FRACTIONAL REVIVALS: UNIVERSALITY IN THE LONG-TERM EVOLUTION OF QUANTUM WAVE PACKETS BEYOND THE CORRESPONDENCE PRINCIPLE DYNAMICS ☆

I.Sh. AVERBUKH and N.F. PERELMAN

Applied Physics Institute, MSSR Academy of Sciences, Kishinev-28 277028, USSR

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The long-term free evolution of wave packets formed by highly excited states of quantum systems performing regular periodic motion in the classical limit is considered. The universal asymptotic scenario of the wave function temporal development is discovered which provides the generation of a certain sequence of initial wave packet fractional revivals. Each revival represents the coherent superposition of macroscopically distinguishable quantum states. The manifestation of the phenomenon in the dynamics of a Rydberg atom excited by a short laser pulse is discussed.

The problem of the relation between the quantum and classical description of physical system dynamics raised already at the dawn of quantum mechanics is still attracting the attention of researchers. Recently this range of questions was intensively studied in connection with the problem of quasiclassical quantization of highly excited multidimensional quantum systems [1], in order to examine the quantum dynamics of systems chaotic in the classical limit [2,3], and due to the arisen experimental possibilities for generation and detection of atomic Rydberg electron wave packets by short laser pulse excitation [4–10]. The problem of the transition from the quantum description to the classical one requires a very careful treatment in the case of bounded systems with a discrete energy spectrum. When $\hbar\to 0$ (in the energy (E) region corresponding to large quantum numbers n) the energy spectrum of systems with the regular periodic dynamics in the classical limit has a quasiequidistant character. The frequency distance $\omega_{n+1,n}$ between the adjacent levels is determined by the inverse period of classical motion T_{cl}: $\omega_{n+1,n}\approx\omega_{cl}=2\pi/T_{cl}$ [11]. As is well known [12,13], however, the large value of the stationary state quantum number does not provide the classical nature of the state. The transition to the classical description requires the consideration of the wave packets' (superpositions of quantum stationary states with different n) evolution. The number Δn of states which form the packet has to be large enough ($\Delta n\to\infty$) to provide the spatial localization of the packet. It should be noticed that the packets containing a small number of states manifest nonclassical behaviour even for large n [14].

On time intervals much smaller than the period of vibrations the discrete character of the spectrum is inessential and the packet moves along the classical trajectory and, generally speaking, spreads. But this spreading is not an irreversible one as in the case of free motion and the packet completely regains its initial shape after the period due to the equidistant character of the spectrum of the states it is formed by. In this sense the packet dynamics may be interpreted by means of quantum beats among the large number of states [15]. This correspondence between the quantum and classical dynamics is retained infinitely long only in the case of strict equality of the energy level spacing (for a harmonic oscillator, for example). In the general case, however, the long-term evolution of the packet is inevitably affected by the rather slight (in the domain of

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high excitations) anharmonicity produced by the ω_{cl} dependence on the energy:

$$\omega_{n+1,n}-\omega_{n,n-1}\approx\hbar\omega_{cl}\frac{\partial\omega_{cl}}{\partial E}\,.$$

The arising quantum dephasing of the stationary state contributions leads to the destruction of the packet in the course of many periods and restricts the duration of its "classical-like" evolution (see e.g. refs. [1,3]),

$$t\ll T_{rev}\,,\quad T_{rev}=2T_{cl}(\hbar|\partial\omega_{cl}/\partial E|)^{-1}\,.$$

In several papers devoted to the numerical investigation of the long-term evolution of Rydberg wave packets [4,5] and to the evolution of some model nonlinear systems [16–20] it was discovered that this dephasing was also not completely irreversible and when $t\approx T_{rev}$ the wave packet again regains its initial form and moves according to the "classical" laws. In refs. [16–20] it was shown that the uncomplete delocalization may be observed even in the region $t<T_{rev}$ and some regular well-localized structures were found at intermediate moments too.

