Some musings on the category of bordisms

I'm writing this set of notes to try pin down to myself the details the underlies the definition of a category of bordisms that is suitable for stating and proving the folklore equivalence between 2-dimensional topological quantum field theories and Frobenius algebras.

1. Oriented bordisms

We start by recalling how the boundary components of an oriented manifold with boundary can be labelled as either "in" out "out".

Let X be an oriented manifold with boundary. Given a boundary point $x \in \partial X$, let's say that a vector $v \in \mathrm{T}_x X$ is

(i) *inward pointing* if there exists a curve $\gamma\colon [0, \varepsilon[\to X$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$;
(ii) *tangent* to ∂X if v belongs to the copy of $\mathrm{T}_x(\partial X)$ that sits inside of $\mathrm{T}_x X$;
(iii) *outward-pointing* if there exists a curve $\gamma\colon]-\varepsilon, 0] \to X$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$.

The following theorem then is true.

1. Theorem. *Let X be an oriented manifold with boundary. There always exists a vector field v along ∂X such that its value $v(x)$ is outward-pointing at every point $x \in \partial X$.*

This result allows us to introduce an orientation on ∂X as follows. First, take an orientation n-form ω for X. Then, consider the interior product $v \lrcorner \omega$, an $(n-1)$-form on X. Lastly, pull $v \lrcorner \omega$ back with the inclusion $\iota^{\partial X}\colon \partial X \hookrightarrow X$. The resulting $(n-1)$-form $\iota^{\partial X,*}(v \lrcorner \omega)$ on ∂X never vanishes, and induces thus an orientation.

To see that this orientation is in accordance with the "Outward Normal First" rule, let ω denote an orientation form for X, and call η the induced orientation form on ∂X. Recall then that a basis $\{v_1, \ldots, v_n\}$ of $\mathrm{T}_x X$ is called *positive* whenever

$$\langle \omega|_x, v_1 \wedge \cdots \wedge v_n \rangle > 0,$$

and compute the quantity $\langle \eta|_x, v_1 \wedge \cdots \wedge v_{n-1} \rangle$ for a basis of $\{v_1, \ldots, v_{n-1}\}$ of $\mathrm{T}_x(\partial X)$.

One can distinguish between the "in" and "out" connected components of the boundary of an oriented manifold as follows. Given $x \in \partial X$, call a vector $w \in \mathrm{T}_x X$ a *positive normal* for ∂X at x if[1] adding it on top of a positive basis of $\mathrm{T}_x(\partial X)$ results in a positive basis of $\mathrm{T}_x X$. The following theorem holds.

2. Theorem. *Let X be a manifold with boundary. Let Σ be a connected component of ∂X. The inward/outward direction of a positive normal vector at a point of Σ does not depend on the particular positive normal vector chosen, nor on the particular point.*

This theorem allows us to separate the boundary components whose positive normals are inward-pointing from those whose positive normals are outward-pointing. The former are called *in-bondaries*, the latter *out-boundaries* of X.

We're now ready to spell out the definition of oriented bordism.

[1] Sloppily, but I hope it's clear what I mean. Here "out top" mean "as the last one".

3. Definition (Oriented bordisms). Let Σ_0 and Σ_1 be two oriented closed $(n-1)$-dimensional manifolds. A *bordism* between Σ_0 and Σ_1 is a triple $(B, \vartheta_0, \vartheta_1)$ where

- B is a compact n-dimensional manifold with boundary;
- ϑ_0 and ϑ_1 are orientation-preserving embeddings
$$\vartheta_0 \colon \Sigma_0 \times [0, \varepsilon[\to M \qquad \vartheta_1 \colon \Sigma_1 \times]1-\varepsilon, 1] \to M$$
such that $\vartheta_0(x, 0)$ belongs to an in-boundary of B for every $x \in \Sigma_0$ and $\vartheta_1(x, 1)$ belongs to an out-boundary of B for every $x \in \Sigma_1$, and such that the restrictions
$$\Sigma_0 \times \{0\} \to \coprod \{\text{in-boundaries of } B\} \qquad \Sigma_1 \times \{1\} \to \coprod \{\text{out-boundaries of } B\}$$
are (orientation-preserving) diffeomorphisms.

We write $(B, \vartheta_0, \vartheta) \colon \Sigma_0 \to \Sigma_1$ or even $B \colon \Sigma_0 \to \Sigma_1$ to denote that $(B, \vartheta_0, \vartheta_1)$ is a bordism from Σ_0 to Σ_1.

Before going on let's recall the notion of collar neighbourhood.

4. Definition (Collars). Let X be a manifold with boundary. Let $\vartheta \colon \partial X \times [0, \varepsilon[\to X$ be an embedding such that

$$\begin{tikzcd}
\partial X \arrow[r, "x \mapsto (x,0)"] \arrow[dr, hook] & \partial X \times [0, \varepsilon[\arrow[d, "\vartheta"] \\
& X
\end{tikzcd}$$

commutes. Then we say that ϑ defines a *collar neighbourhood* of ∂X.

Keep in mind that, despite their name, to specify a collar neighbourhood the entire embedding $\partial X \times [0, \varepsilon[\to X$ is needed, not only its image inside of X.

The following theorem is worth mentioning. It will be useful later when we will "glue" bordism together.

5. Theorem.

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6. Exercise (Constructing bordisms). Let S^1 denote the (counterclockwise-)oriented 1-dimensional sphere, and let's flip its orientation in $\bar{S}^1$. Take the oriented manifold with boundary $B = S^1 \times [0, 1]$. Denote with $\emptyset^2$ the 2-dimensional null manifold. Write all the 2-dimensional bordisms

- $\emptyset^2 \to S^1$;
- $S^1 \to S^1$;
- $S^1 \to S^1$;
- $\bar{S}^1 \to S^1$;
- $S^1 \to \bar{S}^1$;
- $\bar{S}^1 \to \bar{S}^1$;
- $S^1 \sqcup S^1 \to \emptyset^2$;
- $S^1 \sqcup \bar{S}^1 \to \emptyset^2$;
- $\bar{S}^1 \sqcup S^1 \to \emptyset^2$;
- $\emptyset^2 \to S^1 \sqcup S^1$;
- $\emptyset^2 \to S^1 \sqcup \bar{S}^1$;
- $\emptyset^2 \to \bar{S}^1 \sqcup S^1$;

that have B as bordant manifold.