A recursive multiport scheme for implementing Grover's search algorithm with integrated optics (22 Nov 2013)

Grover's algorithm describes a quantum algorithm for searching an unsorted database of N items using $O(\sqrt{N})$ operations, which is a quadratic speedup over known classical search methods. In the simplest scenario, we have a search space of N items and we are supplied with a quantum oracle that can marks the solution to the search problem by shifting the phase of the solution's register. The goal of the algorithm is to find the solution using the smallest number of uses of the oracle.

To start, we prepare the equal superposition state

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle \tag{1}$$

where $N = 2^n$ for computational basis states $|x\rangle$ of n qubits.

Grover's algorithm is characterized by the repeated use of a quantum subroutine known as the Grover operator

$$G = (2|\psi\rangle\langle\psi| - I)O \tag{2}$$

where O is the oracle call and

$$W = 2|\psi\rangle\langle\psi| - I \tag{3}$$

is sometimes called the inversion about the mean.

In this note, we describe a recursive scheme for implementing the Grover inversion with a multiport circuit. A multiport circuit describes a decomposition of a unitary operation into a sequence of qubit operations. The qubit operations can be implemented using optical devices acting on adjacent modes of a multimode inteferometer, also known as an integrated photonic circuit.

If we consider the case of having a unique solution to the search problem, the quantum oracle can be implemented by a single π-phase shift on the optical mode that corresponds to the marked item. Here we focus on a particular way of realizing the Grover inversion W with integrated optics such that the circuit for N items can be used for building the Grover inversion for a $2N$-item search.

Fore a more efficient description of the multiport circuit, we introduce the following optical notation:

1. $B_\epsilon(i,j)$ refers to a beam splitter with reflection coefficient ϵ between modes i and j, where the modes are labeled from top to bottom. Specifically, for the pair of modes, we can choose the overall phases so that in terms of the mode operators we have

$$\begin{pmatrix} a^\dagger_{i,\mathrm{out}} \\ a^\dagger_{j,\mathrm{out}} \end{pmatrix} = \begin{pmatrix} \sqrt{\epsilon} & \sqrt{1-\epsilon} \\ \sqrt{1-\epsilon} & -\sqrt{\epsilon} \end{pmatrix} \begin{pmatrix} a^\dagger_{i,\mathrm{in}} \\ a^\dagger_{j,\mathrm{in}} \end{pmatrix}. \tag{4}$$

   For simplicity, $B(i,j)$ denotes an equal beam splitter ($\epsilon = 1/2$) between modes i and j.

2. $S(i,j)$ refers to a swap operation between modes i and j, and is the same as a beam splitter with $\epsilon = 0$.

3. $P_\theta(i)$ refers to a phase shifter on mode i, which means the photon amplitude for mode i is multiplied by $e^{i\theta}$.

We use the notation as follows: Suppose we have a 3-mode circuit C where we have an equal beam splitter between mode 1 and 2 followed by a $\frac{\pi}{2}$ phase shift on mode 2 and finally a swap operation between modes 2 and 3. Then we would write

$$C = [B(1,2)]\left[P_{\frac{\pi}{2}}(2)\right][S(2,3)]. \tag{5}$$

Fig. 1 shows what circuit C would look like as an integrated photonic circuit.

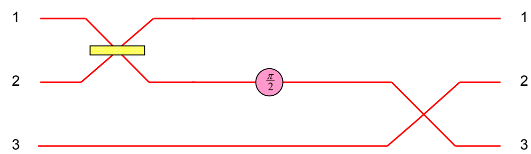

Figure 1: Sample multiport circuit C of Eq. 5 described in the optical multiport notation.

To describe the recursive scheme, it is useful to define the unitary V_N, which describes a certain series of equal beam splitters on N modes. For example, using the optical notation above,

$$V_4(1,2,3,4) = [B(1,2)B(3,4)][B(1,3)B(2,4)] \tag{6}$$

means that first we apply equal beam splitters between the pair of modes (1,2) and (3,4) and follow it with equal beam splitters on modes (1,3) and (2,4). Fig. 3 shows the optical circuit for V_4.

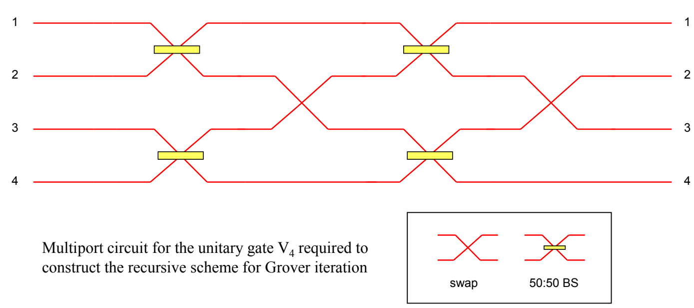

Figure 2: Optical multiport circuit for V_4.

