Stat 516

Homework 5

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1

(a) For the model $Y|\theta \sim \text{Poisson}(E \times \theta)$, where E is the "expected" number of cases, Y is the count of disease cases, and $\theta > 0$ is the relative risk, we have the likelihood function

$$L(\theta) = p_\theta(Y = y) = \frac{(E\theta)^y}{y!} e^{-E\theta}$$

which yields the log likelihood

$$\ell(\theta) = \log L(\theta) = y \log(E\theta) - \log(y!) - E\theta$$

To find Fisher's (expected) information $I(\theta)$, we use $I(\theta) = -\mathbb{E}\left[\ddot{\ell}(\theta)\right]$,

$$\begin{aligned}
\ell(\theta) &= y \log E + y \log \theta - \log(y!) - E\theta \\
\dot{\ell}(\theta) &= \frac{y}{\theta} - E \\
\ddot{\ell}(\theta) &= -\frac{y}{\theta^2} \\
I(\theta) &= -\mathbb{E}\left[\ell''(\theta)\right] = -\mathbb{E}\left[-\frac{Y}{\theta^2}\right] = \frac{1}{\theta^2}\mathbb{E}[Y] = \frac{E}{\theta}
\end{aligned}$$

Maximizing $\ell(\theta)$ to find the MLE, we have

$$\hat{\theta} = \frac{y}{E}$$

The variance of the MLE $\hat{\theta}$ is then given by

$$\text{Var}(\hat{\theta}) = I(\theta)^{-1} = \frac{\theta}{E}$$

(b) If we assume a prior of $\theta \sim \text{Gamma}(a, b)$, we have

$$\begin{aligned}
p(\theta|y) &\propto p(y|\theta)p(\theta) \\
&\propto \theta^y e^{-E\theta} \theta^{a-1} e^{-b\theta} \\
&= \theta^{y+a-1} e^{-(E+b)\theta}
\end{aligned}$$

so we see $\theta|y \sim \text{Gamma}(y + a, E + b)$.

(c) When we see $y = 4$ cases of leukemia with an expected number $E = 0.25$, the MLE is

$$\hat{\theta} = \frac{y}{E} = 16$$

with variance

$$\mathrm{Var}(\hat{\theta}) = \frac{\theta}{E} = 64$$

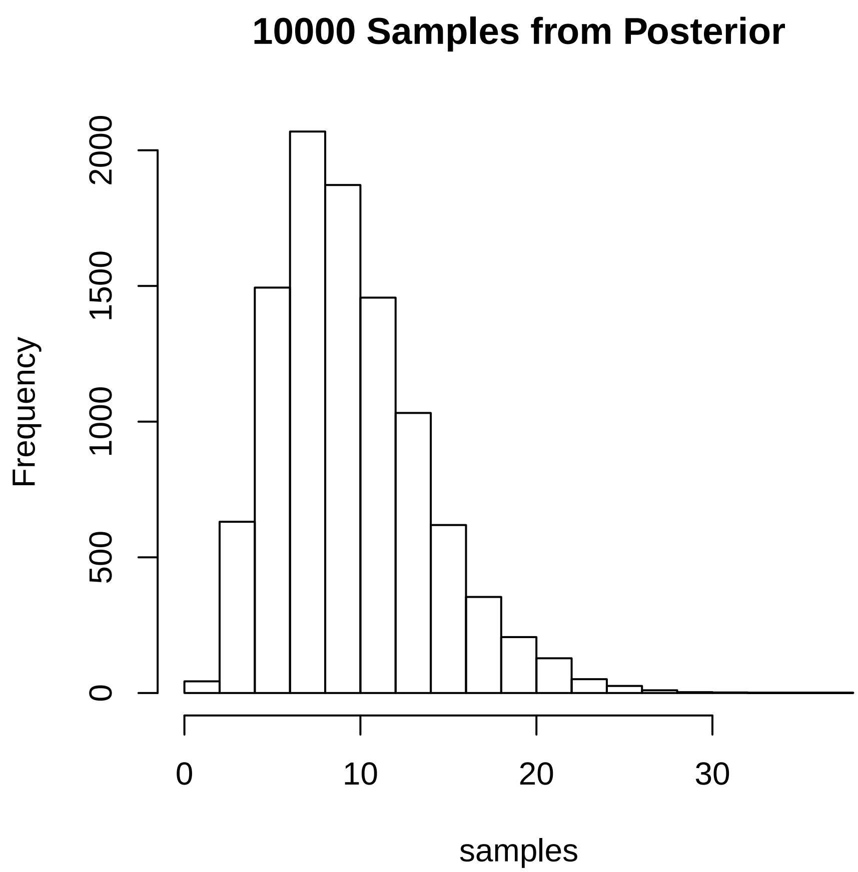
Since the MLE is asymptotically normal, we can approximate the 95% confidence interval with a normal distribution,

