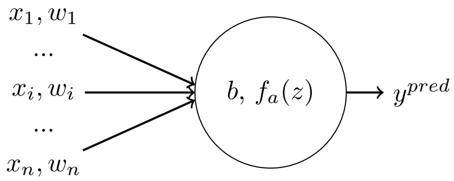
Neuron

Schema of a neuron:



To compute predicted value by the neuron:

$$z = \sum_{i=1}^{n} w_i x_i + b \tag{1}$$

$$y^{pred} = f_a(z) = f_a(\sum_{i=1}^{n} w_i x_i + b) \tag{2}$$

To compute the cost function for m training samples:

$$\begin{aligned} C(w,b) &= \frac{1}{2m} \sum_{j=1}^{m} \left(y_j^{pred} - y_j \right)^2 \\ &= \frac{1}{2m} \sum_{j=1}^{m} \left(f_a(z)_j - y_j \right)^2 \end{aligned} \tag{3}$$

Training the neuron consists on minimizing the cost function. The method used to minimize the cost function is Gradient Descent (`https://en.wikipedia.org/wiki/Gradient_descent`).

$$\mathbf{w}^{n+1} = \mathbf{w}^n - \alpha \nabla C(\mathbf{w}^n, b^n) \tag{4}$$

$$b^{n+1} = b^n - \alpha \nabla C(\mathbf{w}^n, b^n) \tag{5}$$

The equations above compute the gradient descent for the weights and for the bias. α is the so called Learning Rate.
To compute the gradient of the cost function for m training samples, it is necessary to compute the partial derivatives with respect $\mathbf{w}$ and b.
Partial derivative with respect $\mathbf{w}$:

$$\begin{aligned} \frac{\delta C(\mathbf{w},b)}{\delta w_i} &= \left(\frac{1}{2m} \left(f_a(z) - y \right)^2 \right)' \\ &= \frac{1}{m} \left(f_a(z) - y \right) f_a'(z) z' \\ &= \frac{1}{m} \sum_{j=1}^{m} \left(f_a(\mathbf{w^T} \cdot \mathbf{x} + b) - y_j \right) f_a'(\mathbf{w^T} \cdot \mathbf{x} + b)(x_i)_j \end{aligned} \tag{6}$$

In a similar way, the partial derivative with respect b:

$$\frac{\delta C(\mathbf{w},b)}{\delta b} = \frac{1}{m} \sum_{j=1}^{m} \left(f_a(\mathbf{w^T} \cdot \mathbf{x} + b) - y_j \right) f_a'(\mathbf{w^T} \cdot \mathbf{x} + b) \tag{7}$$

Activation functions

Sigmoid

Sigmoid function is:

$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{8}$$

Sigmoid derivative is:

$$\sigma'(x) = \sigma(x) \cdot (1 - \sigma(x)) \tag{9}$$

ReLU

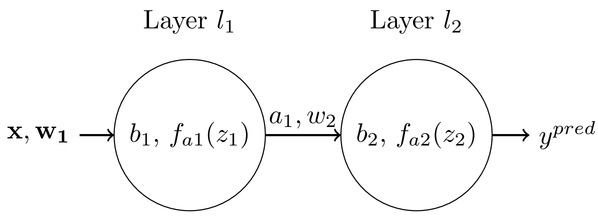
ReLU function is:

$$ReLU(x) = \max(0, x) \tag{10}$$

The derivative of the ReLU function is:

$$ReLU'(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{11}$$

Network



Compute Layer 1:

$$\begin{aligned} z_1 &= \mathbf{w_1^T} \cdot \mathbf{x} + b_1 \\ a_1 &= f_{a1}(z_1) = f_{a1}(\mathbf{w_1^T} \cdot \mathbf{x} + b_1) \end{aligned} \tag{12}$$

Compute Layer 2:

$$\begin{aligned} z_2 &= w_2 a_1 + b_2 \\ y^{pred} &= f_{a2}(z_2) = f_{a2}(w_2 a_1 + b_2) \end{aligned} \tag{13}$$

Network:

$$y^{pred} = f_{a2}(w_2 f_{a1}(\mathbf{w_1^T} \cdot \mathbf{x} + b_1) + b_2) \tag{14}$$

Cost function for m samples:

$$\begin{aligned} C(\mathbf{w}^*, \mathbf{b}^*) &= \frac{1}{2m} \sum_{i=1}^{m} (y^{pred} - y)^2 \\ &= \frac{1}{2m} \sum_{i=1}^{m} (f_{a2}(w_2 f_{a1}(\mathbf{w_1^T} \cdot \mathbf{x} + b_1) + b_2) - y)^2 \end{aligned} \tag{15}$$

