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On Constructing Formal Integrals of a Hamiltonian System Near an Equilibrium Point*

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A procedure is given for constructing formal integrals of a Hamiltonian system of n degrees of freedom near an equilibrium point. The method consists of performing a series of canonical transformations which reduces the Hamiltonian to a normal form. The integrals then appear as quadratic forms. A computer program carries out the construction procedure by manipulating polynomials symbolically. For a given Hamiltonian system, it determines approximate integrals (truncated power series) and expresses them in the original coordinates. In the case of two degrees of freedom, level lines of the intersection of the integral with some appropriate surface are plotted. To illustrate this technique the problem of the existence of a third isolating integral of motion in an axisymmetrical potential is investigated; a comparison is then made between computed orbits and the curves generated by the approximate third integral.

INTRODUCTION

THIS paper is concerned with the construction of formal integrals of a time-independent Hamiltonian system about an equilibrium point. It is known (cf. Poincaré 1957) that such integrals, in general (except H), do not exist. However, recent numerical evidence (Contopoulos 1960, 1963c; Henón and Heiles 1964) shows a regular behavior of the flow, indicating the existence of approximate integrals.

Our object is to give a consistent and constructive procedure for finding formal integrals, i.e., in the form of power series, where we ignore questions of convergence. We give a meaning to these integrals by truncating them, losing thereby the precise constancy of the integral; however, the truncated expressions represent integrals in an approximate sense. The quality of the approximation is then studied. As a model example we choose a problem that Henón and Heiles (1964) investigated. However, this technique has a much wider applicability. In many of the cases studied the degree of approximation was much higher than expected.

As expected, the computation of integrals is formidable. Therefore, computer programs were constructed which carry out symbolically the required polynomial manipulations.

To describe our formal procedure we consider a Hamiltonian system near an equilibrium point,

$$H(x,y)=\sum_{\nu=1}^{n}\frac{\alpha_\nu}{2}(x_\nu^2+y_\nu^2)+\cdots.$$

Birkhoff (1927) treated this problem extensively when the α_ν were rationally independent and showed that a canonical change of variables $(x,y)\rightarrow(\xi,\eta)$ can be found (in formal power series) such that $H=\Gamma(\omega_1,\cdots,\omega_n)$, where $\omega_\nu=\frac{1}{2}(\xi_\nu^2+\eta_\nu^2)$, $\nu=1,\cdots,n$. Clearly, this implies that formally the ω_ν are n-independent integrals.

For our applications, the case of *rationally dependent* α_ν is of main interest. Our result, in this case, can be formulated: If there are r rationally independent relations $(j,\alpha)=0$ with integer vectors j, then there exist n-r formal independent integrals starting with quadratic terms. This general result is discussed and applied to one specific problem of two degrees of freedom. The model problem deals with the motion of a particle in a rotation symmetric gravitational field. Since energy and angular momentum are immediate integrals, the question of a "third integral" becomes important. For this problem orbit calculations are available and therefore appropriate for comparison. [I should like to thank Dr. Carl Heiles for sending me one of the computer programs used in preparation of Henón and Heiles (1964).] Figure 1 contains the level lines of the constructed third integral. The topological structure is quite complicated; note especially the number of islands which is due to low-order resonance. In comparison, Fig. 2 shows the same orbits resulting from direct integration.

Our work is very closely related to the recent work of Contopoulos which we learned about during the preparation of this work. Contopoulos has made an extensive study of the problem of finding new integrals for conservative dynamical systems. In particular, he (see 1960, 1963c) has made a detailed investigation of the existence of a third integral in a galaxy from both a theoretical and numerical point of view. In still other papers (see 1963a, b) he has unified and clarified the existing literature in this area. His work also focuses attention on some very important open problems in mathematics and astronomy; e.g., the sudden breakdown of integrability for large perturbation.

In the papers (1960, 1963c) Contopoulos gives a construction of a third integral. His procedure combines two methods used earlier by Whittaker (1916, 1944). Cherry (1924a, b) uses similar methods, however, he shows the existence of formal integrals by an indirect proof.

In all these previous works the formal integrals I are found by "comparison of coefficients" in the partial differential equation $[I,H]=0$. However, in the resonant case a new difficulty arises. In the comparison of coefficients of I a number of terms remain unspecified. In the

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paper by Cherry (1924b), for instance, it is shown that in a case of two degrees of freedom the arbitrary coefficients introduced at any stage are not determined until the third succeeding stage. However, the choice of these so-far-unspecified terms may interfere with the terms of higher order, making it impossible to proceed. This is the question of the consistency of the construction. It is by no means clear that an arbitrary choice of these terms allows one to continue the construction to all orders. It is one of the main objects of this paper to clarify this point. In fact, we proceed entirely differently, not seeking the integrals at first, but rather transforming the Hamiltonian into a normal form [along the lines of Birkhoff (1927)]. This can be done in a consistent manner. For the system in normal form the integrals appear as quadratic forms as by-products. Of course, to find the integrals in the original coordinates, we have to transform these quadratic forms using the constructed canonical transformations. We also mention that an integral of the above type was constructed by Moser (1958) in proving a stability theorem for a time-dependent periodic Hamiltonian system.

Recently, Kolmogorov (1954a,b), Arnold (1963), and Moser (1962) obtained very important results that show the existence of invariant surfaces for nearly integrable Hamiltonian systems. Since the unperturbed Hamiltonian is integrable, a continuous family of invariant surfaces exists. Their results show that, for small perturbations, the majority of these invariant surfaces are preserved while a small number dissolve or break up. The surfaces that dissolve are densely intertwined with the surfaces that persist showing that no integral exists

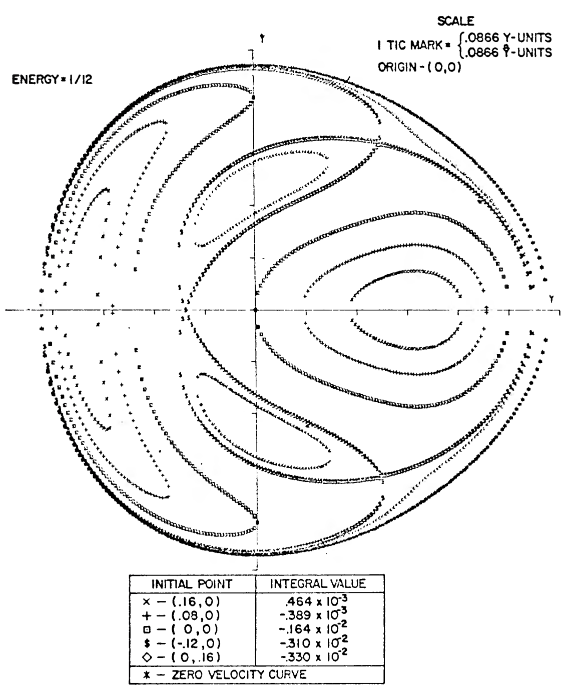


FIG. 1. Level lines of formal integral I for $E=0.08333$.

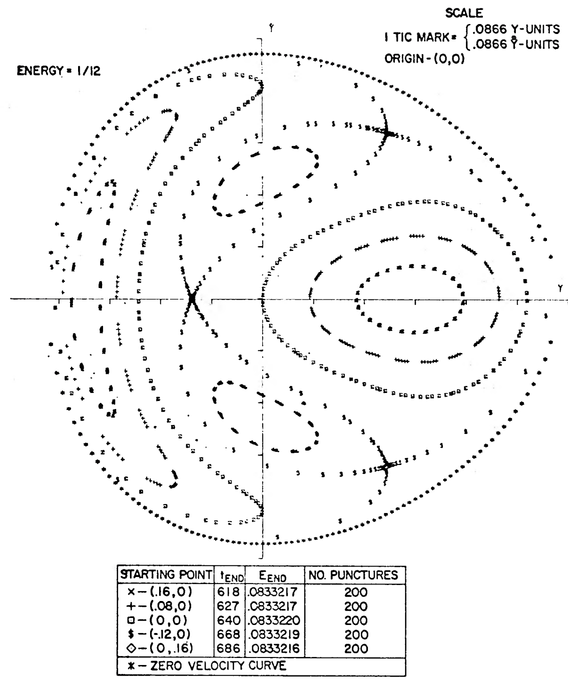


FIG. 2. Corresponding curves of Fig. 1 calculated by forward integration.

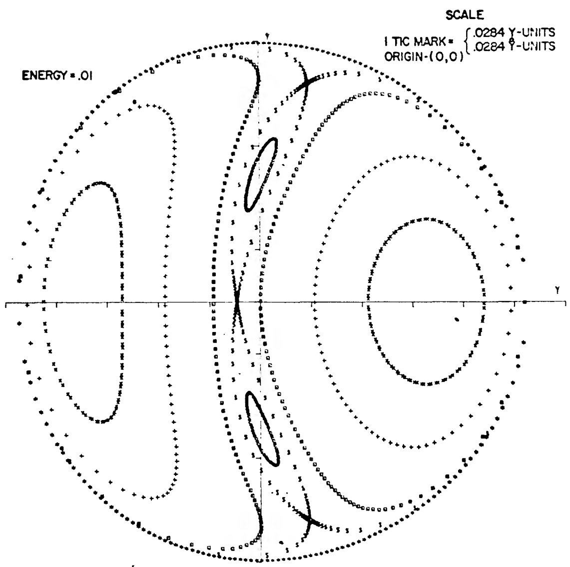
in the strict sense. We emphasize that these results are true only for sufficiently small perturbations and that no concrete results exist in the large. Numerical evidence, however, indicates that the surfaces do continue for large perturbation but then suddenly start to disappear completely. The results of Henón and Heiles are an example of this phenomenon.

The construction of an approximate integral can be viewed as an attempt to find an asymptotic (possibly convergent) representation for this family of invariant surfaces. The approximate integral is a continuous family of surfaces; thus, we seek agreement only where the invariant surfaces exist and say nothing where they do not. In general, the flow is very complicated between the existing invariant surfaces and cannot be described analytically. Numerical integration also fails to detect this behavior except where the surfaces begin to disappear completely.

This paper is divided into four sections. In Sec. 1 we discuss the model problem of Henón and Heiles. The second section describes the construction of formal integrals. The third section describes the important programming aspects of the computer program that implemented the construction procecure. The last section compares the results for the model problem and contains some concluding remarks. An Appendix to Sec. 2 concludes the paper.

1. AN EXAMPLE

In order to explain the motivation for calculating an approximation to a nonexistent integral, it is appro-



STARTING POINT	t_{END}	E_{END}	NO. PUNCTURES
× – (.06,0)	627	.0099988	200
+ – (.03,0)	472	.0099990	150
⊡ – (0,0)	1391	.0099970	440 †
$ – (-.01344,0)	9501	.0099799	3000 †
◇ – (0, .06)	2281	.0099953	720 †
✱ – ZERO VELOCITY CURVE			

† – EVERY ⊡ 2ND, $ 10TH, ◇ 3RD POINT PLOTTED

FIG. 3. Results for $E=0.01$ (forward integration).

priate to consider a concrete problem. As mentioned, Henón and Heiles (1964) discussed a model problem concerning the existence of a third isolating integral of motion in an axisymmetrical potential (for definition, see Wintner 1947, p. 96). This particular problem has recently received the attention of numerous authors (e.g., Contopoulos 1957, 1958, 1960, 1963a, b, c, 1965; Barbanis 1962, 1965; van de Hulst 1962, 1963; Ollongren 1962, 1965), all of whom have tried to settle the question of whether the third integral of galactic motion is isolating or nonisolating.

We consider the motion of a star in a gravitational field which is time-independent and has an axis of symmetry. Introducing cylindrical coordinates r, θ, z, this potential is then a given function $U_g(r,z)$. In a Hamiltonian formulation the conjugate momenta are $p_r=\dot{r}$, $p_\theta=r^2\dot{\theta}$, and $p_z=\dot{z}$, $\cdot\equiv d/dt$, and the Hamiltonian per unit mass, which governs the motion is

$$H=\tfrac{1}{2}[p_r^2+(p_\theta^2/r^2)+p_z^2]+U_g(r,z).$$

The equations of motion (in addition to the three above) are

$$\dot{p}_r=-\frac{\partial}{\partial r}\left(\frac{p_\theta^2}{2r^2}+U_g\right),\quad \dot{p}_\theta=0,\quad \dot{p}_z=-\frac{\partial}{\partial z}U_g. \qquad (1.1)$$

One can easily see (since H does not explicitly depend upon θ) that the z component of angular momentum is a constant of the motion. Calling this constant C and introducing the reduced potential

$$U(r,z)=\tfrac{1}{2}(C^2/r^2)+U_g(r,z), \qquad (1.2)$$

one can reduce the order of the problem from three degrees of freedom to two degrees of freedom. We first note that

$$r=r_0,\quad z=0,$$
$$p_r=0,\quad p_z=0,$$

is a solution to the full system (1.1) with the condition

$$C^2/r_0^3=(\partial U_g/\partial r)_{\substack{r=r_0\\ z=0}}. \qquad (1.3)$$

This solution corresponds to circular motion in the galactic plane, and Eq. (1.3) means that centrifugal and centripetal forces are equal. Substituting x for $r-r_0$ and y for z and using (1.2), the equations of motion become

$$\ddot{x}=-\partial U/\partial x,\quad \ddot{y}=-\partial U/\partial y, \qquad (1.4)$$

where $x=\dot{x}=y=\dot{y}=0$ is an equilibrium point.

