1 MPC Problem Formulation

The Model Predictive Control (MPC) algorithm is based on the solution of an optimization problem over a finite horizon of length N. In general, the formulation of such optimization problem requires a discrete-time representation of the model dynamics. Moreover, in order to cast the MPC problem as a Quadratic Program (QP) one and hence solve it efficiently, the model must also be linear. The models introduced in Section **??** are not only defined in continuous-time, but also highly non-linear. To overcome this issues, a Linear Time Varying (LTV) approximation of the dynamics is here presented. Then, two formulations of the MPC algorithm as a QP problem are introduced.

1.1 Linear Time Varying (LTV) Model

The LTV model is obtained by successive linearization and discretization of the non-linear dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ around a set of nominal state and input trajectories sampled at a fixed regular time intervals T_s:

$$\begin{aligned} \bar{\mathbf{x}}_k &= \bar{\mathbf{x}}(kT_s), \quad k = k_0, \dots, k_0 + N + 1 \\ \bar{\mathbf{u}}_k &= \bar{\mathbf{u}}(kT_s), \quad k = k_0, \dots, k_0 + N \end{aligned}$$

Hence, the system is first linearized around the nominal states and inputs as follows:

$$\begin{gathered} \delta\mathbf{x}(t) = \mathbf{A}_k \delta\mathbf{x}(t) + \mathbf{B}_k \delta\mathbf{u}(t) \\ \mathbf{A}_k = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k}, \quad \mathbf{B}_k = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k} \\ \delta\mathbf{x}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}_k, \quad \delta\mathbf{u}(t) = \mathbf{u}(t) - \bar{\mathbf{u}}_k \end{gathered} \tag{1.1}$$

with $kT_s \leq t < (k+1)T_s$ for $k = k_0, \dots, k_0 + N$. Then, the obtained linear system is discretized using first order forward Euler method:

$$\begin{gathered} \delta\mathbf{x}_{k+1} = \mathbf{A}_{d,k} \delta\mathbf{x}_k + \mathbf{B}_{d,k} \delta\mathbf{u}_k \\ \mathbf{A}_{d,k} = \mathbb{I} + T_s \mathbf{A}_k, \quad \mathbf{B}_{d,k} = T_s \mathbf{B}_k \\ \delta\mathbf{x}_k = \mathbf{x}_k - \bar{\mathbf{x}}_k, \quad \delta\mathbf{u}_k = \mathbf{u}_k - \bar{\mathbf{u}}_k \end{gathered} \tag{1.2}$$

This can eventually be rewritten as:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_{d,k} \mathbf{x}_k + \mathbf{B}_{d,k} \mathbf{u}_k + \mathbf{d}_k \\ \mathbf{d}_k &= \bar{\mathbf{x}}_{k+1} - \mathbf{A}_{d,k} \bar{\mathbf{x}}_k - \mathbf{B}_{d,k} \bar{\mathbf{u}}_k \end{aligned} \tag{1.3}$$

Using this LTV model, the MPC controller is able to solve an optimization problem for each k^{th} time step in the prediction horizon N. In particular, the result of the optimization will be the optimal input sequence that minimizes the cost function in the prediction horizon, from which only the first input will actually be applied. Hence, in successive applications of the algorithm, a nominal input sequence for future linearizations can be obtained as the remaining part of the optimal input sequence[1] while the nominal states can be obtained by applying the nominal inputs to the non-linear dynamics. For the first iteration of the MPC algorithm, the reference input sequence can be used even if it will result in an inaccurate linearization.

1.2 MPC Formulations

The Model Predictive Control problem with the LTV model approximation of the non-linear dynamics is formulated as the optimization problem **??**, which is solved at every time instance t.

$$\begin{aligned} \min_{\mathbf{U}_{t \to t+N|t}} \quad & \sum_{k=t}^{t+N} \left(\Delta\mathbf{x}_k^T \mathbf{Q} \Delta\mathbf{x}_k + \Delta\mathbf{u}_k^T \mathbf{R} \Delta\mathbf{u}_k \right) \\ \text{s.t.} \quad & \mathbf{x}_{k+1|t} = \mathbf{A}_{d,k|t} \mathbf{x}_{k|t} + \mathbf{B}_{d,k|t} \mathbf{u}_{k|t} + \mathbf{d}_{k|t} \\ & \mathbf{d}_{k|t} = \bar{\mathbf{x}}_{k+1|t} - \mathbf{A}_{d,k|t} \bar{\mathbf{x}}_{k|t} - \mathbf{B}_{d,k|t} \bar{\mathbf{u}}_{k|t} \\ & \mathbf{x}_{k|t} \in \mathcal{X}, \quad k = t, \dots, t+N \\ & \mathbf{u}_{k|t} \in \mathcal{U}, \quad k = t, \dots, t+N+1 \end{aligned} \tag{1.4}$$

[1] With the last input duplicated as N nominal inputs are needed.

Where $\mathbf{U}_{t \to t+N|t} = \begin{bmatrix} \mathbf{u}_{t|t} & \dots & \mathbf{u}_{t+N|t} \end{bmatrix}^T$ is the input sequence to be optimized, while $\Delta \mathbf{x}_k = \mathbf{x}_{k|t} - \bar{\mathbf{x}}_k$ and $\Delta \mathbf{u}_k = \mathbf{u}_{k|t} - \bar{\mathbf{u}}_k$ are the differences between the predicted states and the reference trajectory. The matrices $\mathbf{Q}$ and $\mathbf{R}$ are the weighting matrices for the state and input errors, respectively. This general expression of the MPC problem can be further casted into a QP problems in two different formulations: the *dense* formulation and the *sparse* formulation.