Exact solution for quantum self-phase modulation

K. J. Blow, R. Loudon,* and S. J. D. Phoenix

British Telecom Research Laboratories, Martlesham Heath, Ipswich, IP5 7RE, UK

Received September 24, 1990; revised manuscript received January 16, 1991

We discuss the quantum theory of self-phase modulation as applied to optical fibers. We use a formalism that does not rely on a cavity and thereby resolves some anomalous length dependences present in earlier studies. We show that the exact expectation values and variances can be evaluated without the need for linearizing about a classical pump wave. The standard quantum equations for self-phase modulation are generalized in order to remove singularities in some expectation values whose origin is the instantaneous response time approximation.

1. INTRODUCTION

Optical fibers have been successful media for the demonstration of nonlinear optical propagation effects such as self-phase modulation (SPM),[1] solitons,[2,3] all-optical switching,[4-8] and squeezed-light generation.[9,10] The generation of squeezed states in fibers uses cw light where four-wave mixing is the relevant interaction. However, cw light suffers from guided-acoustic-wave Brillouin scattering[11] that acts as a noise source internal to the fiber. In order to avoid guided-acoustic-wave Brillouin scattering, one should use pulses of light that are sufficiently short to dephase the Brillouin process. Pulsed squeezed-light generation has so far been limited to degenerate parametric downconversion,[12] but SPM can also induce squeezing.

The theoretical study of SPM has so far included the study of single-mode excitation,[13-16] where nonclassical effects have been predicted. However, such a theory is difficult to relate to any physical system, since the prediction contains anomalous dependences on the size of the cavity used to define the mode. Results for a cw excitation in a fiber were obtained in the limit of a strong classical pump wave, where squeezing of the vacuum sidebands was studied.[17] SPM has also been studied with stochastic methods,[18] and exact expressions for the field moments were obtained. More recent research[19] has been devoted to the equivalent problem of the propagation of a transverse beam profile in a Kerr medium. In that study a time-dependent Hartree approximation was used, and the difference in the spatial spectrum between coherent and thermal fields was elucidated. Interferometers containing Kerr media have also been studied[20] in the strong classical pump approximation, and the effects of the pump pulse shape were discussed. The problem of quantum soliton propagation in nonlinear dispersive media has also been treated[21-23]; soliton effects are not considered here, as we will neglect the effects of dispersion.

In this paper we will study the effects of SPM in optical fibers with specific regard to the following points. First, we will include a pulsed excitation from the start and not rely on extending the cw results by replacing the cw power P by the pulse $P(t)$. Second, we solve the problem exactly for a coherent-state input that is an arbitrary function of (continuous) time. Third, we show that the exact solution of the standard equations is singular and that this singularity can be removed by the inclusion of the medium response function. Kennedy and Drummond[18] removed the singularity by restricting the number of modes, and we will show that their method and ours are equivalent. These results enable us to recover the previous approximate treatments and to place them on a sound theoretical basis. Finally, we give, as an example, the squeezing obtained for a single pulse as observed in a balanced homodyne detector.

2. QUANTUM PROPAGATION IN A FIBER

The Hamiltonian describing quantum pulse propagation in an optical fiber is given by the quantized version of the standard classical Hamiltonian,[23] which we write in the form

$$\hat{H} = \frac{1}{2}\nu_G\kappa' \int \mathrm{d}t\, \hat{a}^\dagger(t)\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t), \tag{1}$$

where ν_G is the group velocity, κ' is the nonlinear coefficient, and $\hat{a}$ is the field annihilation operator[24] in the narrow-band limit that satisfies the commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t')] = \delta(t - t'). \tag{2}$$

The spatial Heisenberg equation of motion for the operators can now be derived as

$$i\frac{\partial \hat{a}}{\partial z} = \kappa\hat{a}^\dagger\hat{a}\hat{a}. \tag{3}$$

For reasons that will become clear below we will actually work with a generalized version of Eq. (3) that includes a time-dependent response function $g(\tau)$[25]:

$$i\frac{\partial \hat{a}}{\partial x}(t) = \kappa\int_0^\infty \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(t-\tau)\hat{a}(t-\tau)\hat{a}(t), \tag{4}$$

where t is the time in the frame moving with the group velocity of the medium and the normalized variables x and κ are given by

$$x = (n_0\omega_0/c)z, \tag{5}$$

0740-3224/91/081750-07$05.00

$$\kappa = n_2\hbar\omega_0/4\varepsilon_0 n_0^3 cA. \tag{6}$$

The refractive index is defined by $n = n_0 + (n_2/2n_0)|E|^2$, where n_0 is the linear refractive index of the fiber mode and $n_2 = 7.14 \times 10^{-22}$ (m/V)^2 is the Kerr coefficient. ω_0 is the central frequency, and A is the nonlinear effective area of the fiber mode, which we take to be 40 μm^2. These parameters are appropriate for silica-based fibers; at 1.05 μm we have $\kappa = 1.04 \times 10^{-28}$ s, and a distance of 1 km corresponds to $x = 8.7 \times 10^9$, where x is simply the system length in units of mode wave numbers. The response function $g(\tau)$ is subject to the condition

$$\int_0^\infty g(\tau)\mathrm{d}\tau = 1, \tag{7}$$

and in silica the response time τ_0 for the Kerr effect is ~1 fs; thus a simple model of $g(\tau)$ suitable for picosecond time scales is

$$g(\tau) = \begin{cases} 1/\tau_0 & \tau < \tau_0 \\ 0 & \tau > \tau_0 \end{cases}. \tag{8}$$

Equally, we could have used an exponential response function, a derivation of which, from the standard laser equations, can be found in Appendix A. We recover the instantaneous Kerr limit by setting $g(\tau) = \delta(\tau)$, although we could also use a more general function to include the effects of stimulated Raman scattering.[25,26] When the response is not instantaneous, the susceptibility has both a real and an imaginary part, and the system therefore contains some nonlinear absorption that leads to a lack of conservation of the field commutator (2), as discussed in Appendix B. Equation (4) is then no longer valid, and we would need to include noise operators to restore the conservation of the commutator. However, the nonlinear absorption is negligible under the physical conditions that we will use below, and we can neglect the extra noise operators. The exact solution of Eq. (4) is easily shown to be

$$\hat{a}(x,t) = \exp\left[-i\kappa x \int_0^\infty \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t-\tau)\hat{a}(0,t-\tau)\right]\hat{a}(0,t). \tag{9}$$

Note that we do not have the usual problem with time ordering, since Eq. (4) is a differential equation in x and the quantity $\hat{a}^\dagger(t)\hat{a}(t)$ is conserved. Having obtained the exact operator solution of the Heisenberg equation of motion, we can now proceed with calculating the relevant expectation values.