Thus, the problem considered is equivalent to the problem of describing the planar motion of a particle in an arbitrary potential field.

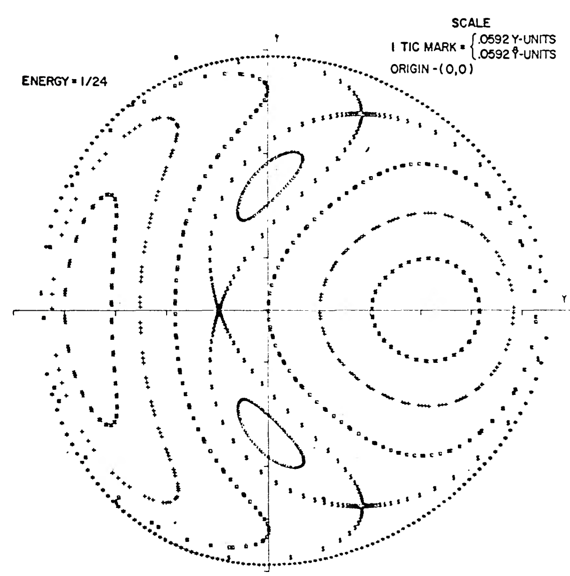
The problem of solving equations (1.4) is equivalent to finding independent integrals of motion. Three integrals exist. One of them, the total energy

$$E=U(x,y)+\tfrac{1}{2}(\dot{x}^2+\dot{y}^2), \qquad (1.5)$$

is clearly isolating. Another integral is, in general, nonisolating, and hence of no physical interest. The existence of a second isolating integral of (1.4) is an open question and is equivalent to the existence of a third isolating integral of motion of (1.1).

In Henón and Heiles (1964) the potential

$$U(x,y)=\tfrac{1}{2}(x^2+y^2+2x^2y-\tfrac{2}{3}y^3) \qquad (1.6)$$



STARTING POINT	t_{END}	E_{END}	NO. PUNCTURES
× – (.12,0)	623	.0416615	200
+ – (.06,0)	629	.0416612	200
⊡ – (0,0)	639	.0416610	200
$ – (-.0588,0)	843	.0416594	260
◇ – (0,.12)	653	.0416611	200
✱ – ZERO VELOCITY CURVE			

FIG. 4. Results for $E=0.04167$ (forward integration).

was chosen and numerical experiments were carried out. Their procedure was the following. Use the energy integral (1.5) to obtain $\dot{x}$ as a function of x, y and $\dot{y}$. The condition that $\dot{x}^2$ be nonnegative defines, for certain E values, a bounded volume

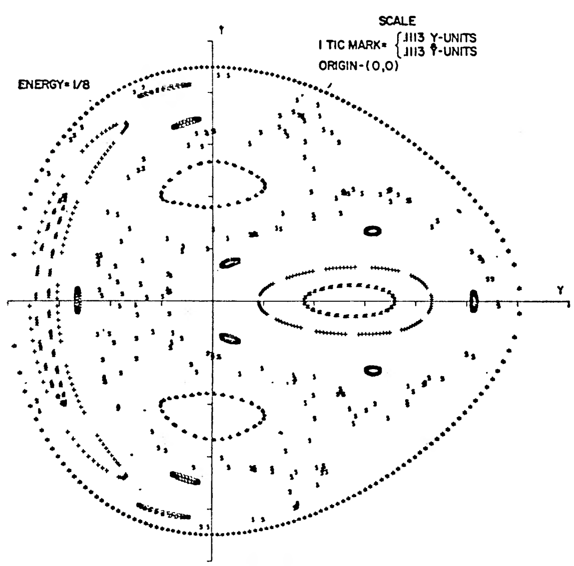
$$U(x,y)+\tfrac{1}{2}\dot{y}^2\leq E. \tag{1.7}$$

Now, if a second isolating integral does not exist, then any single trajectory would probably permeate this volume in a random fashion, whereas the existence of a second isolating integral would mean that the trajectory would lie on some two-dimensional surface. More specifically, consider the bounded area defined by the intersection of the $x=0$ plane with the bounded volume (1.7); i.e.,

$$U(0,y)+\tfrac{1}{2}\dot{y}^2\leq E. \tag{1.8}$$

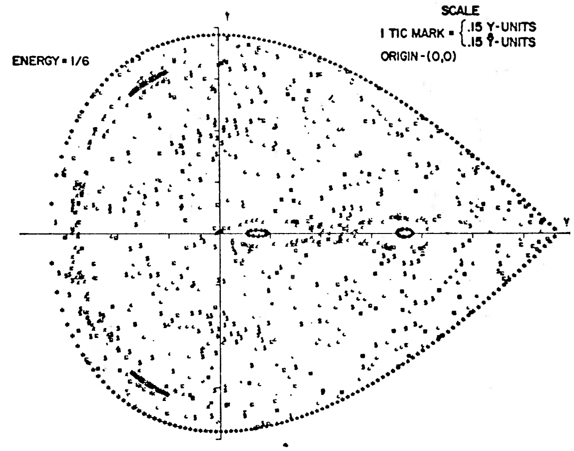
The successive intersections of a given trajectory of (1.4) with the plane $x=0$ will, in general, define an infinite sequence of points P_i, $i=1, 2, \cdots$, which lie in the y-$\dot{y}$ plane. If these points P_i define some pathological set which clearly does not lie on any curve, then no second isolating integral exists. If, however, the points P_i lie on a curve, the second integral is isolating. The passage from point P_i to P_{i+1} defines a mapping, which is completely determined by $U(x,y)$ and E, in the following way. Suppose P_i is known. It defines y and $\dot{y}$; x is zero; $\dot{x}$ is found from (1.5). Using these four values as initial conditions the equations (1.4) can be integrated forward, until x is again equal to zero. The values of y and $\dot{y}$ at this time define P_{i+1}.

The results of the above-mentioned numerical experiments are quite interesting. They suggest strongly that the existence of a second isolating integral depends upon the value of the energy constant E and the initial conditions. For $0\leq E\leq\frac{1}{6}$ and $U(x,y)$ given by (1.6), Eq. (1.7) defines a bounded volume. For $0\leq E\leq 0.12$, the points of intersection P_i defined by any trajectory seem to lie exactly on a curve (see Figs. 2–4). The totality of trajectories forms a one-parameter family of curves which seems to completely fill the available area defined by (1.8). As E increases past 0.12, a set of closed curves is obtained, but these curves no longer fill the whole area (see Fig. 5). In fact, there is a sharp decrease in the area filled by trajectories which lie on curves. The rest of the available area is covered in a random manner by any single trajectory starting in this area; this strongly suggests a nonisolating character for the integral. At $E=\frac{1}{6}$, almost the entire area (1.8) is covered by any single trajectory; this suggests a completely nonisolating nature for the integral (see Fig. 6).



STARTING POINT	t_{END}	E_{END}	NO. PUNCTURES
× – (.2,0)	611	.124983	200
+ – (.1,0)	622	.124984	200
□ – (.02,.08)	802	.124979	250
\$ – (-.17568,0)	674	.124983	200
◇ – (0,.2)	743	.124982	200
✱ – ZERO VELOCITY CURVE			

FIG. 5. Results for $E=0.125$ (forward integration).



STARTING POINT	t_{END}	E_{END}	NO. PUNCTURES
× – (.3,0)	623	.1666444	200
+ – (.08,0)	635	.1666437	200
□ – (0,0)	614	.1666468	200
\$ – (-.222,0)	763	.1666438	200
◇ – (0,.25)	686	.1666451	200
✱ ZERO VELOCITY CURVE			

FIG. 6. Results for $E=0.16667$ (forward integration).

In conclusion, we indicate the motivation that was alluded to in the initial paragraph of this section. The information contained in Figs. 3, 4, and 5 is clearly useful. The original reason for conducting this research was to find a direct method for constructing an approximation to the well-defined curves (or surfaces) that these pictures portray. In Sec. 4 we compare these curves with the level lines of the formal integral (truncated at degree eight) associated with this model problem.

2. NORMAL FORM FOR n DEGREES OF FREEDOM

We consider the system of differential equations

$$\begin{aligned} du/dt&=(\partial H/\partial v)(u,v),\\ dv/dt&=-(\partial H/\partial u)(u,v) \end{aligned} \tag{2.1}$$

generated by a Hamiltonian

$$H(u,v)=H^{(2)}(u,v)+H^{(3)}(u,v)+\cdots,$$

a power series in u and v which is assumed convergent in a neighborhood of $u=v=0$. Here, u and v are n vectors with components u_ν, v_ν, $\nu=1, \cdots, n$, respectively, and

$$H^{(s)}(u,v)=\sum_{|i|+|j|=s} a_{ij}u^i v^j, \quad s=2, 3, \cdots,$$

is a homogeneous polynomial of degree s. Throughout this section we shall use multi-index notation. Thus, i represents an n vector with nonnegative integer components, $|i|=\sum_{\nu=1}^{n}|i_\nu|$ and $u^i=\prod_{\nu=1}^{n}u_\nu^{\nu i}$. The assumption that H begins with quadratic terms implies that the origin $u=v=0$ is an equilibrium point of the system (2.1).

The main object is to simplify the Hamiltonian by introducing appropriate coordinates which are related to the original ones by a canonical transformation in a manner which is similar to diagonalizing a matrix. We can then exhibit, in a natural way, new functions, independent of the Hamiltonian, which are formal integrals of the motion.

We restrict our attention to a case which occurs frequently in mechanics; i.e., we assume that $H^{(2)}(u,v)$ is a positive definite quadratic form. Then, there exists a canonical transformation $(u,v)\rightarrow(x,y)$ which transforms $H^{(2)}(u,v)$ into a normal form

$$H^{(2)}(x,y)=\sum_{\nu=1}^{n}\frac{\alpha_\nu}{2}(x_\nu^2+y_\nu^2), \tag{2.2}$$

where the α_ν are positive. If there were no higher-order terms in H, then the assumption (2.2) would mean that we have a system of n-uncoupled harmonic oscillators with frequencies α_ν, $\nu=1, \cdots, n$. The higher-order terms of H have the effect of coupling these oscillators.

We now distinguish between resonant and nonresonant cases. One has *rth-fold resonance* whenever the α_ν, $\nu=1, \cdots, n$, are connected by r, and only r, linearly independent relations of commensurability, say,

$$\sum_{\nu=1}^{n} a_{\sigma\nu}\alpha_\nu=0, \quad \sigma=1, 2, \cdots, r; \tag{2.3}$$

i.e., $A\alpha=0$, where the r by n matrix $A=(a_{\sigma\nu})$ has integer coefficients and rank r. Nonresonance is defined by $r=0$. Then, the α_ν are rationally independent; i.e., $(\alpha,j)\neq 0$ for $|j|\neq 0$. Here, j represents an n vector of integer components, and the notation (u,v) stands for $\sum_{\nu=1}^{n}u_\nu v_\nu$. Our result is that there exist n-r independent formal integrals of (2.1) whenever (2.3) holds. If $r\neq 0$, these n-r integrals are independent of H while for $r=0$, the Hamiltonian H is a function of these n integrals. Thus, for the model problem ($n=2$), the case of resonance ($r=1$) and nonresonance ($r=0$) merge and give rise to a single new integral.

We now define normal form. Let $H(x,y)$ be a Hamiltonian with $H^{(2)}(x,y)$ given by (2.2). We say that $H(x,y)$ is in *normal form* if $DH(x,y)=0$, where

$$D=\sum_{\nu=1}^{n}\alpha_\nu\left(x_\nu\frac{\partial}{\partial y_\nu}-y_\nu\frac{\partial}{\partial x_\nu}\right). \tag{2.4}$$

Theorem 1: Let $H(x,y)=H^{(2)}(x,y)+H^{(2)}(x,y)+\cdots$ be a Hamiltonian function where $H^{(2)}(x,y)=\sum_{\nu=1}^{n}\frac{1}{2}\alpha_\nu \times(x_\nu^2+y_\nu^2)$. Then there exists a formal canonical transformation

$$x=\xi+\phi(\xi,\eta), \quad y=\eta+\psi(\xi,\eta),$$

that transforms $H(x,y)$ into $\Gamma(\xi,\eta)$, where $\Gamma(\xi,\eta)$ is in a normal form.

Proof: For every s, $s=2, 3, \cdots$ we show there exists a homogeneous polynomial of degree s, $W^{(s)}(x,\eta)$, which generates a canonical transformation

$$\xi=x+(\partial W^{(s)}/\partial\eta)(x,\eta), \quad y=\eta+(\partial W^{(s)}/\partial x)(x,\eta) \tag{2.5}$$

such that the new Hamiltonian $\Gamma(\xi,\eta)$ is in normal form up to degree s. For $s=2$, the identity transformation [i.e., $W^{(2)}(x,\eta)\equiv 0$] suffices since $H^{(2)}(x,y)$ is already in normal form; i.e., $DH^{(2)}(x,y)=\sum\alpha_\nu(x_\nu y_\nu-y_\nu x_\nu)=0$.

