Dynamic Indexes

Algorithmic Design

Alberto Casagrande
Email: `acasagrande@units.it`

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Motivations

A Simple Problem for Registry Office

Let us consider the registry office

For each newborn, they record a set of data e.g., name, birthday, parents, etc.

So, the registry data-set (hopefully) changes quite often

What if they frequently perform a birthday-based search on the data-set? E.g., Find all the baby born a given (variable) day?

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The data-set often changes, thus, array is not the most suitable data structure to achieve this goal

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We need a data structure providing (efficient) support for:

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- searching data
- removing data

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We are aiming to build an index i.e., an auxiliary data structure to "efficiently" perform above operations

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`x.left=NIL`, then x misses the left child

`T.root` is the root of the tree T and `T.root.parent=NIL`

Some Useful $\Theta(1)$ Functions

```
def IS_ROOT(x):
  return x.parent=NIL
enddef

def IS_RIGHT_CHILD(x):
  return ¬IS_ROOT(x) and x.parent.right=x
enddef

def SIBLING(x):     # get x's sibling
  if IS_RIGHT_CHILD(x):
    return x.parent.left
  endif
  return x.parent.right
enddef
```

Some Useful $\Theta(1)$ Functions (Cont'd)

```
def CHILDHOOD_SIDE(x):  # get x's side w.r.t.
                        # its parent
  if IS_RIGHT_CHILD(x):
    return RIGHT
  endif
  return LEFT
enddef

def REVERSE_SIDE(side):    # reverse the side
  if side=LEFT:
    return RIGHT
  endif
  return LEFT
enddef
```

Some Useful Θ(1) Functions (Cont'd 2)

```
def GET_CHILD(x, side):  # get x's child on side
  if side=LEFT:
    return x.left
  endif

  return x.right
enddef
```

Some Useful $\Theta(1)$ Functions (Cont'd 3)

```
def SET_CHILD(x, side, y):  # set x's child
  if side=LEFT:
    x.left ← y
  else:
    x.right ← y
  endif

  if y≠NIL:
    y.parent ← x
  endif
enddef
```

Some Useful $\Theta(1)$ Functions (Cont'd 4)

```
def UNCLE(x):          #get x's uncle
  return SIBLING(x.parent)
enddef

def GRANDPARENT(x):       # get x's granpa
  return x.parent.parent
enddef

def NEW_NODE(v):        # build a new node
  x.key ← v
  x.parent ← NIL
  x.right ← NIL
  x.left ← NIL
enddef
```

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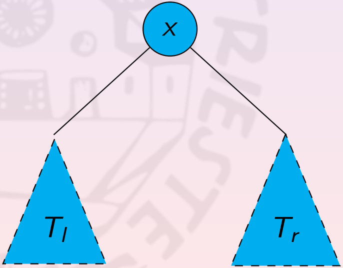
A Binary Search Tree (BST) is a tree s.t.:

- all the keys belong to a totally ordered set w.r.t. $\preceq$



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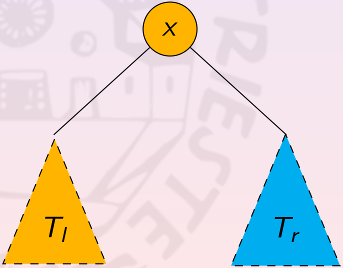

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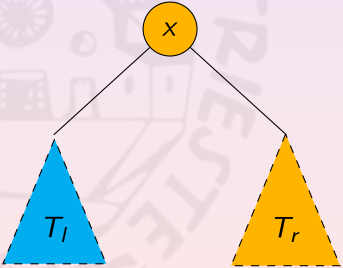
- if x_r is in the right sub-tree of x, then $x.key \preceq x_r.key$



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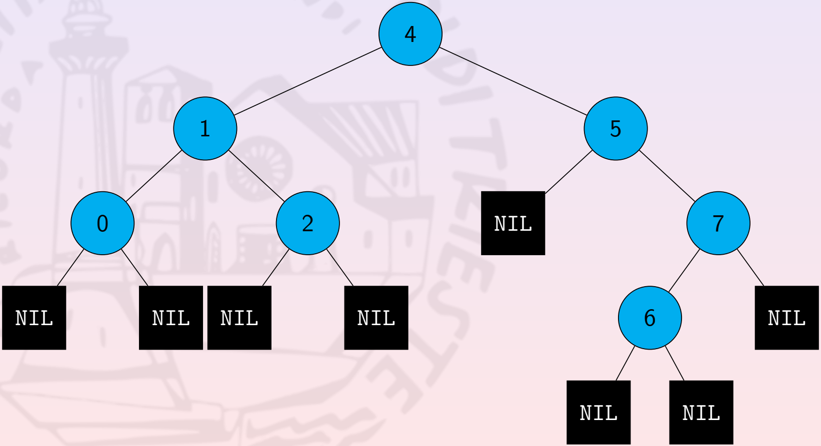

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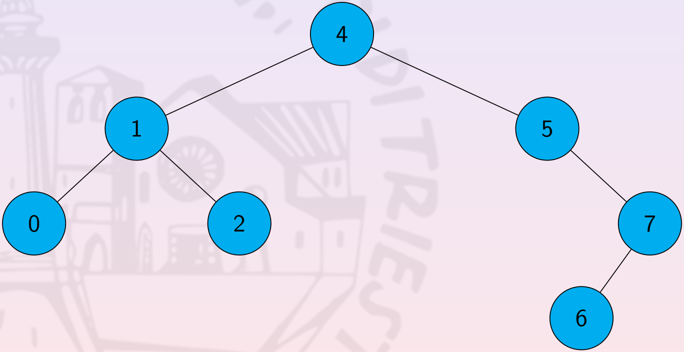


In-Order Walk

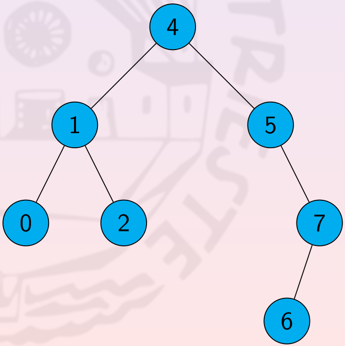
```
def INORDER_WALK_AUX(x):
  if x≠NIL:
    INORDER_WALK_AUX(x.left)
    print x.key
    INORDER_WALK_AUX(x.right)
  endif
endif


def INORDER_WALK(T):
  INORDER_WALK_AUX(T.root)
endif
```

Searching for the Maximum/Minimum

Due to the BST property:

- the minimum key is contained by the first node on the leftmost branch that has not a left child



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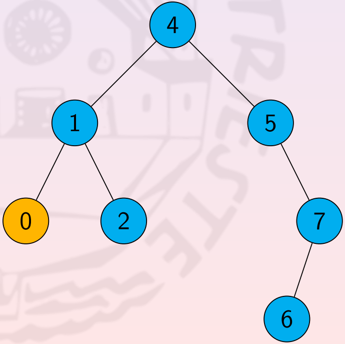

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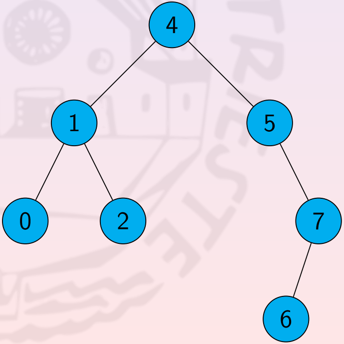

