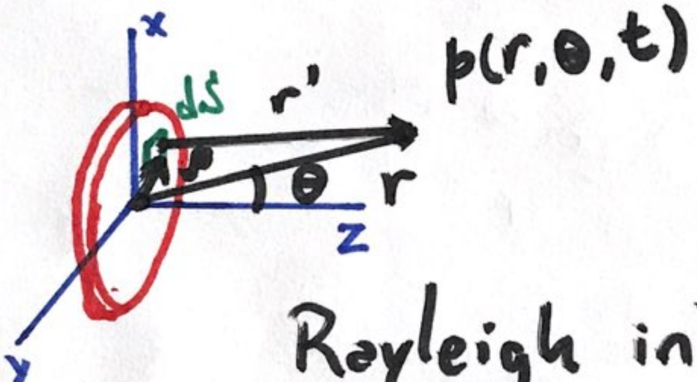
Acoustic Beam Properties

- Circular piston



Rayleigh integral

$$p(\vec{r}) = -\frac{i\omega\rho_0}{2\pi}\int_{S'} v_n(x_0, y_0)\,\frac{e^{ikr'}}{r'}\,dx_0\,dy_0$$

- On the piston axis

$$r' = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}$$

$$r = z\,, \quad x = y = 0$$

$$r' = \sqrt{x_0^2 + y_0^2 + r^2} = \sqrt{\rho^2 + r^2}$$

Normal velocity: $v_n = v_0\, e^{-i\omega t}$

Pressure:

$$p(z) = -\frac{i\omega\rho_0}{2\pi}\, v_0 \int_{S_0} \frac{e^{ik\sqrt{\rho^2+r^2}}}{\sqrt{\rho^2+r^2}}\, \underbrace{dS}_{2\pi\rho\, d\rho}$$

Cont...

$$p(z) = -i\omega\rho_0 v_0 \int_0^a \frac{e^{ik\sqrt{\rho^2+r^2}}}{\sqrt{\rho^2+r^2}}\,\rho\,d\rho$$

(easy) $\rightarrow$ $\frac{d}{d\rho}\left(\frac{e^{ik\sqrt{\rho^2+r^2}}}{ik}\right) =$

$$= \frac{\cancel{2}\cancel{ik}\rho}{\cancel{ik}}\, e^{ik\sqrt{\rho^2+r^2}}\,\frac{1}{\cancel{2}}\,\frac{1}{\sqrt{\rho^2+r^2}}$$

$$= \frac{\rho\, e^{ik\sqrt{\rho^2+r^2}}}{\sqrt{\rho^2+r^2}}$$

We find

$$p(z) = -i\omega\rho_0 V_0 \left.\frac{e^{ik\sqrt{\rho^2+r^2}}}{\underset{\rightarrow \frac{\omega}{c_0}}{ik}}\right|_0^a$$

$$= +\rho_0 c_0 V_0\left(e^{ikr} - e^{ik(\sqrt{r^2+a^2})}\right)$$

$$= \rho_0 c_0 V_0\left(1 - e^{ik(\sqrt{r^2+a^2}-r)}\right)e^{ikr}$$

Cont...

$$p = \rho_0 c_0 V_0 \, e^{ikr}\left(1 - \exp\left(ik\left(\sqrt{r^2+a^2} - r\right)\right)\right)$$

Pressure magnitude

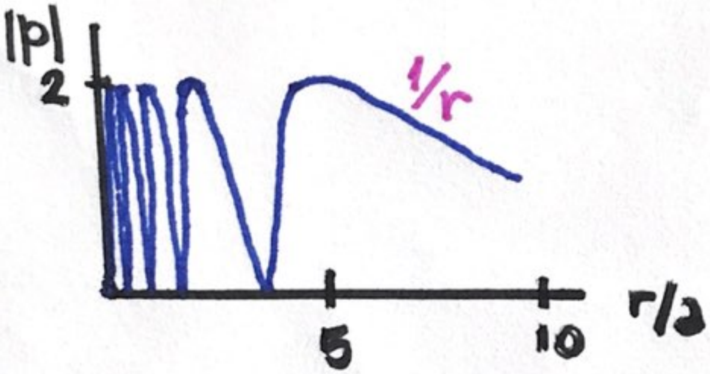
$$p = 2\rho_0 c_0 V_0 \left| \sin\left[\frac{1}{2} kr\left(\sqrt{1+(a/r)^2} - 1\right)\right] \right|$$