Derivative of the cost function:

$$\nabla_{\mathbf{w}^*} C(\mathbf{w}^*, \mathbf{b}^*) = [\delta_{w_1} C(\mathbf{w}^*, \mathbf{b}^*), ..., \delta_{w_i} C(\mathbf{w}^*, \mathbf{b}^*), ..., \delta_{w_n} C(\mathbf{w}^*, \mathbf{b}^*)] \tag{16}$$

$$\nabla_{\mathbf{b}^*} C(\mathbf{w}^*, \mathbf{b}^*) = [\delta_{b_1} C(\mathbf{w}^*, \mathbf{b}^*), ..., \delta_{b_i} C(\mathbf{w}^*, \mathbf{b}^*), ..., \delta_{b_n} C(\mathbf{w}^*, \mathbf{b}^*)] \tag{17}$$

In our case:

$$\nabla_{\mathbf{w}^*} C(\mathbf{w}^*, \mathbf{b}^*) = [\delta_{w_1} C(\mathbf{w}^*, \mathbf{b}^*), \delta_{w_2} C(\mathbf{w}^*, \mathbf{b}^*)] \tag{18}$$

$$\nabla_{\mathbf{b}^*} C(\mathbf{w}^*, \mathbf{b}^*) = [\delta_{b_1} C(\mathbf{w}^*, \mathbf{b}^*), \delta_{b_2} C(\mathbf{w}^*, \mathbf{b}^*)] \tag{19}$$

Let's define some nomencalutre before compute the gradients of the cost function:

$$\mathbf{a_0} = \mathbf{x} \tag{20}$$

$$z_1 = \mathbf{w_1^T} \cdot \mathbf{x} + b_1 \tag{21}$$

$$a_1 = f_{a1}(z_1) = f_{a1}(\mathbf{w_1^T} \cdot \mathbf{x} + b_1) \tag{22}$$

$$z_2 = w_2 f_{a1}(z_1) + b_2 = w_2 f_{a1}(\mathbf{w_1^T} \cdot \mathbf{x} + b_1) + b_2 \tag{23}$$

$$a_2 = f_{a2}(z_2) = w_2 f_{a1}(z_1) + b_2 \tag{24}$$

Let's compute the gradients with respect the weights $\mathbf{w}^*$:

$$\begin{aligned}\delta_{w_1} C(\mathbf{w}^*, \mathbf{b}^*) &= \delta_{w_1}(\frac{1}{2m}\sum_{i=1}^{m}(a_2 - y)^2) \\ &= \frac{1}{m}\sum_{1}^{m}(a_2 - y) f'_{a2}(z_2) w_2 f'_{a1}(z_1)\mathbf{a_0} \end{aligned} \tag{25}$$

$$\begin{aligned}\delta_{w_2} C(\mathbf{w}^*, \mathbf{b}^*) &= \delta_{w_2}(\frac{1}{2m}\sum_{i=1}^{m}(a_2 - y)^2) \\ &= \frac{1}{m}\sum_{1}^{m}(a_2 - y) f'_{a2}(z_2) a_1 \end{aligned} \tag{26}$$

Let's do the same for the bias $\mathbf{b}^*$:

$$\begin{aligned}\delta_{b_1} C(\mathbf{w}^*, \mathbf{b}^*) &= \delta_{b_1}(\frac{1}{2m}\sum_{i=1}^{m}(a_2 - y)^2) \\ &= \frac{1}{m}\sum_{1}^{m}(a_2 - y) f'_{a2}(z_2) w_2 f'_{a1}(z_1) \end{aligned} \tag{27}$$

$$\begin{aligned}\delta_{b_2} C(\mathbf{w}^*, \mathbf{b}^*) &= \delta_{b_2}(\frac{1}{2m}\sum_{i=1}^{m}(a_2 - y)^2) \\ &= \frac{1}{m}\sum_{1}^{m}(a_2 - y) f'_{a2}(z_2) \end{aligned} \tag{28}$$

Let's generalize, and consider the bias b, the component 0 of the weights, so $\mathbf{w_i} = (w_{i0}, w_{i1}, ..., w_{in})$ and $\mathbf{a_i} = (1, a_{i1}, ..., a_{in})$:

$$\delta_{w_i} C(\mathbf{w}) = \frac{1}{2m}\sum_{i=1}^{m}(a_n - y) f'_{an}(z_n) w_n f'_{an-1}(z_{n-1}) w_{n-1} ... f'_{a_i}(z_i) a_i \tag{29}$$