We now assume that H is in normal form up to degree $s-1$. At this stage we have already performed $s-3$ canonical transformations, and the original Hamiltonian H is expressed in terms of variables which are the composite of these $s-3$ transformations. For convenience, we call these variables x and y. We have

$$\begin{aligned}H(x,y)=&H^{(2)}(x,y)+H^{(3)}(x,y)+\cdots\\&+H^{(s-1)}(x,y)+H^{(s)}(x,y)+H^{(s+1)}(x,y)+\cdots\end{aligned}$$

with $H^{(i)}$, $i=2, \cdots, s-1$ in normal form, and we introduce the canonical transformation (2.5), where $W^{(s)}(x,\eta)$ is to be considered an arbitrary homogeneous polynomial of degree s in x and η. Under transformation (2.5), we have (see Goldstein, 1956, p. 241)

$$H(x,\eta+(\partial W^{(s)}/\partial x))=\Gamma(x+(\partial W^{(s)}/\partial\eta), \eta), \tag{2.6}$$

where $\Gamma(\xi,\eta)=\sum_{\nu=2}^{\infty}\Gamma^{(\nu)}(\xi,\eta)$ is the new Hamiltonian. Using (2.5) and (2.6) we see that $\Gamma^{(i)}=H^{(i)}$, $i=2, \cdots, s-1$ so that $\Gamma(\xi,\eta)$ is in normal form up to degree $s-1$. We now show that the determining function $W^{(s)}(x,\eta)$ can be chosen so that $\Gamma^{(s)}(\xi,\eta)$ is in normal form. To do this we equate terms of order s in (2.6) and obtain

$$DW^{(s)}(x,\eta)=-H^{(s)}(x,\eta)+\Gamma^{(s)}(x,\eta), \tag{2.7}$$

where D is given by (2.4) with η_ν in place of y_ν. The partial differential operator D admits the decomposition of its domain $\mathfrak{D}$ into $\mathfrak{D}=\mathfrak{N}\cup\mathfrak{R}$ with $\mathfrak{N}\cap\mathfrak{R}=0$, where $\mathfrak{N}$ and $\mathfrak{R}$ are the null and range spaces, respectively, of the operator D. [By *null space* we mean the space of homogeneous polynomials $P^{(s)}(x,\eta)$ with the property that $DP^{(s)}(x,\eta)=0$. A homogeneous polynomial

$P^{(s)}(x,\eta)$ is in the *range* if there exists a homogeneous polynomial $Q^{(s)}(x,\eta)$ such that $DQ^{(s)}(x,\eta)=P^{(s)}(x,\eta)$.]

To see this, make the canonical substitution

$$x=(1/\sqrt{2})(q+ip), \quad \eta=(i/\sqrt{2})(q-ip), \tag{2.8}$$

and express Eq. (2.7) in the new complex coordinates p_ν, q_ν, $\nu=1, \cdots, n$. Equation (2.7) then becomes

$$\tilde{D}\tilde{W}^{(s)}(p,q)=-\tilde{H}^{(s)}(p,q)+\tilde{\Gamma}^{(s)}(p,q),$$

where

$$\tilde{D}=\sum_{\nu=1}^{n} i\alpha_\nu\left(q_\nu\frac{\partial}{\partial q_\nu}-p_\nu\frac{\partial}{\partial p_\nu}\right),$$

$$\tilde{H}^{(s)}(p,q)=H^{(s)}(x,\eta), \quad \tilde{\Gamma}^{(s)}(p,q)=\Gamma^{(s)}(x,\eta), \tag{2.9}$$

and

$$\tilde{W}^{(s)}(p,q)=W^{(s)}(x,\eta).$$

The above transformation diagonalizes the operator D; i.e., $\tilde{D}\tau=i(\alpha, j-k)\tau$, where $\tau=\beta q^j p^k$ and β is a constant. We note here that the meaning of the decomposition of $\mathfrak{D}$ into the desired form means precisely that the domain $\mathfrak{D}$ can be split so that any element of it can be written uniquely as the sum of an element in the null space and an element in the range space. It is now easily seen that the linear operator D has the above stated decomposition since any term in $\mathfrak{D}$ is a linear combination of monomial terms τ and since each τ is in the null (range) space of D if and only if $(\alpha, j-k)$ equals (does not equal) zero. A further discussion of the solution of equation (2.7) is given in the Appendix.

Using this, we can express $H^{(s)}(x,\eta)$ uniquely as $R^{(s)}(x,\eta)+N^{(s)}(x,\eta)$, where $R^{(s)}\in\mathfrak{R}$ and $N^{(s)}\in\mathfrak{N}$. Now choose $W^{(s)}$ and $\Gamma^{(s)}$ so that $DW^{(s)}(x,\eta)=-R^{(s)}(x,\eta)$, $W^{(s)}\in\mathfrak{R}$ and $\Gamma^{(s)}(x,\eta)=N^{(s)}(x,\eta)$. This choice solves (2.7) uniquely for $W^{(s)}$ and, at the same time, makes $D\Gamma^{(s)}(x,\eta)=0$; i.e., the homogeneous polynomial of degree s of the new Hamiltonian $\Gamma(\xi,\eta)$ is in normal form.

We now complete the proof by showing that the $\Gamma^{(i)}$, $i=s+1, \cdots$ are determined uniquely by H and the now specified $W^{(s)}$. The terms that $H^{(l)}(x, \eta+(\partial W^{(s)}/\partial x)(x,\eta))$ contributes to Eq. (2.6) are of the order l, $l+s-2$, $l+2s-4$, $\cdots$, $l(s-1)$. The general term of order $l-|j|+|j|(s-1)$, $|j|=0, \cdots, l$, is

$$\frac{1}{j!}\frac{\partial^{|j|}H^{(l)}}{\partial\eta^j}\left(\frac{\partial W^{(s)}}{\partial x}\right)^j.$$

Here,

$$\frac{\partial^{|j|}H^{(l)}}{\partial\eta^j}=\frac{\partial^{|j|}H^{(l)}(x,\eta)}{\partial\eta_1^{j_1}\partial\eta_2^{j_2}\cdots\partial\eta_n^{j_n}}, \quad j!=\prod_{\nu=1}^{n}(j_\nu!),$$

and

$$\left(\frac{\partial W^{(s)}}{\partial x}\right)^j=\prod_{\nu=1}^{n}\left(\frac{\partial W^{(s)}}{\partial x_\nu}\right)^{j_\nu}.$$

Similarly, for $\Gamma^{(l)}(x+(\partial W^{(s)}/\partial\eta), \eta)$, the term of order $l-|j|+|j|(s-1)$ is

$$\frac{1}{j!}\frac{\partial^{|j|}\Gamma^{(l)}}{\partial x^j}\left(\frac{\partial W^{(s)}}{\partial\eta}\right)^j.$$

To find all terms of order i in Eq. (2.6), evaluate $l-|j|+|j|(s-1)$ for all $l\geq 2$ and $0\leq|j|\leq l$ and pick out those terms whose degree is equal to i. Doing this results in the equation

$$\Gamma^{(i)}=H^{(i)}+\sum_{\substack{l-|j|+|j|(s-1)=i\\ 1\leq|j|\leq l<i\\ l\geq 2 \quad s\geq 3}}\frac{1}{j!}\left[\frac{\partial^{|j|}H^{(l)}}{\partial\eta^j}\left(\frac{\partial W^{(s)}}{\partial x}\right)^j-\frac{\partial^{|j|}\Gamma^{(l)}}{\partial x^j}\left(\frac{\partial W^{(s)}}{\partial\eta}\right)^j\right], \quad i=2, 3, \cdots. \tag{2.10}$$

Note that for $i=s$ Eq. (2.10) reduces to Eq. (2.7), and that for $i<s$ Eq. (2.10) becomes $\Gamma^{(i)}=H^{(i)}$. Thus, starting with $i=s+1$ we may solve successively for the $\Gamma^{(i)}$ since the only unknown in Eq. (2.10) is $\Gamma^{(i)}$.

It is important to note that the solution to Eq. (2.7) is not unique unless $\mathfrak{N}=0$. We make a unique choice of solution by imposing the condition that $W^{(s)}(x,\eta)\in\mathfrak{R}$. Other choices are clearly available. In fact, our construction procedure exhibits all the possible real transformations that put Γ into normal form. We mention also that even the normal form is not unique. If, however, one makes the additional assumption of nonresonance, then the normal form Γ is unique (see Moser 1956). For an example of a nonunique Γ, see Moser (1955).

We are now ready to prove the existence of formal integrals.

Theorem 2: Suppose that the n frequencies α_ν, $\nu=1, \cdots, n$, are related by r resonance conditions (2.3). Let $\mu=(\mu_1, \cdots, \mu_n)$ be any real vector such that $A\mu=0$. Then, the Hamiltonian system H possesses n-r independent formal integrals of the form

$$I(x,y)=\sum_{\nu=1}^{n}\frac{\mu_\nu}{2}(x_\nu^2+y_\nu^2)+\cdots.$$

In fact, for a system in normal form, Γ, $I^{(2)}(\xi,\eta)=\sum_{\nu=1}^{n}\frac{1}{2}\mu_\nu(\xi_\nu^2+\eta_\nu^2)$ is an integral for any such μ.

Proof: It suffices to prove the statement for a Hamiltonian $\Gamma(\xi,\eta)$ in normal form and to verify that $I^{(2)}(\xi,\eta)$ is an integral. Introduce the canonical transformation (2.8)

$$\xi=(1/\sqrt{2})(q+ip), \quad \eta=(i/\sqrt{2})(q-ip),$$

where ξ, η, q, and p are n vectors, and $i=(-1)^{\frac{1}{2}}$. The equations

$$\dot{\xi}=\Gamma_\eta, \quad \dot{\eta}=-\Gamma_\xi, \tag{2.11}$$

become

$$\dot{q}=\tilde{\Gamma}_p, \quad \dot{p}=-\tilde{\Gamma}_q, \tag{2.12}$$

where the Hamiltonian

$$\tilde{\Gamma}(p,q)=\sum_{\nu=2}^{\infty}\tilde{\Gamma}^{(\nu)}(p,q)\quad\text{and}\quad\tilde{\Gamma}^{(2)}(p,q)=\sum_{\nu=1}^{n}i\alpha_\nu p_\nu q_\nu.$$

Also, the operator D becomes $\tilde{D}$ which is given by (2.9).

We now investigate the form of $\tilde{\Gamma}$ and thus consider any term $\tau=\beta q^j p^k$ of $\tilde{\Gamma}$ where $\beta\neq 0$. Since (2.11) is in normal form, we conclude $\tilde{D}\tau=0$. But $\tilde{D}\tau=i(\alpha,\,j-k)\tau$ so that $\tilde{\Gamma}$ contains only those terms for which $(\alpha,\,j-k)=0$. Thus, $j-k$ must be a linear combination of the rows of A; i.e., $j-k=A^Tc$. Otherwise $(\alpha,\,j-k)=0$ would be another independent resonance condition, which is impossible by assumption (2.3). We conclude that $(j-k,\,\mu)=(A^Tc,\,\mu)=(c,\,A\mu)=0$ since $A\mu=0$. This is equivalent to $\tilde{E}\tau=0$, where

$$\tilde{E}=\sum_{\nu=1}^{n}i\mu_\nu\left(q_\nu\frac{\partial}{\partial q_\nu}-p_\nu\frac{\partial}{\partial p_\nu}\right).$$

Therefore, we have shown that if $\tilde{D}\tilde{\Gamma}=0$, then also $\tilde{E}\tilde{\Gamma}=0$. This shows that

$$\tilde{I}^{(2)}(p,q)=\sum_{\nu=1}^{n}i\mu_\nu p_\nu q_\nu \tag{2.13}$$

is an integral of (2.12) since

$$(d/dt)\tilde{I}^{(2)}(p(t),\,q(t))=[\tilde{I}^{(2)},\,\tilde{\Gamma}]=\tilde{E}\tilde{\Gamma}=0,$$

and thus $\tilde{I}^{(2)}(p(t),q(t))=\text{const}$. Finally, using the inverse of the transformation defined in (2.8), we find that (2.13) becomes $I^{(2)}(\xi,\eta)=\sum_{\nu=1}^{n}\frac{1}{2}\mu_\nu(\xi_\nu^2+\eta_\nu^2)$, which implies that $I^{(2)}(\xi,\eta)$ is an integral of (2.11).

