

# Zeroth-Order Frank-Wolfe Optimization for Black-Box Adversarial Attacks

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## Abstract

*The goal of this project is to compare the behaviour and performance of two Zeroth-Order variants of the Frank-Wolfe Algorithm, aimed at solving constrained optimization problems with a better iteration complexity, especially with respect to oracle queries. We take into consideration: Faster Zeroth-Order Conditional Gradient Sliding (FZCGS) (Gao et al., 2018) and Stochastic Gradient Free Frank Wolfe (SGFFW) (Sahu et al., 2019). The latter algorithm branches off into three slightly different ones, depending on the Stochastic Approximation Technique used, namely: classical Kiefer-Wolfowitz Stochastic Approximation (KWSA) (Kiefer and Wolfowitz, 1952), Random Directions Stochastic Approximation (RDSA) (Nesterov and Spokoiny, 2011; Duchi et al., 2015), and an Improvised RDSA (IRDSA). The theory behind these algorithms is presented, with an emphasis on proving that the performance are guaranteed. Then, the aforementioned algorithms are tested on a black-box adversarial attack on the MNIST dataset.*

## 1. Introduction

The Frank-Wolfe algorithm, also known as the conditional gradient method, is an iterative optimization technique used for constrained convex optimization problems. It was proposed by Marguerite Frank and Philip Wolfe in 1956, and nowadays finds various applications in the field of machine learning. It approximates the objective function by a first-order Taylor approximation. The algorithm iteratively selects a direction that minimizes the linear approximation of the objective function within the feasible set  $C$ . This direction is then combined with the current solution in a convex combination, and the process is repeated until convergence.

In particular, the Frank-Wolfe algorithm excels in constrained optimization problems with a closed convex set  $C$ :

$$\min_{x \in C} f(x)$$

The problem formulation can vary widely; for example, Gao et al. deal with a variant tailored for finite-sum minimization problems, in which the component functions  $f_i(x)$  are summed up as follows:

$$\min_{x \in \Omega} F(x) = \frac{1}{n} \sum_{i=1}^n f_i(x)$$

Sahu et al., on the other hand, in order to estimate the loss function  $f(x)$ , deal with a variant that uses a stochastic zeroth-order oracle. The loss function is therefore defined as:

$$\min_{x \in C} f(x) = \min_{x \in C} \mathbb{E}_y [F(x, y)]$$

This paper addresses the use of the Frank-Wolfe algorithm for a particularly critical constrained optimization problem in Deep Learning, which is the problem of Adversarial Attacks. The objective of an adversarial attack is to find a small enough perturbation of the input able to make the neural network output the wrong prediction, while adhering to constraints inside of the convex set  $C$ . In our case, we use MNIST, so the goal is to find a non-trivial perturbation of the 28x28 black and white image of a hand-written digit. Moreover, as we will see later, we employ a zeroth-order variant of the Frank-Wolfe algorithm, since we don't have access to the full exact gradient.

### 1.1. Deterministic Frank-Wolfe Algorithm

In case first-order information is available in an optimization task, the deterministic version of the Frank-Wolfe algorithm can be a good choice, especially when exact minimization is computationally expensive.

The exact minimization in the first formula in the introduction is approximated through an inexact minimization, where a vector  $v$  satisfies some conditions, while maintaining the same convergence rate.

When full, exact first-order information is available through an incremental first-order oracle (IFO), the Frank-Wolfe algorithm is basically described by the following two formulas:

$$v_t = \arg \min_{v \in \mathcal{C}} \langle h, \nabla f(x_t) \rangle$$

$$x_{t+1} = (1 - \gamma_{t+1}) x_t + \gamma_{t+1} v_t,$$

where

- $f(x_t)$  is the objective function we need to minimize;
- $\mathcal{C}$  is the convex set;
- $\langle \cdot, \cdot \rangle$  is the inner/dot product;
- $v_t$  is the direction we need to take in order to minimize the linear approximation;
- $x_t$  is the current iteration result;
- $h$  is a vector in the same space as  $x_t$ ;
- $\gamma_{t+1} = \frac{2}{t+2}$  is the step size.

## 1.2. Stochastic Frank-Wolfe Algorithm

In case first-order information is not available, and we can only work with zeroth-order information, the stochastic variant of the Frank-Wolfe algorithm can be a good choice.