The recursive scheme works when N is doubled:

$$V_8(1,2,\ldots,8) = [V_4(1,2,3,4)V_4(5,6,7,8)][B(1,5)B(2,6)B(3,7)B(4,8)] \tag{7}$$

$$V_{16}(1,2,\ldots,16) = [V_8(1,2,\ldots,8)V_8(9,10,\ldots,16)][B(1,9)B(2,10)\cdots B(8,16)] \tag{8}$$

which suggests the pattern

$$V_{2N}(1,\ldots,2N) = [V_N(1,\ldots,N)V_N(N+1,\ldots,2N)][B(1,N+1)B(2,N+2)\cdots B(N,2N)] \tag{9}$$

Let W_N denote the Grover inversion for N modes. First consider W_4 given by

$$W_4(1,2,3,4) = [B(1,2)B(3,4)][S(1,3)][B(1,2)B(3,4)][S(1,2)S(3,4)]. \tag{10}$$

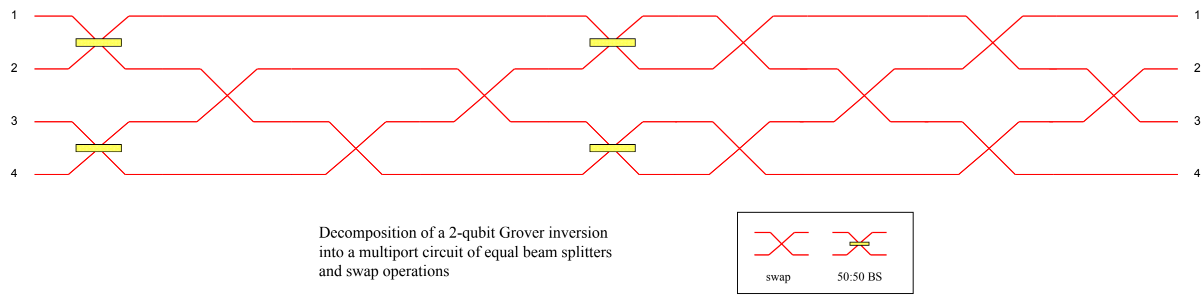

Figure 3: Implementing Grover inversion W_4 with integrated optics.

We use W_4 to build the Grover inversion when the number of modes is doubled:

$$\begin{aligned} W_8(1,2,\ldots,8) = {} & [W_4(1,2,3,4)W_4(5,6,7,8)][V_4(1,2,3,4)V_4(5,6,7,8)] \\ & [S(1,5)][V_4(1,2,3,4)V_4(5,6,7,8)] \qquad (11) \\ W_{16}(1,2,\ldots,16) = {} & [W_8(1,2,\ldots,8)W_8(9,10,\ldots,16)][V_8(1,2,\ldots,8)V_8(9,10,\ldots,16)] \\ & [S(1,9)][V_8(1,2,\ldots,8)V_8(9,10,\ldots,16)]. \qquad (12) \end{aligned}$$

In general, we have

$$\begin{aligned} W_{2N}(1,\ldots,2N) = {} & [W_N(1,\ldots,N)W_N(N+1,\ldots,2N)][V_N(1,\ldots,N)V_N(N+1,\ldots,2N)] \\ & [S(1,N+1)][V_N(1,\ldots,N)V_N(N+1,\ldots,2N)]. \qquad (13) \end{aligned}$$

Figs. 5 and 6 show how the scheme works for implementing W_{16} with optical multiport devices.

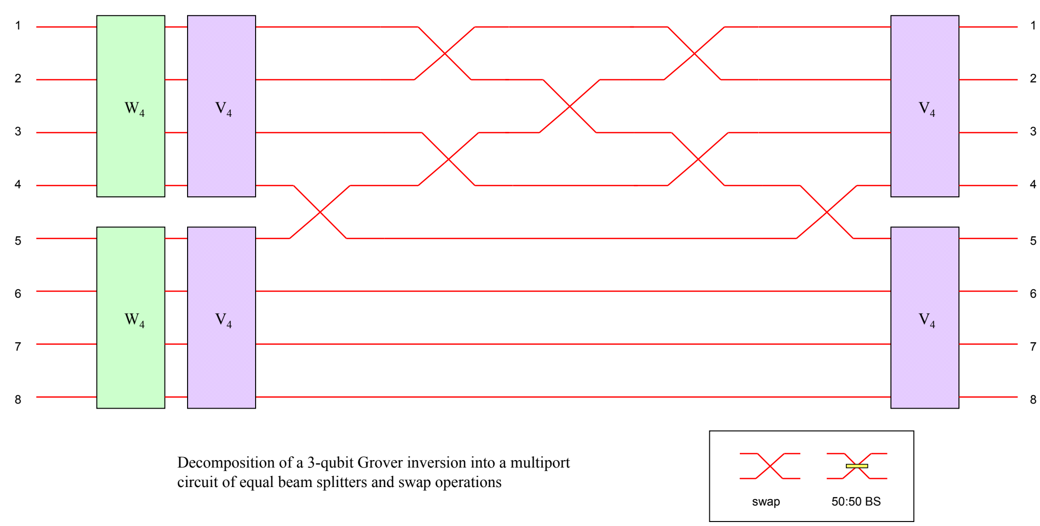

Figure 4: Implementing Grover inversion W_8 with integrated optical circuits for W_4 in Fig. 3 and V_4 in Fig. 2.

