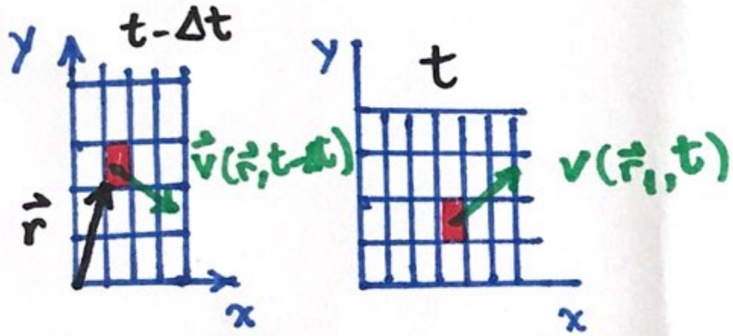
Introduction to Acoustics

- Continuum fluid (Eulerian frame)



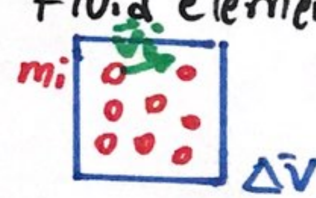
$\vec{r} \equiv$ position vector

$t \equiv$ time

$\vec{v} \equiv$ fluid element velocity

Zoom in

Fluid element



$m_i \equiv$ mass

$\vec{v}_i \equiv$ particle velocity

$\Delta V \equiv$ element volume

Density: $\rho(\vec{r},t) \equiv \frac{1}{\Delta V} \sum_{i=1}^{N(t)} m_i$

Center of mass

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{N(t)} m_i \vec{r}_i}{\sum_{i=1}^{N(t)} m_i} = \frac{\sum_{i=1}^{N(t)} m_i \vec{r}_i}{\rho(\vec{r},t)\Delta V}$$

Velocity field

$$\cancel{\vec{v} = \frac{d\vec{r}_{CM}}{dt} \; i \; N}$$

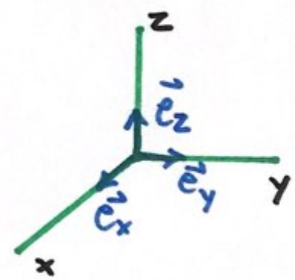
$$\vec{v} = \frac{1}{\rho(\vec{r},t)\Delta \bar{V}} \sum_{i=1}^{N} m_i \vec{v}_i$$

Mathematical preliminaries

Coordinate systems

- Cartesian (x, y, z), unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$
- Curvilinear: cylindrical, spherical

Position vector



$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

Summation rule

$$\vec{r} = \underbrace{r_i \vec{e}_i}_{\sum\limits_{i=x,y,z} r_i \vec{e}_i} \;; \quad r_i = r_x, r_y, r_z = x, y, z$$

$$i = x, y, z$$

Vector operations

- Scalar product

$$\vec{v} \cdot \vec{u} = v_i u_i$$

$$v = |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}^*} = \sqrt{v_i v_i^*} = \sqrt{|\vec{v}|^2}$$

- Cross-product

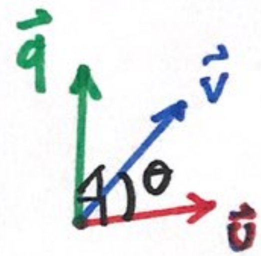
$$\vec{q} = \vec{u} \times \vec{v}$$

$$q_i \equiv \varepsilon_{ijk} u_j v_k$$

Levi-Cevita symbol

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{even permutation} \\ -1, & \text{odd} \quad " \\ 0, & \text{repeated index} \end{cases}$$

Geometrically



$$q = uv \sin\theta$$

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Tensors (second-rank)

→ Multidimensional arrays

A tensor $\overleftrightarrow{T}$ in Cartesian coordinates:

$$\overleftrightarrow{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Tensors can be formed from vector thorough the tensor product $\otimes$

$$\vec{v} \otimes \vec{u} = v_i u_j \, \vec{e}_i \otimes \vec{e}_j = v_i u_j \underbrace{\vec{e}_i \vec{e}_j}_{\text{dyad}}$$

$$\vec{e}_x\vec{e}_x = \vec{e}_1\vec{e}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \vec{e}_x\vec{e}_y = \vec{e}_1\vec{e}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \vec{e}_x\vec{e}_z = \vec{e}_1\vec{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

...

