

## A recursive multiport scheme for implementing quantum Fourier transforms with integrated optics (9 Nov 2013)

Many known quantum algorithms that exhibit exponential speedup over their classical counterparts make use of quantum Fourier transform, which describes a discrete Fourier transform on quantum mechanical amplitudes. Typically, the speedups come from the application of quantum Fourier transform to perform phase estimation, which involves finding approximate eigenvalues of a unitary operator. This is helpful in solving interesting problems like prime factorization or the order-finding problem.

In this note, we describe a recursive scheme for realizing the quantum Fourier transform with an integrated photonic circuit. By recursive, we mean that the multiport circuit for an  $(m - 1)$ -qubit quantum Fourier transform is used to build the multiport circuit for an  $m$ -qubit quantum Fourier transform. Formally, the multiport circuit describes a decomposition of the  $m$ -qubit Fourier matrix into a product of unitary matrices, each of which can be expressed in terms of a sequence of qubit operations on adjacent modes.

A key ingredient for the decomposition of the  $m$ -qubit quantum Fourier transform presented here is the unitary described by the direct sum of a pair of  $(m - 1)$ -qubit Fourier matrices.

Fig. 1 shows an optical multiport circuit for 2-qubit quantum Fourier transform. Fig. 2 shows how we use the 2-qubit circuit for implementing 3-qubit quantum Fourier transform. Note that the circuit is presented such that the generalization to more qubits is apparent.

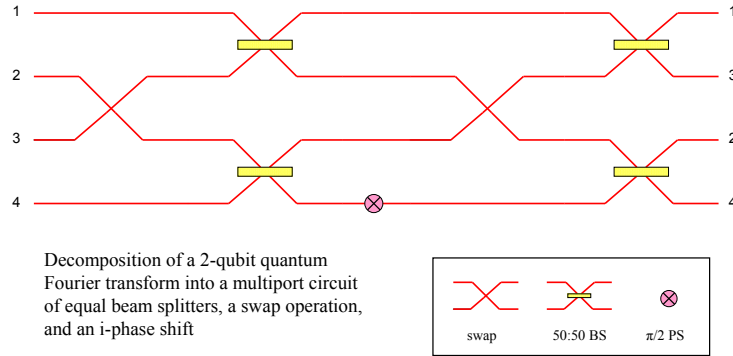


Figure 1: Implementing 2-qubit quantum Fourier transform with an integrated photonic circuit.

In general, the scheme can be described as follows. Let  $d = 2^m$  be the number of optical modes required to represent  $m$  qubits. Let  $F_d$  be the  $d$ -dimensional quantum Fourier transform.

1. Let  $j_n$  denote the label for input mode  $n$ , where  $n$  is counted from top to bottom, and  $k_n$  be the corresponding label for the output modes. The input modes are labeled in alternating fashion, that is,

$$j_{2r-1} = r, \quad j_{2r} = \frac{d}{2} + r, \quad \text{for } r = 1, 2, \dots, \frac{d}{2}. \quad (1)$$

The output mode labels are in the standard order,  $k_n = n$ . For instance, if  $d = 8$  then from top to bottom the input modes are labeled 1, 5, 2, 6, 3, 7, 4, 8. Note that the mode labels correspond to  $m$ -qubit basis states, e.g., for  $d = 8$  we can have  $1 = |000\rangle, 2 = |001\rangle, \dots, 8 = |111\rangle$ .

