



University of Insubria

DIPARTIMENTO DI SCIENZA E ALTA TECNOLOGIA
PhD in Computer Science and Mathematics of Calculus

EXAM OF THE RISM COURSE III

Generalized Solutions in Differential Equations: Theory and Applications

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Exercise 1. Prove that any Non-Archimedean field cannot satisfy the Dedekind's axiom.

Solution. Let be \mathbb{K} a Non-Archimedean field. So $\mathbb{K} \equiv (\mathbb{K}, +, \cdot, \leq)$ is an infinite totally ordered field such that there exists at least an infinitesimal number $\xi \in \mathbb{K} \setminus \{0\}$. We can suppose $\xi > 0$ and we know that, for every $N \in \mathbb{N} \setminus \{0\}$, it holds $\xi < \frac{1}{N}$ (where: $0 \in \mathbb{K}$ denotes the neutral element with respect to $+$, $1 \in \mathbb{K}$ denotes the neutral element with respect to \cdot , $N = 1 + \dots + 1$ (N times) and $\frac{1}{N} = N^{-1}$ w.r.t. \cdot).

Now let's consider the two following countable subsets of \mathbb{K} :

$$A_\xi \doteq \left\{ -\frac{\xi}{n} \mid n \in \mathbb{N} \setminus \{0\} \right\} \quad \text{and} \quad B_\xi \doteq \left\{ -m\xi^2 \mid m \in \mathbb{N} \right\}.$$

Then it's easy to verify that B_ξ is a set of majorant elements for A_ξ in the sense that $(-\xi < 0$ and), for every $n, m \in \mathbb{N} \setminus \{0\}$, $-\frac{\xi}{n} < -m\xi^2$ (because equivalently $\xi < \frac{1}{nm}$).

Finally, let be $x \in \mathbb{K}$, $x > 0$, another generic element among those for which $-x$ is a majorant element for A_ξ : this means that, for every $n \in \mathbb{N} \setminus \{0\}$, $-\frac{\xi}{n} < -x$ (in particular, x is an infinitesimal number). Then, for every $m \in \mathbb{N} \setminus \{0\}$, $-x - m\xi^2$ is a majorant element for A_ξ as well: indeed, for every $n \in \mathbb{N} \setminus \{0\}$,

$$-\frac{\xi}{n} = \left(-\frac{\xi}{2n} \right) + \left(-\frac{\xi}{2n} \right) < -x - m\xi^2.$$

This shows that $-x$ can't be the minimum of all the majorant elements for A_ξ (because $-x - m\xi^2 < -x$) and that, consequently, \mathbb{K} cannot satisfy the Dedekind's axiom. \square

Exercise 2. Find an ultrafunction $u(x)$ such that

$$\oint_{\Gamma} |Du(x)|^2 dx = 1$$

and such that

$$\forall x \in \Gamma, u(x) \sim 0.$$

Solution. Let be Λ an infinite set with $\Lambda \supsetneq \mathbb{R}$ and let denote $\mathcal{L} \equiv \mathcal{L}_\Lambda := \mathcal{P}_{\text{fn}}(\Lambda)$ in such a way that, for every $\lambda \in \mathcal{L}$, λ is a finite subset of Λ . Note that \mathcal{L} is a directed upward set with respect to \subseteq (in fact, we could write n instead of λ thinking about the correspondence $n = |\lambda| \equiv \#\lambda \in \mathbb{N}$).

Thanks to the rings basic theory on maximal ideals, and in particular to the Krull-Zorn lemma, we know that there exists a field $\mathbb{E} \equiv \mathbb{E}_\Lambda \supsetneq \mathbb{R}$ of Euclidean numbers: that is, a totally ordered field $\mathbb{E} \equiv (\mathbb{E}, +, \cdot, \leq)$ such that can be built a surjective homomorphism $J: \mathbb{R}^{\mathcal{L}} \rightarrow \mathbb{E}$ (between ordered algebras).

So, given a net $\varphi: \mathcal{L} \rightarrow \mathbb{R}$, the Λ -limit of φ is by definition

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \doteq J(\varphi) \in \mathbb{E}.$$

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Therefore, in fact, the Λ -limit of nets (always exists and) satisfies all the “basic properties” of an “usual limit”.

Remember also that, for every Euclidean number $\xi \in \mathbb{E}$ which is finite (not infinite), there exists one and only one real number $\text{st}(\xi) \in \mathbb{R}$ such that $\xi \sim \text{st}(\xi)$ in the sense of infinitely closeness: this means that $\xi - \text{st}(\xi)$ is an infinitesimal number of \mathbb{E} (in particular, \mathbb{E} is in any case a Non-Archimedean field).