By employing a Stochastic Zeroth-order Oracle (SZO), the deterministic objective function is substituted by a stochastic objective function  $f(x_t, y_t)$ , with  $y_t$  being a random variable. Therefore, the Stochastic Frank-Wolfe algorithm becomes:

$$v_t = \arg \min_{v \in \mathcal{C}} \langle h, \nabla f(x_t, y_t) \rangle$$

$$x_{t+1} = (1 - \gamma_{t+1}) x_t + \gamma_{t+1} v_t,$$

## 1.3. Zeroth-Order Gradient Estimation

When the gradient of a function is unavailable, it is possible to estimate it using function evaluations, by calculating the function values at selected points. More in detail, we can use the difference of the function value with respect to two random points to estimate the gradient. In our case, we employ the use of the coordinate-wise gradient estimator, as in the Gao et al. paper.

The coordinate-wise gradient estimator is defined as follows:

$$\hat{\nabla} f(\mathbf{x}) = \sum_{j=1}^d \frac{f(\mathbf{x} + \mu_j \mathbf{e}_j) - f(\mathbf{x} - \mu_j \mathbf{e}_j)}{2\mu_j} \mathbf{e}_j$$

where

- $\hat{\nabla} f(\mathbf{x})$  is the estimated gradient of the function  $f$  in  $\mathbf{x}$
- $d$  is the dimensionality of the optimization space

- $\mu_j > 0$  is a smoothing parameter
- $\mathbf{e}_j \in \mathbb{R}^d$  is the basis vector where only the  $j$ -th element is 1 and all others are 0.

## 2. Implemented Algorithms

This project involves the implementation of the following algorithms:

- **FZCGS**: Faster Zeroth-Order Conditional Gradient Sliding Method
- **SGFFW**: Stochastic Gradient-Free Frank-Wolfe with the following gradient approximation schemes:
  - **KWSA**: Kiefer-Wolfowitz stochastic approximation
  - **RDSA**: random directions stochastic approximation
  - **I-RDSA**: improvised random directions stochastic approximation

### 2.1. FZCGS: Faster Zeroth-Order Conditional Gradient Sliding Method

Gao et al. proposes a novel algorithm to optimize the following constrained finite-sum minimization problem:

$$\min_{x \in \Omega} F(x) = \frac{1}{n} \sum_{i=1}^n f_i(x)$$

where

- $\Omega \subset \mathbb{R}^d$  is the closed convex feasible set
- $f_i(x)$  are the various  $n$  component functions, which are smooth and non-convex

FZCGS incorporates an acceleration technique from the non-convex Frank-Wolfe method. (?) It uses gradient estimation and Conditional Gradient Sliding to perform the updates. The convergence rate depends on the choice of the parameters. The pseudocode of the algorithm is presented below.

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**Algorithm 2** Faster Zeroth-Order Conditional Gradient Method (FZCGS)

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**Input:**  $\mathbf{x}_0, q > 0, \mu > 0, K > 0, \eta > 0, \gamma > 0, n$

- 1: **for**  $k = 0, \dots, K - 1$  **do**
- 2:   **if**  $\text{mod}(k, q) = 0$  **then**
- 3:     Sample  $S_1$  without replacement to compute  $\hat{\mathbf{v}}_k = \hat{\nabla} f_{S_1}(\mathbf{x}_k)$
- 4:   **else**
- 5:     Sample  $S_2$  with replacement to compute  $\hat{\mathbf{v}}_k = \frac{1}{|S_2|} \sum_{i \in S_2} [\hat{\nabla} f_i(\mathbf{x}_k) - \hat{\nabla} f_i(\mathbf{x}_{k-1}) + \hat{\mathbf{v}}_{k-1}]$
- 6:   **end if**
- 7:    $\mathbf{x}_{k+1} = \text{condg}(\hat{\mathbf{v}}_k, \mathbf{x}_k, \gamma_k, \eta_k)$
- 8: **end for**

**Output:** Randomly choose  $\mathbf{x}_\alpha$  from  $\{\mathbf{x}_k\}$  and return it

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**Algorithm 3**  $\mathbf{u}^+ = \text{condg}(\mathbf{g}, \mathbf{u}, \gamma, \eta)$  (Qu et al., 2017)