3. COHERENT-STATE EXPECTATION VALUES

We begin with the current generated in a balanced homodyne detector with unit quantum efficiency that is determined by

$$\hat{\imath} = e(P_{\mathrm{LO}}/\hbar\omega)^{1/2}\hat{X}, \tag{10a}$$

where we have assumed a cw coherent local oscillator with power P_{LO} and we define

$$\hat{X} = \hat{a}\exp(-i\phi) + \hat{a}^\dagger\exp(i\phi), \tag{10b}$$

with ϕ as the local oscillator phase [see Eqs. (6.25) and (6.26) of Ref. 24]. Note that we could include an arbitrary shape for the local oscillator, but this would only change the magnitude of the observed squeezing through an overlap integral, as we have previously shown.[24] When the input field is a time-dependent coherent state, which we denote as $|\{\alpha\}\rangle$, and is an eigenstate of the operator $\hat{a}(0,t)$, that is,

$$\hat{a}(0,t)|\{\alpha\}\rangle = \alpha(t)|\{\alpha\}\rangle, \tag{11}$$

the field corresponds to a classical wave whose shape is given by the complex function $\alpha(t)$ and has a photon flux given by $|\alpha(t)|^2$.[24] The expectation value of $\hat{X}$ now becomes

$$\begin{aligned}\langle\hat{X}(x,t)\rangle &= \langle\{\alpha\}|\hat{a}(x,t)\exp(-i\phi) + \hat{a}^\dagger(x,t)\exp(i\phi)|\{\alpha\}\rangle \\ &= \langle\{\alpha\}|\exp\left[-i\kappa x\int_0^\infty \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t-\tau)\right. \\ &\qquad \left.\times\, \hat{a}(0,t-\tau) - i\phi\right]\hat{a}(0,t)|\{\alpha\}\rangle + \text{c.c.},\end{aligned} \tag{12}$$

and we now use the fact that the state $|\{\alpha\}\rangle$ is an eigenstate of the time-dependent annihilation operator to obtain

$$\begin{aligned}\langle\hat{X}(x,t)\rangle = \langle\{\alpha\}|\exp&\left[-i\kappa x\int_0^\infty \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t-\tau)\right. \\ &\left.\times\, \hat{a}(0,t-\tau) - i\phi\right]|\{\alpha\}\rangle\alpha(t) + \text{c.c.}\end{aligned} \tag{13}$$

The expectation value of the exponentiated operator can be evaluated with the generalized normal-ordering theorem,[24] which gives

$$\begin{aligned}\langle\hat{X}\rangle &= \langle\{\alpha\}|:\exp\left(\int_0^\infty \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\}\hat{a}^\dagger(0,t-\tau)\right. \\ &\qquad \left.\times\, \hat{a}(0,t-\tau) - i\phi\right):|\{\alpha\}\rangle\alpha(t) + \text{c.c.} \\ &= \exp\left(\int_0^\infty \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\}|\alpha(0,t-\tau)|^2 - i\phi\right) \\ &\qquad \times\, \alpha(0,t) + \text{c.c.},\end{aligned} \tag{14}$$

where the colons denote normal ordering. This is the exact solution for the homodyne current for an arbitrary response function $g(\tau)$. We can now see why it was necessary to generalize the propagation equation (3) to include a physical response time. If we make the response function $g(\tau)$ a Dirac delta function, then the current contains the term $\exp[-i\kappa x\delta(\tau)]$, which is not an integrable singularity. We can see the origin of this singular contribution by evaluating its exponent for the simple step response of Eqs. (8), and we obtain for a fiber of length L,

$$\kappa x g(\tau) \equiv \phi_{\mathrm{vac}} = \frac{n_2\hbar\omega_0^2 L}{4\varepsilon_0 n_0^2 c^2 A\tau_0} \equiv \frac{\omega_0}{c}\frac{n_2}{2n_0}L|E_0|^2, \tag{15}$$

where $|E_0|^2 = \hbar\omega_0/2\varepsilon_0 n_0^2 V$ is the mean-square vacuum field confined to a volume V. In our case the volume is given by $cA\tau_0/n_0$, whereas the usual value implicit in the single-mode treatments is AL. We note that the traditional approach leads to an anomalous dependence of the results on the quantization length L. The phase shift ϕ_{vac}

is clearly the classical nonlinear phase shift of the rms vacuum field E_0 confined to a volume V propagating in a Kerr medium. Thus, when the response time goes to zero, the rms field fluctuations of the vacuum must diverge, and hence in a nonlinear system we obtain an infinite contribution to the phase shift. Equivalently, we can interpret this as due to the vacuum field fluctuation over a bandwidth $2\pi/\tau_0$ centered on ω_0 [Eqs. (2.8) and (2.10) in Ref. 24]. This interpretation was used by Kennedy and Drummond.[18] These singularities do not appear in approximate treatments such as linearization about a strong classical wave[17,20] or the use of a time-dependent Hartree–Fock wave function.[19]

We can now make the approximation that the response time is much smaller than the time scale of the optical pulse, obtaining

$$\langle \hat{X} \rangle = \exp[-i\phi + |\alpha(0,t)|^2 G(x)]\alpha(0,t) + \text{c.c.}, \tag{16}$$

where

$$G(x) = \int_0^\infty \mathrm{d}\tau \{\exp[-i\kappa x g(\tau)] - 1\}. \tag{17}$$

In silica-based optical fibers the physical parameters satisfy the criterion

$$\kappa x/\tau_0 \ll 1, \tag{18}$$

so that we can expand the exponential in Eq. (17), use the model response function of Eqs. (8), and retain the leading-order term, yielding

$$\langle \hat{X} \rangle = \exp[-i\phi - i\kappa x|\alpha(0,t)|^2]\alpha(0,t) + \text{c.c.}, \tag{19}$$

which gives the semiclassical result for the balanced homodyne current, as we would expect.