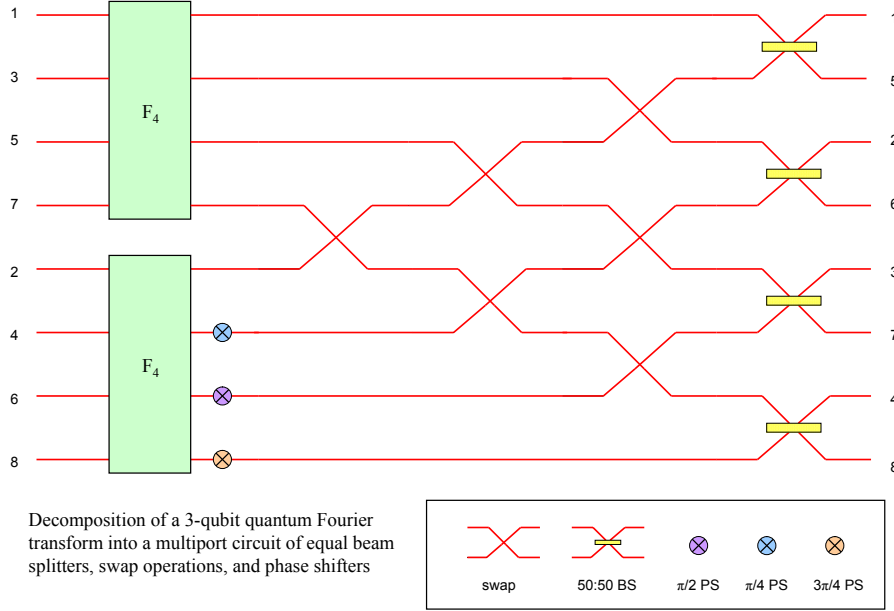


Figure 2: Implementing 3-qubit quantum Fourier transform with an integrated photonic circuit using a pair of 2-qubit quantum Fourier transforms  $F_4$ . Note the pattern of labels for the input and output modes.

2. Form two groups of modes, one with the top  $\frac{d}{2}$  paths, the other with the remaining  $\frac{d}{2}$  paths. For each group apply the  $\frac{d}{2}$ -dimensional quantum Fourier transform. Since we only consider powers of 2, this is just the quantum Fourier transform for  $m - 1$  qubits.
3. For the bottom group, apply the following phase shifts  $V = \text{diag} \left( 1, \omega, \omega^2, \dots, \omega^{\frac{d}{2}-1} \right)$  where  $\omega = e^{2\pi i/d}$ .
4. We apply a series of swap operations, first swapping the 2 middle modes, then swaps on the adjacent pair of modes, and so on until we get the modes into standard order when going from top to bottom.
5. Finally, going from top to bottom we apply an equal beam splitter between consecutive pairs of modes.

Fig. 3 shows how the scheme works for implementing  $F_{16}$  using a pair of  $F_8$ .

The number  $N(m)$  of optical elements needed for implementing the  $m$ -qubit quantum Fourier transform in the above scheme is given by the recursion relation

$$N(m) = 2N(m-1) + \left( \frac{d}{2} - 1 \right) + \frac{d}{2} \left( \frac{d}{2} - 1 \right) + \frac{d}{2} \quad (2)$$

with  $N(1) = 1$ . The first term corresponds to pair of  $(m - 1)$ -qubit quantum Fourier transforms, the second term in parenthesis corresponds to the phase shifters, the third term counts the swap operations (including the swaps needed if we start with input modes in the standard order), and