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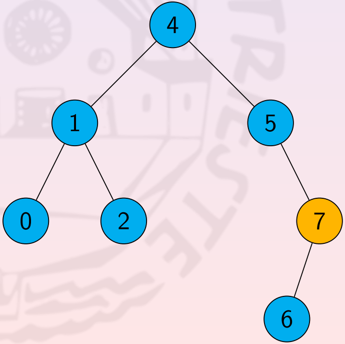

Searching for the Maximum/Minimum: Pseudo-Code

```
def MINIMUM_IN_SUBTREE(x):
  while x.left ≠ NIL:
    x ← x.left
  endif
  return x
endif


def MAXIMUM_IN_SUBTREE(x):
  while x.right ≠ NIL:
    x ← x.right
  endif
  return x
endif
```

Searching for the Maximum/Minimum: Pseudo-Code

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$O(h_T)$

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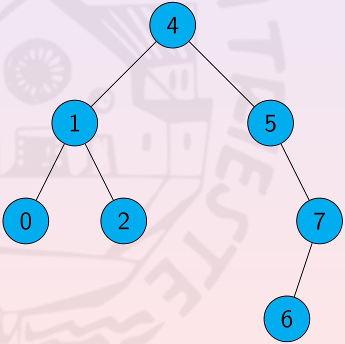
Due to the BST property, the successor w.r.t. $\preceq$ of n is either



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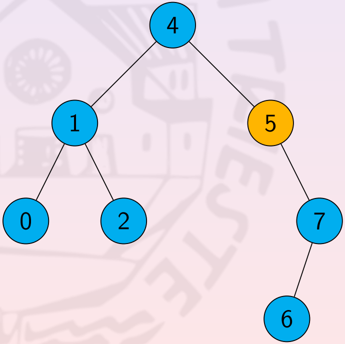
Due to the BST property, the successor w.r.t. $\preceq$ of n is either

- the node containing the minimum in the right sub-tree of n



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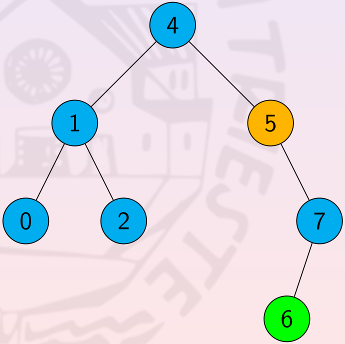
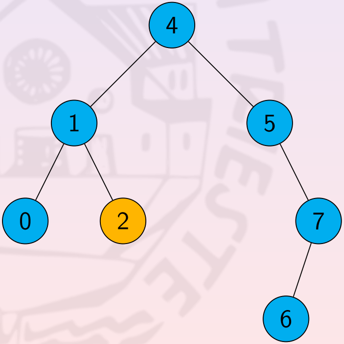


Successor of a Node

Due to the BST property, the successor w.r.t. $\preceq$ of n is either

- the node containing the minimum in the right sub-tree of n or
- the nearest "right-ancestor" of n, if n has not right child



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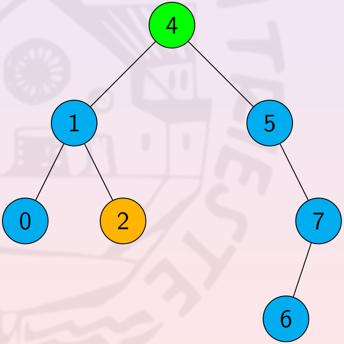

Successor of a Node: Pseudo-Code

```
def SUCCESSOR(x):
  if x.right ≠ NIL:
    return MINIMUM_IN_SUBTREE(x.right)
  endif

  y ← x.parent
  while y≠NIL and IS_RIGHT_CHILD(x):
    x ← y
    y ← x.parent
  endwhile

  return y
enddef
```

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```

$O(h_T)$

Searching for a Value in a BST

To search for v, apply a dichotomic approach from the root x:

- if n is NIL or $x.key = v$, return n
- if $x.key \preceq v$, search on the right sub-tree
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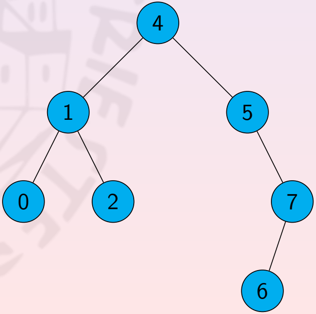
Searching for 3



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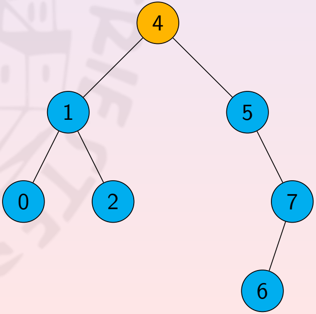

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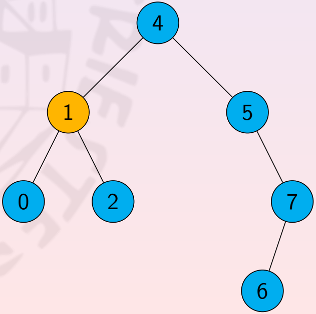

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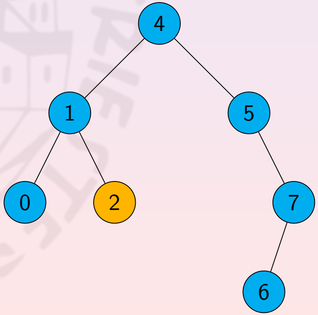

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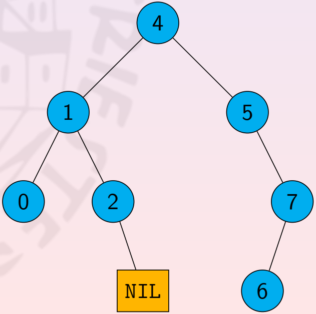


Searching for a Value in a BST: Pseudo-Code

```
def SEARCH_SUBTREE(x, v):
  while x≠NIL:
    if x.key ⪯ v:
      if v ⪯ x.key:
        return x
      endif

      x ← x.right
    else:
      x ← x.left
    endif
  endwhile
enddef
```

Searching for a Value in a BST: Complexity

Each iteration performs $\Theta(1)$ operations

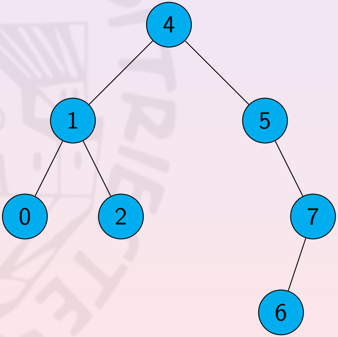
The # of iterations depends on the height h_T of T and on v

The algorithm takes time $O(h_T)$

Inserting a Value in a BST

Browse a branch of T and add a new leaf having v as key

Inserting 3



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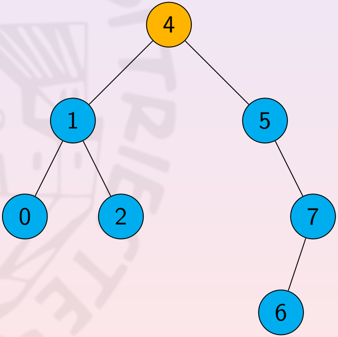

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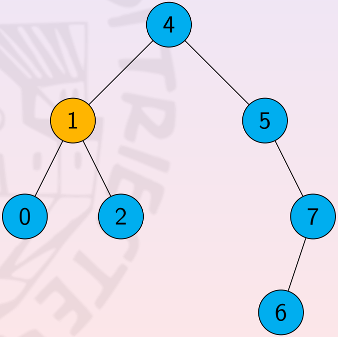