Theorem 2 makes evident certain important facts. When $r=0$, the Hamiltonian Γ is a function of the $\omega_\nu=\frac{1}{2}(\xi_\nu^2+\eta_\nu^2)$, $\nu=1,\cdots,n$, only. Thus any real μ gives rise to an integral. Therefore, the ω_ν form a set of n independent integrals and clearly $\Gamma(\omega)$ is dependent upon them. However, when $r>0$, Γ will, in general, depend upon null space terms other than the ω_ν and thus will be independent of the integrals exhibited in theorem 2. This class of integrals constitutes a set of n-r independent integrals since there are n-r independent vectors which satisfy (2.3). Thus, in the resonant case there are $n-r+1$ independent integrals. Finally, we note that $\Gamma^{(2)}(\xi,\eta)$ is an integral since $A\alpha=0$. Thus, $\Gamma-\Gamma^{(2)}$ is an integral independent of Γ.

3. COMPUTER CONSTRUCTION OF INTEGRALS BY POLYNOMIAL MANIPULATION

The construction of the approximate integrals requires simple but extensive manipulation of polynomials in $2n$ variables. To do these calculations by hand is a horrendous task; consequently, we have sought the aid of a digital computer (IBM 7094). Computer programs have been developed which perform the necessary calculations. The aim of this section is to describe briefly these programs.

The purpose of the programs, of which there are three, is to generate an integral, and then plot its level lines. In the first program a normal form is determined by following the steps of the proof in theorem 1. The input for this program are the coefficients of $H^{(i)}(x,y)$, $i=2,\cdots,m$, where m represents the degree of approximation desired. The program determines for each integer s, $s=3,\cdots,m$, a generating function $W^{(s)}(x,\eta)$ and then a new Hamiltonian function, $\Gamma(\xi,\eta)=\sum_{\nu=2}^{m}\Gamma^{(\nu)}(\xi,\eta)$ in the following way: A subprogram solves Eq. (2.7) by the method outlined in theorem 1 and a second subprogram determines Γ by solving Eq. (2.10). When $W^{(s)}$ and $\Gamma^{(i)}$, $i=s,\cdots,m$, have been determined for a given s, the quantities H, x, y, and s are replaced by Γ, ξ, η and $s+1$, respectively, and the process is repeated.

The second program expresses the integrals, which appear as quadratic forms in the "normal" coordinates, in the original coordinates. This is achieved by successively inverting the canonical transformations generated by the $W^{(s)}$, $s=m,\,m-1,\cdots,3$. The net result is a composite transformation relating the "normal" coordinates to the original coordinates:

$$\xi=x+\phi(x,y),\quad \eta=y+\psi(x,y).$$

The third program plots the integrals. (We do this only for the case $n=2$ and with the restriction that $x_1=0$, the latter resulting in I and H being functions of only x_2, y_2, and y_1.) We first solve the equation

$$H(x_2,y_2,y_1)=h,$$

for y_1, where h is the energy constant. This value is then substituted into the expression for I. We then consider the equation

$$I(x_2,y_2,h)=I_0,$$

for fixed h. By varying I_0, this equation defines y_2 as a function of x_2 and thus describes a family of level curves. We plot these families for a range of values of h.

In these programs the bulk of the work consists in representing, adding, subtracting, multiplying, and differentiating polynomials in $2n$ variables. The program itself consists of nothing more than a systematic use of the above fundamental operations. We describe here only how each of these operations is carried out.

A polynomial Φ in $m=2n$ variables is expressed as a sum of homogeneous polynomials; i.e.,

$$\Phi(x_1,\cdots,x_m)=\sum_{\nu=0}^{p}\Phi^{(\nu)}(x_1,\cdots,x_m),$$

where $\Phi^{(\nu)}(x)=\sum_{|j|=\nu}\alpha_j x^j$. We again use multi-index notation. The polynomial Φ has $C_{m+p,p}$ terms and $\Phi^{(\nu)}$ has $C_{m+\nu-1,\nu}$ terms, where $C_{n,r}=n!/r!(n-r)!$ are simply the binomial coefficients.

In manipulating the homogeneous polynomial $\Phi^{(\nu)}(x)$ we deal only with its coefficients α_j which are stored sequentially in the computer in a lexographic order on

TABLE I. Exponent mapping.

1–(0000)	51–(1003)	101–(1202)	151–(0420)	201–(4002)	251–(1042)	301–(3103)	351–(0233)	401–(1340)	451–(3203)
2–(0001)	52–(1012)	102–(1211)	152–(0501)	202–(4011)	252–(1051)	302–(3112)	352–(0242)	402–(1403)	452–(3212)
3–(0010)	53–(1021)	103–(1220)	153–(0510)	203–(4020)	253–(1060)	303–(3121)	353–(0251)	403–(1412)	453–(3221)
4–(0100)	54–(1030)	104–(1301)	154–(0600)	204–(4101)	254–(1105)	304–(3130)	354–(0260)	404–(1421)	454–(3230)
5–(1000)	55–(1102)	105–(1310)	155–(1005)	205–(4110)	255–(1114)	305–(3202)	355–(0305)	405–(1430)	455–(3302)
6–(0002)	56–(1111)	106–(1400)	156–(1014)	206–(4200)	256–(1123)	306–(3211)	356–(0314)	406–(1502)	456–(3311)
7–(0011)	57–(1120)	107–(2003)	157–(1023)	207–(5001)	257–(1132)	307–(3220)	357–(0323)	407–(1511)	457–(3320)
8–(0020)	58–(1201)	108–(2012)	158–(1032)	208–(5010)	258–(1141)	308–(3301)	358–(0332)	408–(1520)	458–(3401)
9–(0101)	59–(1210)	109–(2021)	159–(1041)	209–(5100)	259–(1150)	309–(3310)	359–(0341)	409–(1601)	459(–3410)
10–(0110)	60–(1300)	110–(2030)	160–(1050)	210–(6000)	260–(1204)	310–(3400)	360–(0350)	410–(1610)	460–(3500)
11–(0200)	61–(2002)	111–(2102)	161–(1104)	211–(0007)	261–(1213)	311–(4003)	361–(0404)	411–(1700)	461–(4004)
12–(1001)	62–(2011)	112–(2111)	162–(1113)	212–(0016)	262–(1222)	312–(4012)	362–(0413)	412–(2006)	462–(4013)
13–(1010)	63–(2020)	113–(2120)	163–(1122)	213–(0025)	263–(1231)	313–(4021)	363–(0422)	413–(2015)	463–(4022)
14–(1100)	64–(2101)	114–(2201)	164–(1131)	214–(0034)	264–(1240)	314–(4030)	364–(0431)	414–(2024)	464–(4031)
15–(2000)	65–(2110)	115–(2210)	165–(1140)	215–(0043)	265–(1303)	315–(4102)	365–(0440)	415–(2033)	465–(4040)
16–(0003)	66–(2200)	116–(2300)	166–(1203)	216–(0052)	266–(1312)	316–(4111)	366–(0503)	416–(2042)	466–(4103)
17–(0012)	67–(3001)	117–(3002)	167–(1212)	217–(0061)	267–(1321)	317–(4120)	367–(0512)	417–(2051)	467–(4112)
18–(0021)	68–(3010)	118–(3011)	168–(1221)	218–(0070)	268–(1330)	318–(4201)	368–(0521)	418–(2060)	468–(4121)
19–(0030)	69–(3100)	119–(3020)	169–(1230)	219–(0106)	269–(1402)	319–(4210)	369–(0530)	419–(2105)	469–(4130)
20–(0102)	70–(4000)	120–(3101)	170–(1302)	220–(0115)	270–(1411)	320–(4300)	370–(0602)	420–(2114)	470–(4202)
21–(0111)	71–(0005)	121–(3110)	171–(1311)	221–(0124)	271–(1420)	321–(5002)	371–(0611)	421–(2123)	471–(4211)
22–(0120)	72–(0014)	122–(3200)	172–(1320)	222–(0133)	272–(1501)	322–(5011)	372–(0620)	422–(2132)	472–(4220)
23–(0201)	73–(0023)	123–(4001)	173–(1401)	223–(0142)	273–(1510)	323–(5020)	373–(0701)	423–(2141)	473–(4301)
24–(0210)	74–(0032)	124–(4010)	174–(1410)	224–(0151)	274–(1600)	324–(5101)	374–(0710)	424–(2150)	474–(4310)
25–(0300)	75–(0041)	125–(4100)	175–(1500)	225–(0160)	275–(2005)	325–(5110)	375–(0800)	425–(2204)	475–(4400)
26–(1002)	76–(0050)	126–(5000)	176–(2004)	226–(0205)	276–(2014)	326–(5200)	376–(1007)	426–(2213)	476–(5003)
27–(1011)	77–(0104)	127–(0006)	177–(2013)	227–(0214)	277–(2023)	327–(6001)	377–(1016)	427–(2222)	477–(5012)
28–(1020)	78–(0113)	128–(0015)	178–(2022)	228–(0223)	278–(2032)	328–(6010)	378–(1025)	428–(2231)	478–(5021)
29–(1101)	79–(0122)	129–(0024)	179–(2031)	229–(0232)	279–(2041)	329–(6100)	379–(1034)	429–(2240)	479–(5030)
30–(1110)	80–(0131)	130–(0033)	180–(2040)	230–(0241)	280–(2050)	330–(7000)	380–(1043)	430–(2303)	480–(5102)
31–(1200)	81–(0140)	131–(0042)	181–(2103)	231–(0250)	281–(2104)	331–(0008)	381–(1052)	431–(2312)	481–(5111)
32–(2001)	82–(0203)	132–(0051)	182–(2112)	232–(0304)	282–(2113)	332–(0017)	382–(1061)	432–(2321)	482–(5120)
33–(2010)	83–(0212)	133–(0060)	183–(2121)	233–(0313)	283–(2122)	333–(0026)	383–(1070)	433–(2330)	483–(5201)
34–(2100)	84–(0221)	134–(0105)	184–(2130)	234–(0322)	284–(2131)	334–(0035)	384–(1106)	434–(2402)	484–(5210)
35–(3000)	85–(0230)	135–(0114)	185–(2202)	235–(0331)	285–(2140)	335–(0044)	385–(1115)	435–(2411)	485–(5300)
36–(0004)	86–(0302)	136–(0123)	186–(2211)	236–(0340)	286–(2203)	336–(0053)	386–(1124)	436–(2420)	486–(6002)
37–(0013)	87–(0311)	137–(0132)	187–(2220)	237–(0403)	287–(2212)	337–(0062)	387–(1133)	437–(2501)	487–(6011)
38–(0022)	88–(0320)	138–(0141)	188–(2301)	238–(0412)	288–(2221)	338–(0071)	388–(1142)	438–(2510)	488–(6020)
39–(0031)	89–(0401)	139–(0150)	189–(2310)	239–(0421)	289–(2230)	339–(0080)	389–(1151)	439–(2600)	489–(6101)
40–(0040)	90–(0410)	140–(0204)	190–(2400)	240–(0430)	290–(2302)	340–(0107)	390–(1160)	440–(3005)	490–(6110)
41–(0103)	91–(0500)	141–(0213)	191–(3003)	241–(0502)	291–(2311)	341–(0116)	391–(1205)	441–(3014)	491–(6200)
42–(0112)	92–(1004)	142–(0222)	192–(3012)	242–(0511)	292–(2320)	342–(0125)	392–(1224)	442–(3023)	492–(7001)
43–(0121)	93–(1013)	143–(0231)	193–(3021)	243–(0520)	293–(2401)	343–(0134)	393–(1223)	443–(3032)	493–(7010)
44–(0130)	94–(1022)	144–(0240)	194–(3030)	244–(0601)	294–(2410)	344–(0143)	394–(1232)	444–(3041)	494–(7100)
45–(0202)	95–(1031)	145–(0303)	195–(3102)	245–(0510)	295–(2500)	345–(0152)	395–(1241)	445–(3050)	495–(8000)
46–(0211)	96–(1040)	146–(0312)	196–(3111)	246–(0700)	296–(3004)	346–(0161)	396–(1250)	446–(3104)	496–(0009)
47–(0220)	97–(1103)	147–(0321)	197–(3120)	247–(1006)	297–(3013)	347–(0170)	397–(1304)	447–(3113)	497–(0018)
48–(0301)	98–(1112)	148–(0330)	198–(3201)	248–(1015)	298–(3022)	348–(0206)	398–(1313)	448–(3122)	498–(0027)
49–(0310)	99–(1121)	149–(0402)	199–(3210)	249–(1024)	299–(3031)	349–(0215)	399–(1322)	449–(3131)	499–(0036)
50–(0400)	100–(1130)	150–(0411)	200–(3300)	250–(1033)	300–(3040)	350–(0224)	400–(1331)	450–(3140)	500–(0045)

the index vector j (see Table I). We define this order $\rhd$ by $j \rhd l$ if the first nonzero component of $j-l$ is positive. Under this ordering the coefficient α_j, $j=(0,\cdots 0,\nu)$ is placed in location one of the coefficient array while α_j, $j=(\nu,0,\cdots,0)$ is stored in location $k=C_{m+\nu-1,\nu}$. The ordering $\rhd$ defines a single-valued mapping, say K, of the vectors j onto the positive integers k; i.e., $k=K(j)$. What we have effected by this mapping is a machine representation of the coefficients of a polynomial in m variables.