As a remarkable situation of this, if for every net $\varphi: \mathcal{L} \rightarrow \mathbb{R}$ we denote $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$ the usual Cauchy-limit of φ when it exists (so, when it belongs to \mathbb{R} , $\forall \varepsilon > 0, \exists \lambda_\varepsilon \in \mathcal{L}: \forall \lambda' \in \mathcal{L}, \lambda' \supseteq \lambda_\varepsilon \Rightarrow |\varphi(\lambda') - \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)| \leq \varepsilon$), and if $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$ is a finite Euclidean number, then $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \in \mathbb{R}$ and

$$\text{st} \left(\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \right) = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda).$$

Let's consider now the interval $\Omega :=]0, 1[\subset \mathbb{R}$ and its natural extension Ω^* in \mathbb{E} , that means the set

$$\Omega^* \doteq J(\Omega^\mathcal{L}) = \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \varphi \in \Omega^\mathcal{L} \right\} \subset \mathbb{E},$$

and let be Γ a hyperfinite grid on Ω : that is, $\Omega \subset \Gamma \subset \Omega^*$ and there exists a family $\{\Gamma_\lambda\}_{\lambda \in \mathcal{L}}$ of finite subsets of Ω , so $\Gamma_\lambda \subset \Omega$ with $|\Gamma_\lambda| < \infty$ for every $\lambda \in \mathcal{L}$, such for which

$$\Gamma = \lim_{\lambda \uparrow \Lambda} \Gamma_\lambda \doteq \left\{ \lim_{\lambda \uparrow \Lambda} x_\lambda \mid \forall \lambda \in \mathcal{L}, x_\lambda \in \Gamma_\lambda \right\}.$$

For instance, we could imagine $\Gamma \equiv \Gamma_\Omega \doteq \lim_{\lambda \uparrow \Lambda} (\Omega \cap \lambda)$.

So let call grid function on Γ any function $u: \Omega^* \rightarrow \mathbb{E}$ such that there exists a family $\{u_\lambda\}_{\lambda \in \mathcal{L}}$ of functions $u_\lambda: \Omega \rightarrow \mathbb{R}$, $\lambda \in \mathcal{L}$, such for which

$$u|_\Gamma = \lim_{\lambda \uparrow \Lambda} u_\lambda$$

in the sense that, for every $x = \lim_{\lambda \uparrow \Lambda} x_\lambda \in \Gamma$ (where $x_\lambda \in \Gamma_\lambda$ for every $\lambda \in \mathcal{L}$), it holds

$$u(x) = \lim_{\lambda \uparrow \Lambda} u_\lambda(x_\lambda).$$

Note that, for every function $f: \Omega \rightarrow \mathbb{R}$, its natural extension $f^*: \Omega^* \rightarrow \mathbb{E}$ on Ω^* given by

$$f^* \left(\lim_{\lambda \uparrow \Lambda} x_\lambda \right) \doteq \lim_{\lambda \uparrow \Lambda} f(x_\lambda), \quad x_\lambda \in \Omega,$$

is trivially a grid function on Γ . About that, let's observe that in fact it would be sufficient to define a grid function u only on Γ , starting from $\{u_\lambda\}_{\lambda \in \mathcal{L}}$ and defining then u on Γ by $u \doteq \lim_{\lambda \uparrow \Lambda} u_\lambda$, to finally take $(u|_\Omega)^*$ as a function defined on the whole set Ω^* .

Now let's consider the algebra $V^\circ(\Omega)$ of ultrafunctions on Γ modelled on the pair $(V(\Omega), \{V_\lambda(\Omega)\}_{\lambda \in \mathcal{L}})$ where $V(\Omega) := C_c^1(\Omega; \mathbb{R})$ and $\{V_\lambda(\Omega)\}_{\lambda \in \mathcal{L}}$ is a directed upward family of finite-dimensional subspaces $V_\lambda(\Omega)$ of $V(\Omega)$ which contains $\text{Span}_{\mathbb{R}}(V(\Omega) \cap \lambda)$, $\lambda \in \mathcal{L}$, and with $\bigcup_{\lambda \in \mathcal{L}} V_\lambda(\Omega) = V(\Omega)$: so $u \in V^\circ(\Omega)$ if and only if u is a grid function on Γ such that, more precisely, there exists a family $\{u_\lambda\}_{\lambda \in \mathcal{L}}$ of functions

$$u_\lambda \in V_\lambda(\Omega), \quad \lambda \in \mathcal{L},$$

such for which $u|_\Gamma = \lim_{\lambda \uparrow \Lambda} u_\lambda$.

