Credit and Weather derivatives Report

Exercise 1: Zero-Coupon Curve via NSS
Exercise 2: Default and Survival Probabilities from CDS Spreads
Exercise 3: Multiname Credit Derivative Pricing with a Gaussian Copula

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1 Exercise 1: Zero-Coupon Bond Curve Using the NSS Model

1.1 Theoretical Framework

The *term structure of interest rates* describes the dependence of risk-free yields on maturity and is fundamental for the valuation of all bonds and derivative securities. In a risk-neutral setting, at time 0 the price of a zero-coupon bond (ZCB) maturing at time t is

$$P(0,t) = \mathbb{E}^{Q}\left[e^{-\int_0^t r_s\,ds}\right],$$

where $(r_t)_{t\geq 0}$ denotes the short interest rate under the risk-neutral measure Q.
In practice, $P(0,t)$ is not observed directly for all maturities but must be inferred from a discrete set of market instruments, such as money market rates, swaps or bond yields.

One approach is to fit a *parametric* functional form for the continuous-compounded zero rate $y(t)$ and then set $P(0,t) = e^{-y(t)t}$. In this exercise we employ the *Nelson-Siegel-Svensson* (NSS) model, a four-factor specification that captures level, slope and curvature features observed in empirical yield curves. For $t > 0$,

$$y(t) = \beta_0 + \beta_1 \frac{1 - e^{-t/\tau_1}}{t/\tau_1} + \beta_2 \left(\frac{1 - e^{-t/\tau_1}}{t/\tau_1} - e^{-t/\tau_1} \right) + \beta_3 \left(\frac{1 - e^{-t/\tau_2}}{t/\tau_2} - e^{-t/\tau_2} \right). \tag{1}$$

The loading functions are chosen so that:

- as $t \to \infty$, $y(t) \to \beta_0$ (long-term level);
- the term multiplying β_1 behaves like a decaying slope factor;
- the terms multiplying β_2 and β_3 generate one or two humps, capturing medium-term curvature.

Once $y(t)$ has been fitted to the market data, the ZCB price follows from

$$P(0,t) = \exp\big(-y(t)t\big). \tag{2}$$

In addition to match observed zero rates (for the short periods below 1 year), the curve must be also consistent with *par swap* quotes at longer maturities.
For a plain vanilla fixed-for-floating swap with maturity n and annual fixed coupon, the fair fixed rate S_n satisfies

$$S_n \sum_{i=1}^{n} P(0,i) = 1 - P(0,n), \tag{3}$$

where $P(0,i)$ are the discount factors at the fixed payment dates and the floating leg is assumed to reset to par at each payment. Under the NSS specification, equation (4) can be evaluated for every integer n using (2), producing a model par rate $S_n(\theta)$ as a function of the parameters $\theta = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$.

1.2 Methodology and Computations

Data preparation

The Euribor swap curve is taken from the sheet `swap curve` in the file `market_data2.xlsx`. The maturities are reported in tenor format ("1M", "3M", "1Y", ..., "30Y"). These are mapped into year fractions according to

$$1\text{M} = \tfrac{1}{12}, \quad 3\text{M} = \tfrac{3}{12}, \quad 6\text{M} = \tfrac{6}{12}, \quad 1\text{Y} = 1, \quad \ldots, \quad 30\text{Y} = 30.$$

Swap rates are given in percentages and converted to decimals; for example, 2.516% becomes 0.02516.

The short segment of the curve ($t < 1$ year) is treated as consisting of *zero* rates, whereas maturities beyond one year are annual fixed-rate par swaps. We obtain

$$\text{18 valid observations:} \quad \text{4 zero maturities } (< 1\text{y}) \quad \text{and} \quad \text{14 par swap maturities } (\geq 1\text{y}).$$

Calibration of the NSS model

The NSS parameters are estimated by solving a non-linear least squares problem that combines information from both the short-end zeros and the par swaps. Let

$$\{(t_i^{(z)}, r_i^{(z)})\}_{i=1}^{N_z}$$

denote the short-end maturities($t_i^{(z)}$) and corresponding zero rates($r_i^{(z)}$), and

$$\{(n_j, S_{n_j}^{(\mathrm{mkt})})\}_{j=1}^{N_s}$$

denote the integer maturities(n_j) and observed par swap rates($S_{n_j}^{(mkt)}$). For a given parameter vector θ, the NSS curve $y(t;\theta)$ is evaluated at each $t_i^{(z)}$, and the model par swap rates $S_{n_j}(\theta)$ are obtained from the par swap identity (see equation (4) below). The calibration minimises the sum of squared errors:

$$\sum_{i=1}^{N_z} \left(y(t_i^{(z)};\theta) - r_i^{(z)}\right)^2 + \sum_{j=1}^{N_s} \left(S_{n_j}(\theta) - S_{n_j}^{(\mathrm{mkt})}\right)^2$$

subject to reasonable bounds on the parameters. The optimisation is performed using the `scipy.optimize.least_squares` routine with the trust region reflective (TRF) algorithm.

For the 2023 Euribor swap curve, the optimal parameter estimates are:

$$\beta_0 = 0.02546,\ \beta_1 = -0.00298,\ \beta_2 = 0.04508,\ \beta_3 = -0.02871, \quad \tau_1 = 0.6672,\ \tau_2 = 1.7066.$$

Bootstrapping zero-coupon bond prices

Once the continuous zero curve $y(t)$ is fitted, discount factors at generic maturities are obtained from $P(0,t) = e^{-y(t)t}$. In order to match the bootstrap structure, however, we reconstruct discount factors at integer maturities from par swap rates using a recursive method.

Maturities below 1 year. For $t < 1$ year, discount factors are obtained directly from observed zero rates:

$$P(0, t_i^{(z)}) = \exp\big(-r_i^{(z)} t_i^{(z)}\big).$$

Maturities of 1 year and above. For integer maturities $n = 1, \ldots, 30$, the NSS curve implies par swap rates $S_n(\theta)$. The par swap identity relates the swap rate to the discount factors:

$$S_n \sum_{i=1}^{n} P(0, i) = 1 - P(0, n) \tag{4}$$

This equation can be rearranged to solve for $P(0, n)$ recursively.

- **Base Case ($n = 1$):**
$$P(0, 1) = \frac{1}{1 + S_1}.$$
- **Recursive Step ($n \geq 2$):** Given the previously calculated discount factors $P(0, 1), \ldots, P(0, n-1)$, the discount factor for year n is:
$$P(0, n) = \frac{1 - S_n \sum_{i=1}^{n-1} P(0, i)}{1 + S_n}.$$

Starting from $n = 1$ and proceeding iteratively up to $n = 30$ yields the full set of discount factors $\{P(0, n)\}_{n=1}^{30}$. Together with the short-end discount factors, this defines the entire ZCB curve on the grid of observed maturities.

1.3 Results

Figure 1 shows the observed market rates (short-end zero quotes and par swap rates) together with the fitted NSS zero curve $y(t)$. The curve interpolates smoothly between the short maturities, where zero rates are directly observed, and the longer maturities, where swap rates are used indirectly through the par condition.

