) and labeled "random reflection."

As the specular limit is approached ($s \to 1$), Eq. (28) approaches

$$r = 1 - \alpha = e^{-\mu} = e^{-AD/v} , \tag{31}$$

which is equivalently written as

$$2AD/3v = -\tfrac{2}{3}\ln(1-\alpha) , \quad s = 1 \tag{32}$$

$$= \tfrac{2}{3}\alpha + \tfrac{1}{3}\alpha^2 + \cdots . \tag{33}$$

Equation (32) is labeled "specular-reflection limit" in Fig. 4. Equation (32) can be arbitrarily closely approached for s sufficiently near the excluded limit $s = 1$.

As s is varied over the range $0 < s < 1$, Eq. (28) generates curves which fill the space between the two solid limiting curves in Fig. 4. For example, the case $s = \frac{7}{9}$ is shown as a dotted line.

From Eqs. (23) and (27) one finds, in the coordinate system of Fig. 1, that the asymptotic distribution is

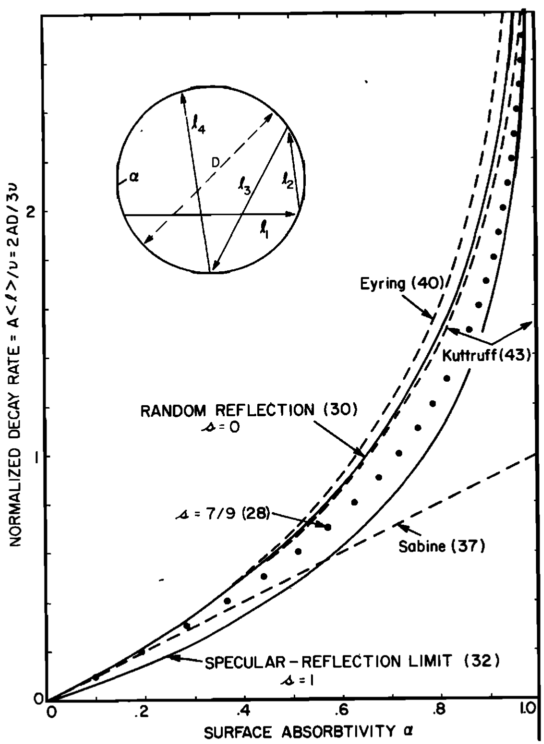

FIG. 4. Decay rate A (= reciprocal of the $1/e$ reverberation time T) for a spherical auditorium of diameter D with uniform isotropic surface absorptivity α. As the surface reflection law, Eq. (12), is varied from random ($s=0$) to specular ($s=1$), the exact solution moves between the solid-curve limits. Intermediate curves, as exemplified by the dotted line for the case $s=\frac{7}{9}$, are given by Eq. (28). Air absorptivity m is neglected, but can be accounted for exactly with Eq. (24). The exact results (solid and dotted lines) and various approximations (dashed lines) are identified by their equation numbers. The sound speed is v.

$$v\psi_a(\rho,\theta)=j_a(\rho,\theta)=[1-sre^{\mu\cos\theta'}]^{-1}e^{\mu L/D} \tag{34}$$

with L and θ' regarded as functions of θ and ρ. Clearly the distribution is inhomogeneous and anisotropic almost everywhere unless the absorption is due entirely to air ($\alpha=0$, $\mu=0$). The assumption of a random asymptotic distribution, or equivalently of the random-sound path length $4V/S$, would therefore have led to an incorrect reverberation time (except in the limit $\alpha\rightarrow 0$, $s\neq 1$).

Physically, Eqs. (32) and (34) can be understood as follows: For s near unity Eq. (31) indicates that e^{μ} is approximately equal to $1/r$, and, thus, the coefficient of the exponential factor in Eq. (34) becomes arbitrarily large near $\theta'=0$. That is, the phonons scatter toward and accumulate near their longest-lifetime paths which are, of course, the diameters of the sphere since those paths maximize the time delay between absorbing surface impacts. Equation (32) may be thought of as the Eyring reverberation time for phonons which travel a distance D, rather than $4V/S=2D/3$, between reflections. Some random reflection ($s\neq 1$) is necessary because otherwise phonons could not happen onto the diameters. For $s=1$, Eq. (32) does not yield the reverberation time if diameters were not included in the solid angle of the source which generated the initial conditions [cf., Eq. (19)].

