World population

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1 Introduction

In this report we will try to find a mathematical model for the world population's growth as a function for the time, namely, given a time t, find a population $p(t)$ as accurate to reality as possible. By using this, we will consider a possible growth of the world inhabitants in the forthcoming years and check the stability in the future.

2 Methodology

In order to carry out such a task, we will use a set of data which give us the population in time-steps of 5 years. Based on that, we will construct a set of discrete derivatives among the points $(t_i, p(t_i))$ with the aim to converting a discrete problem into a continuous one.

This can be assumed since the amount of inhabitants of the planet grows following a continuous rate if we take the intervals large enough. Since we are taking yearly intervals, the risk of having large errors is small. Once we have found the discrete derivatives of the $(p(t_i))$, we will consider as solutions of the population as a function of time the functions fulfilling the IVP

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = pf(p), \\ p(t_0) = p_0, \end{cases} \tag{1}$$

for a given initial value $p(t_0) = p_0$.

Now we need to find an appropriate candidate for $f(p)$. To do so, we will take into account the points of the graph p vs. $\dot{p}/p$. By using the method of least squares we will obtain candidates to $f(p)$. We will consider three different cases.

3 Models and results

From now on, we will write $p_i := p(t_i)$, $\dot{p}_i := \dot{p}(t_i)$. Our set of data is the following

Year	10^9 Inhab
1950	2.519
1955	2.756
1960	2.982
1965	3.335
1970	3.692
1975	4.068
1980	4.435
1985	4.831
1990	5.263
1995	5.674
2000	6.070
2005	6.454

The unit of the time t is years and the population p is 10^9 Inhab or billion of inhabitants. From now on we will not make explicit reference to the units.

3.1 Malthus model

In the so-called Malthus model, the function considered is a constant, namely, $f(p) := a$. Thus, we have to apply the least squares to the points $(p_i, \dot{p}_i/p_i)$ using a constant function. Once we find the constant that matches the best our set of points, we will solve (1) which becomes

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = ap, \\ p(t_0) = p_0. \end{cases} \tag{2}$$

Essentially, this growth-law states that the increase in the population is proportional to its amount. A computation of the points $\dot{p}/p$ and the application of the least squares method give the following function

$$f(p) = 0.01788, \tag{3}$$

and a error sum of squares of $\epsilon = 1.031 \cdot 10^{-4}$. Putting (3) into (2) the equation can be trivially integrated. We take the initial condition $p(1950) = 2.519$, that is, the population in 1950. The function obtained is

$$p(t) = 2.519 \exp\left(-34.87 + 0.01788t\right). \tag{4}$$

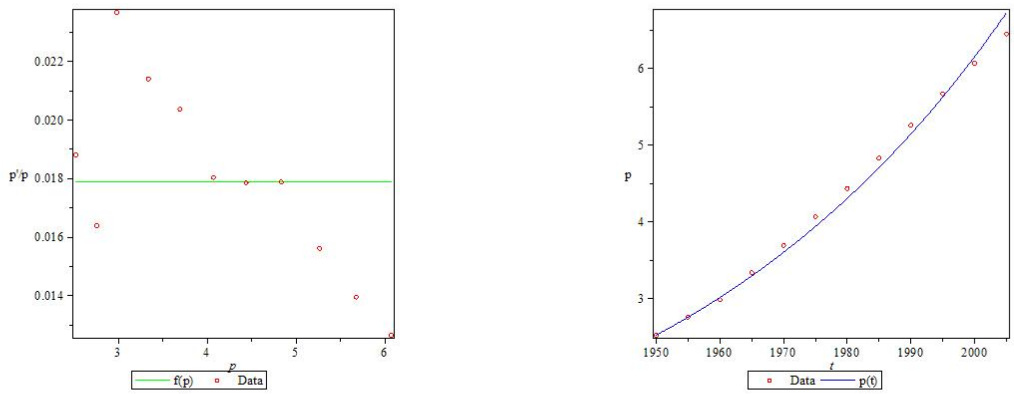

Figure 1: Left: Values of $\dot{p}/p$ vs. p and the function obtained by using the least squares method, (3). Right: Data and our first model for $p(t)$, corresponding to (4).