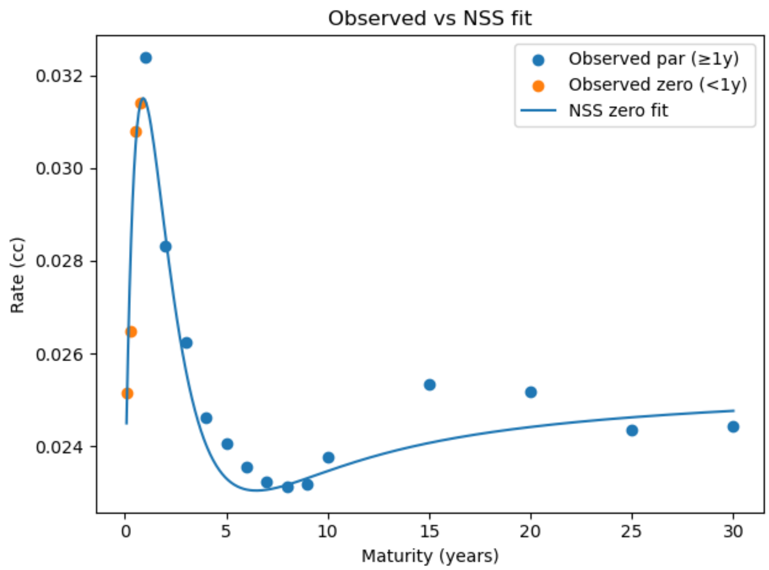


Figure 1: Observed swap and zero rates vs. calibrated NSS curve (Exercise 1).

The discount factors obtained via the recursive formula are plotted in Figure 2. The decreasing shape is consistent with positive interest rates and confirms that the NSS fit, together with the bootstrap relation, produces a risk-free discount curve.

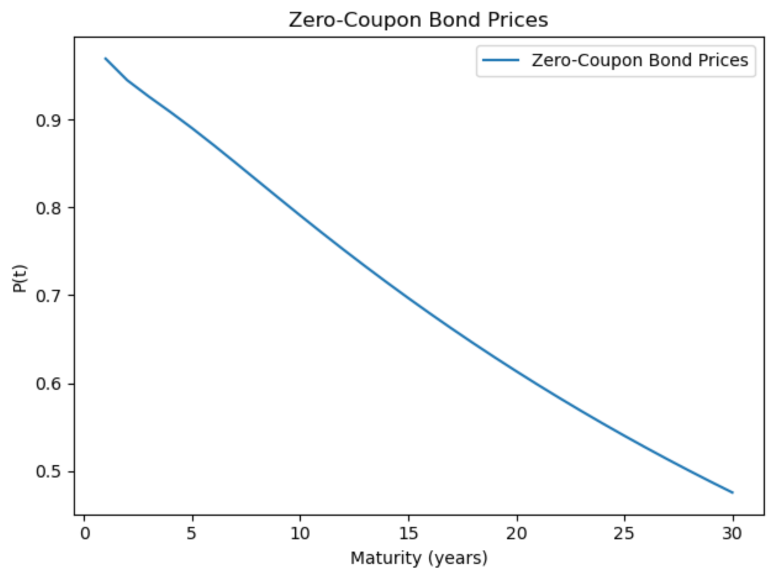


Figure 2: Discount factors obtained via bootstrapping from NSS par rates (Exercise 1).

Finally, implied continuously-compounded yield rates,

$$y_{\text{impl}}(t) = -\frac{\ln P(0,t)}{t},$$

are shown in Figure 3.
By construction, these implied yields are consistent with the discount factors and track closely the NSS curve used as the parametric backbone.

Figure 3: Implied continuously-compounded yield rates from bootstrapped prices.

1.4 Comments and Conclusions

The calibrated NSS model provides a smooth and flexible representation of the Euribor term structure
The estimated parameters generate:

- an upward-sloping short end, reflecting higher 6-9 month rates;
- a medium-term dip around 5-7 years, driven by the curvature parameters β_2 and β_3 and the large τ_1;
- a gradual convergence toward the long-run level β_0 at long maturities.

The bootstrapped discount factors are strictly decreasing in maturity, as expected under positive rates, and the implied yield curve is consistent with the NSS fit.

2 Exercise 2: Bootstrapping Default and Survival Probabilities from CDS Spreads

2.1 Theoretical Framework

In the second exercise, CDS spreads are used to infer the term structure of risk-neutral default probabilities for several reference entities.
We work in a *reduced-form* credit risk framework, in which default is modeled as the first jump time of a Poisson process, according to the time-inhomogeneous intensity-based model.

Let τ be the default time of a given asset.
Under a Cox (or time-inhomogeneous Poisson) process specification, the default intensity $\lambda(t)$ is a non-negative, deterministic function, and

$$\mathbb{P}(\tau > t) = \exp\Big(- \int_0^t \lambda(s)\, ds \Big).$$

The intensity is further specified to be *piecewise constant* between successive CDS maturities:

$$\lambda(t) = \lambda_k, \qquad t \in [T_{k-1}, T_k),$$

where $0 = T_0 < T_1 < \cdots < T_n$ are the quoted CDS maturities. The cumulative hazard and survival probability at T_k are then

$$\Gamma(T_k) = \sum_{i=1}^{k} \lambda_i \Delta t_i, \quad \Delta t_i = T_i - T_{i-1}, \qquad Q(\tau > T_k) = e^{-\Gamma(T_k)}.$$

The corresponding default probability is $Q(\tau \le T_k) = 1 - e^{-\Gamma(T_k)}$.

A CDS with maturity T_k and annual premium rate s_k has two legs:

- the *premium leg*, consisting of periodic fixed payments of s_k on the surviving notional;
- the *protection leg*, consisting of a contingent payment of $\mathrm{LGD} = 1 - R$ at default, where R is the recovery rate.

Under the risk-neutral measure, the time-0 value of the premium leg can be approximated by

$$\mathrm{PV}_{\mathrm{prem}}(s_k) \approx s_k \sum_{i=1}^{k} \Delta t_i \, DF_i \, Q(\tau > T_{i-1}),$$

where $DF_i = P(0, T_i)$ are the discount factors obtained and interpolated in Exercise 1. The protection leg is valued as

$$\mathrm{PV}_{\mathrm{prot}} \approx LGD \sum_{i=1}^{k} DF_i \big[Q(\tau > T_{i-1}) - Q(\tau > T_i)\big].$$

At inception of a par CDS we require

$$\mathrm{PV}_{\mathrm{prem}}(s_k) = \mathrm{PV}_{\mathrm{prot}}.$$

Solving these equations for each maturity yields the term structure of intensities $\{\lambda_k\}$ and hence the survival and default probabilities.

2.2 Methodology and Computations

Risk-free discount curve for CDS pricing

For CDS pricing, we require discount factors $DF_i = P(0, T_i)$ at the specific CDS maturities. Rather than performing a new calibration, we utilize the risk-free term structure derived in Exercise 1, which was calibrated to the 2023 Euribor swap curve. The parameter estimates established in that exercise are:

$$\beta_0 = 0.02546, \quad \beta_1 = -0.00298, \quad \beta_2 = 0.04508,$$
$$\beta_3 = -0.02871, \quad \tau_1 = 0.6672, \quad \tau_2 = 1.7066.$$

These parameters generate the risk-free term structure used to discount both the premium and protection legs. Discount factors are computed via $P(0, t) = e^{-y(t)t}$ and then linearly interpolated to match the exact CDS maturities.

CDS data and spread conversion

The sheet `CDS spread_2023` of `market_data2.xlsx` contains CDS quotes (in basis points) for six assets: Banco Santander, Eni, Ziggo, Lufthansa, Renault and Allianz. For each name and each maturity T_k, we denote the quoted spread by spread_bps_k and convert it to decimals via

$$s_k = \frac{\text{spread\_bps}_k}{10\,000}.$$

These decimal spreads are then used as the premium rates in the valuation equations for the premium and protection legs.