For the usual case of small absorptivity it is worthwhile to expand μ in a power series in $\alpha=1-r$. With $-2mD/3$ omitted from the left-hand side the result through α^2 is, from Eq. (28),

$$\frac{2AD}{3v}=\frac{2\mu}{3}=\alpha+\frac{1}{16}\frac{7-9s}{1-s}\alpha^2+\cdots, \quad 0\leq s<1\,. \tag{35}$$

The coefficient $k=(7-9s)/16(1-s)$ of α^2 thus has the wide possible range $-\infty<k\leq 7/16$ and can be of either sign. The fact that the sphere is suddenly nonrandomizing at $s=1$ is reflected in the breakdown of the power series. That is, the coefficient of α^2 does not exist for $s=1$; the radius of convergence shrinks to $|\alpha|=0$, the point of agreement of Eqs. (33) and (35).

III. DISCUSSION

For a uniformly absorbing sphere of volume $V=\pi D^3/6$ and area $S=\pi D^2$, the mean path length for random sound is $\langle l\rangle=4V/S=2D/3$, and the Sabine reverberation-time expression,[8] as completed theoretically by Franklin,[28]

$$1/T_s=A_s=v\alpha S/4V\,, \tag{36}$$

takes the form

$$2A_sD/3v=\alpha\,. \tag{37}$$

Equation (37) is labeled "Sabine" in Fig. 4. Curiously, although $4V/S$ had been identified as a mean path length in various contexts,[29,30] neither Franklin nor Sabine recognized it as such. Sabine was, however, an advocate of the concept of the mean path length of random sound, but he unfortunately misevaluated it.[11]

It is our contention[31] that if ψ_a and finite T exist and are independent of the initial sound distribution for almost every distribution and if ψ_a (as characterized by its entropy) approaches a random distribution as the absorptivity goes to zero, then the Sabine expression becomes exact in the sense that

$$T/T_s=A_s/A\rightarrow 1\,, \quad \alpha\rightarrow 0 \tag{38}$$

as the absorptivity approaches zero. That is, the Sabine and the exact rates should be tangent at $\alpha=0$.

The example of this paper meets the conditions of Eq. (38) for $0\leq s<1$, and indeed one immediately sees from Eqs. (33), (35), and (37) that Eq. (38) is satisfied unless[32] $s=1$. We regard this as a confirmation of the Sabine-Franklin expression (and of the applicability of the implicit $4V/S$ path length formula) under Sabine's own, rather qualitatively stated, conditions of validity—that "irregular reflection" be present and that the surface absorptivity be sufficiently weak that "the dispersion of sound between all parts of a hall is very rapid in comparison with the total time required for its complete absorption, and that in a very short time after the source has ceased the intensity of the residual sound... is very nearly the same everywhere in the room."[8]

Physically the absence of s from the coefficient of α in Eq. (35), which is essential to the compliance with Eq. (38), is understood as follows: Even if there is only a very small amount of random reflection, i.e., s very nearly equal to unity, Eq. (34) shows that one can always make the absorptivity so small that this random reflection virtually randomizes the asymptotic sound distribution. Graphically, for s near unity, Eq. (28) describes a curve which is close to the specular-limit line in Fig. 4 except near $\alpha = 0$ where the curve moves over to become tangent to the Sabine line.

For the particular case $s = \frac{7}{9}$ ($\approx 78\%$ specular reflection, $\approx 22\%$ random reflection) the Sabine expression also yields the correct coefficient, $k = 0$, of α^2.

It should be mentioned that the Sabine expression will usually fail to satisfy Eq. (38) if the "exact" T is computed from a surface reflection law that violates Eq. (6). Examples of such laws in the recent literature include the case that the scattered phonons are uniformly directed over the hemisphere [Eq. (11)] and the case that phonons are scattered from one component of the surface toward the other components in such a way that the numbers reaching the other components are in proportion to the areas of those components. Sabine's expression and the evaluation $\langle l \rangle = 4V/S$ survive in universes that violate detailed balance, but not in those that violate the second law of thermodynamics. (Violation of the second law results in an asymptotic sound distribution that is not homogeneous and isotropic even with an initially random distribution and no absorption.)