Finally let's choose an ultrafunction $d \in V^\circ(\Omega)$ with $d|_\Gamma = \lim_{\lambda \uparrow \Lambda} d_\lambda$ ($d_\lambda \in V_\lambda(\Omega)$). Then, for every $u \in V^\circ(\Omega)$ with $u|_\Gamma = \lim_{\lambda \uparrow \Lambda} u_\lambda$ ($u_\lambda \in V_\lambda(\Omega)$), the generalized integral on Γ of any u_λ is given by

$$\oint_\Gamma u_\lambda(x) dx \doteq \lim_{\lambda' \uparrow \Lambda} \sum_{a_{\lambda'} \in \Gamma_{\lambda'}} u_\lambda(a_{\lambda'}) d_\lambda(a_{\lambda'}) \in \mathbb{E},$$

and thus, when $\oint_\Gamma u_\lambda(x) dx \in \mathbb{R}$ for every $\lambda \in \mathcal{L}$, the generalized integral on Γ of u is given by

$$\oint_\Gamma u(x) dx \doteq \lim_{\lambda \uparrow \Lambda} \oint_\Gamma u_\lambda(x) dx \equiv \sum_{a \in \Gamma} u(a) d(a) \in \mathbb{E}$$

where the last equality follows by definition of hyperfinite sum on Γ . In particular, for every $a \in \Gamma$, it holds $d(a) = \oint_{\Gamma} \mathbb{1}_a(x) dx$ where $\mathbb{1}_a \equiv \mathbb{1}_{\{a\}}: \Omega^* \rightarrow \{0, 1\}$ is the usual indicator function of $\{a\} \subset \Omega^*$.

Actually, we shall consider $\Gamma = \Gamma_{\Omega}$ and $\{V_{\lambda}(\Omega)\}_{\lambda \in \mathcal{L}}$, $d \in V^{\circ}(\Omega)$ such for which this generalized integral extends the usual Cauchy-Riemann integral; consequently, $\oint_{\Gamma} u_{\lambda}(x) dx = \int_{\Omega} u_{\lambda}(x) dx \in \mathbb{R}$ for every $\lambda \in \mathcal{L}$.

Moreover let's recall the generalized derivative Du as the grid function on Γ determined by

$$Du|_{\Gamma} \doteq \lim_{\lambda \uparrow \Lambda} Du_{\lambda}.$$

Observe that, if we let $|\cdot| := (|\cdot|)^*: \mathbb{E} \rightarrow \mathbb{E}$ denote the natural extension on $\mathbb{R}^* \equiv \mathbb{E}$ of the usual norm $|\cdot|: \mathbb{R} \rightarrow [0, +\infty[$, and similarly about the second power function on \mathbb{R} , then by definitions

$$|Du|^2|_{\Gamma} = \lim_{\lambda \uparrow \Lambda} |Du_{\lambda}|^2$$

and in particular $|Du|^2$ is a grid function on Γ as well.

Finally, keeping in mind the well known Lavrentiev phenomenon (developed with hyperfinite analysis), let's take two sequences $\{V_{\lambda}(\Omega)\}_{\lambda \in \mathcal{L}}$, $\{u_{\lambda}\}_{\lambda \in \mathcal{L}}$ which satisfy the three following properties:

- for every $\lambda \in \mathcal{L}$, u_{λ} and $|Du_{\lambda}|^2$ belong to $V_{\lambda}(\Omega)$ and u_{λ} has compact support that is independent from λ ;
- $\lim_{\lambda \rightarrow \Lambda} u_{\lambda} = 0$ uniformly;
- $\lim_{\lambda \rightarrow \Lambda} \int_{\Omega} |Du_{\lambda}|^2(x) dx = 1$.

Then the ultrafunction $u \in V^{\circ}(\Omega)$ defined by $u|_{\Gamma} \stackrel{\text{def}}{=} \lim_{\lambda \uparrow \Lambda} u_{\lambda}$ demonstrates the desired statement. In fact, on the one hand, $|Du|^2 \in V^{\circ}(\Omega)$ too and, by definitions and assumptions,

$$\oint_{\Gamma} |Du|^2(x) dx = \lim_{\lambda \uparrow \Lambda} \oint_{\Gamma} |Du_{\lambda}|^2(x) dx = 1;$$

on the other hand, for every $x = \lim_{\lambda \uparrow \Lambda} x_{\lambda} \in \Gamma$ (where $x_{\lambda} \in \Gamma_{\lambda}$ for every $\lambda \in \mathcal{L}$), $u(x) \equiv \lim_{\lambda \uparrow \Lambda} u_{\lambda}(x_{\lambda})$ is a finite Euclidean number (because $\{u_{\lambda}\}_{\lambda \in \mathcal{L}}$ is uniformly bounded on Ω) and it has

$$\text{st}(u(x)) \equiv \text{st}\left(\lim_{\lambda \uparrow \Lambda} u_{\lambda}(x_{\lambda})\right) = \lim_{\lambda \rightarrow \Lambda} u_{\lambda}(x_{\lambda}) = 0$$