Premium and protection legs

Let $DF_k = P(0, T_k)$ denote the discount factor at T_k, and let $\Delta t_i = T_i - T_{i-1}$. With $\mathrm{LGD} = 1 - R$ and a fixed $R = 40\%$, the approximate valuation formulas are:

$$\mathrm{PV}_{\mathrm{prem}}(s_k) = s_k \sum_{i=1}^{k} \Delta t_i \, DF_i \, Q(\tau > T_{i-1}),$$

$$\mathrm{PV}_{\mathrm{prot}} = LGD \sum_{i=1}^{k} DF_i \big[Q(\tau > T_{i-1}) - Q(\tau > T_i) \big].$$

These formulas neglect accrual on default between payment dates.

Global bootstrapping of hazard rates

The unknown intensities λ_k are determined by imposing the par conditions at all maturities in increasing order.
The procedure is:

1. Initialise $\lambda_1 = 0$ on $[0, T_1]$, a convention that ensures the first short-term CDS contributes only through the protection leg beyond T_1.

2. For $k = 2, \dots, n$:

   (a) Given $(\lambda_1, \dots, \lambda_{k-1})$, define

   $$f_k(\lambda_k) = \mathrm{PV}_{\mathrm{prem}}(S_k; \lambda_1, \dots, \lambda_{k-1}, \lambda_k) - \mathrm{PV}_{\mathrm{prot}}(\lambda_1, \dots, \lambda_{k-1}, \lambda_k).$$

   (b) Numerically solve $f_k(\lambda_k) = 0$ using Brent's method (a robust bisection-secant hybrid) starting from an initial bracket $[0.0001, 5]$ and expanding the upper bound until a sign change is observed, if necessary.

   (c) In case that a sign change cannot be found, set $\lambda_k = \lambda_{k-1}$ as a fallback in order to maintain monotonicity of the cumulative hazard.

Once $(\lambda_1, \dots, \lambda_n)$ have been determined, we compute

$$\Gamma(T_k) = \sum_{i=1}^{k} \lambda_i \Delta t_i, \qquad Q(\tau > T_k) = e^{-\Gamma(T_k)}, \qquad Q(\tau \le T_k) = 1 - e^{-\Gamma(T_k)}$$

for all maturities T_k.

2.3 Results

For brevity the report will show only the main numerical outputs for each name. In all tables, spreads are in decimal form and hazard rates are the piecewise-constant intensities on the intervals $[T_{k-1}, T_k)$.

Banco Santander

Table 1: Banco Santander – hazard rates and probabilities.

Maturity T_k (y)	Spread s_k	Hazard λ_k	Cum. hazard $\Gamma(T_k)$	Survival $Q(\tau > T_k)$	Default $Q(\tau \leq T_k)$
0.5	0.002413	0.004022	0.002011	0.997991	0.002009
1	0.003028	0.010201	0.005101	0.994912	0.005088
2	0.004080	0.008660	0.013761	0.986333	0.013667
3	0.005123	0.012262	0.026022	0.974313	0.025687
4	0.006190	0.016132	0.042154	0.958722	0.041278
5	0.007259	0.020051	0.062205	0.939690	0.060310
7	0.008503	0.020505	0.103214	0.901934	0.098066
10	0.009567	0.021757	0.168485	0.844944	0.155056
20	0.010692	0.023382	0.402306	0.668776	0.331224
30	0.011557	0.029020	0.692504	0.500322	0.499678

Eni

Table 2: Eni – hazard rates and probabilities.

Maturity	Spread	Hazard	Cum. hazard	Survival	Default
0.5	0.001957	0.003262	0.001631	0.998370	0.001630
1	0.002684	0.009039	0.004520	0.995490	0.004510
2	0.003960	0.008849	0.013369	0.986720	0.013280
3	0.005219	0.013188	0.026557	0.973793	0.026207
4	0.006531	0.018044	0.044601	0.956379	0.043621
5	0.007821	0.022670	0.067271	0.934942	0.065058
7	0.010098	0.028511	0.124293	0.883121	0.116879
10	0.012246	0.032476	0.221720	0.801140	0.198860
20	0.014456	0.036139	0.583110	0.558160	0.441840
30	0.016036	0.050069	1.083795	0.338309	0.661691

Ziggo

Table 3: Ziggo – hazard rates and probabilities.

Maturity	Spread	Hazard	Cum. hazard	Survival	Default
0.5	0.009356	0.015593	0.007797	0.992233	0.007767
1	0.013426	0.045632	0.022816	0.977442	0.022558
2	0.020892	0.049178	0.071994	0.930537	0.069463
3	0.029294	0.083063	0.155057	0.856366	0.143634
4	0.037271	0.115762	0.270819	0.762755	0.237245
5	0.044677	0.149418	0.420237	0.656891	0.343109
7	0.052718	0.162029	0.744294	0.475069	0.524931
10	0.057423	0.169510	1.252826	0.285696	0.714304
20	0.061205	0.169510	2.947931	0.052448	0.947552
30	0.062180	0.169510	4.643035	0.009628	0.990372

Lufthansa

Table 4: Lufthansa – hazard rates and probabilities.

Maturity	Spread	Hazard	Cum. hazard	Survival	Default
0.5	0.009300	0.015500	0.007750	0.992280	0.007720
1	0.011994	0.040715	0.020358	0.979848	0.020152
2	0.013983	0.027170	0.047528	0.953584	0.046416
3	0.015955	0.034253	0.081781	0.921473	0.078527
4	0.019829	0.056074	0.137855	0.871225	0.128775
5	0.023607	0.071224	0.209080	0.811330	0.188670
7	0.026950	0.066936	0.342952	0.709672	0.290328
10	0.029157	0.067658	0.545927	0.579304	0.420696
20	0.031345	0.091534	1.461267	0.231942	0.768058
30	0.032546	0.133288	2.794147	0.061167	0.938833

Renault

Table 5: Renault – hazard rates and probabilities.

Maturity	Spread	Hazard	Cum. hazard	Survival	Default
0.5	0.007316	0.012193	0.006097	0.993922	0.006078
1	0.008313	0.028131	0.014066	0.986033	0.013967
2	0.015059	0.037591	0.051656	0.949655	0.050345
3	0.021588	0.061446	0.113102	0.893059	0.106941
4	0.027344	0.081789	0.194891	0.822924	0.177076
5	0.032782	0.104416	0.299307	0.741331	0.258669
7	0.039574	0.117594	0.534495	0.585965	0.414035
10	0.041303	0.092064	0.810688	0.444552	0.555448
20	0.043026	0.156656	2.377246	0.092806	0.907194
30	0.043688	0.331336	5.690610	0.003378	0.996622

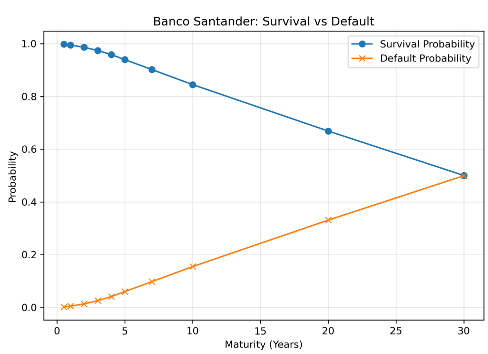
Allianz

Table 6: Allianz – hazard rates and probabilities.