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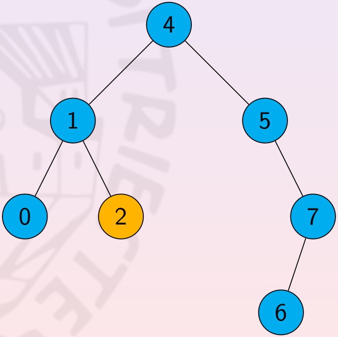

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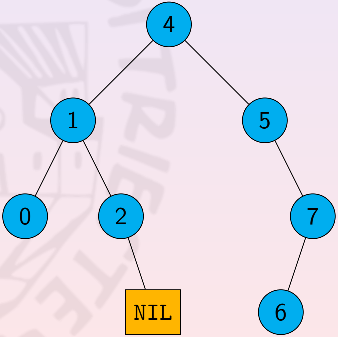

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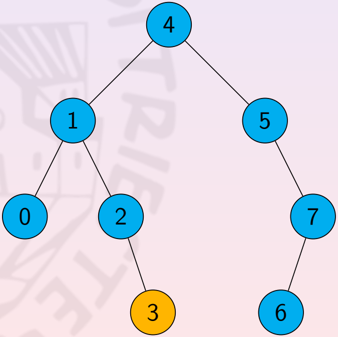

Inserting a Value in a BST: Pseudo-Code

```
def INSERT_BST(T,v): # v is the new value
  x ← T.root
  y ← NIL    # y is x's parent

  # search the right place for z
  while x≠NIL:
    y ← x
    if v ⪯ x.key:
      if x.key ⪯ v:
        return HANDLE_MULTI_INSERT(x,v)
      endif
      x ← x.left
    else:
      x ← x.right
    endif
  endwhile
```

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$O(h_T)$

Inserting a Value in a BST: Pseudo-Code (Cont'd)

```
  # attaching the new node
  x ← NEW_NODE(v)
  if T.root≠NIL:
    if v ≼ y.key:
      SET_CHILD(y, LEFT, x)
    else:
      SET_CHILD(y, RIGHT, x)
    endif
  else:
    T.root ← x
  endif

  return x
enddef
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Inserting a Value in a BST: Pseudo-Code (Cont'd)

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  # attaching the new node
  x ← NEW_NODE(v)
  if T.root≠NIL:
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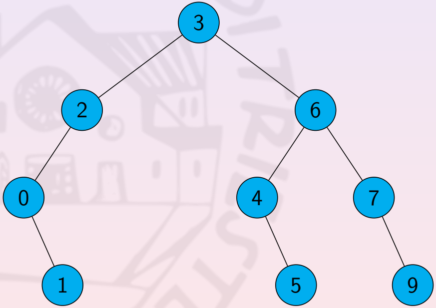
$\Theta(1)$

Removing a Key from a BST

Search the node x containing the key. Either

- x has one child at most



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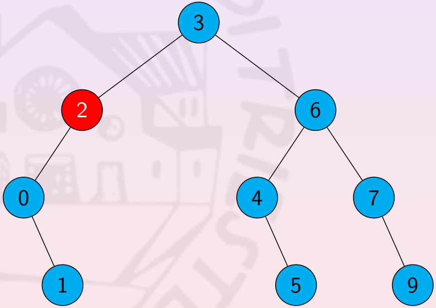

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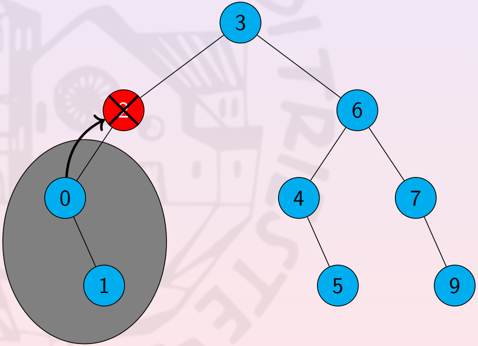

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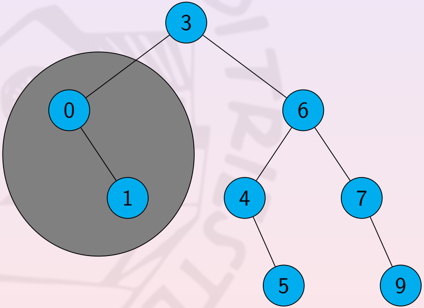

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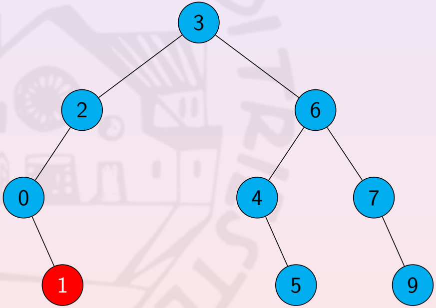

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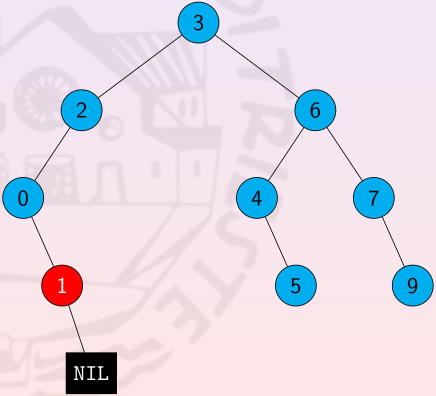

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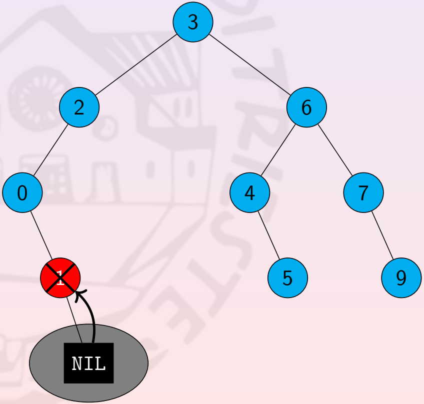

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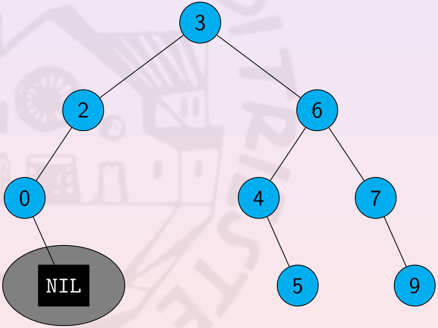

Removing a Key from a BST

Search the node x containing the key. Either

- x has one child at most or
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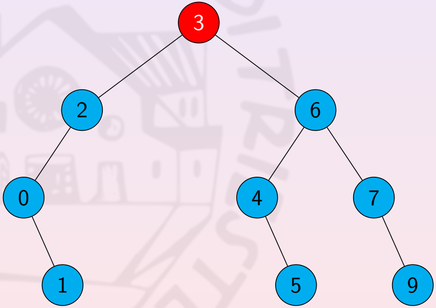

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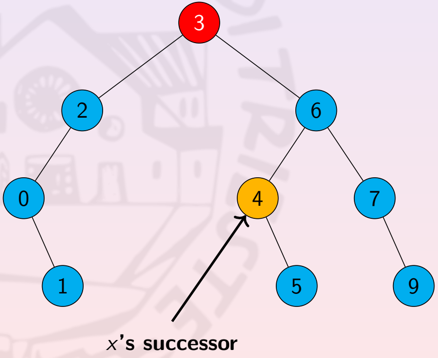

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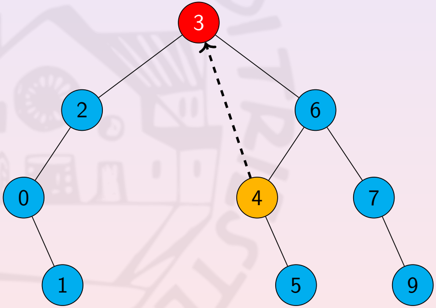

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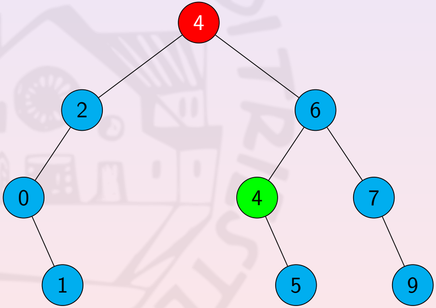

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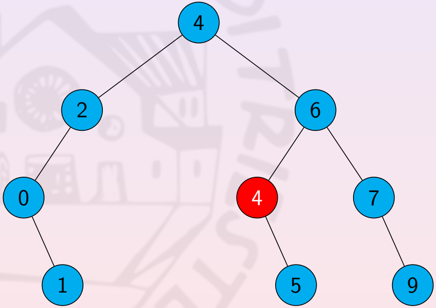

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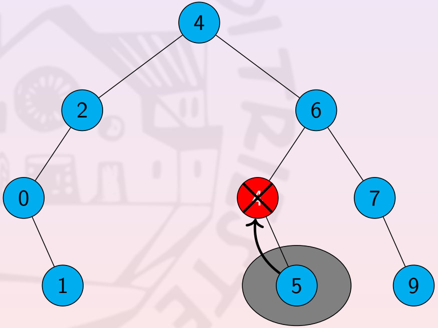

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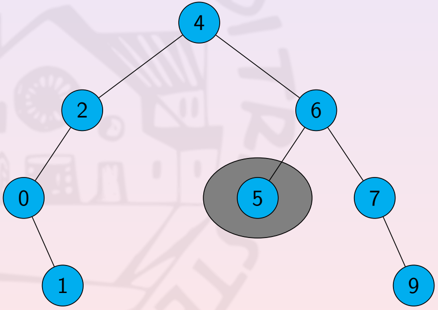

Removing a Key from a BST: Pseudo-Code