(about the last equality, use that x is finite with $\text{st}(x) = \lim_{\lambda \rightarrow \Lambda} x_{\lambda} \in \Omega \cup \{0, 1\}$ and so, even if $\text{st}(x) \notin \{0, 1\}$, $|u_{\lambda}(x_{\lambda})| \leq |u_{\lambda}(x_{\lambda}) - u_{\lambda}(\text{st}(x))| + |u_{\lambda}(\text{st}(x))| = \mathcal{O}(|x_{\lambda} - \text{st}(x)|) + |u_{\lambda}(\text{st}(x))| \rightarrow 0$ for $\lambda \rightarrow \Lambda$). \square

Exercise 3. Study by means of ultrafunctions a PDE which doesn't have solutions in any distributions space.

Solution. Let be $N \in \mathbb{N} \setminus \{0\}$, $\Omega \subset \mathbb{R}_{\mathbf{x}}^N$ a bounded open set, $u_0 \in V_0^{\circ}(\Omega)^L$, $T \in]0, +\infty[$, $I := [0, T]_t$ and $a: \mathbb{R}_u \rightarrow \mathbb{R}$ the quadratic polynomial function defined by, for every $u \in \mathbb{R}$,

$$a(u) = u^2 - u$$

and thus let's consider the associated evolution problem

$$\begin{cases} u \in C^1(I^*, V_0^{\circ}(\Omega)^L) \\ \partial_t^* u = D_{\mathbf{x}} \cdot [a^*(u) D_{\mathbf{x}} u] \text{ on } I^* \times (\Gamma \cap \Omega^*) \\ u(0, \cdot) \equiv u_0(\cdot) \text{ on } \Gamma \cap \Omega^*. \end{cases}$$

It could be shown that there exists an unique global in time ultrafunction solution u such for which, on I^* ,

$$\partial_t^* \oint_{\Gamma} u^2(\mathbf{x}) d\mathbf{x} = \mathcal{O}(1).$$

Proposition. Let's assume that $0 < u < 1$ on $I^* \times (\Gamma \cap \Omega^*)$ and that, on I^* ,

$$\partial_t^* \oint_{\Gamma} u^2(\mathbf{x}) d\mathbf{x} \equiv 0 \quad \text{and} \quad \oint_{\Gamma} |\mathbf{D}_{\mathbf{x}} u|^2(\mathbf{x}) d\mathbf{x} = \mathcal{O}(1).$$

Then $\mathbf{D}_{\mathbf{x}} u \equiv 0$ on $I^* \times (\Gamma \cap \Omega^*)$.

In fact, let's consider the quartic polynomial function $P: \mathbb{R}_u \rightarrow \mathbb{R}$ given by, for every $u \in \mathbb{R}$,

$$P(u) = \frac{u^3}{6} - \frac{u^4}{12} - \frac{u^2}{2}$$

in such a way that $P''(\cdot) \equiv -[a(\cdot) + 1]$ and therefore, on I^* ,

$$\partial_t^* \oint_{\Gamma} P(u) d\mathbf{x} = \oint_{\Gamma} d_u^* P(u) \partial_t^* u d\mathbf{x} = - \oint_{\Gamma} (d_u^*)^2 P(u) a^*(u) |\mathbf{D}_{\mathbf{x}} u|^2 d\mathbf{x} \equiv \oint_{\Gamma} [a^*(u) + 1] a^*(u) |\mathbf{D}_{\mathbf{x}} u|^2 d\mathbf{x}$$

while also

$$\partial_t^* \oint_{\Gamma} P(u) d\mathbf{x} \equiv \partial_t^* \oint_{\Gamma} \left(\frac{u^3}{6} - \frac{u^4}{12} \right) d\mathbf{x} = \oint_{\Gamma} (a^*)^2(u) |\mathbf{D}_{\mathbf{x}} u|^2 d\mathbf{x}$$

and ultimately

$$\oint_{\Gamma} a^*(u) |\mathbf{D}_{\mathbf{x}} u|^2 d\mathbf{x} \equiv 0$$

which is possible if and only if $|\mathbf{D}_{\mathbf{x}} u|^2 \equiv 0$ on $I^* \times (\Gamma \cap \Omega^*)$, and this concludes. \square

References

- [1] V. Benci. *An improved setting for generalized functions: robust ultrafunctions*. Preliminary draft, 2019.