The so-called Eyring expression first published by Schuster and Waetzmann[13-16]

$$1/T_E = A_E = -v(S/4V)\ln(1-\alpha)\ , \tag{39}$$

takes the following form for a sphere

$$2A_E D/3v = -\ln(1-\alpha) \tag{40}$$

$$= \alpha + \tfrac{1}{2}\alpha^2 + \cdots\ . \tag{41}$$

Equation (40) is labeled "Eyring" in Fig. 4. From Eqs. (35) and (41) it is apparent that the Eyring coefficient of α^2 is close to, but just beyond, the random-scattering value, $k = \frac{7}{16}$. Equation (40) does satisfy the Sabine limit, Eq. (38), and does "correctly" yield $A_E = +\infty$ or $T_E = 0$ at $\alpha = 1$ [cf. Eq. (20)]. For small α, Eq. (40) is more accurate than Eq. (37) only for $s < \frac{3}{5}$.

Kuttruff's modification[17] of the Eyring formula,

$$1/T_K = A_K = -v(S/4V)\ln(1-\alpha)[1 + \tfrac{1}{2}\gamma^2 \ln(1-\alpha)]\ , \tag{42}$$

reduces, for the sphere, to

$$2A_K D/3v = -\ln(1-\alpha)[1 + \tfrac{1}{16}\ln(1-\alpha)] \tag{43}$$

$$= \alpha + \tfrac{7}{16}\alpha^2 + \cdots\ , \tag{44}$$

where γ^2, the relative variance in the random-sound mean path length $\langle l^2\rangle/\langle l\rangle^2 - 1$, is readily found to be $\frac{1}{8}$ for a sphere. Equation (43) is labeled "Kuttruff" in Fig. 4. Kuttruff's expression satisfies the Sabine limit, Eq. (38), and yields the correct coefficient of α^2 in Eq. (35) for the particular case $s = 0$ (random reflection only), but it is not, nor did he claim it to be, correct as $\alpha \to 1$ (cf. Fig. 4 where $A_K \to -\infty$ as $\alpha \to 1$). For small α, Eq. (43) is more accurate than Eq. (37) only for $s < \frac{7}{11}$.

In two dimensions there are no qualitative differences. One replaces $4V/S = \langle l \rangle$ in the Sabine, Eyring, and Kuttruff formulas by $\langle l \rangle = \pi V/S$. For a circle of diameter D, the area is $V = \pi D^2/4$, the boundary length is $S = \pi D$, $\langle l \rangle$ is $\pi D/4$, and γ^2 is $32/3\pi^2 - 1$. Again Kuttruff's coefficient of α^2 is precisely the $s = 0$ value.

We have treated the spherical enclosure with variable reflection law for two reasons: On the one hand the problem is simple enough to permit an exact closed-form solution for the reverberation time and thus to avoid the vagaries of approximate solutions. On the other hand it is complicated enough to illustrate the two considerations which dominate the understanding of every enclosure.

The first of these considerations is the distinction between a randomizing and a nonrandomizing enclosure and the demonstration that one cannot ignore this distinction. The sphere is nonrandomizing for purely specular reflection ($s = 1$) and randomizing if there is any random reflection ($0 \le s < 1$). A randomizing enclosure has a unique reverberation time, while for a nonrandomizing enclosure the reverberation time, if it exists, depends upon the initial conditions [cf. Eq. (19)]. Although randomization is achieved by random surface reflection in the example, it can also be achieved with specular reflection only, provided the enclosure shape is of an appropriate form[9,10] more complicated than a sphere or rectangular parallelepiped.

The second consideration derives from the fact that in randomizing enclosures there are two competing effects which influence the reverberation time in opposite directions. Because air is uniformly distributed throughout the enclosure, the exact reverberation-time formula, $1/T = A = mv$ [Eq. (24)], is very simple when the absorption is due to air only. Sabine's formula, Eq. (36), is equally simple because it represents the surface absorption as if it were uniform air absorption. The first competing effect, which shortens the reverberation time from the Sabine value, results from the fact that surface absorption actually occurs at the boundary of the air, not uniformly throughout the air.