In order to perform any of the basic polynomial operations, we must also know the inverse mapping $j=K^{-1}(k)$; i.e., given the positive integer k we must be able to find the unique vector j associated with k. Then, the main idea behind the performance of the basic operation is to use the inverse mapping to translate the machine representation of a polynomial into its standard representation, operate on it, and then use the mapping $k=K(j)$ to translate the result back to machine representation.

Adding and subtracting homogeneous polynomials together is trivial; one merely adds or subtracts corresponding terms in the machine representation of the coefficients.

To multiply

$$\Phi^{(\nu)}=\sum_{|j|=\nu}\alpha_j x^j \quad \text{by} \quad \Psi^{(\mu)}(x_1,\cdots,x_m)=\sum_{|l|=\mu}\beta_l x^l,$$

we perform a double sum over j and l: given $j=(j_1,\cdots,j_m)$ and $l=(l_1,\cdots,l_m)$ we form $j+l=(j_1+l_1,\cdots,j_m+l_m)$ and then find the unique integer k associated with it. The partial product $\alpha_j\beta_l$ is then added into location k of the product. Finally, to compute the partial derivative of $\Phi^{(\nu)}$ with respect to x_σ,

TABLE II. Normal form.

The odd degree terms of the normal form are identically zero

This part of the normal form has degree 2

N(6) = 0.500000	N(7) = −0.	N(8) = 0.500000	N(9) = −0.
N(10) = −0.	N(11) = 0.500000	N(12) = −0.	N(13) = −0.
N(14) = −0.	N(15) = 0.500000		

This part of the normal form has degree 4

N(36) = −0.104167	N(37) = 0.	N(38) = −0.208333	N(39) = 0.
N(40) = −0.104167	N(41) = 0.	N(42) = 0.	N(43) = 0.
N(44) = 0.	N(45) = −0.208333	N(46) = 0.	N(47) = 0.375000
N(48) = 0.	N(49) = 0.	N(50) = −0.104167	N(51) = 0.
N(52) = 0.	N(53) = 0.	N(54) = 0.	N(55) = 0.
N(56) = −1.166667	N(57) = 0.	N(58) = 0.	N(59) = 0.
N(60) = 0.	N(61) = 0.375000	N(62) = 0.	N(63) = −0.208333
N(64) = 0.	N(65) = 0.	N(66) = −0.208333	N(67) = 0.
N(68) = 0.	N(69) = 0.	N(70) = −0.104167	

This part of the normal form has degree 6

N(127) = −0.067998	N(128) = 0.	N(129) = 0.671007	N(130) = 0.
N(131) = −0.787326	N(132) = 0.	N(133) = 0.029225	N(134) = 0.
N(135) = 0.	N(136) = 0.	N(137) = 0.	N(138) = 0.
N(139) = 0.	N(140) = −0.203993	N(141) = 0.	N(142) = 0.734375
N(143) = 0.	N(144) = −0.228299	N(145) = 0.	N(146) = 0.
N(147) = 0.	N(148) = 0.	N(149) = −0.203993	N(150) = 0.
N(151) = 0.063368	N(152) = 0.	N(153) = 0.	N(154) = −0.067998
N(155) = 0.	N(156) = 0.	N(157) = 0.	N(158) = 0.
N(159) = 0.	N(160) = 0.	N(161) = 0.	N(162) = 1.215278
N(163) = 0.	N(164) = −1.118055	N(165) = 0.	N(166) = 0.
N(167) = 0.	N(168) = 0.	N(169) = 0.	N(170) = 0.
N(171) = 1.215278	N(172) = 0.	N(173) = 0.	N(174) = 0.
N(175) = 0.	N(176) = 0.063368	N(177) = 0.	N(178) = −1.015625
N(179) = 0.	N(180) = 0.087674	N(181) = 0.	N(182) = 0.
N(183) = 0.	N(184) = 0.	N(185) = 0.734375	N(186) = 0.
N(187) = −1.015625	N(188) = 0.	N(189) = 0.	N(190) = 0.671007
N(191) = 0.	N(192) = 0.	N(193) = 0.	N(194) = 0.
N(195) = 0.	N(196) = −1.118055	N(197) = 0.	N(198) = 0.
N(199) = 0.	N(200) = 0.	N(201) = −0.228299	N(202) = 0.
N(203) = 0.087674	N(204) = 0.	N(205) = 0.	N(206) = −0.787326
N(207) = 0.	N(208) = 0.	N(209) = 0.	N(210) = 0.029225

This part of the normal form has degree 8

N(331) = −0.077532	N(332) = 0.	N(333) = 0.779975	N(334) = 0.
N(335) = −0.101825	N(336) = 0.	N(337) = −0.915742	N(338) = 0.
N(339) = 0.043590	N(340) = 0.	N(341) = 0.	N(342) = 0.
N(343) = 0.	N(344) = 0.	N(345) = 0.	N(346) = 0.
N(347) = 0.	N(348) = −0.310129	N(349) = 0.	N(350) = 0.875854
N(351) = 0.	N(352) = −1.678323	N(353) = 0.	N(354) = −0.926342
N(355) = 0.	N(356) = 0.	N(357) = 0.	N(358) = 0.
N(359) = 0.	N(360) = 0.	N(361) = −0.465193	N(362) = 0.
N(363) = −0.588219	N(364) = 0.	N(365) = 2.126274	N(366) = 0.
N(367) = 0.	N(368) = 0.	N(369) = 0.	N(370) = −0.310129
N(371) = 0.	N(372) = −0.684097	N(373) = 0.	N(374) = 0.
N(375) = −0.077532	N(376) = 0.	N(377) = 0.	N(378) = 0.
N(379) = 0.	N(380) = 0.	N(381) = 0.	N(382) = 0.
N(383) = 0.	N(384) = C.	N(385) = 2.928144	N(386) = 0.
N(387) = 2.949344	N(388) = 0.	N(389) = 0.021200	N(390) = 0.
N(391) = 0.	N(392) = 0.	N(393) = 0.	N(394) = 0.
N(395) = 0.	N(396) = 0.	N(397) = 0.	N(398) = 5.856289
N(399) = 0.	N(400) = −11.861740	N(401) = 0.	N(402) = 0.
N(403) = 0.	N(404) = 0.	N(405) = 0.	N(406) = 0.
N(407) = 2.928144	N(408) = 0.	N(409) = 0.	N(410) = 0.
N(411) = 0.	N(412) = −0.684097	N(413) = 0.	N(414) = −1.678323
N(415) = 0.	N(416) = −2.757827	N(417) = 0.	N(418) = 0.174362
N(419) = 0.	N(420) = 0.	N(421) = 0.	N(422) = 0.
N(423) = 0.	N(424) = 0.	N(425) = −0.588219	N(426) = 0.
N(427) = 18.859982	N(428) = 0.	N(429) = −2.768427	N(430) = 0.
N(431) = 0.	N(432) = 0.	N(433) = 0.	N(434) = 0.875854
N(435) = 0.	N(436) = −1.678322	N(437) = 0.	N(438) = 0.
N(439) = 0.779975	N(440) = 0.	N(441) = 0.	N(442) = 0.
N(443) = 0.	N(444) = 0.	N(445) = 0.	N(446) = 0.
N(447) = −11.861741	N(448) = 0.	N(449) = 0.042401	N(450) = 0.

TABLE II (*continued*)

$N(451)=$ 0.	$N(452)=$ 0.	$N(453)=$ 0.	$N(454)=$ 0.
$N(455)=$ 0.	$N(456)=$ 2.949344	$N(357)=$ 0.	$N(458)=$ 0.
$N(459)=$ 0.	$N(460)=$ 0.	$N(461)=$ 2.126274	$N(462)=$ 0.
$N(463)=$ −2.768427	$N(464)=$ 0.	$N(465)=$ 0.261543	$N(466)=$ 0.
$N(467)=$ 0.	$N(468)=$ 0.	$N(469)=$ 0.	$N(470)=$ −1.678323
$N(471)=$ 0.	$N(472)=$ −2.757827	$N(473)=$ 0.	$N(474)=$ 0.
$N(475)=$ −0.101825	$N(476)=$ 0.	$N(477)=$ 0.	$N(478)=$ 0.
$N(479)=$ 0.	$N(480)=$ 0.	$N(481)=$ 0.021200	$N(482)=$ 0.
$N(483)=$ 0.	$N(484)=$ 0.	$N(485)=$ 0.	$N(486)=$ −0.926342
$N(487)=$ 0.	$N(488)=$ 0.174362	$N(489)=$ 0.	$N(490)=$ 0.
$N(491)=$ −0.915742	$N(492)=$ 0.	$N(493)=$ 0.	$N(494)=$ 0.
$N(495)=$ 0.043590			

$1\leq\sigma\leq m$, we search out in increasing order over k, using $j=K^{-1}(k)$, those vectors j which have a nonzero component j_σ. For each of these vectors we form the product $j_\sigma\alpha_j$ and store it in the same sequence as it was found, which happens to be the correct machine representation. These products then constitute the coefficients of $\partial\Phi^{(\nu)}/\partial x_\sigma$.

In completion, we mention that all these operations carry over easily to nonhomogeneous polynomials since they are sums of homogeneous polynomials.

4. RESULTS AND CONCLUSIONS

The model problem of Sec. 1 describes low-order resonance. This feature causes the topological nature of the integral curves to be more complicated than the situation in which there are no low-order or nearly low-order resonances. By constructing a truncated integral we determine the topological nature of these curves. To indicate the degree of approximation, we also directly integrate the equations of motion and make appropriate comparisons.

The normal form, generating function, and a formal integral can be expressed as polynomials in the four variables x, y, $\dot{x}$, and $\dot{y}$. These three polynomials are found by using the procedures of Secs. 2 and 3. Tables II, III, and IV are computer listings of the coefficients, up to degree eight, of the normal form, generating function and formal integral, respectively. Table I gives the mapping function $K(j)$; it is included to identify the exponents of the variables of these three polynomials. For example, $\mathrm{I}(196)=-5.493055$ in Table IV represents the monomial $-5.943055x^3y\dot{x}\dot{y}$. The number 196 determines the exponent vector (3 1 1 1) which is found from Table I.

In Tables II and III it is important to keep in mind the coordinates in which the polynomials are expressed. In determining the new integral up to terms of degree

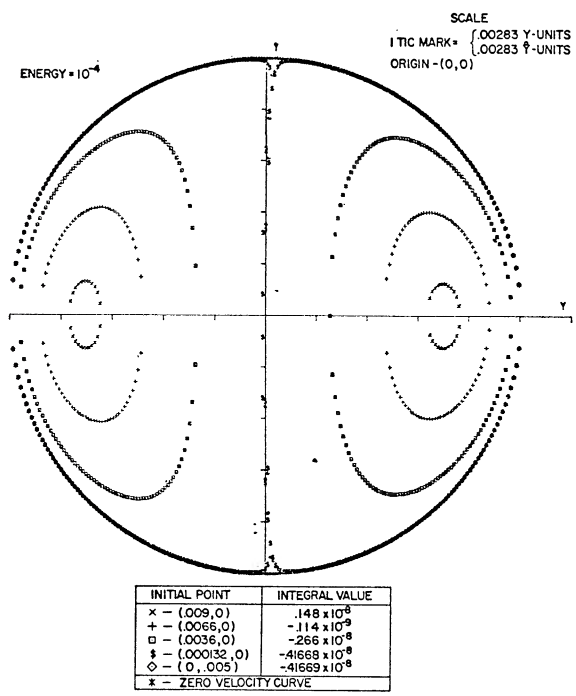


FIG. 7. Level lines of I for $E=0.0001$.

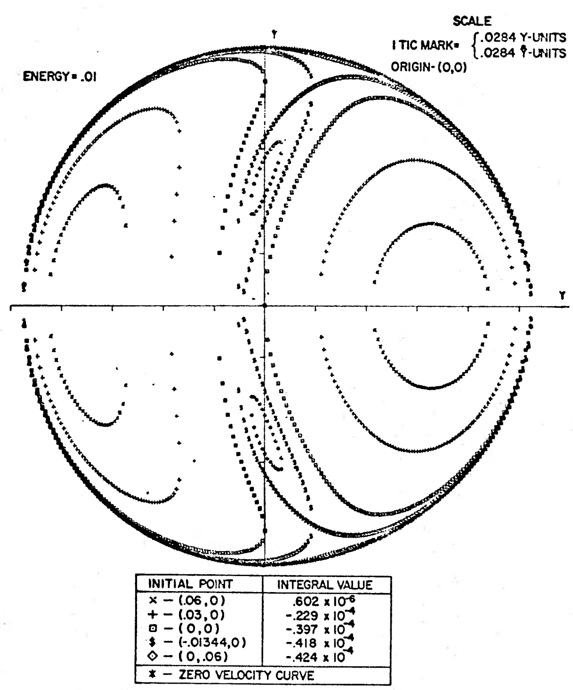


FIG. 8. Level lines of I for $E=0.01$.

TABLE III. Generating function.

