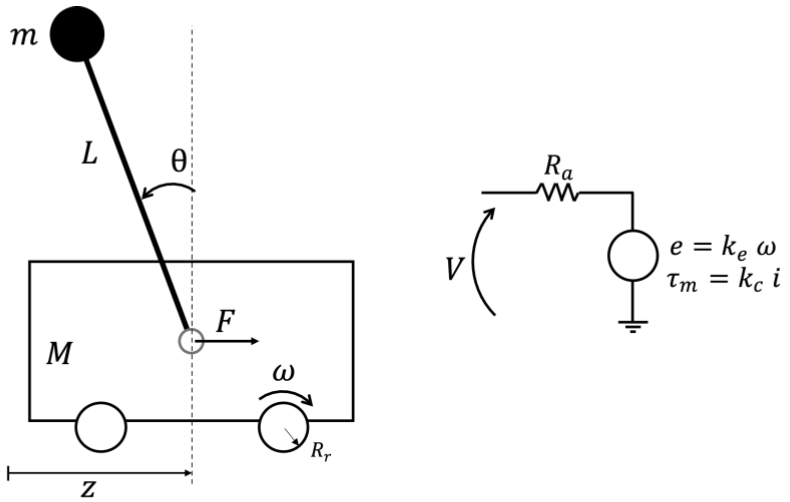
Control of an inverted pendulum

Consider the system depicted in the Figure, composed by a pendulum fixed on a cart and controlled through a DC motor.

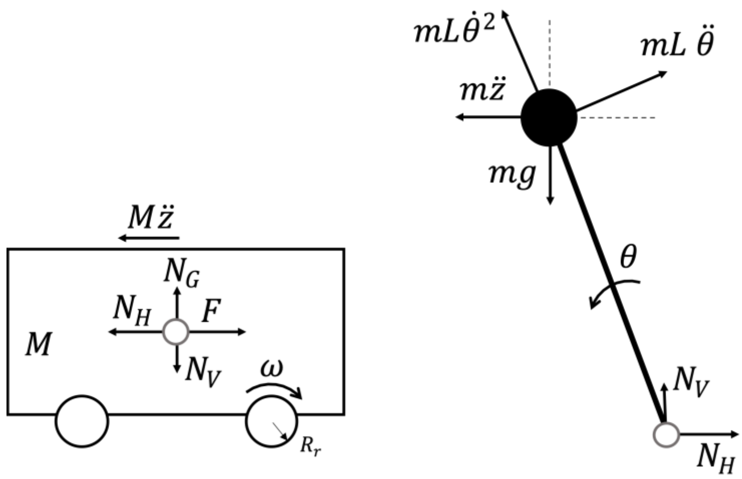


The model parameters are given as
$M = 10\ kg,\ m = 2\ kg,\ L = 1m,\ R_r = 0.1m,\ R_a = 10\ \Omega,\ k_c = 2\frac{V\cdot s}{rad},\ k_e = 2\frac{N\cdot m}{A}$

The goal of this work is to stabilize the pendulum in its vertical position (around $\theta = 0$) acting on the motor voltage V. Measurements of the position z and of angle θ are available.

System Model

To derive the dynamical model the system can be decomposed as follows.



- DC motor:

$$F = \frac{\tau_m}{R_r} = \frac{k_c}{R_r}\cdot\frac{V - k_e\omega}{R_a} = \frac{k_c}{R_a R_r}\cdot\left(V - k_e\frac{\dot{z}}{R_r}\right)$$

- Cart:

$$M\,\ddot{z} + N_H = F$$

- Pendulum:

$$m\ddot{z}\ + mL\dot{\theta}^2\sin\theta - mL\ddot{\theta}\cos\theta - N_H = 0$$
$$mL^2\ddot{\theta} - mL\ddot{z}\ \cos\theta - mgL\sin\theta = 0$$