In this paper it is shown that the observation of such structures in different nonlinear quantum systems is not accidental. It is established that the long-term evolution of quantum wave packets in systems periodic in the classical limit is governed by the universal scenario developing after the stage of correspondence principle dynamics. In the course of the scenario the wave function of the system is subjected to a certain sequence of reconstructions which provide regular well-localized structures of the probability density. We call this phenomenon "fractional revivals" because the form of each condensation of the probability density is directly determined by the shape of the initial packet. The structures observed in refs. [16–19] in concrete physical systems present the particular episodes in the course of the discovered general scenario.

Let us consider the wave packet formed from highly excited discrete states of a quantum system performing finite motion in the energy region $E\approx E_{\bar n}$ ($\bar n\gg1$) corresponding to regular periodic classical dynamics:

$$\psi(\boldsymbol{r},t)=\sum_n c_n\varphi_n(\boldsymbol{r})\exp[-\mathrm{i}(E_n/\hbar)t]\,. \qquad (1)$$

Here $\varphi_n(\boldsymbol{r})$ are the wave functions of the stationary states with energy E_n, and c_n are constants. We assume that at the initial moment $t=0$ the wave packet is well localized in space (its spatial dimension Δx is much smaller than the characteristic size L of the classical orbit with energy $E\approx E_{\bar n}$). From the uncertainty principle it follows that the packet energy width ΔE is of the order of

$$\Delta E\sim v\Delta p\sim\frac{\hbar\omega_{cl}L}{\Delta x}\,,\quad \Delta x\ll L\,,$$

where v, Δp are the characteristic values of the velocity and the momentum uncertainty. It means that the distribution $|c_n|^2$ with a sharp maximum at $n\approx\bar n$ has a width of about $\Delta n\sim L/\Delta x$. For systems close to the harmonic oscillator, for example, $L\sim\sqrt{E/M\omega^2}$ and

$$\Delta n\sim\sqrt{\bar n}\,\frac{\sqrt{\hbar/M\omega}}{\Delta x}$$

(M and ω are the mass and the frequency of the oscillator respectively). For a coherent state of the oscillator $\Delta x\sim\sqrt{\hbar/M\omega}$ and $\Delta n\sim\sqrt{\bar n}\ll\bar n$. So the condition $L/\Delta x\gg1$ necessary for transition to a classical description means $\Delta n\gg1$. The value E_n in the interval Δn near $\bar n$ may be given by

$$E_n=E_{\bar n}+\hbar\omega_{cl}(n-\bar n)+\tfrac12\hbar^2\omega_{cl}\frac{\partial\omega_{cl}}{\partial E_{\bar n}}(n-\bar n)^2+\dots\,. \qquad (2)$$

The packet evolution resembles the dynamics of a classical particle over the time interval $t\gg T_{cl}$ if the following conditions are satisfied:

$$\hbar\left|\frac{\partial\omega_{cl}}{\partial E_{\bar n}}\right|(\Delta n)^2\ll1\,,\quad \text{i.e.}\quad \hbar\left|\frac{\partial\omega_{cl}}{\partial E_{\bar n}}\right|\ll1\,.$$

It restricts the energy width of the packet from above.

As was already mentioned the dephasing mechanism caused by the terms quadratic in $n-\bar n$ in (2) sooner or later reveals itself in the course of time. If, however,

$$\frac{t}{T_{cl}}\ll\frac{1}{\hbar^2}\left|\frac{\partial}{\partial E}\left(\omega_{cl}\frac{\partial\omega_{cl}}{\partial E}\right)\right|^{-1}_{E=E_{\bar n}}\,, \qquad (3)$$

the effect of the next terms in the expansion (2) can be neglected. In the following we shall concentrate on the long-term evolution in the time interval pointed out.

The dephasing under discussion will cause the distortion of the shape of the packet with given Δn when $t \sim T_{\rm rev}/(\Delta n)^2$, i.e. when the additional phase shifts between different energy components in (1) inside the packet width become of the order of unity. However, when $t = T_{\rm rev}$ the excess phases caused by the term nonlinear in $n-\bar{n}$ in (2), are exactly the integer multiples of 2π. It provides complete restoration of the initial pulse form. For $t > T_{\rm rev}$ the "classical-like" evolution of the packet begins again. This phenomenon found in ref. [4] was called wave packet revival (see also ref. [21]).