In this case, we do not obtain a finite limit for the population, that is,

$$\lim_{t \to +\infty} p(t) = +\infty.$$

Roughly speaking, the population grows unbounded and there is no limit for it as the time goes by.

3.2 Verhulst model - I

In the model purposed by Verhulst, the function $f(p)$ is a straight line, that is, $f(p) := a - bp$. Equation (1) therefore becomes

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = p(a - bp), \\ p(t_0) = p_0. \end{cases} \tag{5}$$

This means that the growth ratio of the population is indeed proportional to its amount and to a factor that compensates for the competition, since the more inhabitants there are, the greater the competition for resources is, and therefore, the amount of inhabitants decreases. Proceeding as before, we now obtain

$$f(p) = 0.02604 - 0.001967p, \tag{6}$$

with $\epsilon = 4.650 \cdot 10^{-5}$. We have remarkably reduced the error, namely, in one order of magnitude. This is an indicator that the new least square function is a better approximation to our data. Using (6) in (5) we integrate the equation, again, taking as initial condition $p(1950) = 2.519$. The function obtained is

$$p(t) = \frac{0.2912}{0.02199 + 1.0542 \cdot 10^{23} \exp\left(-0.02604t\right)}. \tag{7}$$

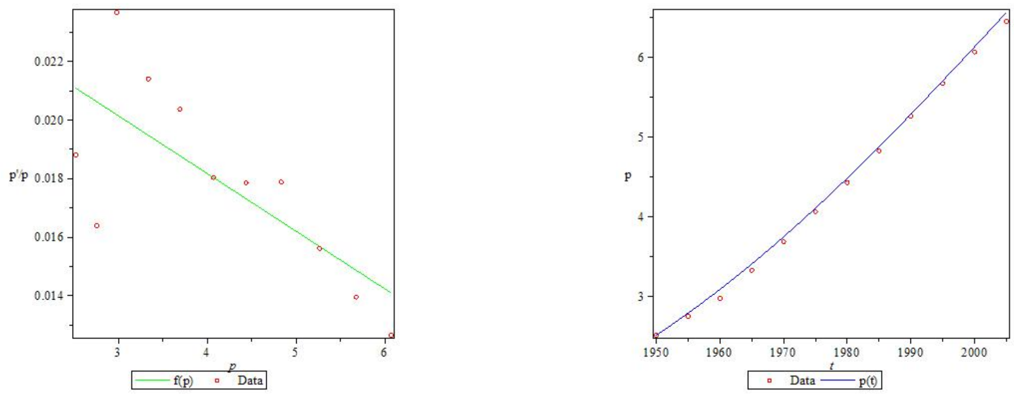

Figure 2: Left: Values of $\dot{p}/p$ vs. p and the function obtained by using the least squares method, (6). Right: Data and our first model for $p(t)$, corresponding to (7).

In this case, if we consider the asymptotic behavior of the function $p(t)$ obtained above, we observe that indeed this time we have a finite limit

$$\lim_{t \to +\infty} p(t) = 13.24$$

According to this model, the population will converge to approximately 13.24 billion of inhabitants.

3.3 Verhulst model - II

Here we will use the same kind of function $f(p)$ as above. Nevertheless, when doing the least squares method we will take less data than before, since so far we have been considering data which contains the numbers right after WW II. Consequently, we will filter the data until we obtain a uniform population growth. This will take place approximately in 1965.