Far from the source $r/a \gg 1$

$$\sqrt{1 + (a/r)^2} \simeq 1 + \frac{1}{2}\left(\frac{a}{r}\right)^2$$

$$\Rightarrow p_{far} = 2\rho_0 c_0 V_0 \left| \sin\left(\frac{ka}{4}\frac{a}{r}\right) \right|$$

Pressure plot



$$p_{far} = 2\rho_0 c_0 v_0 \left| \sin\left(\frac{ka}{4}\frac{a}{r}\right)\right| \quad (r \gg a)$$

Consider $\frac{ka}{4}\frac{a}{r} \ll 1$

$$p_{far} \simeq \rho_0 c_0 v_0 \frac{ka}{2}\frac{a}{r}$$

Condition: $\frac{ka}{4}\frac{a}{r} = 1 \quad \therefore \quad r = \frac{ka^2}{4} = \frac{2\pi a^2}{4\lambda}$

$r = \frac{\pi a^2}{2\lambda}$; $r < \frac{a^2}{\lambda}$ (Freshel zone)

$r \gg \frac{a^2}{\lambda}$ (Fraunhofer zone)

Fresnel zone

Pressure zeros:

$$\frac{1}{2} kr\left(\sqrt{1+(a/r)^2} - 1\right) = n\pi \quad ; \quad n = 1, 2, \ldots$$

Solve for r:

$$kr_n = \frac{(ka)^2 - (2n\pi)^2}{4n\pi}$$

$$kr_1 = \frac{(ka)^2 - 4\pi^2}{4\pi} > 0 \quad ; \quad ka > 2\pi$$

$$\frac{ka}{2\pi} > 1$$

$$(ka)^2 - (2n\pi)^2 > 0$$

$$\Rightarrow n < \frac{ka}{2\pi}$$

[Mathematica]

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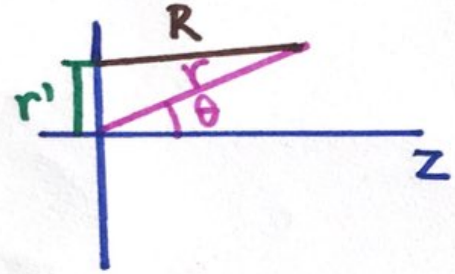
[Mathematica]

Farfield region



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r' = \sqrt{x_0^2 + y_0^2}$$

$$\vec{R} = |\vec{r} - \vec{r}'|$$

In spherical coordinates (r, θ, φ)

$$R = \sqrt{r^2 + r'^2 - 2rr'\cos(\varphi - \varphi')\sin\theta}$$

Fraunhofer approximation $r' \ll r$

$$R \simeq r - r'\sin\theta\cos(\varphi - \varphi')$$

Back to the Rayleigh integral

$$p = -\frac{i\omega\rho_0 v_0}{2\pi}\int_{S_0} \frac{e^{ikR}}{R}\, r'dr'd\varphi'$$

$$= -\frac{i\omega\rho_0 v_0}{2\pi}\, e^{ikr}\int_0^{2\pi}\int_0^{a} \frac{e^{-ikr'\sin\theta\cos(\varphi-\varphi')}}{\underset{R\approx r}{r}}\, r'dr'd\varphi'$$

$$= -\frac{ik\rho_0 c_0 v_0}{2\pi}\,\frac{e^{ikr}}{r}\int_0^{2\pi}\int_0^{a} e^{-ikr'\sin\theta\cos(\varphi-\varphi')}\, r'dr'd\varphi'$$

Bessel identity

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\varphi}\, d\varphi \;, \quad J_0(-x) = J_0(x)$$

$$p = -\frac{ik}{\text{[redacted]}} \rho_0 c_0 v_0 \frac{e^{ikr}}{r} \int_0^a r' \text{[redacted]} J_0(kr'\sin\theta)\, dr'$$

Using $\int x J_0(x)\,dx = x J_1(x)$; $J_1(0) = 0$

$$p = -ik \rho_0 c_0 v_0 \frac{e^{ikr}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Zero of $J_1(x)$: $x_1 = 3.83171$

$$ka\sin\theta = 3.83171$$

$$ka >> 1 \Rightarrow \sin\theta << 1$$

$$\Rightarrow \theta \simeq \frac{3.83171}{ka} = \frac{3.83171\lambda}{2\pi a}$$

$$= 0.61\frac{\lambda}{a}$$