```
def TRANSPLANT(T,x,y): # replace x by y
  if IS_ROOT(x):
    T.root ← y

    if y ≠ NIL:
      y.parent ← NIL
    endif
  else:                  # x has a parent
    x_side ← CHILDHOOD_SIDE(x)

    # attach y in place of x
    SET_CHILD(x.parent, x_side, y)
  endif
enddef
```

Removing a Key from a BST: Pseudo-Code (Cont'd)

```
def REMOVE_BST(T,x): # remove x.key from T and
                     # return a removed node
  if x.left=NIL:          # if x lacks of left child
    TRANSPLANT(T,x,x.right)
    return x
  endif

  if x.right=NIL:         # if x lacks of right child
    TRANSPLANT(T,x,x.left)
    return x
  endif

  y ← MINIMUM_IN_SUBTREE(x.right)
  x.key ← y.key
  return REMOVE_BST(T,y) # y lacks of left child
enddef
```

Remove a Node from a BST: Complexity

`TRANSPLANT` costs $\Theta(1)$, while `MINIMUM_IN_TREE` $O(h_T)$

So, if x has at most one child, removing it costs $\Theta(1)$

In the general case, removing x takes time $O(h_T)$

Summarizing BSTs. . .

Creation	$\Theta(1)$
Insertion	$O(h_T)$
Search	$O(h_T)$
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Summarizing BSTs...

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BSTs cost more than single-linked lists (insertion $\Theta(1)$)

Balancing BSTs

The minimum height for a binary tree having n nodes is $\lceil \log_2 n \rceil$

We aim to balance the trees i.e., bring their heights to $O(\log n)$

How to do it?

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LOCALLY: balance only the "unbalanced" part of the tree How to know if it is unbalanced? How to handle branches' lengths?

Red-Black Trees

RBTs: Definition

Are BSTs satisfying the following conditions:

- each node is either a RED or a BLACK node
- the tree's root is BLACK
- all the leaves are BLACK `NIL` nodes
- all the RED nodes must have BLACK children
- for each node x, all the branches from x contain the same $\#$ of black nodes

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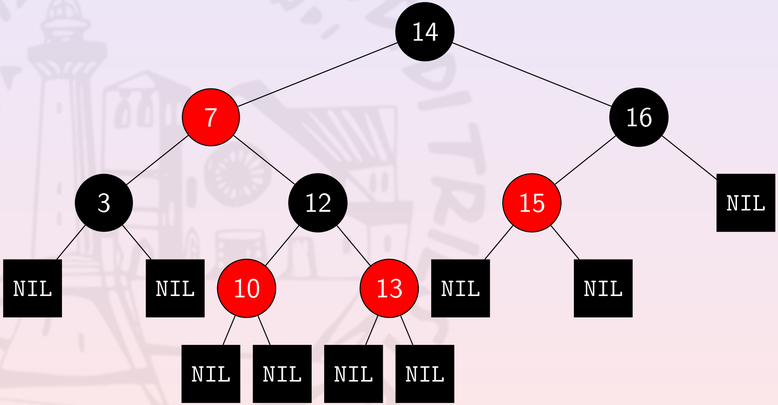
- for each node x, all the branches from x contain the same # of black nodes

$BH(x)$ will be the # of BLACK nodes below x in any branch

RBTs: An Example



Another Useful $\Theta(1)$ Function

```
def COLOR(x):
  if x=NIL:
    return BLACK
  endif

  return x.color
enddef
```

How "Tall" Are RB-Trees?

Theorem (Heights of a RB-Tree)

Any RBT with n internal nodes has height at most $2\log_2(n+1)$

Proof Sketch:

- prove that the sub-tree rooted in x has at least $2^{BH(x)} - 1$ internal nodes
- $BH(x)$ is at least half of x's height h then

$$n \geq 2^{h/2} - 1$$

How "Tall" Are RB-Trees?

The ratio between x's height and $BH(x)$ is topped by 2

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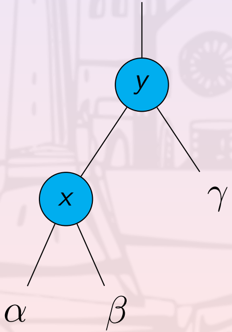
Rotating a Sub-Tree

Rotations are operations on the tree structure. They preserve the BST property.



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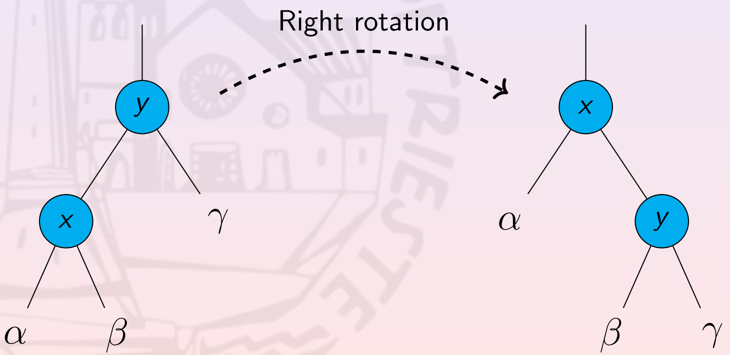

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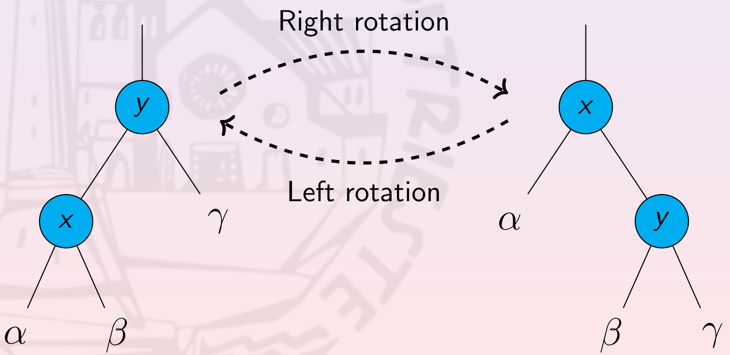

Rotating a Sub-Tree: Pseudo-Code

