Flexible Modelling of Diel and Other Periodic Variation in Hidden Markov Models

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Contents

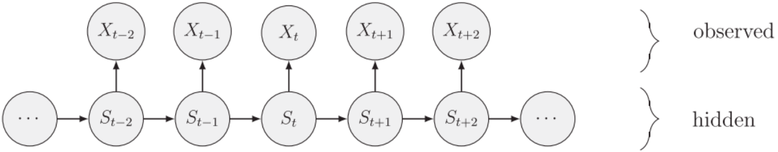
1. Hidden Markov Model

Motivation

- Understanding key patterns of animal movement
- Investigating drivers of animal movement and behavior

HMM

Hidden Markov Models (HMMs) are statistical models for time series data involving two stochastic processes:



1. the observation process $x_1, ..., x_T$, the time series
2. the latent state process $S_1, ..., S_T$, not observed

HMM

In the basic framework of HMM

- X_t is the state-dependent process
- each observation x_t is assumed to be generated by one of N possible distributions $f_j(x)$
- the latent state process S_t selects which distributions is active at any time
- S_t is assumed to be a Markov process with N states

Markov Property

Being S_t a Markov process, it satisfies the **Markov property**

$$f(S_t|S_1, ..., S_{t-1}) = f(S_t|S_{t-1})$$

that is, the state at time t depends only on the state at time $t-1$. Moreover, the Markov property implies that S_t is fully characterized by:

- $\delta_j^{(1)} = P(S_1 = j)$, the initial state probabilities
- $\Gamma^{(t)} = [\gamma_{ij}^{(t)}]$, the transition probability matrix, where $\gamma_{ij}^{(t)} = P(S_t = j|S_{t-1} = i)$

2. Modelling Periodic Variations

Problem: Periodicity

In many real applications, time series data are often characterized by **periodicities**, such as diel variations (recurrent patterns over a 24 hours period).

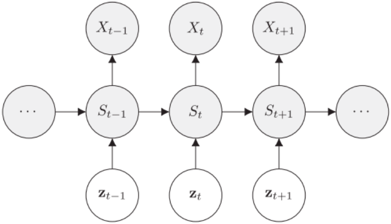
Ignoring the periodicity of the data can **invalidate inference**, for instance the standard errors might be underestimated due to autocorrelation.

Moreover, it may be of interest, instead, to model such periodic variations to comprehensively **understand the process dynamics**, for instance some intra-day behavioral patterns.

Periodic variations can be effectively modeled within the framework of HMM by including **temporal covariates** in the model.

HMM with Covariates

Incorporating covariates in HMM can be done in the **state-transition probabilities**.



Expressing state transition probabilities as function of covariates to infer how state switching depends on external factors.

HMM with Covariates

The covariance-dependence on the categorical distribution of states at time t is typically modeled using a **multinomial logistic regression**, which is achieved by applying the inverse multinomial logit link to each row of the transition probability matrix

$$\gamma_{i1}^{(t)} = logit^{-1}(\tau_{ij}^{(t)}) = \frac{e^{\tau_{ij}^{(t)}}}{\sum_{k=1}^{N} e^{\tau_{ij}^{(t)}}}$$

and the general form of the linear predictor for $\tau_{ij}^{(t)}$ is given by

$$\tau_{ij}^{(t)} = z_t' \beta^{(ij)} = \beta_0^{(ij)} + \beta_1^{(ij)} z_{t1} + \ldots + \beta_p^{(ij)} z_{tp} \quad (1)$$

HMM with Seasonality

A special type of covariate is **seasonality**, that is a variation that repeats over a specif time period, e.g. within-day or within-year.

Modelling this type of periodicity can be achieved by using:

- calendar features (e.g. time of the day)
- **trigonometric functions**, with period equal to the cycle length
- **cyclic splines**

Trigonometric Modelling

When the aim is to model periodic patterns in the state-transition dynamics, the linear predictor (1) can be extended by including **trigonometric functions** with the desired periodicity.

$$\tau_{ij}^{(t)} = z_t' \beta^{(ij)} + \sum_{k=1}^{K} \omega_k^{(ij)} \; sin\left(\frac{2\pi kt}{l}\right) + \psi_k^{(ij)} \; cos\left(\frac{2\pi kt}{l}\right)$$

where l is the period length (i.e. number of sequential observations to complete a cycle).

The flexibility is of the model increases with K, the number of periodic components.

Cyclic Splines Modelling

To avoid making any assumption on the functional form of the periodic effect, a **nonparametric** modelling of the periodic effect can be obtained replacing trigonometric functions with **splines**.

$$\tau_{ij}^{(t)} = z_t' \beta^{(ij)} + \sum_{q=1}^{Q} a_q^{(ij)} \; B_q$$

where $a_q^{(ij)}$ are the scaling coefficients and B_q are basis splines wrapped at the desired periodicity (l).

A large value of Q is used to guarantee sufficient flexibility and overfitting is avoided by including a penalty term (P-spline approach).

3. Model Estimation

EM Algorithm

The model is estimated using the **Expectation-Maximization** (EM) algorithm.

The missing data are taken to be the hidden states and the complete-data log-likelihood for the HMM is given by

$$\mathcal{L}_c(\theta) = log\ \delta_{S1}^{(1)} + \sum_{t=2}^{T} log\ \gamma_{S_{t-1},S_t}^{(1)} + \sum_{t=1}^{T} log\ f_{S_t}(x_t)$$

where θ is the set of parameters necessary to define $\delta_j^{(1)}$, $\Gamma^{(t)}$ and the state-dependent distributions $f_j(x)$.

EM Algorithm

The parameter estimates are derived:

- Choosing **starting values** for the parameter set θ necessary to define state-dependent distributions and transition probabilities
- **E–step** compute conditional expectations of missing data (i.e. the latent states), given the data and current parameter values
- **M–step** update parameters based on the CDLL, replacing the functions of the states by their conditional expectations calculated in the E–step

The E-M steps are repeated until convergence.

4. Conclusions

Conclusions

A particular class of HMM to model periodic variations is introduced that:

- makes use of splines instead of trigonometric functions
- increases flexibility, which might be helpful to reveal interesting patterns
- applications show that using sine and cosine waves is often enough

Bibliografy

Carlina C. Feldmann, Sina Mews, Angelica Coculla, Ralf Stanewsky & Roland Langrock (2023), 'Flexible Modelling of Diel and Other Periodic Variation in Hidden Markov Models', Journal of Statistical Theory and Practice, 17:45.

Thank you!