In order to calculate the squeezing effects of SPM, we must now calculate the second moment of $\hat{X}$:

$$\begin{aligned}\langle\{\alpha\}|\hat{X}(x,t)\hat{X}(x,t')|\{\alpha\}\rangle = \langle\{\alpha\}|\hat{a}(x,t)\hat{a}(x,t')\exp(-2i\phi) \\ + \hat{a}^\dagger(x,t)\hat{a}^\dagger(x,t')\exp(2i\phi) + \hat{a}^\dagger(x,t)\hat{a}(x,t') \\ + \hat{a}(x,t)\hat{a}^\dagger(x,t')|\{\alpha\}\rangle.\end{aligned} \tag{20}$$

We begin by calculating the first term, which, apart from the phase factor, is

$$\begin{aligned}V_1 &= \langle\{\alpha\}|\hat{a}(x,t)\hat{a}(x,t')|\{\alpha\}\rangle \\ &= \langle\{\alpha\}|\exp\left[-i\kappa x\int \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t-\tau)\hat{a}(0,t-\tau)\right]\hat{a}(0,t) \\ &\quad\times \exp\left[-i\kappa x\int \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t'-\tau)\hat{a}(0,t'-\tau)\right]\hat{a}(0,t')|\{\alpha\}\rangle \\ &= \langle\{\alpha\}|\exp\left[-i\kappa x\int \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t-\tau)\hat{a}(0,t-\tau)\right]\hat{a}(0,t) \\ &\quad\times \exp\left[-i\kappa x\int \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t'-\tau)\hat{a}(0,t'-\tau)\right]|\{\alpha\}\rangle\alpha(t').\end{aligned} \tag{21}$$

In order to proceed, we must commute the last two operators, and to do this we use

$$\hat{a}(t)f[\hat{a}^\dagger\hat{a}(t')] = f[\hat{a}^\dagger\hat{a}(t') + \delta(t-t')]\hat{a}(t) \tag{22}$$

to obtain

$$\begin{aligned}V_1 &= \alpha(t)\alpha(t')\langle\{\alpha\}| \\ &\quad\times \exp\left[-i\kappa x\int \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t-\tau)\hat{a}(0,t-\tau)\right] \\ &\quad\times \exp\left[-i\kappa x\int \mathrm{d}\tau\, g(\tau)\hat{a}^\dagger(0,t'-\tau)\hat{a}(0,t'-\tau) \right. \\ &\quad\left. + i\kappa x g(t'-t)\right]|\{\alpha\}\rangle.\end{aligned} \tag{23}$$

The exponential operators can now be replaced with their normally ordered versions, which gives

$$\begin{aligned}V_1 &= \alpha(t)\alpha(t')\exp[-i\kappa x g(t'-t)] \\ &\quad\times \langle\{\alpha\}|:\exp\left(\int \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\}\right. \\ &\quad\left.\times\, \hat{a}^\dagger(0,t-\tau)\hat{a}(0,t-\tau)\right): \\ &\quad\times \exp\left(\int \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\}\right. \\ &\quad\left.\times\, \hat{a}^\dagger(0,t'-\tau)\hat{a}(0,t'-\tau)\right):|\{\alpha\}\rangle \\ &= \alpha(t)\alpha(t')\exp[-i\kappa x g(t'-t)]\langle\{\alpha\}| \\ &\quad\times \exp\left(\int \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\}\alpha^*(0,t-\tau)\hat{a}(0,t-\tau)\right) \\ &\quad\times \exp\left(\int \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\}\right. \\ &\quad\left.\times\, \hat{a}^\dagger(0,t'-\tau)\alpha(0,t'-\tau)\right)|\{\alpha\}\rangle.\end{aligned} \tag{24}$$

We can now combine the operators in Eq. (23) into a single exponential using the standard theorem[27]

$$\exp(\hat{c})\exp(\hat{d}) = \exp\left(\hat{c} + \hat{d} + \frac{1}{2}[\hat{c},\hat{d}]\right), \tag{25}$$

which applies to $\hat{a}$ and $\hat{a}^\dagger$ since they commute with their commutator. The commutator of the two exponents is easily shown to be

$$\int \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\} \\ \times \{\exp[-i\kappa x g(\tau - t + t')] - 1\}|\alpha(0,t-\tau)|^2.$$

Again, using the fact that the response time is short compared with the pulse width, we can write the commutator in the simple form

$$|\alpha(t)|^2 H_1(x, t-t'),$$

where

$$H_1(x,s) = \int_0^\infty \mathrm{d}\tau\{\exp[-i\kappa x g(\tau)] - 1\} \\ \times \{\exp[-i\kappa x g(\tau - s)] - 1\}. \tag{26}$$

The matrix element (20) can now be evaluated, and it is

$$\begin{aligned}&\langle\{\alpha\}|\hat{a}(x,t)\hat{a}(x,t')|\{\alpha\}\rangle \\ &\quad= \alpha(t)\alpha(t')\exp\{-i\kappa x g(t'-t) + [|\alpha(t)|^2 + |\alpha(t')|^2]G(x) \\ &\quad\quad + |\alpha(t)|^2 H_1(x,t-t')\}.\end{aligned} \tag{27}$$