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- 1:  $\mathbf{u}_1 = \mathbf{u}, t = 1$
- 2:  $\mathbf{v}_t$  be an optimal solution for
$$V_{\mathbf{g}, \mathbf{u}, \gamma}(\mathbf{u}_t) = \max_{\mathbf{x} \in \Omega} \langle \mathbf{g} + \frac{1}{\gamma}(\mathbf{u}_t - \mathbf{u}), \mathbf{u}_t - \mathbf{x} \rangle$$
- 3: If  $V_{\mathbf{g}, \mathbf{u}, \gamma}(\mathbf{u}_t) \leq \eta$ , return  $\mathbf{u}^+ = \mathbf{u}_t$ .
- 4: Set  $\mathbf{u}_{t+1} = (1 - \alpha_t)\mathbf{u}_t + \alpha_t\mathbf{v}_t$  where  $\alpha_t = \min\{1, \frac{\langle \frac{1}{\gamma}(\mathbf{u} - \mathbf{u}_t) - \mathbf{g}, \mathbf{v}_t - \mathbf{u}_t \rangle}{\frac{1}{\gamma} \|\mathbf{v}_t - \mathbf{u}_t\|^2}\}$ .
- 5: Set  $t \leftarrow t + 1$  and goto step 2.

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The results of FZCGS from the paper are:

- When choosing the right set of parameters for FZCGS, the expected squared norm of the gradient estimation error converges as per the given rate
- The amortized function queries oracle complexity is  $O\left(\frac{n^{1/2}d}{\epsilon}\right)$ .
- The linear oracle complexity is  $O\left(\frac{1}{\epsilon^2}\right)$ .

In the paper, the experimental results demonstrate the superiority of FZCGS over baseline methods in terms of convergence speed and performance, also thanks to the acceleration technique that this algorithm employs.

## 2.2. Stochastic Gradient-Free Frank-Wolfe Algorithm

Sahu et al. propose a stochastic zeroth-order Frank-Wolfe algorithm for the following stochastic optimization problem, particularly significant in deep learning:

$$\min_{x \in C} f(x) = \min_{x \in C} \mathbb{E}_y[F(x; y)]$$

where  $C \in \mathbb{R}^d$  is a closed convex set.

To solve this kind of problem, one can employ projection and projection-free methods. SGFFW uses a zeroth-order oracle while focusing on a projection-free stochastic variant of Frank-Wolfe. It uses gradient approximation schemes and addresses challenges such as non-smoothness and variance. Using an averaging trick to ensure the stability of the algorithm, the updates are the following:

$$\begin{aligned} d_t &= (1 - \rho_t) d_{t-1} + \rho_t g(x_t, y_t) \\ v_t &= \arg \min_{s \in C} \langle s, d_t \rangle \\ x_{t+1} &= (1 - \gamma_{t+1}) x_t + \gamma_{t+1} v_t \end{aligned}$$

Employing a gradient approximation techniques or another influences the algorithm's dimension dependence and the computational cost, also in terms of query number and time.

- Kiefer-Wolfowitz Stochastic Approximation (KWSA):

$$\mathbf{g}(\mathbf{x}_t; \mathbf{y}) = \sum_{i=1}^d \frac{F(\mathbf{x}_t + c_t \mathbf{e}_i; \mathbf{y}) - F(\mathbf{x}_t; \mathbf{y})}{c_t} \mathbf{e}_i$$

- Random Direction Stochastic Approximation (RDSA):  
Sample  $\mathbf{z}_t \sim \mathcal{N}(0, \mathbf{I}_d)$ ,

$$\mathbf{g}(\mathbf{x}_t; \mathbf{y}, \mathbf{z}_t) = \frac{F(\mathbf{x}_t + c_t \mathbf{z}_t; \mathbf{y}) - F(\mathbf{x}_t; \mathbf{y})}{c_t} \mathbf{z}_t$$

- Improvised Random Direction Stochastic Approximation (I-RDSA):  
Sample  $\{\mathbf{z}_{i,t}\}_{i=1}^m \sim \mathcal{N}(0, \mathbf{I}_d)$

$$\mathbf{g}(\mathbf{x}_t; \mathbf{y}, \mathbf{z}_t) = \frac{1}{m} \sum_{i=1}^m \frac{F(\mathbf{x}_t + c_t \mathbf{z}_{i,t}; \mathbf{y}) - F(\mathbf{x}_t; \mathbf{y})}{c_t} \mathbf{z}_{i,t}$$

Below, the pseudocode of the algorithm is presented:

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**Algorithm 2** Stochastic Gradient Free Frank Wolfe