This part of the generating function has degree 3

W(16) = −0.222222	W(17) = 0.	W(18) = 0.666667	W(19) = 0.
W(20) = 0.	W(21) = −0.	W(22) = 0.	W(23) = −0.333333
W(24) = 0.	W(25) = −0.	W(26) = −0.	W(27) = 0.
W(28) = −0.	W(29) = −0.	W(30) = 0.666667	W(31) = 0.
W(32) = 0.333333	W(33) = 0.	W(34) = −0.	W(35) = 0.

This part of the generating function has degree 4

W(36) = −0.	W(37) = −0.	W(38) = −0.	W(39) = −0.
W(40) = −0.	W(41) = 0.118056	W(42) = −0.	W(43) = 0.118056
W(44) = 0.	W(45) = −0.	W(46) = −0.	W(47) = 0.
W(48) = 0.048611	W(49) = −0.	W(50) = −0.	W(51) = 0.
W(52) = 0.118056	W(53) = 0.	W(54) = 0.118056	W(55) = 0.
W(56) = 0.	W(57) = −0.	W(58) = 0.	W(59) = 0.048611
W(60) = 0.	W(61) = −0.	W(62) = −0.	W(63) = 0.
W(64) = 0.048611	W(65) = −0.	W(66) = −0.	W(67) = 0.
W(68) = 0.048611	W(69) = 0.	W(70) = −0.	

This part of the generating function has degree 5

W(71) = −0.059259	W(72) = 0.	W(73) = 0.118519	W(74) = 0.
W(75) = 0.177778	W(76) = 0.	W(77) = 0.	W(78) = 0.
W(79) = 0.	W(80) = 0.	W(81) = −0.	W(82) = −0.148148
W(83) = 0.	W(84) = −1.096296	W(85) = 0.	W(86) = 0.
W(87) = −0.	W(88) = 0.	W(89) = −0.074074	W(90) = 0.
W(91) = −0.	W(92) = −0.	W(93) = −0.	W(94) = −0.
W(95) = −0.	W(96) = −0.	W(97) = −0.	W(98) = 0.770370
W(99) = 0.	W(100) = 0.770370	W(101) = −0.	W(102) = 0.
W(103) = −0.	W(104) = −0.	W(105) = −1.540741	W(106) = 0.
W(107) = 0.622222	W(108) = 0.	W(109) = −0.325926	W(110) = 0.
W(111) = 0.	W(112) = −0.	W(113) = 0.	W(114) = 1.688889
W(115) = 0.	W(116) = −0.	W(117) = −0.	W(118) = 0.
W(119) = −0.	W(120) = −0.	W(121) = 0.711111	W(122) = 0.
W(123) = −0.488889	W(124) = 0.	W(125) = −0.	W(126) = 0.

This part of the generating function has degree 6

W(127) = −0.	W(128) = −0.	W(129) = −0.	W(130) = −0.
W(131) = −0.	W(132) = −0.	W(133) = 0.	W(134) = 0.029972
W(135) = −0.	W(136) = −0.541908	W(137) = −0.	W(138) = 0.230589
W(139) = −0.	W(140) = −0.	W(141) = −0.	W(142) = −0.
W(143) = −0.	W(144) = −0.	W(145) = 0.040525	W(146) = −0.
W(147) = 0.264902	W(148) = 0.	W(149) = −0.	W(150) = −0.
W(151) = 0.	W(152) = 0.000651	W(153) = −0.	W(154) = −0.
W(155) = 0.	W(156) = −0.270954	W(157) = 0.	W(158) = 0.461179
W(159) = 0.	W(160) = −0.070337	W(161) = 0.	W(162) = 0.
W(163) = 0.	W(164) = 0.	W(165) = 0.	W(166) = 0.
W(167) = −1.926745	W(168) = 0.	W(169) = 0.771074	W(170) = 0.
W(171) = 0.	W(172) = −0.	W(173) = 0.	W(174) = −0.216941
W(175) = 0.	W(176) = −0.	W(177) = −0.	W(178) = −0.
W(179) = −0.	W(180) = −0.	W(181) = 0.264902	W(182) = −0.
W(183) = −0.408227	W(184) = 0.	W(185) = −0.	W(186) = −0.
W(187) = 0.	W(188) = −0.433883	W(189) = −0.	W(190) = −0.
W(191) = 0.	W(192) = 0.771074	W(193) = 0.	W(194) = −0.128199
W(195) = 0.	W(196) = 0.	W(197) = −0.	W(198) = 0.
W(199) = 0.291425	W(200) = 0.	W(201) = −0.	W(202) = −0.
W(203) = 0.	W(204) = 0.145713	W(205) = −0.	W(206) = −0.
W(207) = 0.	W(208) = −0.071880	W(209) = 0.	W(210) = −0.

This part of the generating function has degree 7

W(211) = −0.100823	W(212) = 0.	W(213) = 0.100832	W(214) = 0.
W(215) = 0.504115	W(216) = 0.	W(217) = 0.302469	W(218) = 0.
W(219) = 0.	W(220) = 0.	W(221) = 0.	W(222) = 0.
W(223) = 0.	W(224) = −0.	W(225) = 0.	W(226) = −0.319959
W(227) = 0.	W(228) = −1.628230	W(229) = 0.	W(230) = 3.718230
W(231) = 0.	W(232) = 0.	W(233) = 0.	W(234) = 0.
W(235) = 0.	W(236) = −0.	W(237) = −0.332819	W(238) = 0.
W(239) = −2.323930	W(240) = 0.	W(241) = 0.	W(242) = −0.
W(243) = 0.	W(244) = −0.111625	W(245) = 0.	W(246) = −0.
W(247) = −0.	W(248) = −0.	W(249) = −0.	W(250) = −0.
W(251) = −0.	W(252) = 0.	W(253) = −0.	W(254) = −0.
W(255) = 3.019012	W(256) = −0.	W(257) = −6.528229	W(258) = −0.
W(259) = 0.505761	W(260) = −0.	W(261) = −0.	W(262) = −0.
W(263) = −0.	W(264) = 0.	W(265) = −0.	W(266) = 2.728724

TABLE III (*continued*)

$W(267)=0.$	$W(268)=2.728724$	$W(269)=-0.$	$W(270)=0.$
$W(271)=-0.$	$W(272)=-0.$	$W(273)=-1.365391$	$W(274)=0.$
$W(275)=-1.070823$	$W(276)=0.$	$W(277)=4.409793$	$W(278)=0.$
$W(279)=0.454115$	$W(280)=0.$	$W(281)=0.$	$W(282)=0.$
$W(283)=0.$	$W(284)=0.$	$W(285)=-0.$	$W(286)=-0.071975$
$W(287)=0.$	$W(288)=-4.054197$	$W(289)=0.$	$W(290)=0.$
$W(291)=-0.$	$W(292)=0.$	$W(293)=1.477016$	$W(294)=0.$
$W(295)=-0.$	$W(296)=-0.$	$W(297)=-0.$	$W(298)=-0.$
$W(299)=-0.$	$W(300)=-0.$	$W(301)=-0.$	$W(302)=1.305349$
$W(303)=0.$	$W(304)=1.305350$	$W(305)=-0.$	$W(306)=0.$
$W(307)=-0.$	$W(308)=-0.$	$W(309)=-0.612593$	$W(310)=0.$
$W(311)=1.684218$	$W(312)=0.$	$W(313)=-0.306893$	$W(314)=0.$
$W(315)=0.$	$W(316)=-0.$	$W(317)=0.$	$W(318)=1.170720$
$W(319)=0.$	$W(320)=-0.$	$W(321)=-0.$	$W(322)=0.$
$W(323)=-0.$	$W(324)=-0.$	$W(325)=0.752798$	$W(326)=0.$
$W(327)=-0.417822$	$W(328)=0.$	$W(329)=-0.$	$W(330)=0.$

This part of the generating function has degree 8

$W(331)=-0.$	$W(332)=-0.$	$W(333)=-0.$	$W(334)=-0.$
$W(335)=-0.$	$W(336)=-0.$	$W(337)=-0.$	$W(338)=-0.$
$W(339)=-0.$	$W(340)=0.008409$	$W(341)=-0.$	$W(342)=-0.661534$
$W(343)=-0.$	$W(344)=2.293017$	$W(345)=-0.$	$W(346)=-0.671215$
$W(347)=0.$	$W(348)=-0.$	$W(349)=-0.$	$W(350)=-0.$
$W(351)=-0.$	$W(352)=-0.$	$W(353)=-0.$	$W(354)=0.$
$W(355)=0.007327$	$W(356)=-0.$	$W(357)=-2.209426$	$W(358)=-0.$
$W(359)=1.626700$	$W(360)=-0.$	$W(361)=-0.$	$W(362)=-0.$
$W(363)=-0.$	$W(364)=-0.$	$W(365)=-0.$	$W(366)=-0.003865$
$W(367)=-0.$	$W(368)=1.703762$	$W(369)=0.$	$W(370)=-0.$
$W(371)=-0.$	$W(372)=0.$	$W(373)=-0.016989$	$W(374)=-0.$
$W(375)=-0.$	$W(376)=0.$	$W(377)=0.687140$	$W(378)=0.$
$W(379)=-2.245241$	$W(380)=0.$	$W(381)=0.709310$	$W(382)=0.$
$W(383)=0.007517$	$W(384)=0.$	$W(385)=0.$	$W(386)=0.$
$W(387)=0.$	$W(388)=0.$	$W(389)=0.$	$W(390)=-0.$
$W(391)=0.$	$W(392)=-0.787568$	$W(393)=0.$	$W(394)=2.586817$
$W(395)=0.$	$W(396)=-0.469068$	$W(397)=0.$	$W(398)=0.$
$W(399)=0.$	$W(400)=0.$	$W(401)=0.$	$W(402)=0.$
$W(403)=-6.774576$	$W(404)=0.$	$W(405)=2.172781$	$W(406)=0.$
$W(407)=0.$	$W(408)=-0.$	$W(409)=0.$	$W(410)=0.647901$
$W(411)=0.$	$W(412)=-0.$	$W(413)=-0.$	$W(414)=-0.$
$W(415)=-0.$	$W(416)=-0.$	$W(417)=-0.$	$W(418)=0.$
$W(419)=0.665837$	$W(420)=-0.$	$W(421)=-3.526445$	$W(422)=-0.$
$W(423)=-0.348828$	$W(424)=-0.$	$W(425)=-0.$	$W(426)=-0.$
$W(427)=-0.$	$W(428)=-0.$	$W(429)=-0.$	$W(430)=0.725697$
$W(431)=-0.$	$W(432)=-2.689558$	$W(433)=0.$	$W(434)=-0.$
$W(435)=-0.$	$W(436)=0.$	$W(437)=-2.699803$	$W(438)=-0.$
$W(439)=-0.$	$W(440)=0.$	$W(441)=-1.429931$	$W(442)=0.$
$W(443)=2.158575$	$W(444)=0.$	$W(445)=-0.254947$	$W(446)=0.$
$W(447)=0.$	$W(448)=0.$	$W(449)=0.$	$W(450)=0.$
$W(451)=0.$	$W(452)=-2.978129$	$W(453)=0.$	$W(454)=2.986776$
$W(455)=0.$	$W(456)=0.$	$W(457)=-0.$	$W(458)=0.$
$W(459)=-4.251212$	$W(460)=0.$	$W(461)=-0.$	$W(462)=-0.$
$W(463)=-0.$	$W(464)=-0.$	$W(465)=-0.$	$W(466)=2.028495$
$W(467)=-0.$	$W(468)=-3.094387$	$W(469)=0.$	$W(470)=-0.$
$W(471)=-0.$	$W(472)=0.$	$W(473)=3.487962$	$W(474)=-0.$
$W(475)=-0.$	$W(476)=0.$	$W(477)=2.497514$	$W(478)=0.$
$W(479)=-0.484938$	$W(480)=0.$	$W(481)=0.$	$W(482)=-0.$
$W(483)=0.$	$W(484)=1.936553$	$W(485)=0.$	$W(486)=-0.$
$W(487)=-0.$	$W(488)=0.$	$W(489)=-0.902317$	$W(490)=-0.$
$W(491)=-0.$	$W(492)=0.$	$W(493)=-0.237428$	$W(494)=0.$
$W(495)=-0.$			

eight, six coordinate transformations of the form

$$x^{(i+1)}=x^{(i)}+\frac{\partial W^{(i+3)}}{\partial y^{(i+1)}}(x^{(i)},y^{(i+1)}),$$

$$y^{(i)}=y^{(i+1)}+\frac{\partial W^{(i+3)}}{\partial x^{(i)}}(x^{(i)},y^{(i+1)}), \qquad i=0,\,1,\,\cdots,\,5.$$

are performed. In this notation, the original coordinates x, y, $\dot{x}$, $\dot{y}$ are given by $x^{(0)}=(x,y)$, $y^{(0)}=(\dot{x},\dot{y})$ and the final coordinates ξ, η, $\dot{\xi}$, $\dot{\eta}$ by $x^{(6)}=(\xi,\eta)$, $y^{(6)}=(\dot{\xi},\dot{\eta})$. The coordinates of Table II are the final ones, while in Table III they vary; i.e., $x^{(i)}$, $y^{(i+1)}$ are the coordinates of $W^{(i+3)}$, $i=0$, 1, $\cdots$, 5.