A similar calculation for the expectation value of the number operator in Eq. (20) gives

$$\langle\{\alpha\}|\hat{a}^{\dagger}(x,t)\hat{a}(x,t')|\{\alpha\}\rangle = \alpha^{*}(t)\alpha(t')\exp\{|\alpha(t)|^{2}G(x) + |\alpha(t')|^{2}G^{*}(x) + |\alpha(t)|^{2}H_{2}(x,t-t')\}, \quad (28)$$

where

$$H_{2}(x,s) = \int_{0}^{\infty} d\tau\{\exp[-i\kappa x g(\tau)] - 1\} \times \{\exp[i\kappa x g(\tau - s)] - 1\}. \quad (29)$$

The new effective response functions H_1 and H_2 can be evaluated analytically for the model response function in Eqs. (8), and they are

$$H_{1}(x,s) = [\exp(-i\kappa x/\tau_{0}) - 1]^{2}(\tau_{0} - |s|)\Theta(\tau_{0} - |s|), \quad (30)$$

$$H_{2}(x,s) = |\exp(i\kappa x/\tau_{0}) - 1|^{2}(\tau_{0} - |s|)\Theta(\tau_{0} - |s|), \quad (31)$$

where $\Theta(t)$ is the unit step function. We note that the response functions H_1 and H_2 are also singular for an instantaneous response time of the nonlinearity. These singularities are related to those occurring in quantum electrodynamics, since they arise from vacuum phase shifts. In our case we achieve the removal of the singularities by including the physical response of the medium.

The results of this section can also be obtained with the complete set of noncontinuous operators that we recently introduced,[24] and this calculation is outlined in Appendix C.

4. SQUEEZING SPECTRUM

We now have all the required matrix elements to enable us to calculate the squeezing spectrum. The noise spectrum is defined as

$$S(\omega,\tau) = \frac{1}{T}\int_{\tau-T/2}^{\tau+T/2} dt \int_{\tau-T/2}^{\tau+T/2} dt'\, V(t,t')\exp[i\omega(t-t')], \quad (32)$$

where

$$V(t,t') = \langle\{\alpha\}|\hat{X}(t)\hat{X}(t')|\{\alpha\}\rangle - \langle\{\alpha\}|\hat{X}(t)|\{\alpha\}\rangle\langle\{\alpha\}|\hat{X}(t')|\{\alpha\}\rangle \quad (33)$$

and the normalization has been chosen so that $S(\omega,\tau) = 1$ in the shot noise limit. τ is the time at which the spectrum is measured, and T models the integration time of the spectrum analyzer. Using Eqs. (16), (20), (27), (28), and (33), one can easily show the variance of $\hat{X}$ to be

$$V(t,t') = \alpha(t)\alpha(t')\exp(2i\phi)\exp\{G(x)[|\alpha(t)|^{2} + |\alpha(t')|^{2}]\} \times \{\exp[i\kappa x g(t'-t) + |\alpha(t')|^{2}H_{1}(x,t-t')] - 1\} + \alpha^{*}(t)\alpha(t')\exp[G(x)|\alpha(t)|^{2} + G^{*}(x)|\alpha(t')|^{2}] \times \{\exp[|\alpha(t')|^{2}H_{2}(x,t-t')] - 1\} + \text{c.c} + \delta(t-t'). \quad (34)$$

We can immediately see from expressions (30) and (31) for the functions H_1 and H_2, respectively, that the variance is nearly a delta function in the sense that $V(t,t')$ is nonzero only for $|t - t'| < \tau_0$. The implication for the squeezing spectrum (32) is that for frequencies that satisfy the inequality

$$\omega < 2\pi/\tau_{0} \quad (35)$$

the spectrum is independent of ω. The upper limit in relation (35) is of the order of 10^{15} Hz, which is well outside the bounds of the narrow-band assumption. Thus, for our purposes, the squeezing produced by SPM in fibers is broadband, and we need to calculate the spectrum only at $\omega = 0$. In practice, the upper limit to the squeezing bandwidth will be set by dispersion.

5. RESULTS AND DISCUSSION

In this section we compare the squeezing obtained for two pulse shapes: a square pulse, which corresponds to the cw results,[17] and a hyperbolic-secant-shaped pulse.[20] In most cases the measurement time τ corresponds to the peak of the pulse, and the integration time T equals the width of the square pulse, which is chosen to be 100 ps. The two pulses are chosen so that they have the same peak power and the same total photon number. The input peak power used in these calculations was 0.66 W so that the classical phase shift would be π at a distance of 1 km. We note that, for the parameters used here, neglecting the dispersion gives a good approximation.

In Fig. 1 we show the evolution of the normally ordered spectrum (i.e., $S - 1$) for the two pulses as they propagate along the fiber with a fixed value of the local oscillator phase. Squeezing is observed in both pulses at short distances. However, the square pulse [Fig. 1(a)] shows recurrence of the observed squeezing, with the next minimum at approximately a classical self-phase shift of π. The hyperbolic secant pulse [Fig. 1(b)] shows no such recurrence, and this is due to the effect of the pulse shape. The ordinate in Fig. 1 can also be interpreted as the pulse intensity at a fixed distance. The squeezing observed at a constant value of the local oscillator phase reappears when the noise ellipse has rotated through π. This is further illustrated in Fig. 2, where we show the squeezing spectrum as a function of the local oscillator phase when the peak classical phase shift is π. The square pulse [Fig. 2(a)] shows a high degree of squeezing, whereas the hyperbolic-secant-shaped pulse [Fig. 2(b)] shows excess noise at all values of the phase. The degree of squeezing is more easily seen in Figs. 2(c) and 2(d), where we plot $10\log_{10}(S)$ as a function of the local oscillator phase; the optimum squeezing ob-

