Exercise 1

X continuous r.v. with PDF

$$f_X(x) = \begin{cases} 5/32 \; x^4 & , x \in (0,2] \\ 0 & ; \text{otherwise} \end{cases}$$

2. $Y(x) \stackrel{def}{=} X^2 \Rightarrow X(Y) = \sqrt{Y}$ on $(0,2]$

$$\Rightarrow f_Y(y) = \begin{cases} 5/32 \; y^2 \cdot 1/2\sqrt{y} = 5/64 \; y^{3/2} \\ 0 \end{cases}$$

1. Cumulative distribution function of Y

$$F_Y(y) = \int_{-\infty}^{y} \frac{5}{64} y'^{3/2} \, dy'$$

If $y \in (0,4]$

$$F_Y(y) = \frac{5}{64} \int_0^y y'^{3/2} \, dy'$$

$$= \frac{\cancel{5}}{\underset{32}{\cancel{64}}} \frac{\cancel{2}}{\cancel{5}} y^{5/2}$$

$$= \frac{y^{5/2}}{32}$$

If $y \leq 0$:

$F_Y \equiv 0$

If $y > 4$:

$$F_Y(y) = \frac{5}{64} \int_0^4 y'^{3/2} \, dy'$$

$$= \frac{\cancel{5}}{\underset{2^5}{\cancel{64}}} \frac{\cancel{2}}{\cancel{5}} 2^{\cancel{2} \cdot 5/\cancel{2}}$$

$= 1$ as it should be

Other way (without using 2.)

$$F_{X^2}(x) := P(X^2 \le x) = P(x \in [-x, x])$$

If $x \in (0,2]$:

$$F_{X^2}(x) = \int_{-x}^{x} f_X(x)\, dx$$

$$= \int_0^x \frac{5}{32} x^4 \, dx$$

$$= \frac{\cancel{5}}{32} \frac{1}{\cancel{5}} x^5$$

$$\Rightarrow F_Y(y) \underset{X(Y)=Y^{1/2}}{=} \frac{y^{5/2}}{32}$$

If $x < 0$: $F_{X^2}(x) = 0 \Rightarrow F_Y(y) = 0$

If $x > 2$:

$$F_{X^2}(x) = \int_{-x}^{x} f_X(x)\, dx$$

$$= \int_0^2 f_X(x)\, dx = 1$$

$$\Rightarrow F_Y(y) = 1$$

3. $\mathbb{E}[Y] = \mathbb{E}[X^2]$

$$= \int_0^2 \frac{5}{32} x^6 \, dx$$

$$= \frac{5}{\cancel{32}} \frac{1}{7} 2^{\cancel{7}\,2}$$

$$= 20/7$$

Exercise 2

$$f(x,y) = \begin{cases} \frac{15}{4}x^2 & , y \in [0, 1-x^2] \ , \ x \in [-1,1] \\ 0 & , \text{otherwise} \end{cases}$$

- Marginal PDF of X

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$= \int_0^{1-x^2} \frac{15}{4}x^2 \,dy$$

$$= \frac{15}{4}x^2 \int_0^{1-x^2} dy$$

$$= \frac{15}{4}x^2(1-x^2)$$

- Marginal PDF of Y

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

$y \in [0, 1-x^2] \Rightarrow$ boundaries of Y domain as fn of x are $y = 1-x^2$

$\Rightarrow x^2 = 1-y$

$\Rightarrow$ boundaries of X domain as fn of y are $x = \pm\sqrt{1-y}$

$\Rightarrow x \in [-\sqrt{1-y}, \sqrt{1-y}]$

$$\Rightarrow f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4}x^2\,dx$$

$$= \frac{15}{4} \frac{x^3}{3} \Big|_{-\sqrt{1-y}}^{\sqrt{1-y}}$$

$$= \frac{15}{6} \sqrt{1-y}$$

<u>Exercise 3</u>

$$f(x,y) = \begin{cases} 6e^{-(2x+3y)} & , x,y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

1. $f(x,y) \overset{x,y\geq 0}{=} 6e^{-2x} e^{-3y} = f_X(x) f_Y(y)$, $f_X(x) = \underset{\substack{\uparrow \\ \text{so } f_X(x) \text{ is normalized}}}{2} e^{-2x}$, ___