In Sec. 3 we described how the integral I can be used to find level curves. These level curves should approximate the "curves" obtained by numerical integration. We choose a representative set of six energy values 10^{-4}, 0.01, 1/24, 1/12, 1/8, and 1/6. For each of these values a family of level curves (Figs. 1, 7–11) is plotted along

TABLE IV. Integral.

This part of the integral has degree 4

$I(36)=-0.104167$	$I(37)=-0.$	$I(38)=-0.208333$	$I(39)=-0.$
$I(40)=-0.104167$	$I(41)=-0.$	$I(42)=-0.$	$I(43)=-0.$
$I(44)=-0.$	$I(45)=-0.208333$	$I(46)=-0.$	$I(47)=0.375000$
$I(48)=-0.$	$I(49)=-0.$	$I(50)=-0.104167$	$I(51)=-0.$
$I(52)=-0.$	$I(53)=-0.$	$I(54)=-0.$	$I(55)=-0.$
$I(56)=-1.166667$	$I(57)=-0.$	$I(58)=-0.$	$I(59)=-0.$
$I(60)=-0.$	$I(61)=0.375000$	$I(62)=-0.$	$I(63)=-0.208333$
$I(64)=-0.$	$I(65)=-0.$	$I(66)=-0.208333$	$I(67)=-0.$
$I(68)=-0.$	$I(69)=-0.$	$I(70)=-0.104167$	

This part of the integral has degree 5

$I(71)=-0.$	$I(72)=-0.$	$I(73)=-0.$	$I(74)=-0.$
$I(75)=-0.$	$I(76)=-0.$	$I(77)=-0.$	$I(78)=-0.$
$I(79)=-2.333333$	$I(80)=-0.$	$I(81)=0.777778$	$I(82)=-0.$
$I(83)=-0.$	$I(84)=-0.$	$I(85)=-0.$	$I(86)=0.138889$
$I(87)=-0.$	$I(88)=-1.027778$	$I(89)=-0.$	$I(90)=-0.$
$I(91)=0.138889$	$I(92)=-0.$	$I(93)=2.333333$	$I(94)=-0.$
$I(95)=-0.777778$	$I(96)=-0.$	$I(97)=-0.$	$I(98)=-0.$
$I(99)=-0.$	$I(100)=-0.$	$I(101)=-0.$	$I(102)=-1.166667$
$I(103)=-0.$	$I(104)=-0.$	$I(105)=-0.$	$I(106)=-0.$
$I(107)=-0.$	$I(108)=-0.$	$I(109)=-0.$	$I(110)=-0.$
$I(111)=1.916667$	$I(112)=-0.$	$I(113)=0.750000$	$I(114)=-0.$
$I(115)=-0.$	$I(116)=-0.277778$	$I(117)=-0.$	$I(118)=-1.166667$
$I(119)=-0.$	$I(120)=-0.$	$I(121)=-0.$	$I(122)=-0.$
$I(123)=-0.$	$I(124)=-0.$	$I(125)=-0.416667$	$I(126)=-0.$

This part of the integral has degree 6

$I(127)=-0.111400$	$I(128)=-0.$	$I(129)=2.874132$	$I(130)=-0.$
$I(131)=-2.473090$	$I(132)=-0.$	$I(133)=0.245081$	$I(134)=-0.$
$I(135)=-0.$	$I(136)=-0.$	$I(137)=-0.$	$I(138)=-0.$
$I(139)=-0.$	$I(140)=-0.334201$	$I(141)=-0.$	$I(142)=1.494792$
$I(143)=-0.$	$I(144)=-2.448785$	$I(145)=-0.$	$I(146)=-0.$
$I(147)=-0.$	$I(148)=-0.$	$I(149)=-0.334201$	$I(150)=-0.$
$I(151)=1.537326$	$I(152)=-0.$	$I(153)=-0.$	$I(154)=-0.157697$
$I(155)=-0.$	$I(156)=-0.$	$I(157)=-0.$	$I(158)=-0.$
$I(159)=-0.$	$I(160)=-0.$	$I(161)=-0.$	$I(162)=8.506944$
$I(163)=-0.$	$I(164)=-0.048611$	$I(165)=-0.$	$I(166)=-0.$
$I(167)=-0.$	$I(168)=-0.$	$I(169)=-0.$	$I(170)=-0.$
$I(171)=1.506944$	$I(172)=-0.$	$I(173)=-0.$	$I(174)=-0.$
$I(175)=-0.$	$I(176)=-1.379340$	$I(177)=-0.$	$I(178)=-4.921875$
$I(179)=-0.$	$I(180)=0.735243$	$I(181)=-0.$	$I(182)=-0.$
$I(183)=-0.$	$I(184)=-0.$	$I(185)=-3.828125$	$I(186)=-0.$
$I(187)=-1.421875$	$I(188)=-0.$	$I(189)=-0.$	$I(190)=0.818576$
$I(191)=-0.$	$I(192)=-0.$	$I(193)=-0.$	$I(194)=-0.$
$I(195)=-0.$	$I(196)=-5.493055$	$I(197)=-0.$	$I(198)=-0.$
$I(199)=-0.$	$I(200)=-0.$	$I(201)=0.662326$	$I(202)=-0.$
$I(203)=0.540799$	$I(204)=-0.$	$I(205)=-0.$	$I(206)=-1.334201$
$I(207)=-0.$	$I(208)=-0.$	$I(209)=-0.$	$I(210)=-0.014178$

This part of the integral has degree 7

$I(211)=-0.$	$I(212)=-0.$	$I(213)=-0.$	$I(214)=-0.$
$I(215)=-0.$	$I(216)=-0.$	$I(217)=-0.$	$I(218)=-0.$
$I(219)=0.000000$	$I(220)=-0.$	$I(221)=2.780555$	$I(222)=-0.$
$I(223)=1.853704$	$I(224)=-0.$	$I(225)=-0.926852$	$I(226)=-0.$
$I(227)=-0.$	$I(228)=-0.$	$I(229)=-0.$	$I(230)=-0.$
$I(231)=-0.$	$I(232)=0.222801$	$I(233)=-0.$	$I(234)=-1.800231$
$I(235)=-0.$	$I(236)=2.825116$	$I(237)=-0.$	$I(238)=-0.$
$I(239)=-0.$	$I(240)=-0.$	$I(241)=0.445602$	$I(242)=-0.$
$I(243)=-0.500694$	$I(244)=-0.$	$I(245)=-0.$	$I(246)=0.222801$
$I(247)=-0.$	$I(248)=-2.780555$	$I(249)=-0.$	$I(250)=-1.853704$
$I(251)=-0.$	$I(252)=0.926852$	$I(253)=-0.$	$I(254)=-0.$
$I(255)=-0.$	$I(256)=-0.$	$I(257)=-0.$	$I(258)=-0.$
$I(259)=-0.$	$I(260)=-0.$	$I(261)=5.026388$	$I(262)=-0.$
$I(263)=0.178241$	$I(264)=-0.$	$I(265)=-0.$	$I(266)=-0.$
$I(267)=-0.$	$I(268)=-0.$	$I(269)=-0.$	$I(270)=-4.086574$
$I(271)=-0.$	$I(272)=-0.$	$I(273)=-0.$	$I(274)=-0.$
$I(275)=-0.$	$I(276)=-0.$	$I(277)=-0.$	$I(278)=-0.$
$I(279)=-0.$	$I(280)=-0.$	$I(281)=-3.448958$	$I(282)=-0.$
$I(283)=-9.143749$	$I(284)=-0.$	$I(285)=-0.846644$	$I(286)=-0.$
$I(287)=-0.$	$I(288)=-0.$	$I(289)=-0.$	$I(290)=4.141666$
$I(291)=-0.$	$I(292)=-4.031481$	$I(293)=-0.$	$I(294)=-0.$

TABLE IV (*continued*)

$I(295)=-0.222801$	$I(296)=-0.$	$I(297)=5.026389$	$I(298)=-0.$
$I(299)=0.178241$	$I(300)=-0.$	$I(301)=-0.$	$I(302)=-0.$
$I(303)=-0.$	$I(304)=-0.$	$I(305)=-0.$	$I(306)=2.294444$
$I(307)=-0.$	$I(308)=-0.$	$I(309)=-0.$	$I(310)=-0.$
$I(311)=-0.$	$I(312)=-0.$	$I(313)=-0.$	$I(314)=-0.$
$I(315)=-0.490972$	$I(316)=-0.$	$I(317)=0.656250$	$I(318)=-0.$
$I(319)=-0.$	$I(320)=-1.114005$	$I(321)=-0.$	$I(322)=-1.993055$
$I(323)=-0.$	$I(324)=-0.$	$I(325)=-0.$	$I(326)=-0.$
$I(327)=-0.$	$I(328)=-0.$	$I(329)=-0.668403$	$I(330)=-0.$

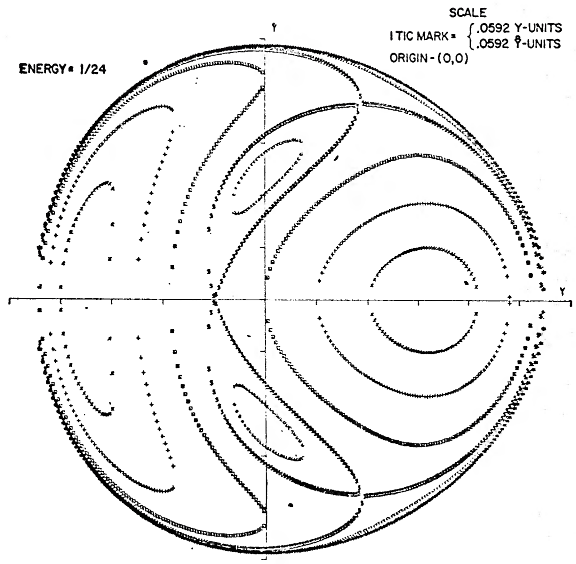
This part of the integral has degree 8

$I(331)=-0.170969$	$I(332)=-0.$	$I(333)=2.270465$	$I(334)=-0.$
$I(335)=-0.041033$	$I(336)=-0.$	$I(337)=-2.325175$	$I(338)=-0.$
$I(339)=0.157291$	$I(340)=-0.$	$I(341)=-0.$	$I(342)=-0.$
$I(343)=-0.$	$I(344)=-0.$	$I(345)=-0.$	$I(346)=-0.$
$I(347)=-0.$	$I(348)=-0.683875$	$I(349)=-0.$	$I(350)=2.708145$
$I(351)=-0.$	$I(352)=-4.349445$	$I(353)=-0.$	$I(354)=-2.489305$
$I(355)=-0.$	$I(356)=-0.$	$I(357)=-0.$	$I(358)=-0.$
$I(359)=-0.$	$I(360)=-0.$	$I(361)=-1.025813$	$I(362)=-0.$
$I(363)=-6.724503$	$I(364)=-0.$	$I(365)=6.318200$	$I(366)=-0.$
$I(367)=-0.$	$I(368)=-0.$	$I(369)=-0.$	$I(370)=-0.832409$
$I(371)=-0.$	$I(372)=-2.533865$	$I(373)=-0.$	$I(374)=-0.$
$I(375)=-0.319503$	$I(376)=-0.$	$I(377)=-0.$	$I(378)=-0.$
$I(379)=-0.$	$I(380)=-0.$	$I(381)=-0.$	$I(382)=-0.$
$I(383)=-0.$	$I(384)=-0.$	$I(385)=8.206500$	$I(386)=-0.$
$I(387)=8.534759$	$I(388)=-0.$	$I(389)=0.328260$	$I(390)=-0.$
$I(391)=-0.$	$I(392)=-0.$	$I(393)=-0.$	$I(394)=-0.$
$I(395)=-0.$	$I(396)=-0.$	$I(397)=-0.$	$I(398)=26.608369$
$I(399)=-0.$	$I(400)=-37.679093$	$I(401)=-0.$	$I(402)=-0.$
$I(403)=-0.$	$I(404)=-0.$	$I(405)=-0.$	$I(406)=-0.$
$I(407)=5.354647$	$I(408)=-0.$	$I(409)=-0.$	$I(410)=-0.$
$I(411)=-0.$	$I(412)=-1.832785$	$I(413)=-0.$	$I(414)=-4.349444$
$I(415)=-0.$	$I(416)=-7.139654$	$I(417)=-0.$	$I(418)=0.629165$
$I(419)=-0.$	$I(420)=-0.$	$I(421)=-0.$	$I(422)=-0.$
$I(423)=-0.$	$I(424)=-0.$	$I(425)=-4.175660$	$I(426)=-0.$
$I(427)=70.817255$	$I(428)=-0.$	$I(429)=-13.328321$	$I(430)=-0.$
$I(431)=-0.$	$I(432)=-0.$	$I(433)=-0.$	$I(434)=8.661386$
$I(435)=-0.$	$I(436)=5.008194$	$I(437)=-0.$	$I(438)=-0.$
$I(439)=1.185357$	$I(440)=-0.$	$I(441)=-0.$	$I(442)=-0.$
$I(443)=-0.$	$I(444)=-0.$	$I(445)=-0.$	$I(446)=-0.$
$I(447)=-44.476003$	$I(448)=-0.$	$I(449)=8.998186$	$I(450)=-0.$
$I(451)=-0.$	$I(452)=-0.$	$I(453)=-0.$	$I(454)=-0.$
$I(455)=-0.$	$I(456)=8.106981$	$I(457)=-0.$	$I(458)=-0.$
$I(459)=-0.$	$I(460)=-0.$	$I(461)=5.468585$	$I(462)=-0.$
$I(463)=-10.779478$	$I(464)=-0.$	$I(465)=1.175460$	$I(466)=-0.$
$I(467)=-0.$	$I(468)=-0.$	$I(469)=-0.$	$I(470)=-13.172360$
$I(471)=-0.$	$I(472)=-12.772061$	$I(473)=-0.$	$I(474)=-0.$
$I(475)=-1.095894$	$I(476)=-0.$	$I(477)=-0.$	$I(478)=-0.$
$I(479)=-0.$	$I(480)=-0.$	$I(481)=2.752334$	$I(482)=-0.$
$I(483)=-0.$	$I(484)=-0.$	$I(485)=-0.$	$I(486)=-1.538687$
$I(487)=-0.$	$I(488)=0.813347$	$I(489)=-0.$	$I(490)=-0.$
$I(491)=-2.646548$	$I(492)=-0.$	$I(493)=-0.$	$I(494)=-0.$
$I(495)=-0.045795$			