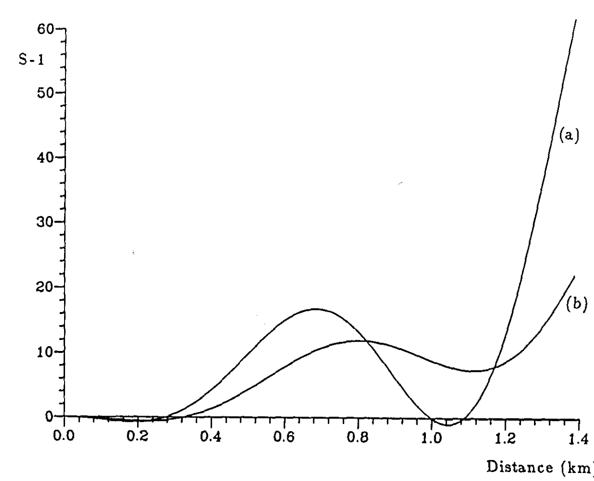

Fig. 1. Evolution of the normally ordered spectrum as a function of distance down the fiber for (a) a square pulse and (b) a hyperbolic secant pulse.

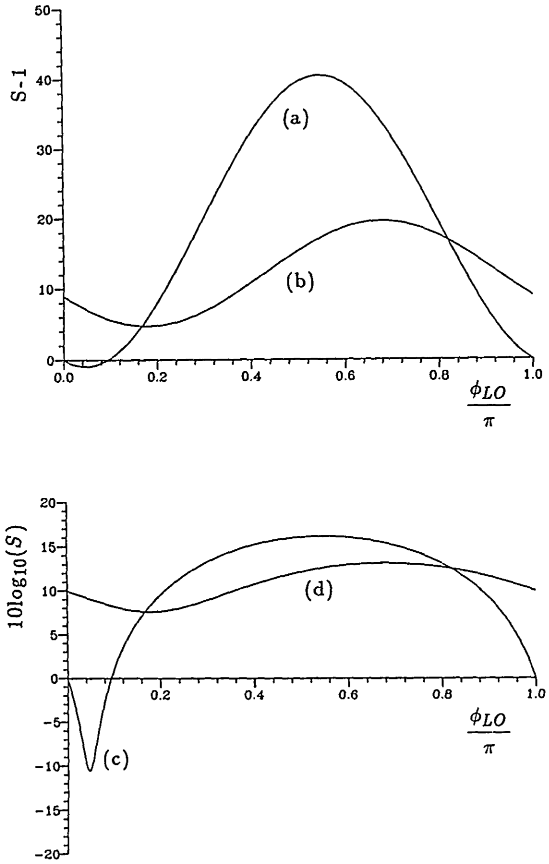

Fig. 2. Dependence of the normally ordered squeezing spectrum on the local oscillator phase for (a) a square pulse and (b) a hyperbolic secant pulse. Curves (c) and (d) show the same results as those for curves (a) and (b), but the total noise is plotted logarithmically.

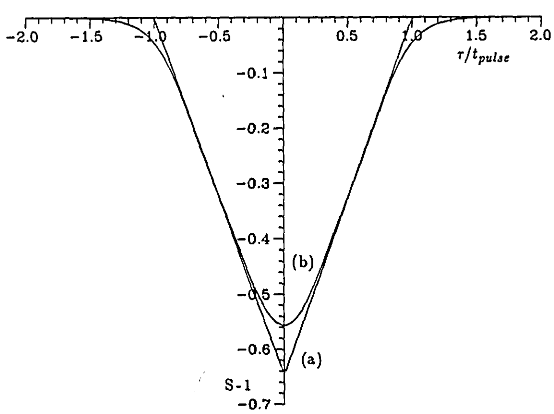

Fig. 3. Time dependence of the squeezing spectrum for a fixed local oscillator width and phase as a function of the time difference between the signal and the local oscillator.

tained for this length of fiber is ~11 dB. We can understand this difference in the squeezing spectra for the two pulse shapes by considering the time dependence of the squeezing. Since the squeezing is essentially instantaneous, the pulse acquires a noise ellipse whose angle varies along the pulse, and therefore, since the local oscillator has a constant phase, the spectrum contains some contributions with reduced noise and some with enhanced noise. The net result is that the oscillation in the evolution of the noise is washed out and the squeezing is destroyed at higher intensities. This pulse-shape mechanism is extremely well known in all-optical switching, where it leads to a degradation in the cw contrast ratio.[5] Shirasaki and Haus[20] showed that by adjusting the local oscillator phase, one may recover the observed squeezing. This method works by increasing the overlap integral between the local oscillator and the coherent pump pulse shapes.[24]

In Fig. 3 we show the normally ordered spectrum obtained with a square local oscillator pulse as a function of the time delay between the signal and the local oscillator. When the pulses are far apart, they do not overlap and we observe the shot noise limit. The square pulse [Fig. 3(a)] shows a triangular dependence on time and is simply due to the overlap integral between two squares. The hyperbolic secant pulse shows a lower degree of squeezing at the peak but is slightly broader than the square pulse and is a result of the pulse-shape effects, as we have just described.

6. CONCLUSIONS

We have calculated the exact quantum solution for an arbitrary time-dependent coherent-state pulse propagating in a lossless and dispersion-free optical fiber including an arbitrary response function for the medium. We used a simple model for the response function and introduced approximations appropriate for pulses that are much longer than the medium response time. We identified a singularity in the vacuum phase shift that was associated with the use of contact terms in the nonlinear response and showed how these could be removed by the inclusion of the physical response function of the silica fiber. The effects of pulse shape on the squeezing spectrum were calculated and were shown to be similar to those occurring in all-optical switching. Our results can be extended to include the effects of stimulated Raman scattering, and this will be covered in a future publication.