$$\hat{\theta} \pm 1.96 \times \sqrt{\mathrm{Var}(\hat{\theta})} = 16 \pm 1.96 \times 8 \approx (0.32, 31.68)$$

(d) To find the a and b which give a gamma prior with 90% interval $[0.1, 10]$, we use the R function `optim`,

```
priorch <- function(x, q1, q2, p1, p2) {
  (p1 - pgamma(q1, x[1], x[2]))^2 + (p2 - pgamma(q2, x[1], x[2]))^2
}
opt <- optim(par=c(1,1), fn=priorch, q1=0.1, q2=10, p1=0.05, p2=0.95)
a <- opt$par[1]
b <- opt$par[2]
```

yielding $a = 0.8405$ and $b = 0.2679$.
Using this prior and the data from part (c), we arrive at a posterior of $\theta|y \sim \mathrm{Gamma}(4.8405, 0.5179)$. Sampling from this distribution 1000 times, we get the following histogram,



A 95% credible interval using the 0.025- and 0.975-quantiles is $(2.9642, 19.3296)$.

(e) Using the 95% confidence interval from the MLE and its variance, we would say there is *not* evidence of excess risk for these data, because the value $\theta = 1$ corresponding to "null" risk is contained in the asymptotic 95% confidence interval. From the Bayesian analysis, we would say there *is* evidence of excess risk for these data, because the 95% credible interval does not contain $\theta = 1$ ("null" risk).

2

(a) MLEs of p_{12} p_{21} are given by

$$\hat{p}_{12} = \frac{n_{12}}{n_{1+}} = \frac{223}{588} \approx 0.3793$$
$$\hat{p}_{21} = \frac{n_{21}}{n_{2+}} = \frac{250}{456} \approx 0.5482$$

The asymptotic variances of $\hat{p}_{12}$ and $\hat{p}_{21}$ are then

$$\widehat{\mathrm{Var}(\hat{p}_{12})} \approx \frac{\hat{p}_{12}(1-\hat{p}_{12})}{n_{1+}} \approx 4.0037 \times 10^{-4}$$
$$\widehat{\mathrm{Var}(\hat{p}_{21})} \approx \frac{\hat{p}_{21}(1-\hat{p}_{21})}{n_{2+}} \approx 5.4314 \times 10^{-4}$$

Using these estimates, we can form 95% asymptotic confidence intervals using the asymptotic normality of the MLEs,

$$\mathrm{CI}_{12} = \hat{p}_{12} \pm 1.96 \times \sqrt{\widehat{\mathrm{Var}(\hat{p}_{12})}} \approx (0.34, 0.4185)$$
$$\mathrm{CI}_{21} = \hat{p}_{21} \pm 1.96 \times \sqrt{\widehat{\mathrm{Var}(\hat{p}_{21})}} \approx (0.5026, 0.5939)$$

(b) With independent uniform priors, we have $p_{12} \sim \mathrm{Beta}(1,1)$ and $p_{21} \sim \mathrm{Beta}(1,1)$. The posterior distributions are then

$$p_{12}|n_{1\cdot} \sim \mathrm{Beta}(n_{12}+1, n_{11}+1)$$
$$p_{21}|n_{2\cdot} \sim \mathrm{Beta}(n_{21}+1, n_{22}+1)$$

The posterior medians are then

$$\tilde{p}_{12} \approx \frac{n_{12}+1-1/3}{n_{1+}-2/3} \approx 0.3795$$
$$\tilde{p}_{21} \approx \frac{n_{21}+1-1/3}{n_{2+}-2/3} \approx 0.5481$$