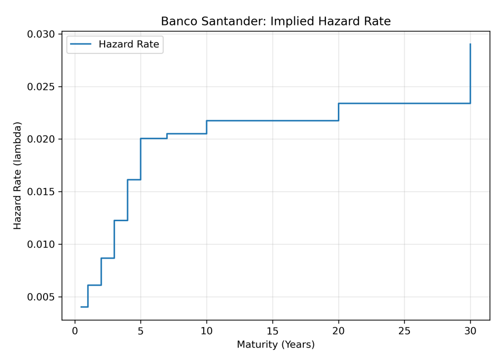
Maturity	Spread	Hazard	Cum. hazard	Survival	Default
0.5	0.001484	0.002473	0.001237	0.998764	0.001236
1	0.002011	0.006769	0.003385	0.996621	0.003379
2	0.002743	0.005855	0.009240	0.990803	0.009197
3	0.003467	0.008336	0.017576	0.982578	0.017422
4	0.004152	0.010603	0.028178	0.972215	0.027785
5	0.004829	0.012992	0.041170	0.959666	0.040334
7	0.005835	0.014650	0.070469	0.931956	0.068044
10	0.006886	0.016833	0.120970	0.886061	0.113939
20	0.008001	0.017725	0.298224	0.742135	0.257865
30	0.008896	0.022953	0.527754	0.589928	0.410072

Graphs

Figures 4–9 display, for each company, the term structure of survival and default probabilities together with the piecewise-constant hazard rate curve. The survival and default curves are monotone in maturity, and the hazard functions exhibit the expected upward or flattening patterns depending on the credit quality implied by the spreads.

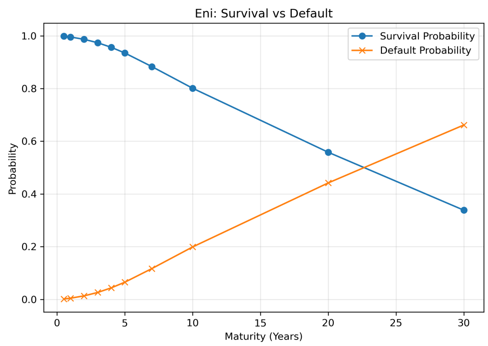


(a) Survival and default probabilities.

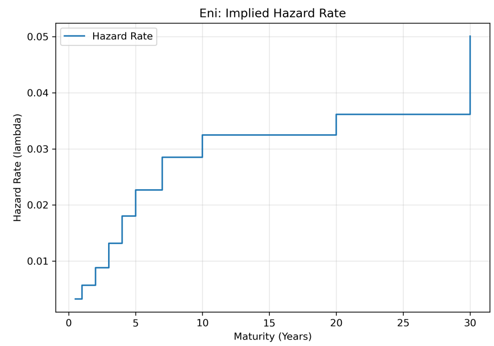


(b) Hazard rate term structure.

Figure 4: Banco Santander.

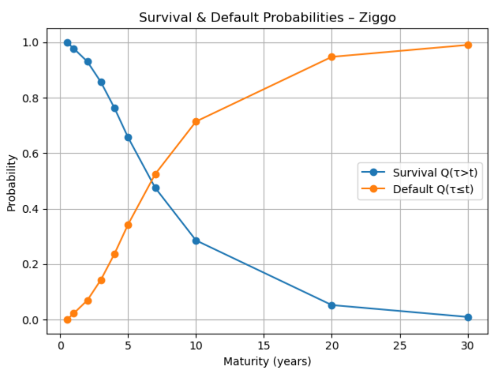


(a) Survival and default probabilities.

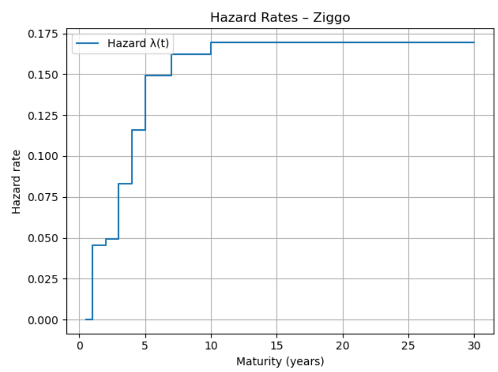


(b) Hazard rate term structure.

Figure 5: Eni.

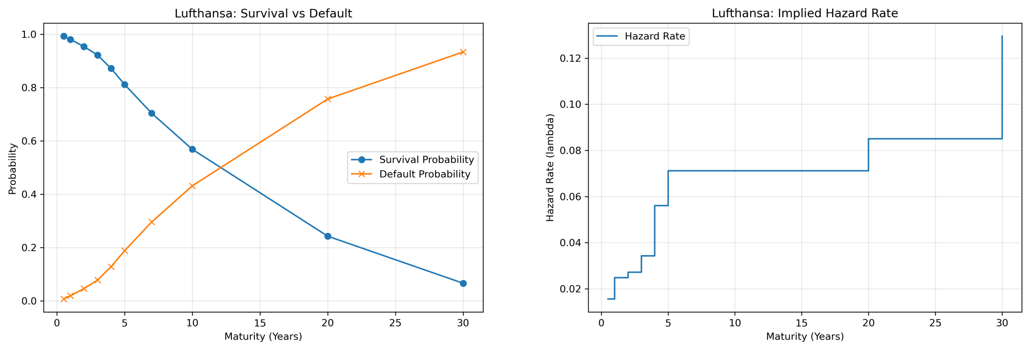


(a) Survival and default probabilities.



(b) Hazard rate term structure.

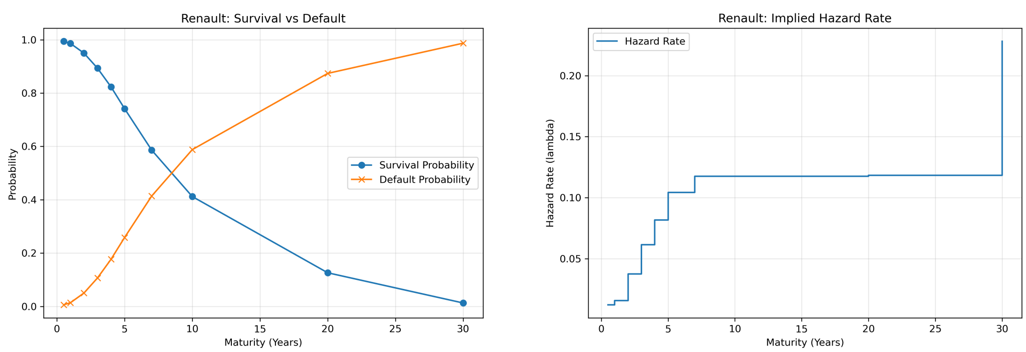
Figure 6: Ziggo.



(a) Survival and default probabilities.

(b) Hazard rate term structure.

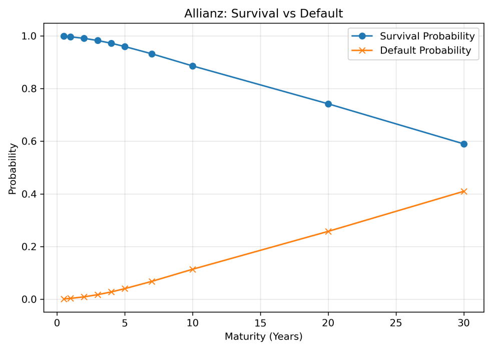
Figure 7: Lufthansa.



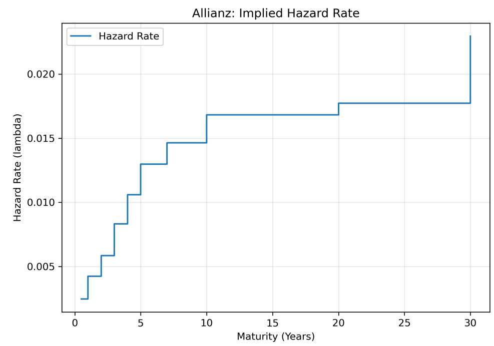
(a) Survival and default probabilities.