```
def ROTATE(T,x,side):
  r_side ← REVERSE_SIDE(side)

  y ← GET_CHILD(x, r_side)
  TRANSPLANT(T,x,y)

  beta ← GET_CHILD(y, side)
  TRANSPLANT(T,beta,x)

  SET_CHILD(x,r_side,beta) # move beta
enddef
```

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  TRANSPLANT(T,beta,x)

  SET_CHILD(x,r_side,beta) # move beta
enddef
```

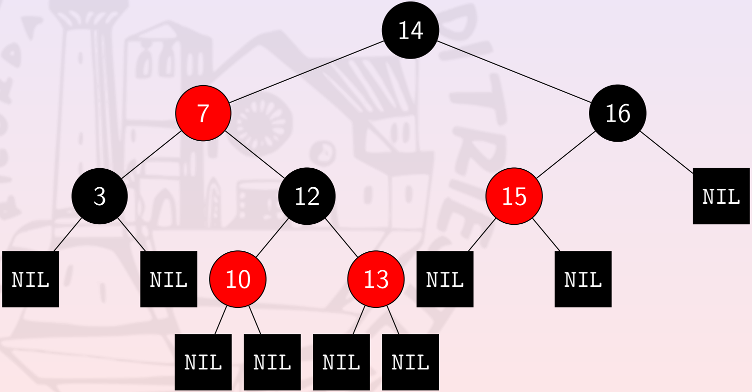
$\Theta(1)$

Inserting a New Node

Requires:

- inserting as in BST
- RED-coloring the node
- fixing up RB-Tree properties

Inserting a New Node: Example



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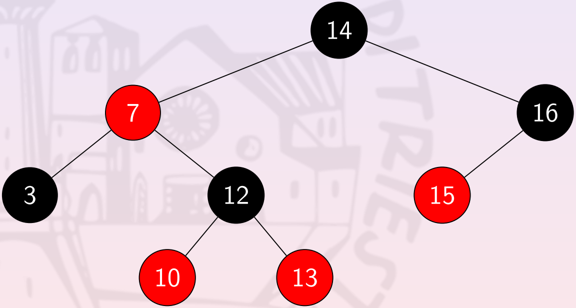

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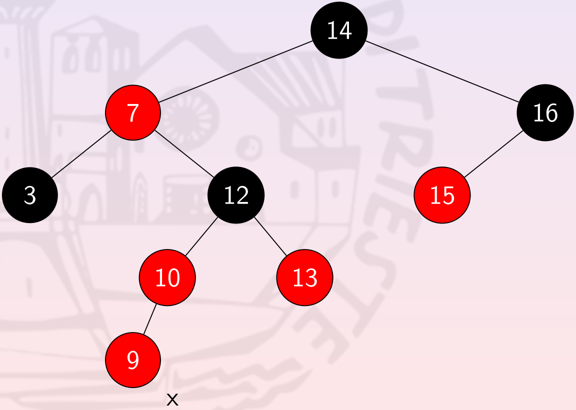
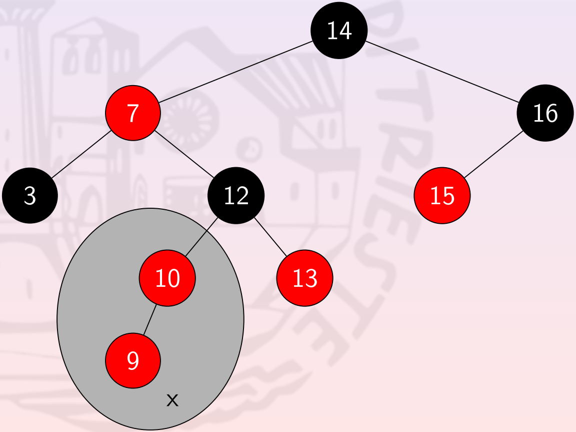


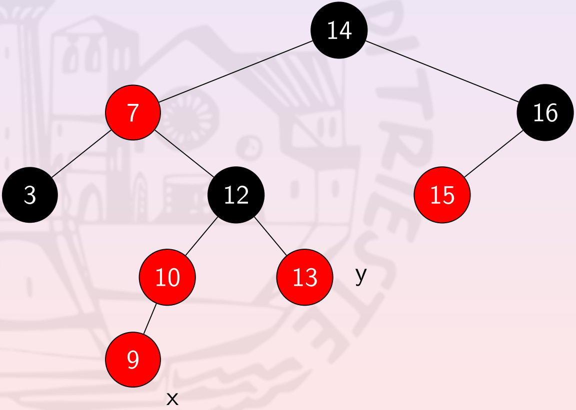
Inserting a New Node: Example

x's parent may be RED. How to fix it?



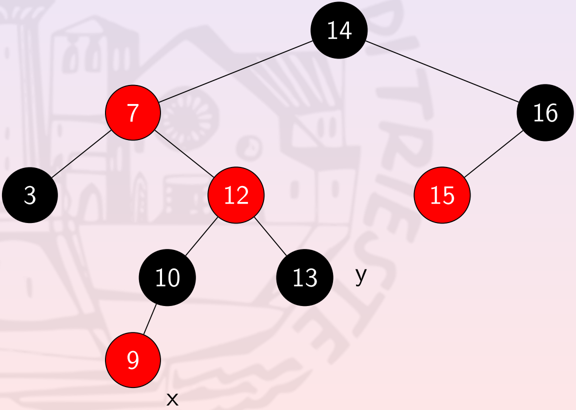
Inserting a New Node: Example

Case 1: x's uncle (y) is RED...



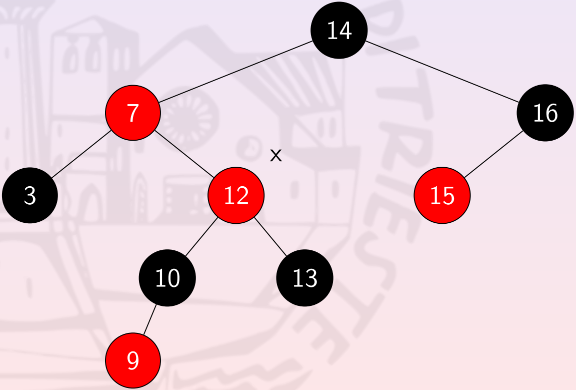
Inserting a New Node: Example

Case 1: x's uncle (y) is RED. . . RED-color x's granpa and BLACK-color x's parent and y.