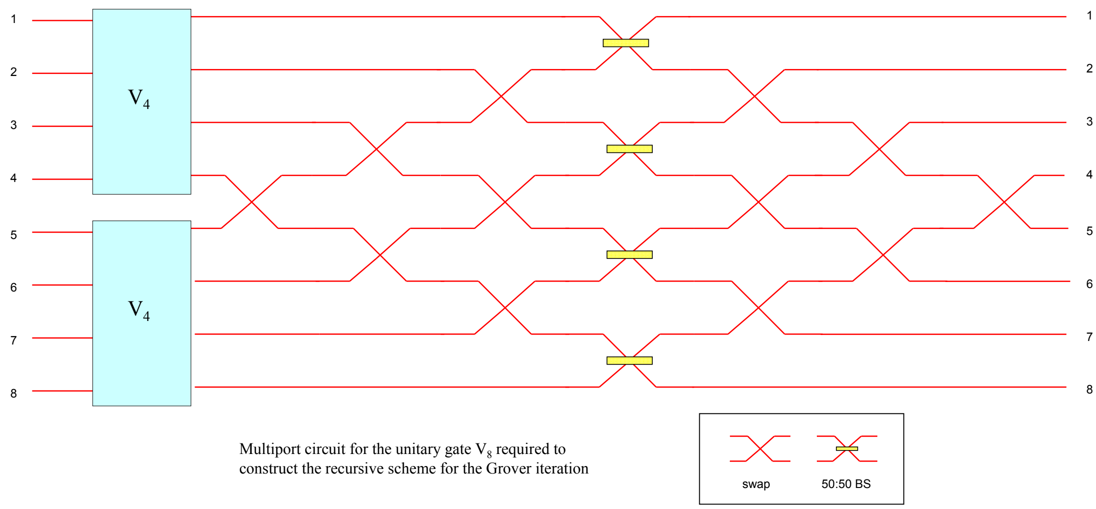

Figure 5: Optical multiport circuit for V_8.

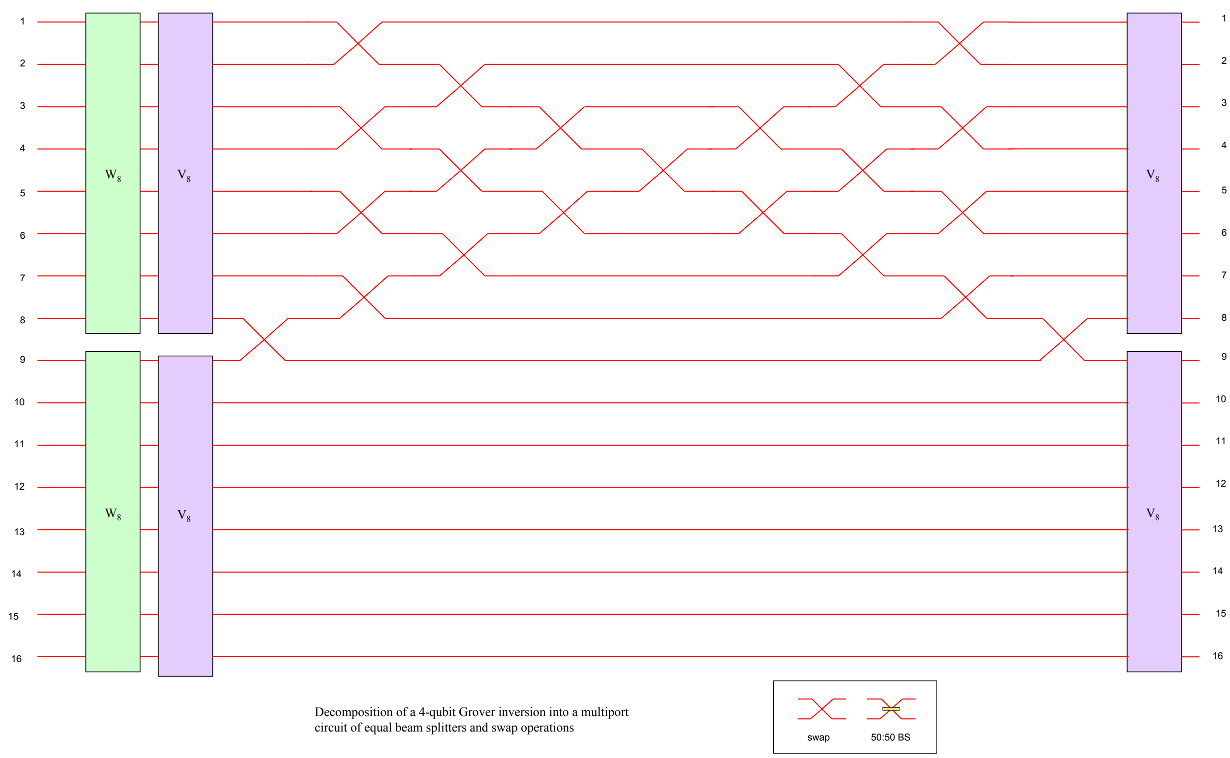

Figure 6: Implementing Grover inversion W_{16} with multiport optical circuits for W_8 and V_8.