(b) Hazard rate term structure.

Figure 8: Renault.



(a) Survival and default probabilities.



(b) Hazard rate term structure.

Figure 9: Allianz.

2.4 Comments and Conclusions

The bootstrapped intensity curves and their corresponding survival and default probabilities are consistent with the CDS spreads and the underlying risk-free term structure constructed in Exercise 1.
Key features are:

- **Allianz** exhibits the lowest hazard rates and highest survival probabilities across maturities, reflecting its relatively tight CDS spreads and perceived high credit quality.

- **Ziggo** and **Renault** show high and increasing hazard rates, with 30-year default probabilities above 95%, consistent with their wide spreads and lower implied credit quality.

- **Banco Santander**, **Eni** and **Lufthansa** lie in between, with moderate hazard rates and long–term default probabilities between roughly 40% and 75%.

The piecewise-constant intensity structure implies that changes in the hazard rate occur only at CDS maturities, but survival curves remain smooth and monotonically decreasing. From the assignment perspective, Exercise 2 shows explicitly how to translate quoted CDS spreads into a term structure of risk–neutral default probabilities using a time–inhomogeneous Poisson model and a consistent risk–free curve.

3 Exercise 3: Multiname Credit Derivatives and Gaussian Copula Simulation

3.1 Theoretical Framework

The third exercise focuses on the pricing of a multiname credit derivative whose payoff depends on the joint survival of several obligors. Specifically, we consider a contract that pays

$$1_{\{\tau_i > 5,\ \forall i = 1, \dots, 5\}},$$

that is, one Euro at maturity $T = 5$ years provided that none of the five reference entities defaults before T.
Denoting by $P(0, 5)$ the 5-year risk-free discount factor and assuming independence between interest rates and default times, the time-0 fair value is

$$\mathrm{FV} = P(0,5)\,\mathbb{E}^Q\left[1_{\{\tau_i > 5\ \forall i\}}\right] = P(0,5)\,\mathbb{P}^Q(\tau_1 > 5, \dots, \tau_5 > 5).$$

The assignment fixes $P(0, 5) = 0.95$.

For each name i, the default time τ_i is modeled as the first jump of a Poisson process with constant intensity λ_i, so that

$$\mathbb{P}(\tau_i > t) = e^{-\lambda_i t}.$$

Unlike Exercise 2, where $\lambda(t)$ is piecewise constant in t, here each λ_i is taken as a single constant calibrated to the most recent CDS spread.

Dependence between the default times is introduced using a *Gaussian copula*. In general, by Sklar's theorem any multivariate distribution with continuous margins can be expressed in terms of its univariate marginal distribution and a copula capturing dependence.
The Gaussian copula is obtained by:

1. mapping marginal default probabilities to standard normal scores;
2. assuming a multivariate normal structure for these scores with some correlation matrix C;
3. mapping back to $[0, 1]$ via the univariate normal CDF.

Formally, if $X = (X_1, \dots, X_d) \sim N(0, C)$ and $U_i = \Phi(X_i)$, where Φ is the standard normal CDF, then $U = (U_1, \dots, U_d)$ has uniform marginals and dependence structure given by the Gaussian copula with parameter C.

In the assignment, for each name i we construct a time series of "5-year default probabilities" $p_i(t)$ from CDS spreads via

$$p_i(t) = 1 - \exp\left(-\frac{\mathrm{CDS}_i(t)}{1 - \mathrm{REC}_i}\right), \qquad \mathrm{REC}_i = 40\%. \tag{5}$$

These probabilities are not directly used as $Q(\tau_i \leq 5)$, but rather are historical probabilities of default, used to estimate the copula parameter C through empirical pseudo-data, extracting the correlation structure of the dependence between the entities, separating the joint co-movements (the copula) from the individual levels of default risk (the marginals), which are instead calibrated to the current spot market spreads.

3.2 Methodology and Computations

CDS time series and default-probability transformation

We use the Excel file `CDS_time_series.xlsx`, which contains daily CDS spreads (in basis points) for five entities: *ENI, Unicredit, Volkswagen, Allianz* and *Iberdrola*. After removing rows with missing values, the spreads (in bps) are converted to decimal form, $\mathrm{CDS}_i(t)$, and for each date the transformation (5) is applied:

$$p_i(t) = 1 - \exp\left(-\frac{\mathrm{CDS}_i(t)}{1 - \mathrm{REC}}\right), \qquad \mathrm{REC} = 0.4.$$

This yields time series of "5-year default probabilities" which condense the information contained in the CDS spreads. The first five observations appear as:

	ENI	Unicredit	Volkswagen	Allianz	Iberdrola
t_1	0.01212	0.02250	0.01351	0.00576	0.01044
t_2	0.01198	0.02195	0.01325	0.00578	0.01036
t_3	0.01197	0.02226	0.01340	0.00611	0.01038
t_4	0.01202	0.02272	0.01370	0.00648	0.01056
t_5	0.01193	0.02217	0.01311	0.00621	0.01031

The small magnitude of these numbers (typically between 0.5% and 2.5%) is consistent with investment-grade 5-year CDS spreads.

Empirical copula construction

Given the time series $p_i(t)$, the empirical copula is constructed in two steps:

From probabilities to pseudo-uniforms. For each name i separately, the values $p_i(t)$ are converted into pseudo-uniforms via their empirical CDF. Concretely, if the time index t takes N values and

$$\operatorname{rank}(p_i(t))$$

denotes the rank of $p_i(t)$ among these N values (average rank in case of ties), we set

$$U_i(t) = \frac{\operatorname{rank}(p_i(t))}{N+1}.$$

By construction $U_i(t) \in (0,1)$, and for large N each U_i is close to uniform on $(0,1)$.

From pseudo-uniforms to Gaussian copula. To fit a Gaussian copula, we apply the inverse standard normal CDF componentwise:

$$Z_i(t) = \Phi^{-1}(U_i(t)).$$

The vectors $Z(t) = (Z_1(t), \ldots, Z_5(t))$ are then used to estimate the correlation matrix C via the sample correlation:

$$C = \operatorname{Corr}\big(Z_1, \ldots, Z_5\big).$$

The resulting matrix is

$$C = \begin{pmatrix} 1 & 0.742 & 0.849 & 0.619 & 0.045 \\ 0.742 & 1 & 0.844 & 0.723 & 0.470 \\ 0.849 & 0.844 & 1 & 0.714 & 0.232 \\ 0.619 & 0.723 & 0.714 & 1 & 0.527 \\ 0.045 & 0.470 & 0.232 & 0.527 & 1 \end{pmatrix}.$$

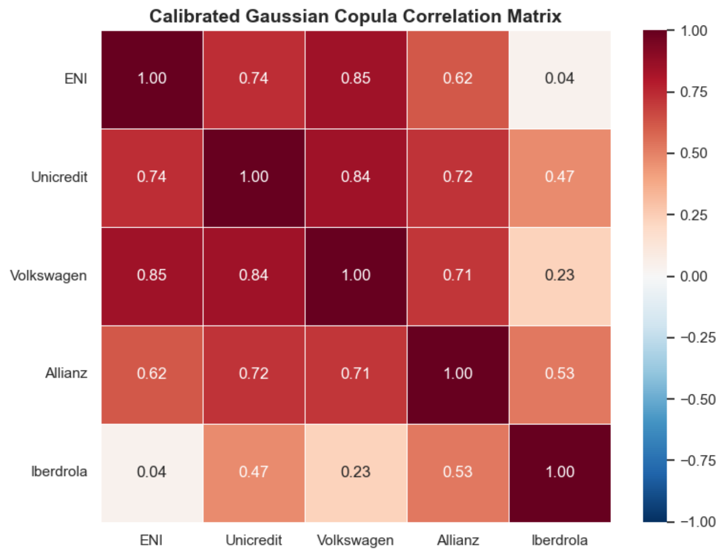


Figure 10: Heatmap of the calibrated correlation matrix C.