with the corresponding family obtained by forward integration (Figs. 2–6). These latter figures were obtained by using a Runge–Kutta single precision integration formula with a step size of 0.05. It can be seen that there is less agreement as the energy increases. Nevertheless, except in those cases when the trajectory does not lie on any "curve", there is always a qualitative agreement. In these cases (see Figs. 10 and 11) the integral I still gives rise to a curve. The integral I does not describe the chain of five small islands in Fig. 5 (⊡ curve), and it appears that neither would higher-order approximations. To describe this fine structure one would probably have to make an expansion in a neighborhood of the stable periodic orbit associated with this island chain. Note also that the integral I describes a companion curve to the island curve (◊ curve) in Figs. 1, 7–11. This is because a separation of the integral curves occurs on one side of the separatrix curve ($ curve), and so an orbit lies on either one of these separated curves but not on both.

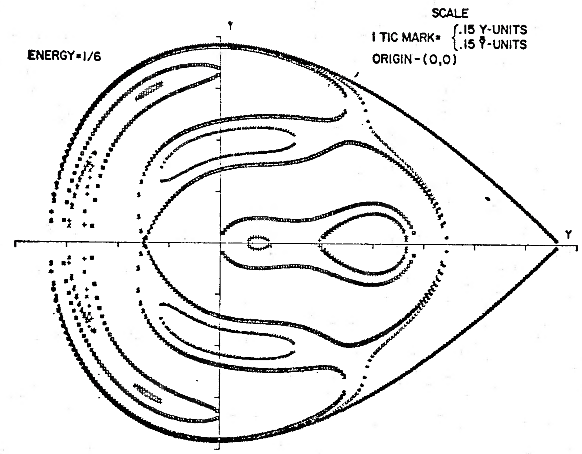
Figures 12 and 13 with $E=1/12$ show the level curves of the integral I when only terms to degree six and seven, respectively, are included. It can be seen that the approximation becomes better as the degree increases. However, it is an open question whether the integral I is a convergent or an asymptotic representation of the integral surfaces. It is our intention to calculate approximations higher then eighth degree to see if the convergent behavior of I continues.

It is also important to mention the amount of time



INITIAL POINT	INTEGRAL VALUE
× – (.12 ,0)	$.979 \times 10^{-4}$
+ – (.06,0)	$-.230 \times 10^{-3}$
⊡ – (0 ,0)	$-.574 \times 10^{-3}$
$ – (-.0588,0)	$-.736 \times 10^{-3}$
◇ – (0 ,.12)	$-.778 \times 10^{-3}$
✱ – ZERO VELOCITY CURVE	

FIG. 9. Level lines of I for $E=0.04167$.

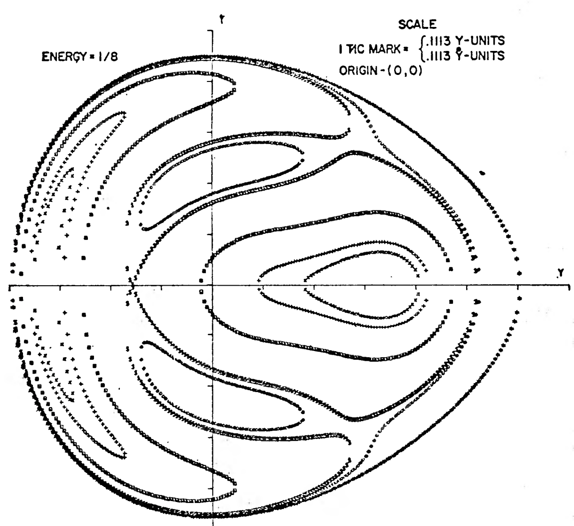


INITIAL POINT	INTEGRAL VALUE
× – (.3 ,0)	$.205 \times 10^{-2}$
+ – (.08,0)	$.188 \times 10^{-2}$
⊡ – (0,0)	$-.555 \times 10^{-3}$
$ – (-.222,0)	$-.149 \times 10^{-1}$
◇ – (0, .25)	$-.169 \times 10^{-1}$
✱ – ZERO VELOCITY CURVE	

FIG. 11. Level lines of I for $E=0.16667$.

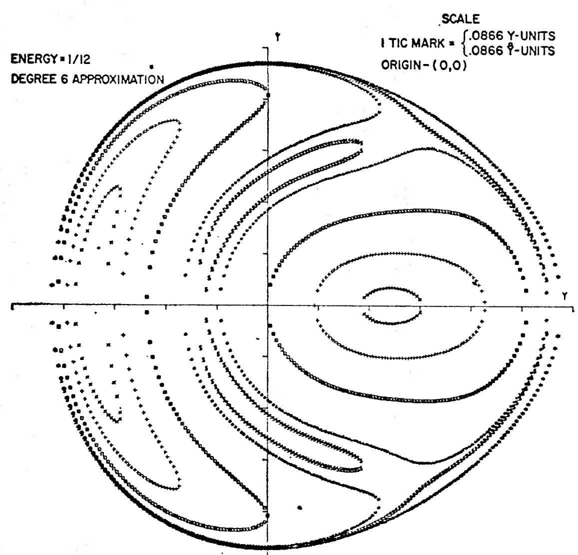
both problems consume in obtaining the data points for the level curves. In doing this, one must take into account the initial expense of calculating the integral and then preparing it for plotting. For the model problem, it took 3.52 min to compute the normal form and 5.00 min to perform the inverse transformations that express I in the original coordinates. To prepare the integral for plotting took 0.10 min. These times are excessive since, as yet, we have not attempted to optimize this program.

The data points in Figs. 1, 7–11 each take 0.04 min to calculate. In contrast, the data points in Figs. 2–6 take 7.59, 1.86, 1.72, 1.84, and 1.77 min, respectively. The explanation for the increased time in the case of Fig. 3 is that the twist angle α (see Henón, Heiles 1964,



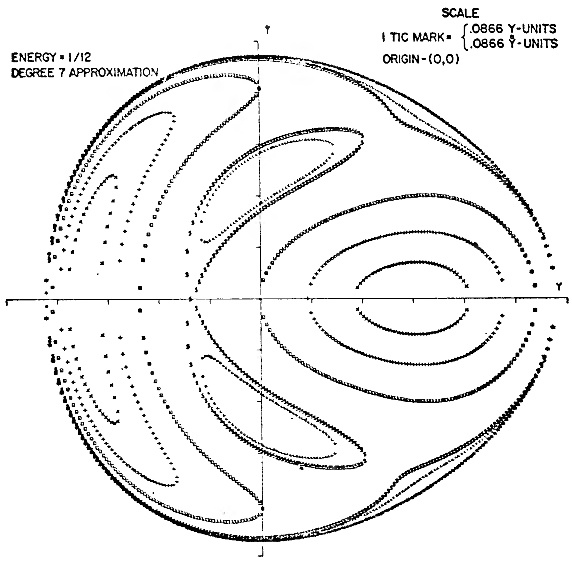
INITIAL POINT	INTEGRAL VALUE
× – (.2 ,0)	$.109 \times 10^{-2}$
+ – (.1 ,0)	$.240 \times 10^{-3}$
⊡ – (.02,.08)	$-.298 \times 10^{-2}$
$ (-.17568,0)	$.759 \times 10^{-2}$
◇ (0, .2)	$-.817 \times 10^{-2}$
✱ ZERO VELOCITY CURVE	

FIG. 10. Level lines of I for $E=0.125$.



INITIAL POINT	INTEGRAL VALUE
× – (.16,0)	$.715 \times 10^{-3}$
+ – (.08,0)	$-.164 \times 10^{-3}$
⊡ – (0,0)	$-.176 \times 10^{-2}$
$ – (-.12,0)	$-.336 \times 10^{-2}$
◇ – (0,.16)	$-.320 \times 10^{-2}$
✱ – ZERO VELOCITY CURVE	

FIG. 12. Level lines of I for $E=0.08333$ (truncated at degree 6).



INITIAL POINT	INTEGRAL VALUE
x − (.16,0)	.507 x 10^{-3}
+ − (.08,0)	−.434 x 10^{-3}
□ − (0,0)	−.176 x 10^{-2}
$ − (−.12,0)	−.309 x 10^{-2}
◇ − (0,.16)	−.320 x 10^{-2}
x − ZERO VELOCITY CURVE	

FIG. 13. Level lines of I for $E=0.08333$ (truncated at degree 7).

p. 75) approaches zero with the energy E. In fact, we did not plot the companion of Fig. 7 since it would have taken 17 min to calculate the x curve alone. To calculate the equivalent $ curve would have taken about 500 min. One then sees that the best approximated curves by the integral I are the most costly to calculate by numerical integration. Generally speaking, one can expect this type of behavior since the smaller the energy the smaller the degree of ergodicity (the latter effectively means that mapping points do not move rapidly). Even in the other cases (α relatively large), there is still a large saving in calculation time.

In this paper we have given a consistent and constructive proof of the existence of formal integrals of a Hamiltonian system of n degrees of freedom near an equilibrium point. We have emphasized that one should transform the Hamiltonian into a normal form instead of seeking the integrals directly as in the method of comparison of coefficients. By doing this, integrals appear as quadratic forms. Our main result is that even in the resonance case a Hamiltonian system can be brought into a normal form in a consistent manner. A program for carrying this out has been written for an IBM 7094 computer which makes feasible the calculation of these new integrals. It determines approximate integrals and expresses them in the original coordinates. We have tested the usefulness of these integrals by studying the problem of the existence of a third isolating integral in an axisymmetric potential. It is found that curves generated by the new integral compare very well with the curves determined from orbits calculated by forward integration.

Recently, Conley (1963) found some new long periodic solutions for the plane-restricted three-body problem. His work amounted to the study of a Hamiltonian system near an equlibrium point in the resonance case. As another application of our procedure we intend to determine the nature of these solutions by constructing a new integral for this system.

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APPENDIX

Consider the equation

$$DW^{(s)}(x,y)=-H^{(s)}(x,y)+\Gamma^{(s)}(x,y), \qquad \text{(A1)}$$

where $H^{(s)}$ is given, $W^{(s)}$ and $\Gamma^{(s)}$ are to be determined appropriately, and

$$D=\sum_{\nu=1}^{n}\alpha_\nu\left(y_\nu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial y_\nu}\right).$$

Our point of view is that D is a linear operator defined on a function space; viz., the space $\mathcal{V}^{(s,n)}$ of all homogeneous polynomials of degree s in $2n$ variables. [It is easily checked that the dimension of this space is $m=(2n+s-1)!/(2n-1)!s!$ and that the functions $x^k y^l$, $|k|+|l|=s$, form a basis.] Letting $\mathfrak{N}$ and $\mathfrak{R}$ denote the null space and range, respectively, of D on $\mathcal{V}^{(s,n)}$, we have shown in Sec. 2 that $\mathcal{V}^{(s,n)}=\mathfrak{N}\oplus\mathfrak{R}$.

The solution of (A1) may be viewed as follows: In matrix notation, Eq. (A1) becomes

$$Ay=b, \tag{A2}$$

where A is an m by m matrix and represents the operator D, y is an unknown m-column vector and represents the m unknown coefficients of $W^{(s)}$, and b is a given m vector representing the sum $-H^{(s)}+\Gamma^{(s)}$. It is well known (see Hildebrand 1956, p. 29) that Eq. (A2) possesses a solution if and only if b is orthogonal to all vector solutions of the transposed homogeneous set of equations $A^T x=0$. Thus, to be able at all to solve (A2), we must adjust b (specify $\Gamma^{(s)}$) so that it becomes a linear combination of the columns of A. This adjustment is equivalent to making the null space terms of $H^{(s)}$ and $\Gamma^{(s)}$ equal.