The other competing effect, which acts to lengthen the reverberation time, is the tendency of sound to accumulate on the paths of longer life. Because we chose to use an incidence-angle-independent absorptivity, sphere diameters are the longest-life paths in the example. (In a square room with two opposing weakly absorbing walls and two strongly absorbing walls, the longer-life paths are, of course, between the weakly absorbing walls, etc.) The accumulation is driven by the surface absorption, but it is more or less successfully resisted by the randomizing strength of the enclosure as determined by the roughness of the surface, irregularities in the enclosure shape, and the presence of furniture, statues, or other diffusors.[33]

In a weakly absorbing sphere with less than $\frac{2}{9}$ random reflection and, hence, more than $\frac{7}{9}$ specular reflection ($\frac{7}{9} < s < 1$), Eq. (35) shows that the accumulation effect is sufficiently strong that the reverberation time is longer than the Sabine value—up to 50% longer in this sphere[32] and unlimitedly longer in other enclosures.[11] With more than $\frac{2}{9}$ random scattering, the boundary effect dominates, and the reverberation time is shorter than the Sabine value. If the surface absorptivity exceeds $\alpha \approx 0.58$ (the intersection of the two lower curves in Fig. 4), the boundary effect is so strong that the reverberation time is always shorter than the Sabine value for the sphere.

The fact that the Eyring and Kuttruff formulas are very close to the random-reflection curve in Fig. 4 is thus understood as follows: These two formulas consider the fact that surface absorption actually occurs only at the air boundary, but even so they are necessarily approximations because their derivations ignore the universal tendency of sound to seek out the longer-lived paths and become nonrandom. Recognizing this limitation Eyring and Kuttruff indicated that their expressions are most accurate for well-randomized sound. But how is one to know when the asymptotic sound distribution is well randomized? This is a difficult question, in general, but one that is answered for a simple case by the present paper. (Incidentally, the uniformly absorbing sphere is an example favorable for the accuracy of the Eyring and Kuttruff formulas. In enclosures of more complex shape and less uniformly distributed absorptivity, a given amount of surface roughness can be much less successful in randomizing the sound.)

The Eyring formula, in the broad way it is usually presented, is typical of attempts to do the impossible; i.e., to improve upon Sabine's formula without invoking any additional information. The Sabine-Franklin expression, Eq. (36), yields the correct coefficient of α in Eq. (35) without requiring the value of s, but no one can find the coefficient of α^2, or even its sign, without introducing s or some equivalent information about the randomizing strength of the enclosure.

IV. SUMMARY

The geometrical-acoustics reverberation time and asymptotic sound distribution are found exactly for a uniformly absorbing spherical enclosure with a surface that is continuously variable from smooth to rough. This result is interpreted as a confirmation of the Sabine reverberation-time expression and of the applicability of the underlying $4V/S$ mean-path-length formula under Sabine's stated conditions of irregular reflection and weak absorptivity ($\alpha \rightarrow 0$). The result also emphasizes the difference between randomizing and nonrandomizing enclosures and the fact that nonrandomizing enclosures do not have unique reverberation times nor mean path lengths. The competing effects of boundary absorption and derandomization by accumulation on long-life paths are identified and seen to make characteristic corrections of opposite sign to the Sabine expression. It is shown that the Eyring and Kuttruff formulas, which consider only the first of these competing effects, cannot be said to be more accurate than the Sabine expression unless one introduces the additional information that the enclosure is quite randomizing or absorbing.

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[32]The fact that our enclosure becomes nonrandomizing at $s=1$ shows up a a noncommuting limit. That is, from Eqs. (33), (35), and (37) one finds

$$\lim_{s\to 1}\lim_{\alpha\to 0}(A/A_S)=1$$

whereas

$$\lim_{\alpha\to 0}\lim_{s\to 1}(A/A_S)=\tfrac{2}{3}$$

[33]In general, depending upon its shape and location, a diffusor or other introduced object, even though nonabsorbing, may increase or decrease the randomness (entropy) of the asymptotic distribution and may increase or decrease the reverberation time. However, for the more usual room geometries and absorptivity locations, the time is decreased; i.e., "any obstacle which may tend to break up the wave and interfere with the reflection through the axis of the room will serve to lessen the resonance" [J. Henry, Can. J. Sci. Lit. Hist. 2, 130–140 (1857)].