Derivative

$f(x) \equiv$ function (1D)

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x,y,z) \equiv$ 3D function

$$\partial_x f = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$\partial_y f = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

$$\partial_z f = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

Gradient

$$\boldsymbol{\nabla} \equiv \vec{e}_x \partial_x + \vec{e}_y \partial_y + \vec{e}_z \partial_z = \vec{e}_i \partial_i$$

Laplacian

$$\boldsymbol{\nabla}^2 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \partial_i \partial_i$$

Total derivative : $F(r(t), t)$

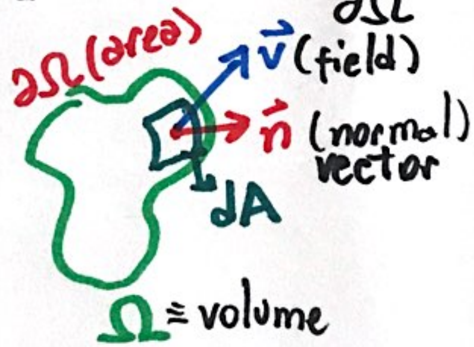
$$\frac{dF}{dt} = \partial_t F + (\partial_t r_i) \partial_i F = \partial_t F + (\vec{v} \cdot \boldsymbol{\nabla}) F$$

Divergence

$$\vec{\nabla}\cdot\vec{v} \equiv \partial_x v_x + \partial_y v_y + \partial_z v_z = \partial_i v_i$$

Gauss integration theorem

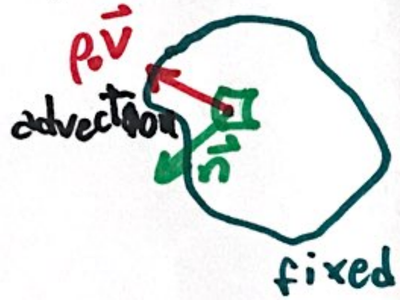
$$\int_\Omega \vec{\nabla}\cdot\vec{v}\, dV = \oint_{\partial\Omega} \vec{n}\cdot\vec{v}\, dA$$



Governing equations (linear approximation)

- Conservation of mass

$$\partial_t \rho + \nabla \cdot \rho_0 \vec{v} = 0 \quad ; \quad \rho_T = \rho_0 + \rho$$



- Navier-Stokes equation (momentum conservation)

$$\rho_0 \partial_t \dot{v}_i = -\underbrace{\partial_i p}_{\text{pressure}} + \partial_j \underbrace{\sigma_{ij}}_{\text{stress tensor}} + \rho g_i$$

Stress tensor

$$\sigma_{ij} = \eta\,(\partial_j v_i + \partial_i v_j) + (\beta - 1)\,\eta\,(\partial_k v_k)\,\delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$\eta \equiv$ dynamic shear viscosity

$\beta \equiv$ bulk-to-shear viscosity ratio

Thermodynamic relation

$$p \equiv p(\rho, s_0) \quad \text{(adiabatic process)} \quad s = s_0$$

$s_0 \rightarrow$ entropy

In vector notation

$$\begin{cases} \partial_t \rho + \rho_0 \nabla \cdot \vec{v} = 0 & (1) \\ \rho_0 \partial_t \vec{v} + \nabla p = \eta \nabla^2 \vec{v} + \beta\eta \nabla(\nabla \cdot \vec{v}) & (2) \\ p = p(\rho) = c_0^2 \rho + p_0 \text{ (ambient pressure)} \end{cases}$$

Taking the time derivative of (1) and using (2), (3) we obtain

$$\partial_t^2 \rho = c_0^2 \left[1 + \frac{(1+\beta)\eta}{\rho_0 c_0^2} \partial_t \right] \nabla^2 \rho_1$$

Consider a time-harmonic field

$$\rho(\vec{r},t) = \rho(\vec{r})e^{-i\omega t}; \quad p(\vec{r},t) = p(\vec{r})e^{-i\omega t}$$

$$\vec{v}(\vec{r},t) = v(\vec{r})e^{-i\omega t}$$

Using $\rho = \dfrac{p}{c_0^2}$

$$\Rightarrow \quad \nabla^2 p + k^2 p = 0$$

$K = (1+i\gamma)\dfrac{\omega}{c_0}$, with $\gamma = \dfrac{(1+\beta)}{2\rho_0 c_0^2}\eta\omega \simeq 10^{-5}$ (water)

$\beta = 1/3$ (air)

$\gamma \ll 1$

Loss-less approximation

$$\boxed{\nabla^2 p + k^2 p = 0 \qquad k = \frac{\omega}{c_0}}$$

Fluid velocity: $\boxed{\vec{v} = \dfrac{\nabla p}{i\rho_0 \omega}}$

Boundary condition