Proceeding as before but taking only the values from 1965, we obtain the following least squares function

$$f(p) = 0.03258 - 0.003266p. \tag{8}$$

By taking less data, the error has again dropped an order of magnitude, being now $\epsilon = 3.660 \cdot 10^{-6}$. The equation (5) can be integrated as before by using (8), but now taking the initial condition imposed by the population in 1965, that is, $p(1965) = 3.335$. The function obtained is

$$p(t) = \frac{8.948}{0.8969 + 1.142 \cdot 10^{28} \exp\left(-0.03258t\right)} \tag{9}$$

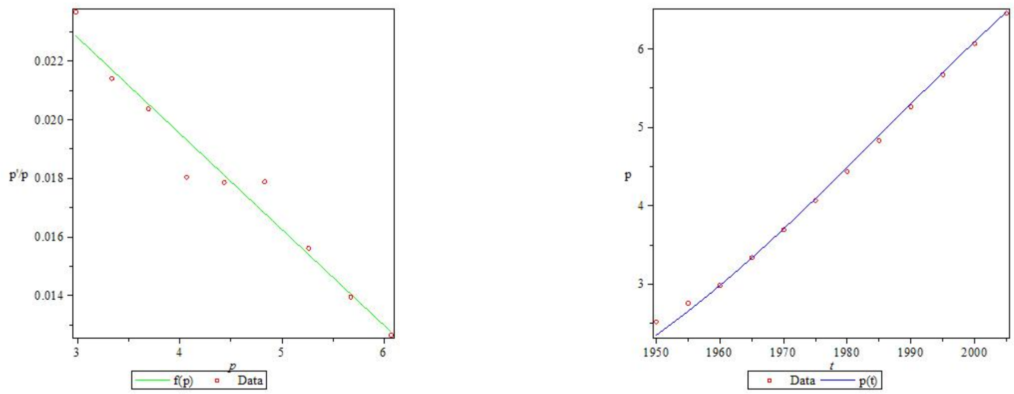

Figure 3: Left: Values of $\dot{p}/p$ vs. p and the function obtained by using the least squares method, (8). Right: Data and our first model for $p(t)$, corresponding to (9).

By construction of p, we again have that the population converges when time goes by, in this case we have

$$\lim_{t\to+\infty} p(t) = 9.977$$

Which means that the population will stabilize in about 9.977 billion of inhabitants.

4 Conclusions

A first observation is the fact that the Malthus model for the population is not reasonable from sensible perspective, since it omits the fact of competition and finite resources, so in fact the variation of population in time cannot be as the one described in (2). As a result one obtains a non-sense, which is that the population grows unbounded.

A good and easy way to solve this issue is by adding a term that plays the role of population decrease. In the Verhulst approach we have discussed this with two successful results. Firstly, we have obtained a much better approximation by using the least squares method with a straight line. Secondly, we have seen that indeed the model gives a finite population in the limit of time going to infinity.

A further analysis using the same model turned out to be even better and the population has a finite limit once more, but this time less than the one obtained previously with the same model. As a matter of fact, using less data to predict the behavior of the system has made us gain accuracy in the dates close to the present day, which should be a better description and basis for a population's prediction of the upcoming years.

Even though more complex approaches for $f(p)$ can be thought of, this might have some side effects, namely, (1) could not be integrated easily in the sense of being given in terms of elemental functions. This could complicate its study and analysis and it might not bring anything meaningful, since having achieved a model which describes a finite population on the Earth as the time goes to infinity is probably one of the most important things.

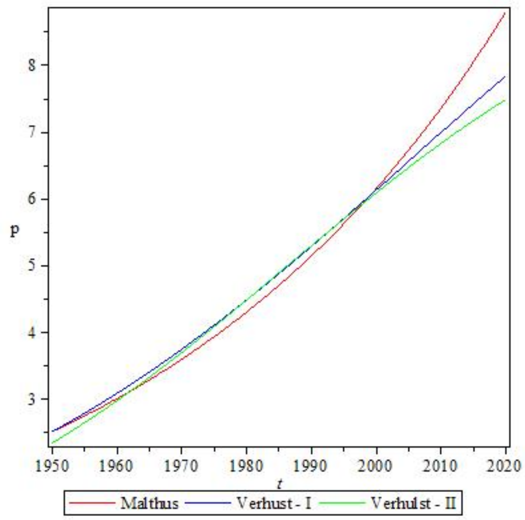

Figure 4: The three different approaches we have been through represented all together, describing $p(t)$ for (4), (7) and (9)