Let us consider the packet evolution over the time scale $0 < t < T_{\rm rev}$. The expression (1) can be rewritten as

$$\psi(\boldsymbol{r}, t) = \sum_{k=-\infty}^{\infty} c_k \varphi_k(\boldsymbol{r}) \times \exp[-2\pi i(kt/T_{\rm cl} + k^2 t/T_{\rm rev})] , \quad k = n - \bar{n} . \tag{4}$$

Here and below the energy is counted from $E = E_{\bar{n}}$. Let us examine now the sum (4) at moments $t \approx (m/n)T_{\rm rev}$, where m and n are mutually prime integers. The additional phase increments produced by terms quadratic in k in (4) are equal to $2\pi\theta k$, where

$$\theta_k = \frac{m}{n} k^2 \pmod 1 . \tag{5}$$

It is easily shown that the quantities θ_k form sequences periodic in k. Indeed, let l be the minimal period of θ_k. Then $\theta_k = \theta_{k+l}$ for arbitrary k, i.e.

$$\frac{m}{n} k^2 = \frac{m}{n}(k+l)^2 \pmod 1 . \tag{6}$$

Eq. (6) gives the necessary and sufficient conditions for the existence of period l:

$$\frac{2ml}{n} = 0 \pmod 1 , \qquad \frac{ml^2}{n} = 0 \pmod 1 . \tag{7}$$

For odd n the conditions (7) are fulfilled when $l = n$. For even n (and odd m respectively) the first of the conditions (7) can be satisfied also by $l = n/2$. The latter then leads to $mn/4 = 0 \pmod 1$.

Thus $l = n/2$ when n is an integer multiple of 4 and $l = n$ in all other cases. So near $t \approx (m/n)T_{\rm rev}$ the terms quadratic in k in (4) lead to additional (with respect to the moment $t = 0$) phase factors which are l-periodic in k. We note that phase factors with the same periodicity in k appear in the spectral expansion of the packets following the correspondence principle dynamics (when the nonlinear phases are ignored) and shifted in time by an integer multiple of $T_{\rm cl}/l$ with respect to the initial packet. It allows us to suppose that near $t = (m/n)T_{\rm rev}$ the system wave function may be presented in the form

$$\psi(\boldsymbol{r}, t) = \sum_{s=0}^{l-1} a_s \psi_{\rm cl}(\boldsymbol{r}, t + (s/l)T_{\rm cl}) , \tag{8}$$

where

$$\psi_{\rm cl}(\boldsymbol{r}, t) = \sum_{k=-\infty}^{\infty} c_k \varphi_k(\boldsymbol{r}) \exp(-2\pi i k t/T_{\rm cl}) \tag{9}$$

describes the "classically-like" evolving packet, a_s are constants. The expression (8) can be immediately obtained using the fact that the l-periodic sequence $\exp(-2\pi i\theta_k)$ may be expanded in l fundamental sequences,

$$\exp(-2\pi i k s/l) , \quad s = 0, 1, ..., l-1 ,$$

with the same period,

$$\exp(-2\pi i\theta_k) = \sum_{s=0}^{l-1} a_s \exp[-2\pi i(s/l)k] , \quad k = 0, 1, ..., l-1 . \tag{10}$$

Multiplying the expansion (10) by $\exp[2\pi i(q/l)k]$ (q is an integer) and using the summation of both parts of the equality over all k one gets

$$a_q = \frac{1}{l} \sum_{k=0}^{l-1} \exp(-2\pi i\theta_k + 2\pi i k q/l) . \tag{11}$$

By means of the explicit form of the θ_k and their l-periodicity one can obtain using a shift by 1 in the summation index in (11) the following relation:

$$a_q = \exp(2\pi i m/n - 2\pi i q/l) a_{q'} , \quad q' = (q + 2ml/n) \pmod l . \tag{12}$$