A numerical check of positive-definiteness based on the eigenvalues of C shows that

$$\lambda_{\min}(C) \approx 1.08 \times 10^{-1} > 0,$$

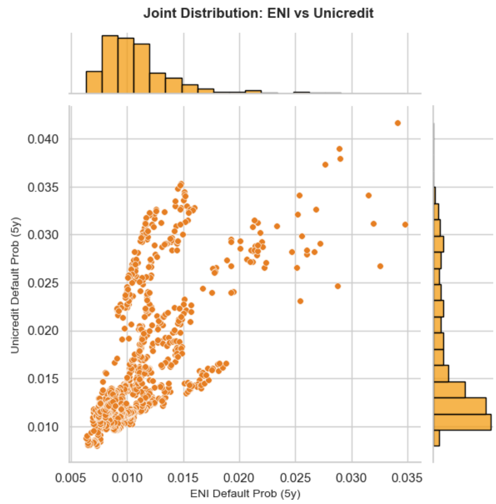
so that C is a valid correlation matrix. Scatterplots of selected pairs (U_i, U_j) further confirm the strong positive dependence between ENI, Unicredit, Volkswagen and Allianz; Iberdrola instead exhibits weaker correlation with the others.

To visually inspect the dependence, we plot the joint distributions of selected pairs. Figure 11 shows pairs with strong correlation (> 0.7), characterized by a tight, linear-like relationship. Figure 12 shows pairs involving Iberdrola, where the scatter is more dispersed due to lower correlation.

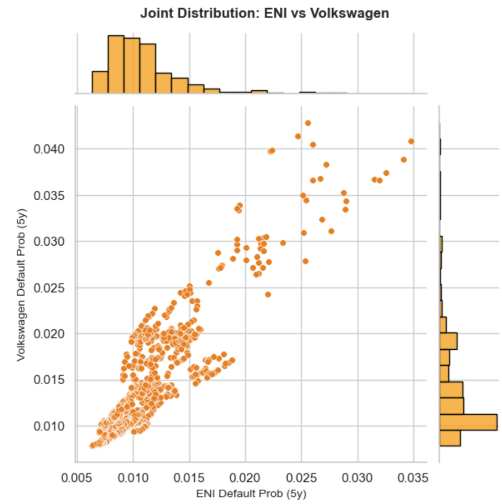
Pairwise Joint Distributions

To visualize the dependence structure, we examine the joint distributions of all entity pairs.

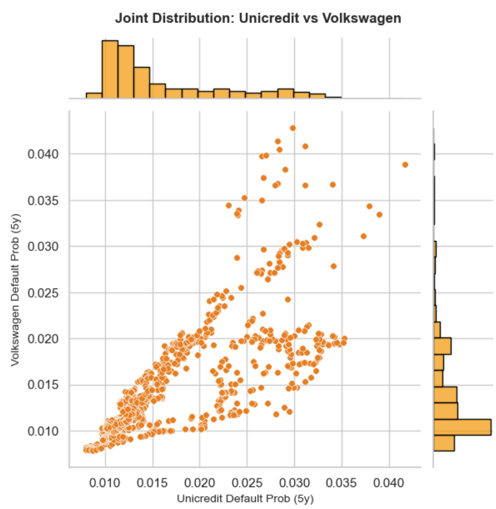
Figure 11 displays the relationships between ENI, Unicredit, Volkswagen, and Allianz. These pairs exhibit **strong positive dependence,** characterized by tight, diagonal clustering. This visual evidence supports the high correlation coefficients (ranging from 0.62 to 0.85) obtained in the calibration.