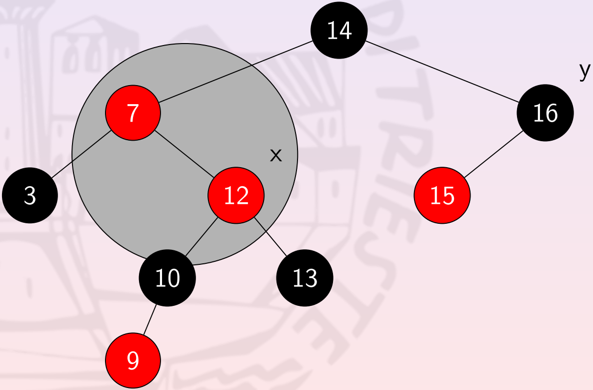
Inserting a New Node: Example

Case 1: x's uncle (y) is RED… RED-color x's granpa and BLACK-color x's parent and y. New x is former x's granpa



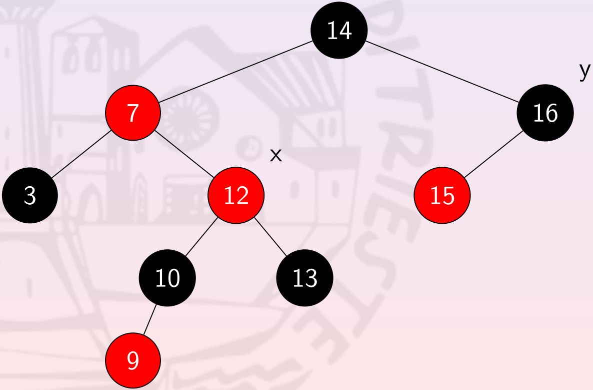
Inserting a New Node: Example

Still facing problems, but x's uncle is BLACK (not Case 1)



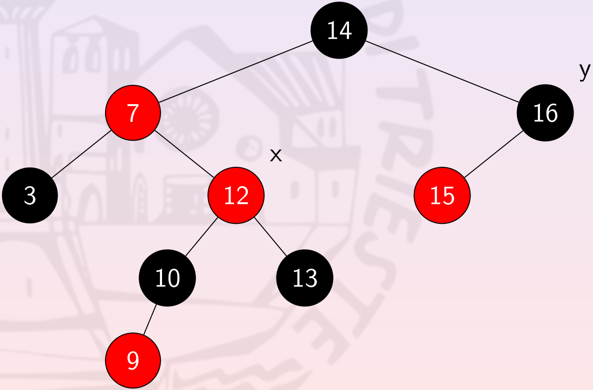
Inserting a New Node: Example

Case 2: y is BLACK and y and x are on the same side.



Inserting a New Node: Example

Case 2: y is BLACK and y and x are on the same side. Rotate on x's parent on the opposite side.



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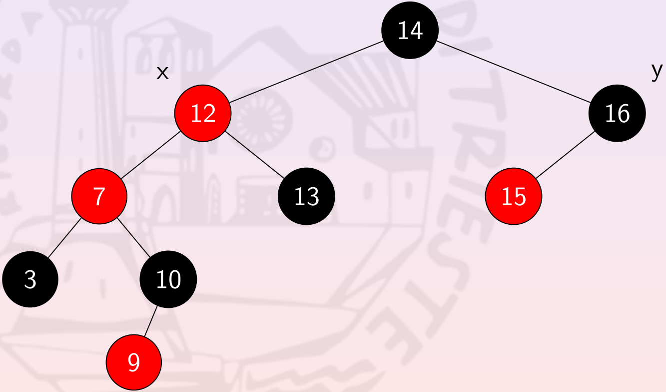
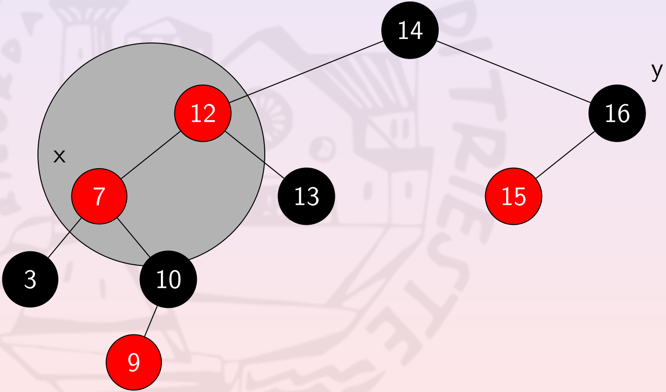
Case 2: y is BLACK and y and x are on the same side. Rotate on x's parent on the opposite side. New x is former x's parent



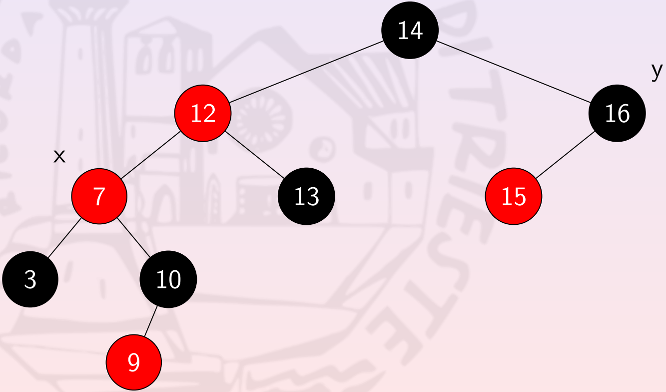
Inserting a New Node: Example

Still facing problems, but



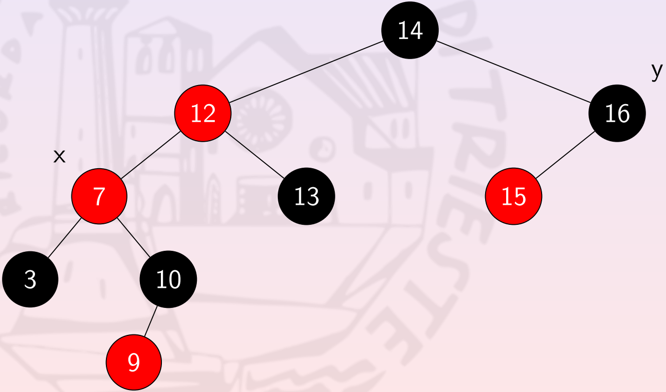
Inserting a New Node: Example

Still facing problems, but y is still BLACK (no Case 1)



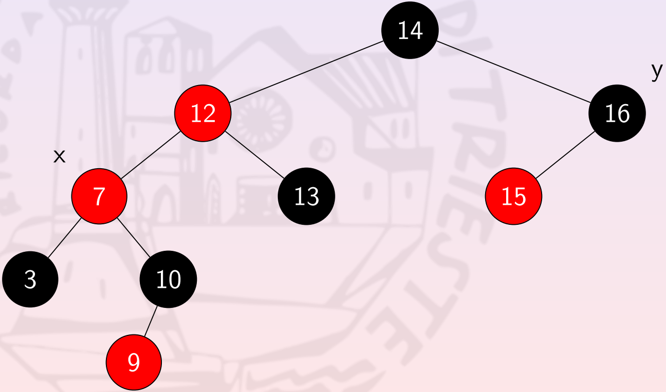
Inserting a New Node: Example

Still facing problems, but y is still BLACK (no Case 1) and x and y are on different sides (no Case 2)



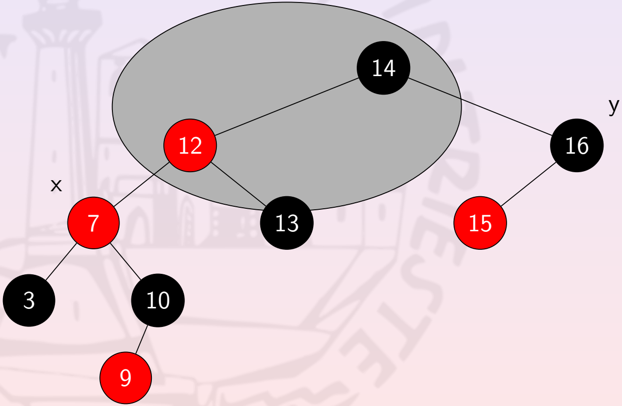
Inserting a New Node: Example

Case 3: y is BLACK and y and x are on different sides.