If n is an odd integer then $l = n$ and an n-fold application of (12) demonstrates the equality of all a_q moduli. When n is an integer multiple of 4, $l=n/2$, m is odd and expression (12) relates again all a_q values providing their moduli being equal. If $n=2$ (mod 4) (12) connects separately the a_q values with odd and even indices. All the coefficients with even indices happen to be equal to zero. Indeed, let us consider the magnitude of a_0 which belongs to this group. Using the expression (11) for $q=0$ with the summation index shifted by integer $n/2$ we have

$$a_0=\frac{1}{l}\sum_{k=0}^{l-1}\exp[-2\pi i(m/n)k^2]$$
$$=\frac{1}{l}\sum_{k'=0}^{l-1}\exp[-2\pi i(m/n)(k'+n/2)^2]=-a_0\,.$$

So $a_0=0$ as all other a_q with even q. The magnitude of the modulus of r non-zero coefficients a_q ($r=n/2$ for n even and $r=n$ for odd ones) may be obtained using (11) and

$$r|a_q|^2=\sum_{s=0}^{l-1}|a_s|^2=1\,,\quad |a_q|=1/\sqrt{r}\,.$$

So for any irreducible rational fraction m/n the initial wave packet splits near $t=(m/n)T_{\rm rev}$ into r spatially separated packet-fractions performing a periodic evolution by means of the correspondence principle with relative time shift $T_{\rm cl}/r$ ($r=n/2$ for even n and $r=n$ for odd n) with respect to each other. We have called such a structure the fractional revival of the initial wave packet of order m/n. Naturally this structure would be well emphasized only for non-overlapping split packets, i.e. when $r<L/\Delta x\sim\Delta n$. The better the quasiclassical conditions are fulfilled the higher splitting can be observed.

Let us consider now some particular structures.

(1) When $t\approx\frac{1}{2}T_{\rm rev}$ ($l=n=2$, $r=1$, $m=1$), $a_0=0$, $a_1=1$, i.e.

$$\psi(\boldsymbol{r},t)=\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{1}{2}T_{\rm cl})\,. \tag{13}$$

Expression (13) represents the initial packet shifted by half a classical period. It is worthwhile to mention that just this type of revival (of order $\frac{1}{2}$ in our notation) was discovered in ref. [4] (see also ref. [5]).

(2) When $t\approx T_{\rm rev}$ ($l=r=n/2=2$, $m=1$)

$$\psi(\boldsymbol{r},t)=\frac{1}{\sqrt{2}}[e^{-i\pi/4}\psi_{\rm cl}(\boldsymbol{r},t)$$
$$+e^{i\pi/4}\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{1}{2}T_{\rm cl})]\,. \tag{14}$$

Expression (14) presents essentially a non-classical object formed by two correlated packets macroscopically separated within the classical orbit. For the case when the $\psi_{\rm cl}(\boldsymbol{r},t)$ are coherent oscillator states similar objects called generalized coherent states were studied in ref. [22]. The possibility of such state generation in nonlinear optical systems for the observation of macroscopic quantum effects in optics was considered in refs. [18,20].

Let us note that a similar structure occurs at $t\approx\frac{3}{4}T_{\rm rev}$ ($l=r=n/2$, $m=3$).

(3) Let $t\approx\frac{1}{3}T_{\rm rev}$ ($l=r=n=3$, $m=1$). Then

$$\psi(\boldsymbol{r},t)=\tfrac{1}{3}(1+2e^{-2i\pi/3})$$
$$\times\{\psi_{\rm cl}(\boldsymbol{r},t)+e^{2i\pi/3}[\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{1}{3}T_{\rm cl})$$
$$+\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{2}{3}T_{\rm cl})]\}\,. \tag{15}$$

Such structures formed by three packets arise also at $t/T\approx\frac{1}{6},\frac{2}{3},\frac{5}{6}$.

(4) When $t\approx\frac{1}{8}T_{\rm rev}$ ($l=r=n/2=4$, $m=1$)

$$\psi(\boldsymbol{r},t)=\tfrac{1}{2}e^{-i\pi/4}\{[\psi_{\rm cl}(\boldsymbol{r},t)-\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{1}{2}T_{\rm cl})]$$
$$+e^{i\pi/4}[\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{1}{4}T_{\rm cl})+\psi_{\rm cl}(\boldsymbol{r},t+\tfrac{3}{4}T_{\rm cl})]\}\,. \tag{16}$$

Analogous structures containing four packets were also observed in numerical investigation of the quantum dynamics of some model nonlinear system [16].