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**Require:** Input, Loss Function  $F(x)$ , Convex Set  $\mathcal{C}$ , number of directions  $m$ , sequences  $\gamma_t = \frac{2}{t+8}$ ,

$$(\rho_t, c_t)_{RDSA} = \left( \frac{4}{d^{1/3}(t+8)^{2/3}}, \frac{2}{d^{3/2}(t+8)^{1/3}} \right)$$

$$(\rho_t, c_t)_{I-RDSA} = \left( \frac{4}{(1+\frac{d}{m})^{1/3}(t+8)^{2/3}}, \frac{2\sqrt{m}}{d^{3/2}(t+8)^{1/3}} \right)$$

$$(\rho_t, c_t)_{KWSA} = \left( \frac{4}{(t+8)^{2/3}}, \frac{2}{d^{1/2}(t+8)^{1/3}} \right).$$

**Output:**  $\mathbf{x}_T$  or  $\frac{1}{T} \sum_{t=0}^{T-1} \mathbf{x}_t$ .

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1: Initialize  $\mathbf{x}_0 \in \mathcal{C}$ 
2: for  $t = 0, 2, \dots, T-1$  do
3:   Compute
      KWSA:
       $\mathbf{g}(\mathbf{x}_t; \mathbf{y}) = \sum_{i=1}^d \frac{F(\mathbf{x}_t + c_t \mathbf{e}_i; \mathbf{y}) - F(\mathbf{x}_t; \mathbf{y})}{c_t} \mathbf{e}_i$ 
      RDSA: Sample  $\mathbf{z}_t \sim \mathcal{N}(0, \mathbf{I}_d)$ ,
       $\mathbf{g}(\mathbf{x}_t; \mathbf{y}, \mathbf{z}_t) = \frac{F(\mathbf{x}_t + c_t \mathbf{z}_t; \mathbf{y}) - F(\mathbf{x}_t; \mathbf{y})}{c_t} \mathbf{z}_t$ 
      I-RDSA: Sample  $\{\mathbf{z}_{i,t}\}_{i=1}^m \sim \mathcal{N}(0, \mathbf{I}_d)$ ,
       $\mathbf{g}(\mathbf{x}_t; \mathbf{y}, \mathbf{z}_t) = \frac{1}{m} \sum_{i=1}^m \frac{F(\mathbf{x}_t + c_t \mathbf{z}_{i,t}; \mathbf{y}) - F(\mathbf{x}_t; \mathbf{y})}{c_t} \mathbf{z}_{i,t}$ 
4:   Compute  $\mathbf{d}_t = (1 - \rho_t) \mathbf{d}_{t-1} + \rho_t \mathbf{g}(\mathbf{x}_t, \mathbf{y}_t)$ 
5:   Compute  $\mathbf{v}_t = \operatorname{argmin}_{\mathbf{s} \in \mathcal{C}} \langle \mathbf{s}, \mathbf{d}_t \rangle$ ,
6:   Compute  $\mathbf{x}_{t+1} = (1 - \gamma_t) \mathbf{x}_t + \gamma_t \mathbf{v}_t$ .
7: end for
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### 2.2.1 Primal Gap

Sahu et al. state the main results involving the different gradient approximation schemes for the primal (suboptimality) gap, which provide a characterization of

$$E[f(\mathbf{x}_t) - f(\mathbf{x}^*)]$$

Given the sequence  $\gamma_t = \frac{2}{t+8}$ :

- RDSA gradient approximation scheme:

$$E[f(\mathbf{x}_t) - f(\mathbf{x}^*)] = O\left(d^{1/3}(t+9)^{1/3}\right)$$

- I-RDSA gradient approximation scheme:

$$E[f(\mathbf{x}_t) - f(\mathbf{x}^*)] = O\left(\frac{d}{m^{1/3}}(t+9)^{1/3}\right)$$

- KWSA gradient approximation scheme:

$$E[f(\mathbf{x}_t) - f(\mathbf{x}^*)] = O\left(\frac{1}{(t+9)^{1/3}}\right)$$

The dimension dependence of the primal gap is quantified to be  $d^{1/3}$ . At the same time, the dependence on iterations  $O(T^{-1/3})$ , matches that of the stochastic Frank-Wolfe which has access to first-order information. The improvement of the rates for I-RDSA and KWSA are at the cost of extra directional derivatives at each iteration. The number of queries to the SZO to obtain a primal gap of  $\epsilon$  is given by  $O(\frac{d}{\epsilon^3})$ , where the dimension dependence is consistent with zeroth-order schemes and cannot be improved on.