Inserting a New Node: Example

Case 3: y is BLACK and y and x are on different sides. Invert x's pa and granpa colors



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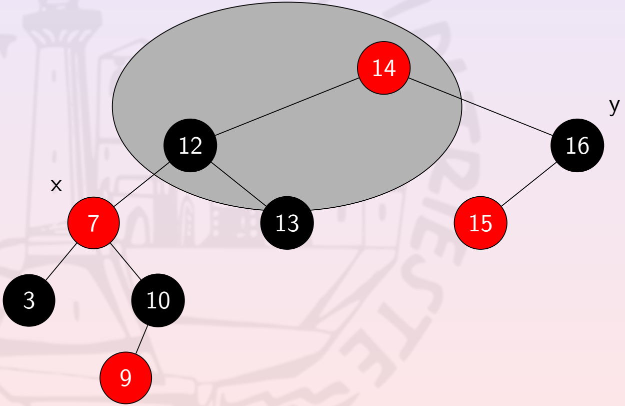
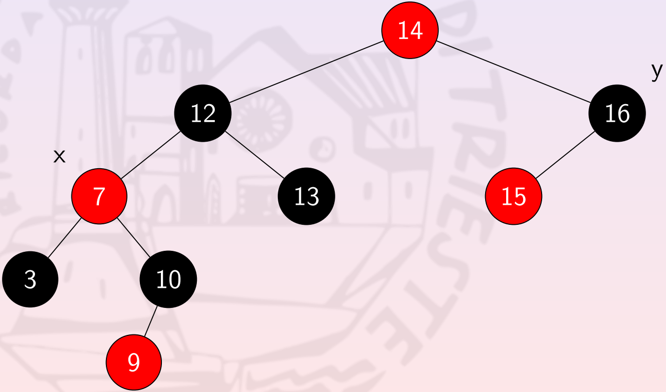


Inserting a New Node: Example

Case 3: y is BLACK and y and x are on different sides. Invert x's pa and granpa colors and rotate on x's granpa on y's side



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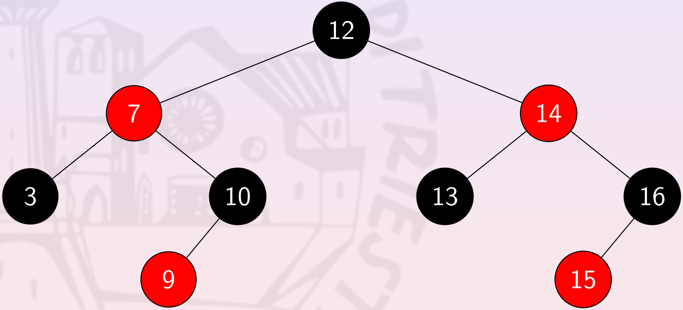

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Case 2 brings to Case 3

Case 3 solves the problem

In the worst case, the algorithm keeps repeating Case 1 steps along the insertion branch and the complexity is $O(\log n)$

Inserting a New Node: Code

```
def INSERT_RBTREE(T, v):
  x ← INSERT_BST(T, v)
  x.color ← RED

  FIX_INSERT_RBTREE(T, x)
enddef

def FIX_INSERT_RBT_CASE1(T, x):
  UNCLE(x).color ← BLACK
  x.parent.color ← BLACK
  GRANDPARENT(x).color ← RED

  return GRANDPARENT(x)
endif
```

Inserting a New Node: Code (Cont'd)

```
def FIX_INSERT_RBT_CASE2(T,x):
  x_side ← CHILDHOOD_SIDE(x)

  p ← x.parent
  ROTATE(T,p,REVERSE_SIDE(x_side))

  return p
endif
```

Inserting a New Node: Code (Cont'd 2)

```
def FIX_INSERT_RBT_CASE3(T,x):
  g ← GRANDPARENT(z)

  x.parent.color ← BLACK
  g.color ← RED

  x_side ← CHILDHOOD_SIDE(x)

  ROTATE(T,g,REVERSE_SIDE(x_side))
endif
```

Inserting a New Node: Code (Cont'd 3)

```
def FIX_INSERT_RBTREE(T,x):
  while (¬IS_ROOT(x) and
        (¬IS_ROOT(x.parent) or x.parent.color=RED)):
    if COLOR(UNCLE(x))=RED:
      z ← FIX_INSERT_RBT_CASE1(T,x)
    else:
      if (CHILDHOOD_SIDE(x) ≠
            CHILDHOOD_SIDE(x.parent)):
        z ← FIX_INSERT_RBT_CASE2(T,x)
      endif
      FIX_INSERT_RBT_CASE3(T,x)
    endif
  endwhile

  T.root.color ← BLACK
enddef
```

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Removing a key as in BST removes also a node y which is replaced by its former child x

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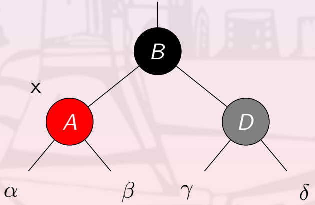
In particular, $BH(x) = BH(w) - 1$ where w is x's sibling

The fixing procedure iteratively balances BH on the sub-tree rooted on x's parent

Removing a Key: Case 0

x is RED

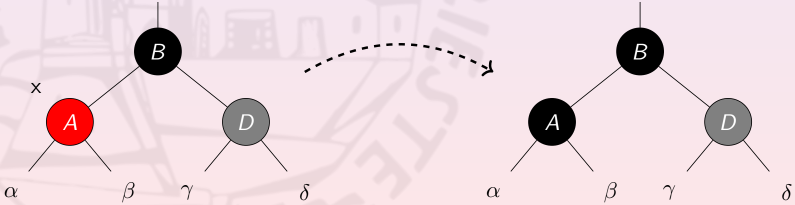


Removing a Key: Case 0

x is RED

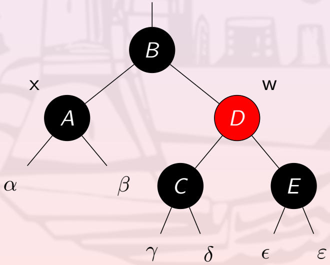
- BLACK-color x

$BH(x)$ is increased by 1 and the tree has been fixed



Removing a Key: Case 1

x's sibling is RED

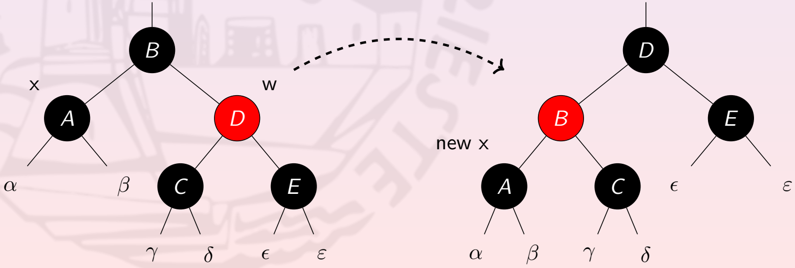


Removing a Key: Case 1

x's sibling is RED

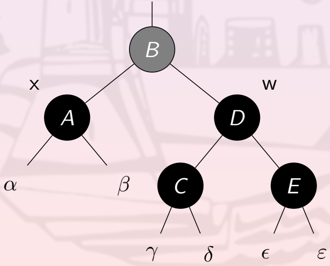
- invert colors in x's parent and sibling
- rotate x's parent on x's side