   $\Rightarrow X, Y$ independent

2. $\mathbb{E}[Y \mid X > 2] \overset{X,Y \text{ indep}}{=} \int_{\mathbb{R}} 3ye^{-3y}\, dy$

$$= 3 \int_0^{\infty} ye^{-3y}\, dy$$

$$= -3 \frac{d}{da} \int_0^{\infty} e^{-ay}\, dy \Big|_{a=3}$$

$$= -3 \frac{d}{da} \frac{1}{a} \Big|_{a=3}$$

$$= +3 \frac{1}{a^2} \Big|_{a=3} = \frac{1}{3}$$

3. $P(X > Y) = P(X - Y > 0) = 1 - P(X < Y)$

   $P(X < y) = F_X(y)$

$$= \int_{-\infty}^{y} f_X(x)\, dx$$

$$= \int_0^y f_X(x)\, dx$$

$$= 2 \int_0^y e^{-2x}\, dx$$

$$= -\cancel{2}\,\frac{1}{\cancel{2}}\, e^{-2x}\Big|_0^y = 1 - e^{-2y}$$

$$P(X < Y) = \int_0^\infty f_Y(y)\,(1 - e^{-2y})\, dy$$

with y PDF because it's not uniform!
must weight CDF of x wrt single value of y on every possible value of y

$$= \int_0^\infty 3\,(e^{-3y} - e^{-5y})\, dy$$

$$\int_0^\infty e^{-3y}\, dy = +\frac{1}{3}, \quad \int_0^\infty e^{-5y}\, dy = +\frac{1}{5}$$

$$= 3\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{5}$$

$$\Rightarrow P(X > Y) = 1 - \frac{2}{5} = \frac{3}{5}$$

Excercise 4

X, Y continuous r.v.

$$f(x,y) = \begin{cases} x + 3y^2/2, & x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

- MAP of X given $Y = y$

$$\underbrace{p(x \mid y)}_{\text{Posterior of } X} = p(y \mid x)\, \underbrace{f_X(x)}_{\text{Prior of } X}$$

Treat x as the parameter

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \left(x + \frac{3}{2}y^2\right)dy =$$

$$= x + \frac{3}{2}\int_0^1 y^2\,dy = x + \frac{3}{2}\left.\frac{y^3}{3}\right|_0^1 = x + \frac{1}{2}$$

$$\Rightarrow f(x|y) = \frac{f(x,y)\,\cancel{f_X(x)}}{\cancel{f_X(x)}} = \left(x + \frac{3}{2}y^2\right)$$

$$x_{MAP}(y) = \underset{x\in[0,1]}{\operatorname{argmax}}\left(x + \frac{3}{2}y^2\right) = 1 + \frac{3}{2}y^2$$

Since $1 = \max\limits_{y\in[0,1]} y^2 \Rightarrow x_{MAP} = 1$

- ML of X given $Y = y$

Maximum Likelihood is

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x + 3/2\,y^2}{x + 1/2}$$

$$\Rightarrow x_{ML} = \underset{x\in[0,1]}{\operatorname{argmax}} \frac{2x + 3y^2}{2x + 1}$$

- $2x + 3y^2 \geq 2x + 1 \iff y^2 \geq 1/3$

  $\iff y \geq 1/\sqrt{3}$

$$\frac{d}{dx}\left(\frac{2x + 3y^2}{2x + 1}\right) = \frac{2(2x+1) - 2(2x + 3y^2)}{(2x+1)^2} = \frac{2 - 6y^2}{(2x+1)^2} \underset{y\geq\frac{1}{\sqrt{3}}}{\leq} 0$$

$\Rightarrow x_{ML} = 0$

- $2x + 3y^2 < 2x + 1 \iff y < 1/\sqrt{3}$

$$\frac{d}{dx}\left(\frac{2x + 3y^2}{2x + 1}\right) = \frac{2 - 6y^2}{(2x+1)^2} \underset{y<\frac{1}{\sqrt{3}}}{>} 0 \Rightarrow x_{ML} = 1$$

Exercise 5

Following a binary classification framework:

$\mathcal{H}_{lin} \stackrel{def}{:=} \{ \mathbb{1}_{\vec{k}\cdot(\vec{x},1)\geq 0}(\vec{x}) \mid \vec{k} \in \mathbb{R}^{d+1} \}$