APPENDIX A: ATOMIC MODEL FOR THE NONLINEAR RESPONSE FUNCTION

Consider a collection of N two-level atoms with transition frequency ω_0 and longitudinal and transverse relaxation rates $\gamma_\parallel$ and $\gamma_\perp$, respectively. If we ignore propagation effects for the moment, the equations of motion of the coupled atom–field system are

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -i\omega\alpha + Ngj^-, \tag{A1}$$

$$\frac{\mathrm{d}j^-}{\mathrm{d}t} = -(\gamma_\perp + i\omega_0)j^- - g\alpha D, \tag{A2}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t} = -\gamma_\parallel(D - 1) + 2g(\alpha^* j^- + \alpha j^+), \tag{A3}$$

where α is the complex amplitude of the field, j^- is the atomic transition dipole moment, D is the difference in occupation probability between the atomic ground and excited states, and g is the atom–field coupling.

We look for a solution of the equations of motion in which the atomic transition is driven at the frequency ω, and Eq. (A2) accordingly gives

$$j^- = -\frac{g\alpha D}{\gamma_\perp + i(\omega_0 - \omega)}. \tag{A4}$$

Substitution into Eq. (A3) gives

$$\frac{dD}{dt} = -\gamma_\parallel(D - 1) - \frac{4g^2\gamma_\perp|\alpha|^2 D}{(\omega_0 - \omega)^2 + {\gamma_\perp}^2}. \tag{A5}$$

This equation may be solved in the form of a power series in the field amplitude, and for the treatment of SPM it suffices to work to order $|\alpha|^2$ when D can be replaced by unity in the final term on the right-hand side of Eq. (A5). The solution is then

$$D(t) = 1 - \frac{4g^2\gamma_\perp}{(\omega_0 - \omega)^2 + {\gamma_\perp}^2}\int_{-\infty}^{t} d\tau \exp[-\gamma_\parallel(t - \tau)]|\alpha(\tau)|^2. \tag{A6}$$

We define a response function

$$g(\tau) = \gamma_\parallel \exp(-\gamma_\parallel \tau), \tag{A7}$$

and with an appropriate change of variable, Eq. (A6) can be written as

$$D(t) = 1 - \frac{4g^2\gamma_\perp}{\gamma_\parallel[(\omega_0 - \omega)^2 + {\gamma_\perp}^2]}\int_0^\infty d\tau\, g(\tau)|\alpha(t - \tau)|^2. \tag{A8}$$

The field equation (A1) is now converted with the use of Eqs. (A4) and (A8) to

$$\frac{d\alpha(t)}{dt} = i\alpha(t)\frac{Ng^2(\omega_0 - \omega + i\gamma_\perp)}{(\omega_0 - \omega)^2 + {\gamma_\perp}^2}D(t), \tag{A9}$$

where a factor $\exp(-i\omega t)$ has been amalgamated into the field amplitude. In the notation used here, the standard expression for the linear susceptibility of the atomic system is

$$\chi^{(1)}(\omega) = \frac{2Ng^2}{\omega}\frac{\omega_0 - \omega + i\gamma_\perp}{(\omega_0 - \omega)^2 + {\gamma_\perp}^2}, \tag{A10}$$

and the component of the third-order nonlinear susceptibility appropriate to SPM in a material of cross-sectional area A is

$$\chi^{(3)}(\omega) = -\frac{8\varepsilon_0 cNAg^4}{3\hbar\omega^2}\frac{\omega_0 - \omega + i\gamma_\perp}{[(\omega_0 - \omega)^2 + {\gamma_\perp}^2]^2}\frac{\gamma_\perp}{\gamma_\parallel}. \tag{A11}$$

The field equation (A9) thus becomes

$$\frac{d\alpha(t)}{dt} = \frac{i\omega\alpha(t)}{2}\left[\chi^{(1)}(\omega) + \frac{3\hbar\omega}{\varepsilon_0 cA}\chi^{(3)}(\omega)\int_0^\infty d\tau\, g(\tau)|\alpha(t - \tau)|^2\right]. \tag{A12}$$

The above derivations ignore the field propagation, which can now be included by a suitable modification of Eq. (A12). A standard procedure then leads to the propagation equation

$$\frac{d\alpha(z, t)}{dz} = -i\kappa'\int_0^\infty d\tau\, g(\tau)|\alpha(z, t - \tau)|^2\alpha(z, t), \tag{A13}$$

where

$$\kappa' = 3\hbar\omega^2\chi^{(3)}(\omega)/2\varepsilon_0 c^2 A. \tag{A14}$$

Thus, with the change of variable defined in Eq. (5) and the introduction of the Kerr coefficient $n_2 = 6\chi^{(3)}(\omega)$, the nonlinear coefficient κ' is transformed to

$$\kappa = n_2\hbar\omega/4\varepsilon_0 cA. \tag{A15}$$

This is identical to Eq. (6) apart from linear refractive index factors not included in the calculation immediately above.

APPENDIX B: POSITION DEPENDENCE OF THE BASIC COMMUTATOR

It is straightforward to show with the use of Eq. (9) that

$$[\hat{a}(x, t), \hat{a}^\dagger(x, t')] = \delta(t - t') + (\exp\{i\kappa x[g(t' - t) - g(t - t')]\} - 1)\hat{a}^\dagger(x, t')\hat{a}(x, t), \tag{B1}$$

where the second term on the right-hand side represents a progressive departure from the correct value of the commutator, which is given by Eq. (2). For the input coherent state $|\{\alpha\}\rangle$ defined in Eq. (11) the second term above provides an additional contribution

$$\alpha^*(t')\alpha(t)\exp[G(x)|\alpha(t)|^2 + G^*(x)|\alpha(t')|^2] \times (\exp\{i\kappa x[g(t' - t) - g(t - t')]\} - 1) \tag{B2}$$

to the quadrature variance $V(t, t')$ evaluated in Eq. (34). This additional contribution has a negligible effect in the regime where the time scale of the coherent excitation is much longer than the material response time. Thus, for the model response function of Eqs. (8), the term in the noise spectrum calculated in accordance with Eq. (32) that results from expression (B2) is proportional to $\sin^2(\kappa x/2\tau_0)$. The squeezing effects illustrated and discussed in Section 5 occur in the first order of the small quantity $\kappa x/\tau_0$. Therefore the term by which the commutator in Eq. (B1) departs from its initial value has no significant effect on the squeezing spectra.