A 95% credible interval for each parameter can then be obtained from the 0.025- and 0.975-quantiles of each parameter's respective posterior,

$$\mathrm{CI}_{12} \approx (0.3409, 0.4192)$$
$$\mathrm{CI}_{21} \approx (0.5023, 0.5933)$$

(c) Using a likelihood ratio test, we want to test the null hypothesis H_0 that the weather on each day is independent versus H_1 that the weather on one day depends on the weather the previous day. Under H_0, $\hat{p}_j = n_j/n$, whereas under H_1, $\hat{p}_{ij} = n_{ij}/n_{i+}$. Using the likelihood ratio test statistic, we have

$$\begin{aligned} T &= 2(\hat{\ell}_1 - \hat{\ell}_0) \\ &= 2\sum_{i=1}^{2}\sum_{j=1}^{2} n_{ij} \log\left(\frac{\hat{p}_{ij}}{\hat{p}_j}\right) \\ &= 2\sum_{i=1}^{2}\sum_{j=1}^{2} n_{ij} \log\left(\frac{n_{ij}/n_{i+}}{n_j/n}\right) \\ &\approx 25.0195 \end{aligned}$$

Under the null, $T \sim \chi^2_1$, and we have $P(T > 25.0195) \approx 5.6753 \times 10^{-7}$, so the likelihood ratio test would reject the null hypothesis that the weather on each day is independent.
Under the Bayesian paradigm, we compare the independence assumption with prior $p_1 \sim \text{Beta}(1,1)$ versus the Markov assumption with priors $p_{12} \sim \text{Beta}(1,1)$ and $p_{21} \sim \text{Beta}(1,1)$. Under the independence assumption, the posterior for p_1 is

$$p_1|n \sim \text{Beta}(n_1+1, n_2+1)$$

while under the Markov assumption we have, as before,

$$\begin{aligned} p_{12}|n_{1\cdot} &\sim \text{Beta}(n_{12}+1, n_{11}+1) \\ p_{21}|n_{2\cdot} &\sim \text{Beta}(n_{21}+1, n_{22}+1) \end{aligned}$$

To find the Bayes factor, we must compute $\Pr(y|H_0)$ and $\Pr(y|H_1)$. For H_0, we have

$$\begin{aligned} \Pr(y|H_0) &= \binom{n}{n_1} \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \frac{\Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma(n+2)} \\ &= \binom{n}{n_1} \frac{\Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma(n+2)} \\ &\approx 9.5694 \times 10^{-4} \end{aligned}$$

while under H_1, we have

$$\begin{aligned} \Pr(y|H_1) &= \int_0^1 \int_0^1 p_{12}^{n_{12}}(1-p_{12})^{n_{11}} p_{21}^{n_{21}}(1-p_{21})^{n_{22}} \left(\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\right)^2 dp_{12}dp_{21} \binom{n_{1+}}{n_{12}} \binom{n_{2+}}{n_{21}} \\ &= \binom{n_{1+}}{n_{12}} \binom{n_{2+}}{n_{21}} \int_0^1 \int_0^1 p_{12}^{n_{12}}(1-p_{12})^{n_{11}} p_{21}^{n_{21}}(1-p_{21})^{n_{22}} dp_{12}dp_{21} \\ &= \binom{n_{1+}}{n_{12}} \binom{n_{2+}}{n_{21}} \int_0^1 p_{12}^{n_{12}}(1-p_{12})^{n_{11}} dp_{12} \times \int_0^1 p_{21}^{n_{21}}(1-p_{21})^{n_{22}} dp_{21} \\ &= \binom{n_{1+}}{n_{12}} \binom{n_{2+}}{n_{21}} \frac{\Gamma(n_{12}+1)\Gamma(n_{11}+1)}{\Gamma(n_{1+}+2)} \cdot \frac{\Gamma(n_{21}+1)\Gamma(n_{22}+1)}{\Gamma(n_{2+}+2)} \\ &\approx 3.7151 \times 10^{-6} \end{aligned}$$

The Bayes factor is then

$$\text{BF} = \frac{\Pr(y|H_0)}{\Pr(y|H_1)} \approx 257.5818$$

Using the Kass and Raftery suggestions for intervals of Bayes Factors, we note that $1/\text{BF} \approx 0.0039 < 1$, so we would say there is *not* sufficient evidence against the null hypothesis of independence.