### 2.2.2 Dual Gap

The paper also quantifies the dual gap with the same gradient approximation schemes

- RDSA gradient approximation scheme:

$$E\left[\min_{t=0,\dots,T-1} G(\mathbf{x}_t)\right] \leq \frac{7(F(\mathbf{x}_0) - F(\mathbf{x}^*))}{2T} + \frac{LR^2 \ln(T+7)}{T} + \frac{Q_0 + R\sqrt{2Q}}{2T^{2/3}}$$

- I-RDSA gradient approximation scheme:

$$E\left[\min_{t=0,\dots,T-1} G(\mathbf{x}_t)\right] \leq \frac{7(F(\mathbf{x}_0) - F(\mathbf{x}^*))}{2T} + \frac{LR^2 \ln(T+7)}{T} + \frac{Q_{ir} + R\sqrt{2Q_{ir}}}{2T^{2/3}}$$

- KWSA gradient approximation scheme:

$$E\left[\min_{t=0,\dots,T-1} G(\mathbf{x}_t)\right] \leq \frac{7(F(\mathbf{x}_0) - F(\mathbf{x}^*))}{2T} + \frac{LR^2 \ln(T+7)}{T} + \frac{Q_{kw} + R\sqrt{2Q_{kw}}}{2T^{2/3}}$$

The dimension dependence of the Frank-Wolfe duality gap is quantified to be  $d^{1/3}$ . At the same time, the dependence on iterations,  $O(T^{-1/3})$ , matches that of the primal gap and hence follows that the number of queries to the SZO to obtain a Frank-Wolfe duality gap of  $\epsilon$  is given by  $O(\frac{d}{\epsilon^3})$ . In particular, it is also asserted that the initial conditions are forgotten as  $O(1/T)$ .

The algorithm's effectiveness is demonstrated in black-box optimization scenarios, specifically in optimizing complex forward models from physics without analytical expressions. The proposed algorithm converges to a first-order stationary point, showcasing its applicability in challenging black-box optimization problems.

The paper introduces a stochastic zeroth order FrankWolfe algorithm suitable for non-smooth optimization problems. The algorithm's reliance on hard-to-estimate parameters is minimal, making it practical for real-world applications. Convergence rates for the primal gap and Frank-Wolfe duality gap are quantified, highlighting its efficiency. Experimental results on various datasets further support its efficacy.

### 3. Frank-Wolfe for Adversarial Attacks on MNIST

#### 3.1. Adversarial Attacks

An Adversarial Attacks is a deliberate manipulation of input data with the intention of causing a machine learning model to make a mistake or produce incorrect outputs. The goal of an adversarial attack is to perturb the input in a way that is not easily noticeable to a human observer, but can mislead the model into making errors. In particular, when it comes to black-box attacks, the attacker has limited or no knowledge of the model's internal structure and parameters. These attacks can be performed on deep neural networks models; in our case, the model is MNIST, designed for the classification of handwritten digits. The MNIST dataset consists of 28x28 pixel grayscale images of handwritten digits (0 through 9). The objective of our experiment is to compare the optimization of an adversarial attack using zeroth-order Frank-Wolfe algorithms in a black-box setting. The task is to find a non-trivial adversarial perturbation such that the DNN model makes the incorrect prediction.

#### 3.2. Methods

We follow the setup (Liu et al., 2019) with the nn-carlini pretrained Deep Neural Network for the MNIST dataset; this repository is also used by Gao et al. for their experiments, so starting from it as a base for the implementation of the optimization algorithms seemed obvious.

#### 3.3. Implementation Differences from Reference Papers

For time and computational constraints, there are some implementative differences between the theoretical papers and our actual implementation.

- In FZCGS the gradient is calculated over a set of randomized directions, same as ZOSCGS, instead of over all the directions built by basis vectors; this is done because while less "precise" over the same amount of epochs, calculating the gradient over all basis vector directions requires too many queries to the oracle and time per epoch. As of the current implementation, a 200 iterations-long optimization run takes about 12 hours to complete.
- In the `condg` (conditional gradient) step for FZCGS, we added an additional stopping condition on the size of  $\alpha$ . Otherwise, it would have taken a lot just for one iteration. (say it better).

### 4. Results

#### 4.1. FZCGS

#### 4.2. SGFFW

### 5. References

List and number all bibliographical references in 9-point Times, single-spaced, at the end of your paper. When referenced in the text, enclose the citation number in square brackets, for example [?]. Where appropriate, include the name(s) of editors of referenced books.