APPENDIX C: USE OF NONCONTINUOUS OPERATORS

In this appendix we outline the calculation of the matrix element in Eq. (21) using the noncontinuous operators. The field operator can be expressed as[24]

$$\hat{a}(t) = \sum_j \phi_j(t)\hat{c}_j \equiv \boldsymbol{\phi}\cdot\hat{\mathbf{c}}, \tag{C1}$$

where the functions ϕ_j form a complete set and we have introduced an obvious vector notation for convenience. The output field operator [Eq. (9)] now becomes

$$\hat{a}(x, t) = \exp(i\hat{\mathbf{c}}^\dagger\boldsymbol{\Gamma}\hat{\mathbf{c}})\boldsymbol{\phi}\cdot\hat{\mathbf{c}}, \tag{C2}$$

where

$$\Gamma_{jk} = \kappa x\int_0^\infty d\tau\, \phi_j^*(t - \tau)g(\tau)\phi_k(t - \tau). \tag{C3}$$

The required matrix element is therefore

$$V_1 = \langle\{\alpha\}|\hat{a}(x, t)\hat{a}(x, t')|\{\alpha\}\rangle = \langle\boldsymbol{\alpha}|\exp(i\hat{\mathbf{c}}^\dagger\boldsymbol{\Gamma}\hat{\mathbf{c}})\boldsymbol{\phi}\cdot\hat{\mathbf{c}}\exp(i\hat{\mathbf{c}}^\dagger\boldsymbol{\Gamma}'\hat{\mathbf{c}})\boldsymbol{\phi}'\cdot\hat{\mathbf{c}}|\boldsymbol{\alpha}\rangle, \tag{C4}$$

where the primes indicate that the quantities are functions of t' rather than t and $\boldsymbol{\alpha}$ is the vector of expansion coefficients of $\alpha(t)$ in the basis $\phi_j(t)$. We can now insert the (over)completeness relation for the noncontinuous coherent states, obtaining

$$V_1 = (\boldsymbol{\phi}' \cdot \boldsymbol{\alpha}) \int d^2\boldsymbol{\beta} \int d^2\boldsymbol{\gamma} \langle \boldsymbol{\alpha} | \exp(i\hat{\mathbf{c}}^\dagger \boldsymbol{\Gamma} \hat{\mathbf{c}}) | \boldsymbol{\beta} \rangle \times \langle \boldsymbol{\beta} | \boldsymbol{\phi} \cdot \hat{\mathbf{c}} | \boldsymbol{\gamma} \rangle \langle \boldsymbol{\gamma} | \exp(i\hat{\mathbf{c}}^\dagger \boldsymbol{\Gamma}' \hat{\mathbf{c}}) | \boldsymbol{\alpha} \rangle, \quad \text{(C5)}$$

where the integrals are defined as

$$\int d^2\boldsymbol{\beta} = \int \frac{d^2\beta_0}{\pi} \int \frac{d^2\beta_1}{\pi} \cdots \int \frac{d^2\beta_i}{\pi} \cdots. \quad \text{(C6)}$$

All the matrix elements in Eq. (C5) can now be evaluated with the use of the normal-ordering theorem; we obtain

$$V_1 = \boldsymbol{\phi}' \cdot \boldsymbol{\alpha} \int d^2\boldsymbol{\beta} \int d^2\boldsymbol{\gamma} \exp(\boldsymbol{\alpha}^\dagger \mathbf{G} \boldsymbol{\beta} + \boldsymbol{\gamma}^\dagger \mathbf{G}' \boldsymbol{\alpha} - \boldsymbol{\alpha}^\dagger \cdot \boldsymbol{\alpha} - \boldsymbol{\beta}^\dagger \cdot \boldsymbol{\beta} - \boldsymbol{\gamma}^\dagger \cdot \boldsymbol{\gamma} + \boldsymbol{\alpha}^\dagger \cdot \boldsymbol{\beta} + \boldsymbol{\beta}^\dagger \cdot \boldsymbol{\gamma} + \boldsymbol{\gamma}^\dagger \cdot \boldsymbol{\alpha}) \boldsymbol{\phi} \cdot \boldsymbol{\gamma}. \quad \text{(C7)}$$

If we look at the $\boldsymbol{\beta}$-dependent parts of the exponent, we can write

$$\boldsymbol{\alpha}^\dagger \mathbf{G} \boldsymbol{\beta} - \boldsymbol{\beta}^\dagger \cdot \boldsymbol{\beta} + \boldsymbol{\alpha}^\dagger \cdot \boldsymbol{\beta} + \boldsymbol{\beta}^\dagger \cdot \boldsymbol{\gamma} = -(\boldsymbol{\beta}^\dagger - \boldsymbol{\alpha}^\dagger \cdot \mathbf{G} - \boldsymbol{\alpha}^\dagger) \times (\boldsymbol{\beta} - \boldsymbol{\gamma}) + \boldsymbol{\alpha}^\dagger \mathbf{G} \boldsymbol{\gamma} + \boldsymbol{\alpha}^\dagger \boldsymbol{\gamma}. \quad \text{(C8)}$$

Now, since the integrals in Eq. (C6) are over the whole complex plane, we can move the origins of $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^\dagger$ independently, so that the integral reduces to that of a Gaussian over the whole plane; thus

$$\int d\boldsymbol{\beta}' d\boldsymbol{\beta}'^\dagger \exp(-\boldsymbol{\beta}'^\dagger \cdot \boldsymbol{\beta}') = 1. \quad \text{(C9)}$$

The remaining exponent in Eq. (C6) can also be reduced to a quadratic form in $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\dagger$, which we evaluate using the same procedure to obtain

$$V_1 = (\boldsymbol{\phi}' \cdot \boldsymbol{\alpha})[\boldsymbol{\phi} \cdot (\mathbf{G}' \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha})] \times \exp(\boldsymbol{\alpha}^\dagger \mathbf{G} \mathbf{G}' \boldsymbol{\alpha} + \boldsymbol{\alpha}^\dagger \mathbf{G} \boldsymbol{\alpha} + \boldsymbol{\alpha}^\dagger \mathbf{G}' \boldsymbol{\alpha}), \quad \text{(C10)}$$

which is simply the discrete version of Eq. (27).