Re-arranging the terms to remove the dependency on N_H we get

$$(M+m)\ddot{z} + mL\dot{\theta}^2 \sin\theta - mL\ddot{\theta}\ \cos\theta = \frac{k_c}{R_a R_r}\cdot\left(V - k_e\frac{\dot{z}}{R_r}\right)$$
$$\ddot{\theta} - \frac{\cos\theta}{L}\ddot{z} - \frac{g}{L}\sin\theta = 0$$

Isolating $\ddot{z}$ and $\ddot{\theta}$ we get

$$\begin{cases} \ddot{z} = \dfrac{1}{M+m-m\cos^2\theta}\left(-\dfrac{k_c k_e}{R_a R_r^2}\dot{z} - mL\,\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta + \dfrac{k_c}{R_a R_r}V\right) \\ \ddot{\theta} = \dfrac{g}{L}\sin\theta + \dfrac{\cos\theta}{L(M+m-m\cos^2\theta)}\left(-\dfrac{k_c k_e}{R_a R_r^2}\dot{z} - mL\,\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta + \dfrac{k_c}{R_a R_r}V\right) \end{cases}$$

In state-space form, denoting $x_1 = z,\ x_2 = \dot{z},\ x_3 = \theta,\ x_4 = \dot{\theta},\ u = V$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{M+m-m\,cos^2 x_3}\left(-\dfrac{k_e k_c}{R_a R_r^2}x_2 - mLx_4^2\sin x_3 + mg\sin x_3\cos x_3 + \dfrac{k_c}{R_a R_r}u\right) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{g}{L}\sin x_3 + \dfrac{\cos x_3}{L(M+m-m\,cos^2 x_3)}\left(-\dfrac{k_e k_c}{R_a R_r^2}x_2 - mLx_4^2\sin x_3 + mg\sin x_3\cos x_3 + \dfrac{k_c}{R_a R_r}u\right) \end{cases}$$

Equilibrium and linearization

We are now interested in linearizing the system model around the vertical pendulum position with zero input, i.e. $\bar{x} = (0,0,0,0)$ and $\bar{u} = 0$; the resulting linearized model is the following

$$\begin{cases} \delta\dot{x}_1 = \delta x_2 \\ \delta\dot{x}_2 = -\dfrac{k_e k_c}{M R_a R_r^2}\delta x_2 + \dfrac{mg}{M}\delta x_3 + \dfrac{k_c}{M R_a R_r}\delta u \\ \delta\dot{x}_3 = \delta x_4 \\ \delta\dot{x}_4 = -\dfrac{k_e k_c}{LM R_a R_r^2}\delta x_2 + \dfrac{(m+M)g}{LM}\delta x_3 + \dfrac{k_c}{LM\ R_a R_r}\delta u \end{cases}$$

Substituting the parameters values the linearized system model can be written as

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -4 & 1.962 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & 11.772 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0.2 \end{bmatrix}\delta v$$
$$\delta y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\delta x$$

Tasks

Given the nonlinear model and its linearization the goal is to compute a controller that stabilizes the pendulum in the vertical position without incurring in large oscillations of the cart.

1. Linearize the system with Simulink, using the Time-Based Linearization block.

   Hint: Select all the four states x_1, x_2, x_3 and x_4 (in the correct order) as outputs of the linearized system. This will ensure that the states of the identified system match the real system states. Then, overwrite the C matrix to consider the fact that only the position and the angle are measured.

2. Analyze the linearized system. Then, enlarge the system to ensure robust asymptotic zero-error regulation of as many outputs as possible.