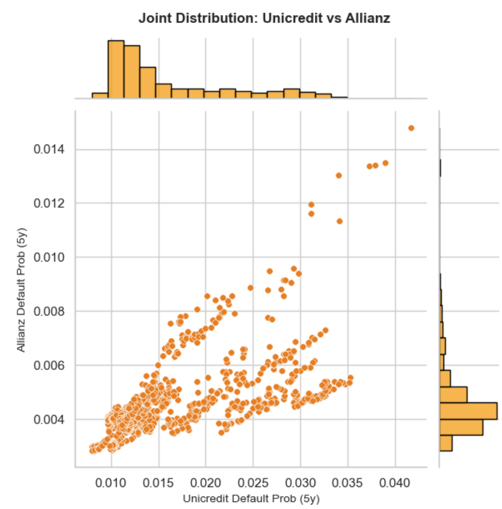
(a) ENI vs Unicredit



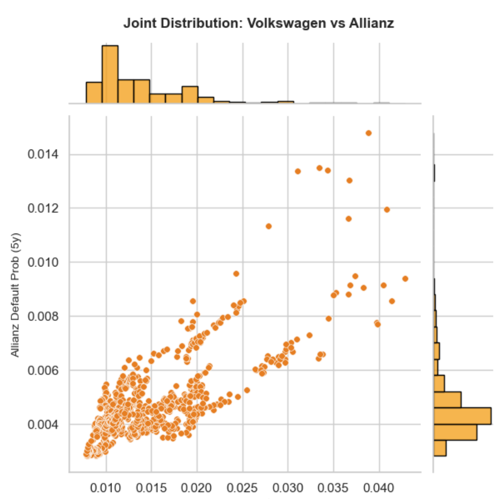
(b) ENI vs Volkswagen



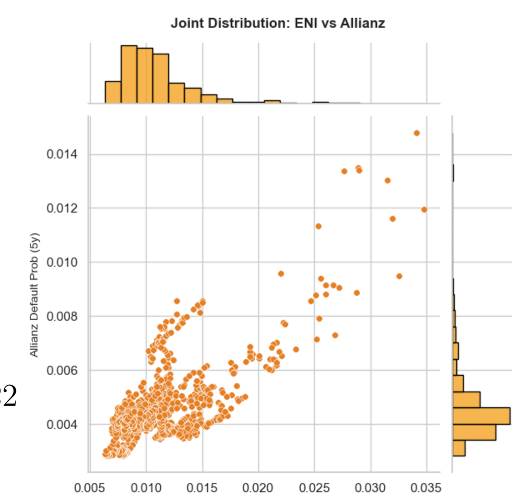
(c) Unicredit vs Volkswagen



(d) Unicredit vs Allianz



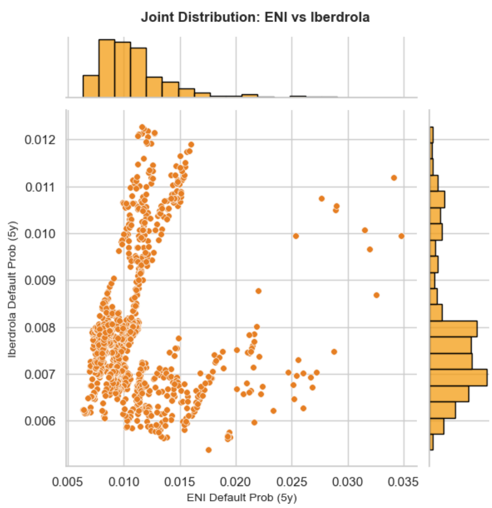
(e) Volkswagen vs Allianz



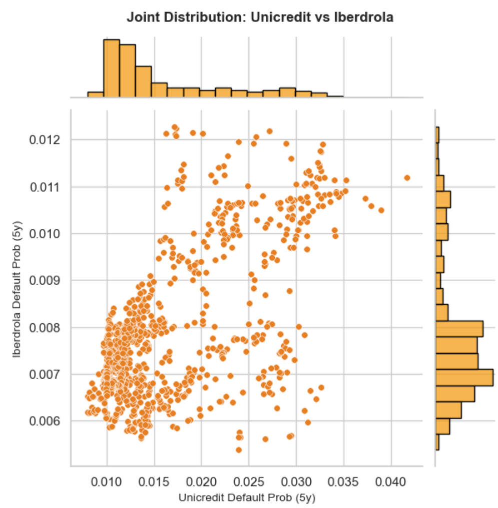
(f) ENI vs Allianz

Figure 11: Joint distributions for the strongly correlated core entities.

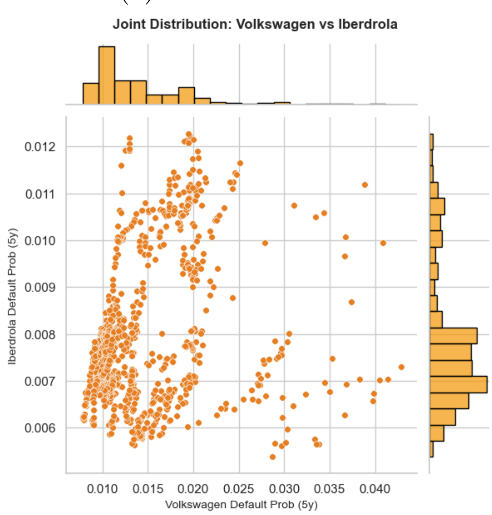
Figure 12 isolates the pairs involving **Iberdrola**. In contrast to the core group, these scatter plots are significantly more dispersed. This confirms that Iberdrola behaves with greater independence (lower correlation), providing a degree of diversification benefit to the basket.



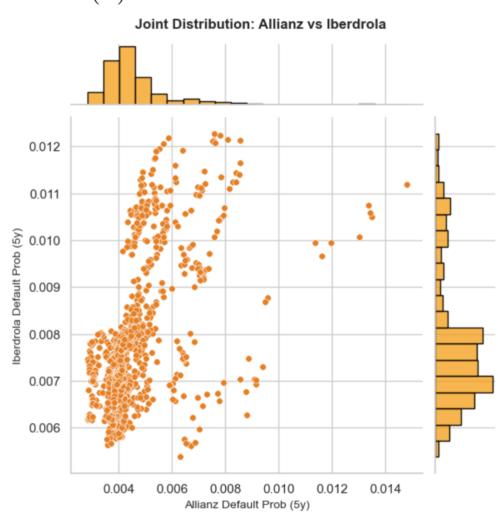
(a) ENI vs Iberdrola



(b) Unicredit vs Iberdrola



(c) Volkswagen vs Iberdrola



(d) Allianz vs Iberdrola

Figure 12: Joint distributions involving Iberdrola. The scattered pattern reflects the lower correlation structure observed in the calibration matrix.

Constant hazard rates

The marginal default intensities λ_i are calibrated from the last observed CDS spreads using the *credit triangle* approximation, which relates a flat hazard rate, a flat CDS spread and a constant recovery rate. If $\mathrm{CDS}_i^{\mathrm{last}}$ denotes the last CDS quote in decimal form for name i, the approximation

$$\mathrm{CDS}_i^{\mathrm{last}} \approx \lambda_i(1-R)$$

leads to

$$\lambda_i \approx \frac{\mathrm{CDS}_i^{\mathrm{last}}}{1-R}.$$

Applying this with $R = 40\%$ yields

$$\lambda = (0.01495,\ 0.02169,\ 0.02456,\ 0.00818,\ 0.01116),$$

for ENI, Unicredit, Volkswagen, Allianz and Iberdrola, respectively. These values correspond to annual default intensities between roughly 0.8% and 2.5%, consistent with the level of the spreads.

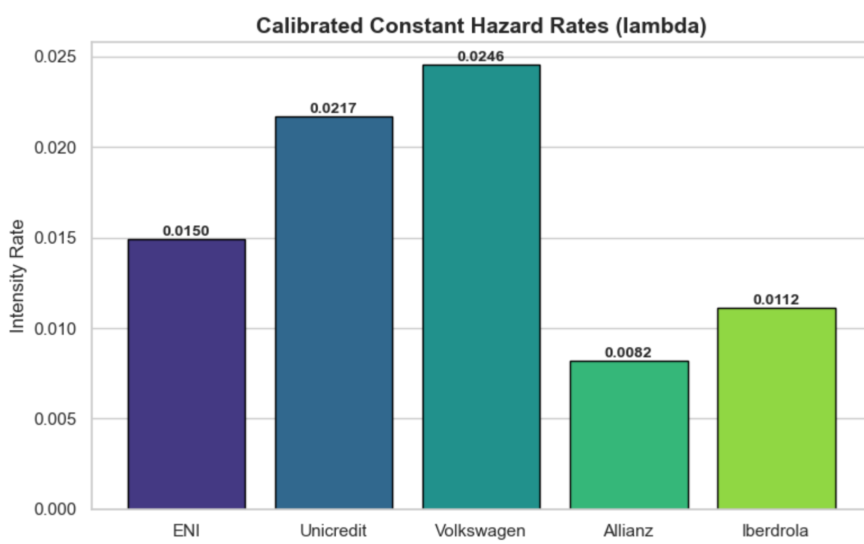


Figure 13: Calibrated constant hazard rates (λ_i) for each entity.

Gaussian copula simulation of default times

Given the correlation matrix C and the marginal intensities λ_i, the joint distribution of default times is simulated as follows:

1. Draw $X \sim N(0, C)$, a 5-dimensional Gaussian vector.
2. Map to uniforms via $U_i = \Phi(X_i)$.
3. For each i, set
$$\tau_i = -\frac{\ln(1 - U_i)}{\lambda_i}.$$
Since $U_i \sim \text{Uniform}(0, 1)$, it follows that τ_i has the exponential distribution with parameter λ_i, consistent with a constant-intensity Poisson model.
4. Record the indicator
$$I = 1_{\{\tau_i > 5 \ \forall i\}},$$
which equals one if all names survive to $T = 5$ and zero otherwise.

This procedure is repeated for $N_{\text{MC}} = 200{,}000$ Monte Carlo simulation paths. The Monte Carlo estimator of the basket survival probability is

$$\widehat{p}_{\text{all}} = \frac{1}{N_{\text{MC}}} \sum_{\ell=1}^{N_{\text{MC}}} I_\ell,$$

and the fair value is

$$\widehat{\text{FV}} = P(0, 5)\, \widehat{p}_{\text{all}}.$$

Under standard Monte Carlo theory, the standard error of $\widehat{p}_{\text{all}}$ is

$$\text{SE}(\widehat{p}_{\text{all}}) = \sqrt{\frac{\widehat{p}_{\text{all}}(1 - \widehat{p}_{\text{all}})}{N_{\text{MC}}}},$$

and the standard error of $\widehat{\text{FV}}$ is simply $P(0, 5)\, \text{SE}(\widehat{p}_{\text{all}})$.