*Permanent address, Physics Department, Essex University, Colchester, CO4 3SQ, UK.

REFERENCES

1. P. N. Kean, K. Smith, and W. Sibbett, "Spectral and temporal investigation of self-phase-modulation and stimulated Raman scattering in a single-mode optical fibre," Proc. Inst. Electr. Eng. Part J **134**, 163–170 (1987).
2. L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, "Experimental observation of picosecond pulse narrowing and solitons in optical fibers," Phys. Rev. Lett. **45**, 1095–1098 (1980).
3. L. F. Mollenauer and K. Smith, "Demonstration of soliton transmission over more than 4000 km in fiber with loss periodically compensated by Raman gain," Opt. Lett. **13**, 675–677 (1988).
4. N. J. Doran, D. S. Forrester, and B. K. Nayar, "Experimental investigation of all-optical switching in a fiber loop mirror device," Electron. Lett. **25**, 267–268 (1989).
5. K. J. Blow, N. J. Doran, and B. K. Nayar, "Experimental demonstration of optical soliton switching in an all-fiber nonlinear Sagnac interferometer," Opt. Lett. **14**, 754–756 (1989).
6. S. R. Friberg, A. M. Weiner, Y. Silberberg, B. G. Sfez, and P. W. Smith, "Femtosecond switching in a dual-core fiber nonlinear coupler," Opt. Lett. **13**, 904–906 (1988).
7. M. N. Islam, S. P. Dijaili, and J. P. Gordon, "Modulation-instability-based fiber interferometric switch near 1.5 μm," Opt. Lett. **13**, 518–520 (1988).
8. M. N. Islam, E. R. Sunderman, R. H. Stolen, W. Pleibel, and J. R. Simpson, "Soliton switching in a fiber nonlinear loop mirror," Opt. Lett. **14**, 811–813 (1989).
9. R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. DeVoe, and D. F. Walls, "Broad-band parametric deamplification of quantum noise in an optical fiber," Phys. Rev. Lett. **57**, 691–694 (1986).
10. B. L. Schumaker, S. H. Perlmutter, R. M. Shelby, and M. D. Levenson, "Four mode squeezing," Phys. Rev. Lett. **58**, 357–360 (1987).
11. R. M. Shelby, M. D. Levenson, and P. W. Bayer, "Guided acoustic-wave Brillouin scattering," Phys. Rev. B **31**, 5244–5252 (1985).
12. R. E. Slusher, P. Grangier, A. LaPorta, B. Yurke, and M. J. Potasek, "Pulsed squeezed light," Phys. Rev. Lett. **59**, 2566–2569 (1987).
13. R. Tanas and S. Kielich, "Self-squeezing of light propagating through nonlinear optically isotropic media," Opt. Commun. **45**, 351–356 (1984).
14. M. D. Levenson, R. M. Shelby, and S. H. Perlmutter, "Squeezing of classical noise by nondegenerate four-wave mixing in an optical fiber," Opt. Lett. **10**, 514–516 (1985).
15. M. Kitagawa and Y. Yamamoto, "Number-phase minimum-uncertainty state with reduced number uncertainty in a Kerr nonlinear interferometer," Phys. Rev. A **34**, 3974–3988 (1986).
16. G. S. Agarwal, "Visibility index as a simple measure of the quantum effects in light propagation through a nonlinear dispersive element," Opt. Commun. **62**, 190–192 (1987).
17. M. J. Potasek and B. Yurke, "Squeezed-light generation in a medium governed by the nonlinear Schrödinger equation," Phys. Rev. A **35**, 3974–3977 (1987).
18. T. A. B. Kennedy and P. D. Drummond, "Quantum-field superposition via self-phase modulation of coherent wave-packets," Phys. Rev. A **38**, 1319–1326 (1988).
19. E. M. Wright, "Quantum theory of self-phase modulation," J. Opt. Soc. Am. B **7**, 1142–1146 (1990).
20. M. Shirasaki and H. A. Haus, "Squeezing of pulses in a nonlinear interferometer," J. Opt. Soc. Am. B **7**, 30–34 (1990).
21. S. J. Carter, P. D. Drummond, M. D. Reid, and R. M. Shelby, "Squeezing of quantum solitons," Phys. Rev. Lett. **58**, 1841–1844 (1987).
22. P. D. Drummond and S. J. Carter, "Quantum field theory of squeezing in solitons," J. Opt. Soc. Am. B **4**, 1565–1573 (1987).
23. Y. Lai and H. A. Haus, "Quantum theory of solitons in optical fibers. I. Time-dependent Hartree approximation," Phys. Rev. A **40**, 844–853 (1989).
24. K. J. Blow, R. Loudon, S. J. D. Phoenix, and T. J. Shepherd, "Continuum fields in quantum optics," Phys. Rev. A **42**, 4102–4114 (1990).
25. K. J. Blow and D. Wood, "Theoretical description of transient stimulated Raman scattering in optical fibers," IEEE J. Quantum Electron. **25**, 2665–2673 (1989).
26. R. H. Stolen, J. P. Gordon, W. J. Tomlinson, and H. A. Haus, "Raman response function of silica-core fibers," J. Opt. Soc. Am. B **6**, 1159–1166 (1989).
27. R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).