3. Design an LQ control law for the enlarged system (assuming that the state is measured). Test the closed-loop performances in Simulink, considering

$$x(0) = [0, 0, 0.2, 0]'$$
$$z^0(t) = 0.25 \cdot \text{step}(t - 30)$$

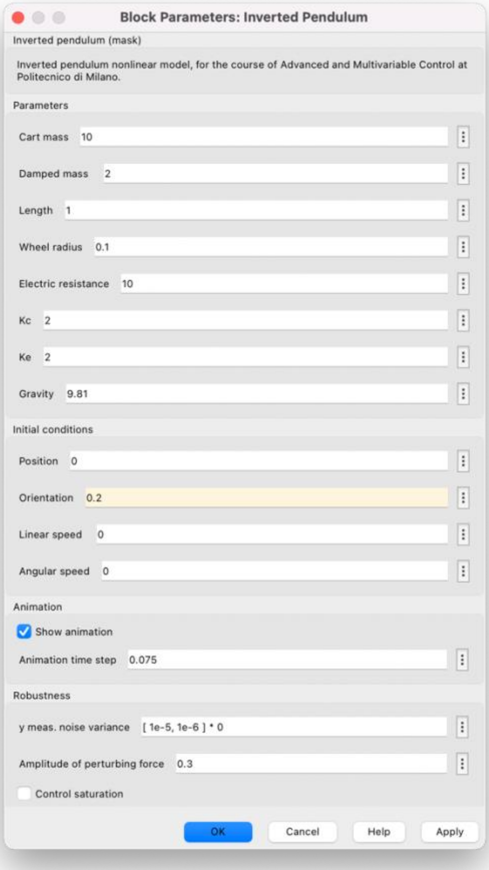
   Tune the Q and R matrices to achieve the smoothest possible performances.

4. Consider now that only the position of the cart and the angle of the pendulum are measured, i.e. $y = [x_1, x_3]'$. Design a Kalman Filter and test the resulting LQG control scheme in Simulink (with the same $x(0)$ and $z^0(t)$).

5. Test the robustness of the control system to:
   - White noise acting on the system outputs
   - Perturbation forces applied horizontally to the mass m
   - Saturation of the control variable to $\pm 24V$

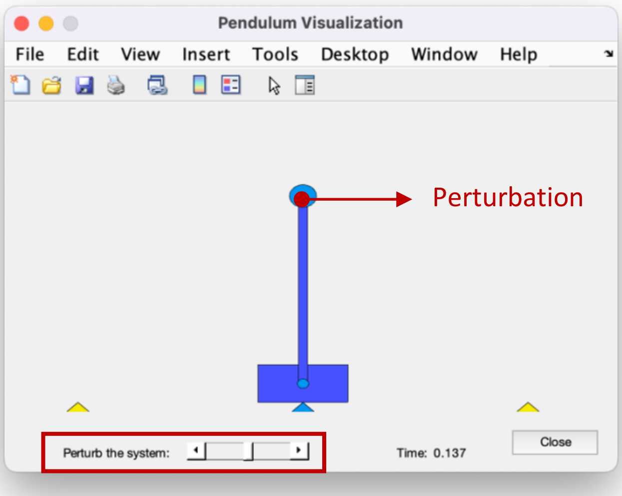
   Hint: Use the system mask and the pendulum animation to introduce these disturbances.

Inverted pendulum parameters



It is possible to configure the Simulink model of the inverted pendulum from the block mask. One can change:

- The physical/electrical parameters of the system
- The initial conditions
- Enable/disable the 2D animation of the pendulum (disable it if the simulation on your pc is too slow)
- The time step of the animation (the lower the time step, the slower the simulation; default: 0.075)
- The variance of the white noise on the outputs' measurements ([0, 0] means no noise)
- The maximum perturbing force that is possible to apply on the mass m through the animation's slider (default: 0.3)
- Enable/disable the saturation of the control variable (at $\pm 24V$)



If the Animation is enabled, the Pendulum Visualization figure should appear. Note that:

- The blue triangle corresponds to $z = 0$
- The left and right yellow triangles correspond to $z = -L$ and $z = L$, respectively
- At the bottom, a force can be applied to perturb the system by applying a force on the mass m: If the slider is moved, an increasing force is applied in the corresponding direction. To remove the disturbance, move the slider to the middle position