3.3 Results

Using $N_{\text{MC}} = 200{,}000$ paths, the simulation yields:

$$\widehat{p}_{\text{all}} = 0.7961 \qquad \text{SE}(\widehat{p}_{\text{all}}) = 0.000903.$$

The estimated probability that *all* five names survive to five years is therefore about 79.61%, with a Monte Carlo uncertainty on the order of 0.09 percentage points. Multiplying by $P(0, 5) = 0.95$ gives the fair value

$$\widehat{\text{FV}} = 0.95 \times 0.7961 = 0.756295, \qquad \text{SE}(\widehat{\text{FV}}) = 0.000887.$$

Thus, the present value of the payoff is approximately 0.7562955 Euro, with a standard error below one thousandth.

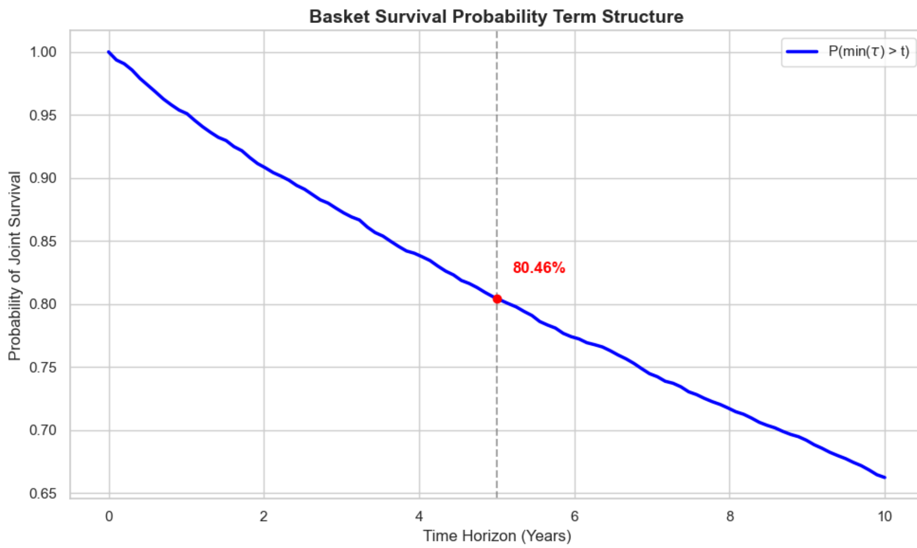


Figure 14: Simulated Basket Survival Probability Term Structure ($P(\min(\tau) > t)$). The red dot indicates the probability at maturity $T = 5$. **N.B. The survival probability shown in the graph is different since it is estimated through Monte Carlo simulation**

Diagnostic histograms of the pseudo-uniform variables U_i confirm that they are close to uniform on $(0, 1)$, as required for a copula-based analysis. Scatterplots of (U_i, U_j) for selected pairs show positively correlated clouds, consistent with the relatively large entries in the correlation matrix C for ENI, Unicredit, Volkswagen and Allianz.

Monte Carlo Diagnostics

To verify the robustness of the simulation, we analyze the convergence of the estimator and the distribution of loss scenarios.

Figure 15 demonstrates the stability of the Monte Carlo estimator. The Fair Value estimate converges rapidly and stabilizes well before $N = 200,000$ iterations, confirming that the sample size is sufficient for the required precision.

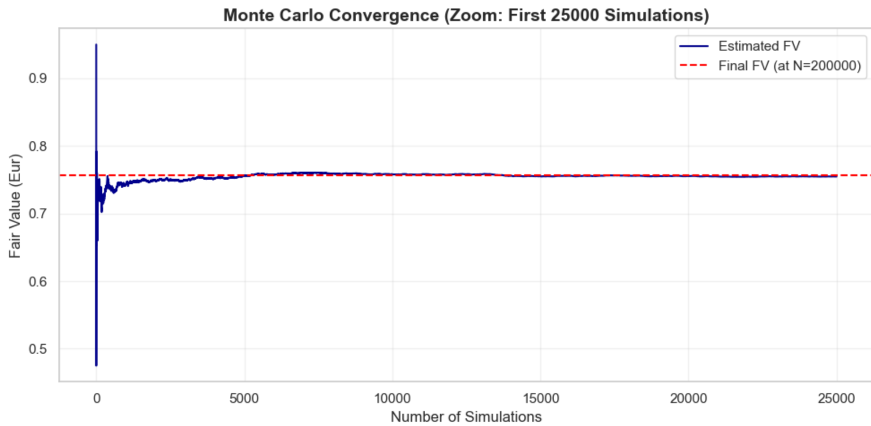


Figure 15: Convergence of the Fair Value estimate as a function of simulation steps.

Figure 16 illustrates the distribution of the number of defaults within the basket at the 5-year horizon. Due to the strong positive correlation, the distribution is effectively bimodal: in $\approx 79.61\%$ of scenarios, **zero** defaults occur (the "survival" case). However, if defaults do occur, it is relatively common to see multiple simultaneous defaults (2 or more) rather than just a single isolated default. This "tail clustering" is a direct consequence of the Gaussian copula dependence structure.

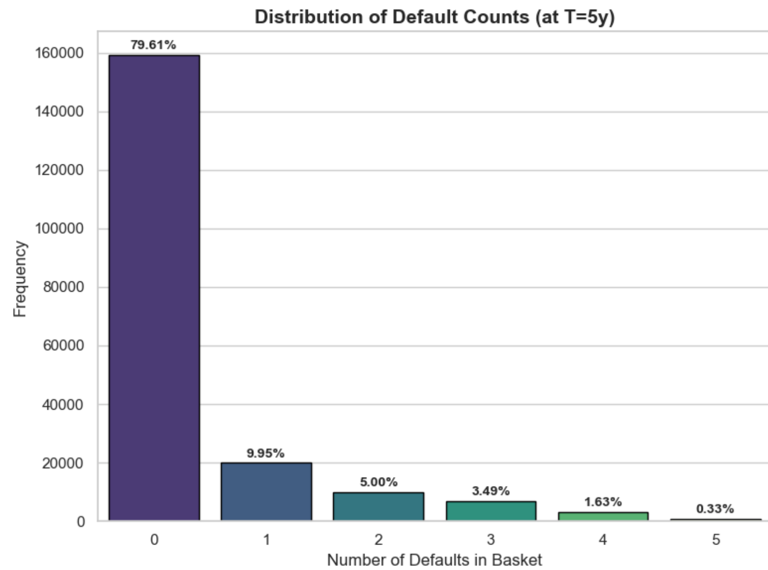


Figure 16: Distribution of the number of defaulting entities per scenario. The high probability of zero defaults drives the derivative's value.

3.4 Comments and Conclusions

Exercise 3 moved from *single* default term structures to a *multivariate* credit model. The key features are:

- a mapping from CDS spreads to "5-year default probabilities" $p_i(t)$, which serve as inputs for copula estimation;
- an empirical Gaussian copula fitted to pseudo-uniform observations derived from the $p_i(t)$ series;
- marginal constant intensities λ_i calibrated from the most recent CDS spreads via the credit triangle;
- Monte Carlo simulation of correlated exponential default times under the Gaussian copula specification.

The resulting fair value of the joint-survival payoff is economically plausible: entities with higher spreads contribute higher marginal intensities, and the strong positive dependence encoded in C lowers the probability of joint survival relative to an independence assumption. The Monte Carlo standard errors are very small, confirming the numerical robustness of the estimate.