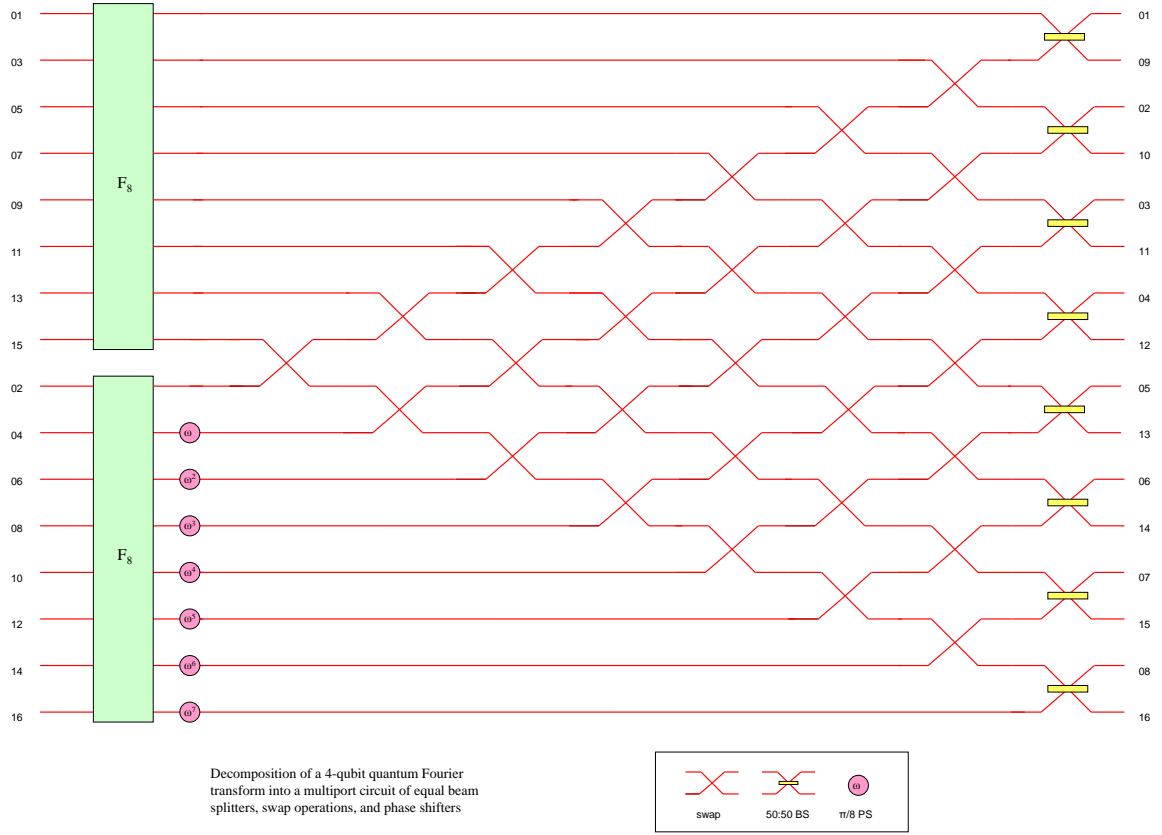


Figure 3: Implementing 4-qubit quantum Fourier transform with an integrated photonic circuit using a pair of 3-qubit quantum Fourier transforms  $F_8$ . The phase shifters are given by  $\omega = e^{\pi i/8}$ .

the last term counts the equal beam splitters for adjacent pairs of modes. As a function of  $d$ , we have

$$N(d) = \frac{d^2 - 3d + 2 + d \log_2(d)}{2} \quad (3)$$

so the total number of beam splitters and phase shifters needed is quadratic in the number of optical modes.

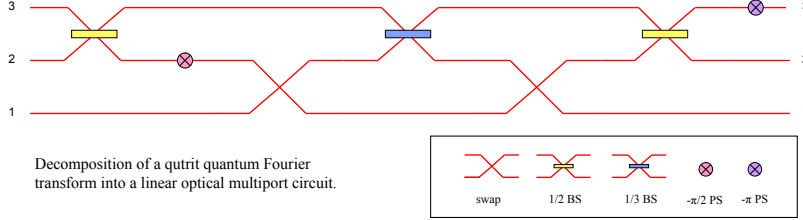


Figure 4: Implementing qutrit quantum Fourier transform with an integrated photonic circuit.

The recursive scheme works in the same way when doubling the dimension of quantum Fourier transforms. That is, given a way to implement the Fourier matrix  $F_n$ , the above scheme can be used to build the quantum Fourier transform for  $d = n2^m$ . For example, Fig. 4 is an optical circuit for the qutrit quantum Fourier transform, a pair of which can be used to realize the 6-dimensional quantum Fourier transform.

This is actually not so surprising since the scheme is just a direct optical translation of a particular matrix factorization of Fourier matrices apparently known since Gauss <sup>1</sup>:

$$F_{2n} = \begin{pmatrix} I_n & D_n \\ I_n & -D_n \end{pmatrix} \begin{pmatrix} F_n & 0 \\ 0 & F_n \end{pmatrix} \begin{pmatrix} \text{even-odd} \\ \text{shuffle} \end{pmatrix} \quad (4)$$

where  $D_n = \text{diag}(1, \omega, \dots, \omega^n)$  and  $\omega = e^{2\pi i/n}$ .

<sup>1</sup>G. Strang, *Bull. Amer. Math. Soc.* **28** (1993) 288-305.