Wave packets of atomic electron Rydberg states excited by a short laser pulse may serve as an interesting object for observation of fractional revivals. The problem of generation and detection of Rydberg wave packets has been intensively studied recently experimentally as well as theoretically [4–7,9,10]. One of the methods of packet dynamics detection is the monitoring of the atomic radiation. During the stage of the packet's "classical" evolution along the Kepler orbit the radiation will be presented by periodic pikes (with the period of the classical motion) corresponding to the moments of nearest approach to the nucleus where the acceleration is maximal. The creation of the above mentioned structures as a result of the packet fractional revivals may reveal itself

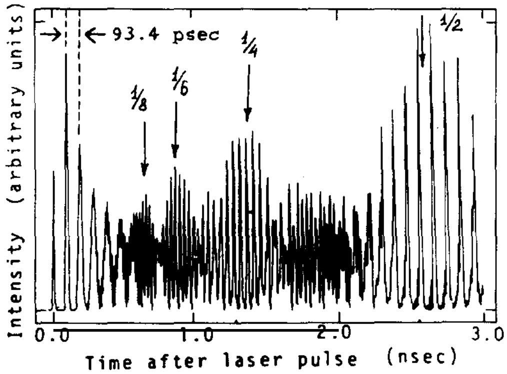

Fig. 1. The emission of an atom excited by a short laser pulse to highly excited Rydberg states (according to ref. [4]). By arrows we have shown the fractional revivals of the initial wave packet.

in the arising of radiation pikes following each other two-fold, three-fold, four-fold more frequently compared with the classical predictions. It is interesting to compare the described asymptotic scenario with the results of a detailed numerical study [4] of the emission of the Rydberg atom, excited by a laser pulse with a duration ~10 ps, to energy states with principal quantum number near $\bar{n}=85$ where $T_{\rm cl}\approx 94$ ps (see fig. 1). The sharp peaks of radiation reproducing the form of the exciting pulse at the initial stage of the evolution reflect the correspondence principle dynamics (see also ref. [15]). The authors of ref. [4] revealed and explained the restoration of the initial time development of the radiation after ~35 periods of classical motion (see fig. 1). From the point of view of the present paper the phenomenon is due to a revival of order $\frac{1}{2}$ ($T_{\rm rev}=5.2$ ns) in our classification. The intermediate region in fig. 1, described in refs. [4,5] as "a complex pattern of quantum beats" has in fact a well defined structure. By the arrows in fig. 1 we have shown the time moments $\frac{1}{8}T_{\rm rev}$, $\frac{1}{6}T_{\rm rev}$, $\frac{1}{4}T_{\rm rev}$, $\frac{1}{2}T_{\rm rev}$. The intensity pikes near the moments pointed out correspond to the fractional revivals of order $\frac{1}{8}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{2}$ with periods of repetition $\frac{1}{4}T_{\rm cl}$, $\frac{1}{3}T_{\rm cl}$, $\frac{1}{2}T_{\rm cl}$, $T_{\rm cl}$ respectively. The asymmetry of fig. 1 with respect to the moment $\frac{1}{4}T_{\rm rev}$ must be attributed to the higher order terms in the expansion of E_n in $n-\bar{n}$ for the chosen parameter values.

Thus in this paper we have described the universal behaviour of wave packets formed from highly excited states of quantum systems which perform a regular periodic motion in the classical limit. It was shown that in the course of the long-term evolution these superposition states after the well-known stage of the correspondence principle dynamics are subjected to a universal sequence of fractional revivals with the creation of correlated sets of localized components distributed along the classical orbit. Such objects present macroscopically distinguishable quantum states whose properties continue to attract attention in view of the fundamentals of quantum mechanics. Generation and detection of these states have recently become a real experimental problem [9,10,18,20]. As follows from the present work states of such type inevitably arise in a wide class of quantum systems of different physical nature in the course of the long-term evolution of an arbitrary "classical-like" initial state.

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