$D_{VC}(\mathcal{H}_{lin}) \stackrel{def}{:=} \max \{ m \mid \exists C \subseteq \mathbb{R}^d, |C| = m$ s.t. $\mathcal{H}_{lin}$ shatters $C \}$

□ $k_0 = 0 \Rightarrow$ hyperplanes contain origin

$\Rightarrow D_{VC}(\mathcal{H}_{lin}|_{k_0=0}) = d$

proof

$f(\vec{x}) \stackrel{def}{:=} \mathbb{1}_{\vec{k}\cdot\vec{x}}(\vec{x})$

$S \stackrel{def}{:=} \{\vec{x}_1, \_, \vec{x}_d\}$ s.t. $\vec{x}_i = \vec{e}_i$ std basis in $\mathbb{R}^d$

$\Rightarrow S$ can be shattered by f since, given

$y \stackrel{def}{:=} \{y_1, \_, y_d\} \in \{0, +1\}^d$

set of all possible labellings,

$\exists f$, $f(\vec{x}) \stackrel{def}{:=} \mathbb{1}_{y_i \vec{x}_i \cdot \vec{x}}(\vec{x})$ s.t. $f(\vec{x}_j) = y_j \quad \forall x_j \in S$

$\Rightarrow D_{VC}(\mathcal{H}_{lin}|_{k_0=0}) \geq d$ (LOWER BOUND)

Assume $\exists S' \stackrel{def}{:=} \{\vec{x}_1, \_, \vec{x}_{d+1}\} \subset \mathbb{R}^d$ s.t. it can be shattered by $\mathcal{H}_{lin}$

$\Rightarrow \forall$ possible label y_k in $\{0, +1\}^{d+1}$ ($\to 2^{d+1}$ possible labels) $\exists$ a $\vec{w}_k$ s.t. $\mathbb{1}(\vec{w}_k \cdot \vec{x}) = y_k \quad \forall x \in S'$.

$H \stackrel{def}{:=} XW$, $X := (\vec{x}_1, \_, \vec{x}_{d+1})^T$, $W := (\vec{w}_1, \_, \vec{w}_{2^{d+1}})$

$$= \begin{pmatrix} \vec{x}_1 \cdot \vec{w}_1 & - & \vec{x}_1 \cdot \vec{w}_{2^{d+1}} \\ | & & | \\ \vec{x}_{d+1} \cdot \vec{w}_1 & - & \vec{x}_{d+1} \cdot \vec{w}_{2^{d+1}} \end{pmatrix}$$

$\Rightarrow$ Rows of H are lin inden:

$\nexists \vec{\alpha} \neq 0$ s.t. $\vec{\alpha}^T H = \vec{0}^T$ because under the hyp of $\mathcal{H}_{lin}$ shattering S', $\exists$ a k s.t.

$\text{sign}(X\vec{w}_k) = \text{sign}(\vec{a}) \Rightarrow \vec{a}^T X \vec{w}_k > 0$ and $\vec{0}^T$ is not reproducible

$\Rightarrow$ Rank of H is $d+1$.

But $\text{Rank}(H) = \min(\text{Rank}(X), \text{Rank}(W))$ and $\text{Rank}(X) = d \Rightarrow$ Contraddiction

$\Rightarrow D_{VC}(\mathcal{H}_{up}|_{k_0=0}) < d+1$ (UPPER BOUND)

$\Rightarrow D_{VC}(\mathcal{H}_{up}|_{k_0=0}) = d$

- $k_0 \neq 0$:

One could always rewrite

$f(\vec{x}) = \mathbb{1}_{\vec{k}\cdot\vec{x}+k_0}(\vec{x}) = \mathbb{1}_{\vec{k}'\cdot\vec{x}'}(\vec{x}')$,

$\vec{k}' := \begin{bmatrix} \vec{k} \\ k_0 \end{bmatrix}, \vec{x}' := \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} \in \mathbb{R}^{d+1}$

$\Rightarrow$ One could redef the prob of computing $D_{VC}(\mathcal{H}_{un})$ in $\mathbb{R}^d$ as the prob of computing $D_{VC}(\mathcal{H}_{un}|_{k_0=0})$ in $\mathbb{R}^{d+1}$

$\Rightarrow D_{VC}(\mathcal{H}_{un}) = d+1$