$BH(x)$ does not change



Removing a Key: Case 2

x's sibling and nephews are BLACK

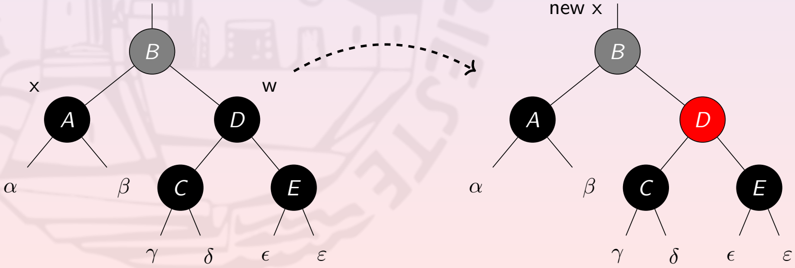


Removing a Key: Case 2

x's sibling and nephews are BLACK

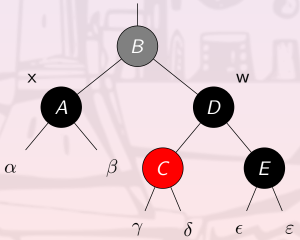
- RED-color x's sibling

$BH(x)$ does not change, while the BLACK height of both x's parent and sibling are decreased by 1



Removing a Key: Case 3

Among x's sibling and nephews, only the nephew on x's side is RED

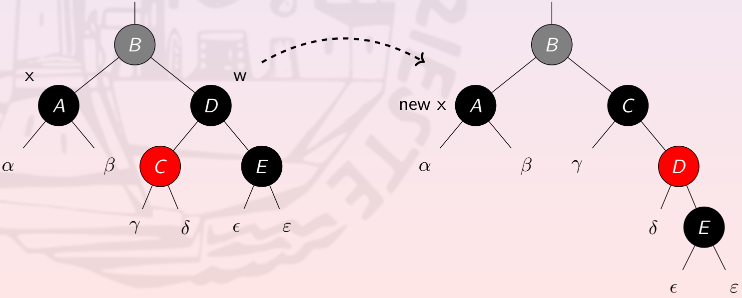


Removing a Key: Case 3

Among x's sibling and nephews, only the nephew on x's side is RED

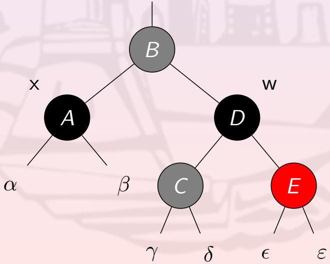
- rotate x's sibling on the opposite side w.r.t. x
- invert colors in both old and new siblings of x

The BLACK height of both x and x's parent does not change



Removing a Key: Case 4

The x's nephew on the opposite side w.r.t. x is RED

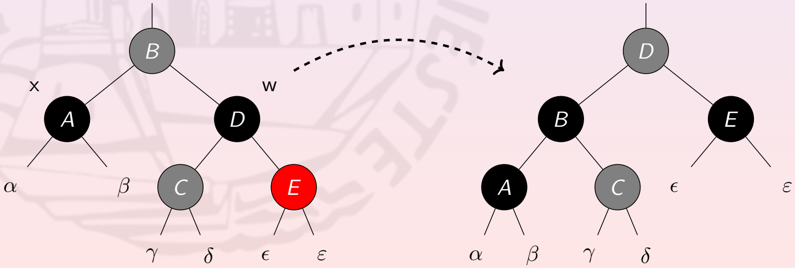


Removing a Key: Case 4

The x's nephew on the opposite side w.r.t. x is RED

- switch colors of x's parent and sibling
- BLACK-color the x's nephew on the opposite side w.r.t. x
- rotate x's parent on x's side

The BLACK height of both x and x's parent does not change



Removing a Key: Some Considerations

All the case transformation procedures takes time $\Theta(1)$

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Both Case 0 and Case 4 transformation procedures fix the RB-Tree

Case 1 cannot occur twice in-row

Case 2 pushes the problem one step towards the root

If Case 2 occurs after Case 1, Case 0 occurs next

Case 3 transformation procedure brings to Case 4

Removing a Key: Some Considerations

In the worst case scenario, Case 2 is repeated until the problem is pushed up to the root ($O(\log n)$ times)

If this is the case, we have decreased the BLACK height of the tree and the problem is no more a problem

Removing a Key: Code

```
def REMOVE_RBTREE(T, z):
  y ← REMOVE_BST(T, z)

  if y.color = BLACK: # fix from its replacement
    if y.left = NIL
      x ← y.right
    else:
      x ← y.left
    endif
    FIX_REMOVE_RBTREE(T, x)
  endif

  return y
enddef
```

Removing a Key: Code (Cont'd 2)

```
def FIX_REMOVE_RBT_CASE1(T,x):
   SIBLING(x).color ← BLACK
   x.parent.color ← RED

   ROTATE(T,x,CHILDHOOD_SIDE(x))
endif

def FIX_REMOVE_RBT_CASE2(T,x):
   SIBLING(x).color ← RED
   return x.parent
endif
```

Removing a Key: Code (Cont'd 3)

```
def FIX_REMOVE_RBT_CASE3(T,x):
   x_side ← CHILDHOOD_SIDE(x)
   r_side ← REVERSE_SIDE(x_side)

  w ← GET_CHILD(w, r_side)
  GET_CHILD(w, x_side).color ← BLACK
  w.color ← RED

  ROTATE(T,w, r_side)
endif
```

Removing a Key: Code (Cont'd 4)

```
def FIX_REMOVE_RBT_CASE4(T,x):
  x_side ← CHILDHOOD_SIDE(x)
  r_side ← REVERSE_SIDE(x_side)

  w ← GET_CHILD(w, r_side)
  GET_CHILD(w, r_side).color ← BLACK
  w.color ← x.parent.color
  x.parent.color ← BLACK

  ROTATE(T,x.parent, x_side)
endif
```

Removing a Key: Code (Cont'd 5)

```
def FIX_REMOVE_RBT(T,x):
  while x ≠ T.root and x.color ≠ RED:
    w ← SIBLING(x)
    if w = RED:
      x ← FIX_REMOVE_RBT_CASE1(T,x)
    else:
      x_side ← CHILDHOOD_SIDE(x)
      r_side ← REVERSE_SIDE(x_side)
```

Removing a Key: Code (Cont'd 6)

```
            if GET_CHILD(w, r_size) == RED:
              FIX_REMOVE_RBT_CASE4(T,x)

              return
            else:
              if GET_CHILD(w, x_side) == RED:
                FIX_REMOVE_RBT_CASE3(T,x)
              else:
                FIX_REMOVE_RBT_CASE2(T,x)
              endif
        endif
    endwhile
enddef
```