Distribución Beta

Es una distribución usada para modelar proporciones, ya que solo ~~tiene para~~ acepta variables aleatorias entre 0 y 1 ($0 < x < 1$). Su función de densidad es:

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{resto} \end{cases}$$

Aclaración función gamma

$\Gamma(z)$ es la función gamma, que es una extensión del término factorial a los números reales y complejos.

$$\boxed{\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt} \quad \forall z \in \mathbb{C} \text{ tal que } \mathrm{Re}(z) > 0$$

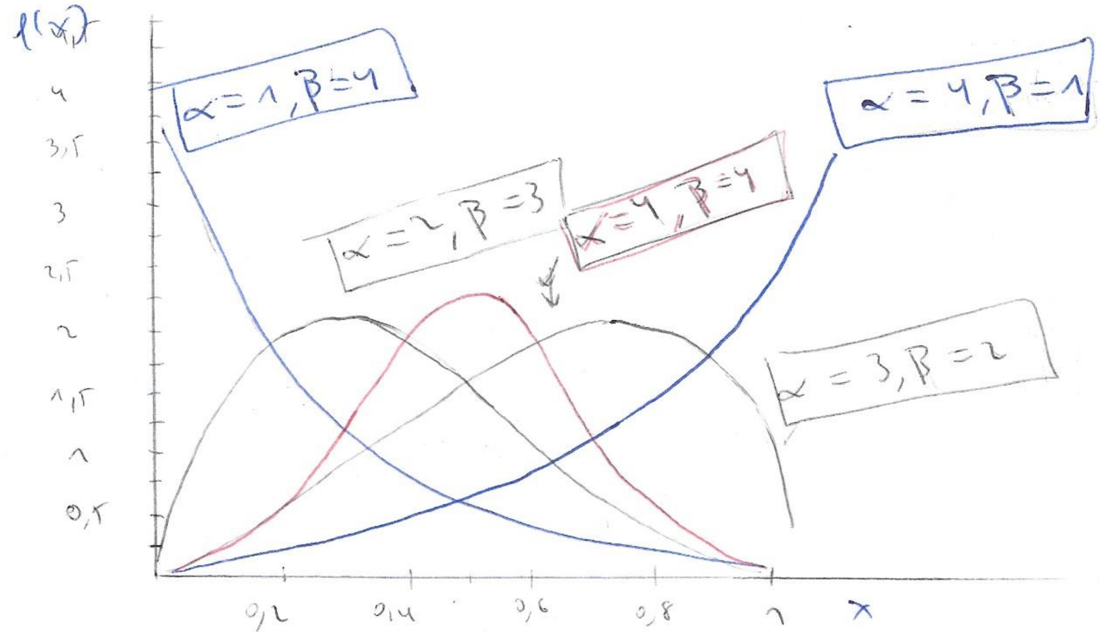
Si n es un número entero positivo (si $n \in \mathbb{Z}^+$) entonces es equivalente a:

$$\boxed{\Gamma(n) = (n-1)!} \quad \forall n \in \mathbb{Z}^+$$

Ejemplo: $\Gamma(1) = \int_0^\infty t^{1-1} \cdot e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_{t=0}^{t=\infty} = 1 = (1-1)! = 0! = 1$

$\Gamma(2) = \int_0^\infty t^{2-1} \cdot e^{-t}\, dt = \int_0^\infty t \cdot e^{-t}\, dt = (2-1)! = 1$

Es una distribución muy utilizada para modelar datos de manera empírica (mirando la forma a la que se parece), ya que tiene muchas posibles formas, en función de α y β.



Momentos de la Distribución Beta

Sea $X \sim Beta(\alpha, \beta)$, sus momentos principales son:

$$\boxed{E[X] = \mu = \frac{\alpha}{\alpha + \beta}}$$

$$\boxed{var[X] = \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

~~Manera alternativa de esc~~

Función Beta

Es una función ~~??~~ relacionada con la función gamma Γ.

$$\boxed{B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}} \qquad \alpha>0, \beta>0$$

Propiedades de la función Beta

1. $\boxed{B(\alpha,\beta) = B(\beta,\alpha)}$
2. $\boxed{B(\alpha,1) = \frac{1}{\alpha}}$
3. $\boxed{B(\alpha+1,\beta) = \frac{\alpha}{\alpha+\beta}\cdot B(\alpha,\beta)} \boxed{= \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}dx}$

Manera alternativa de representar la función de densidad

Si $X \sim \text{Beta}(\alpha,\beta)$; su función de densidad es:

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1 \\ 0 & \text{en el resto} \end{cases}$$

con $B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Cálculo de los momentos de una distribución Beta

Sea $X \sim Beta(\alpha, \beta)$:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^1 x \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx =$$

$$= \int_0^1 x \cdot \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx =$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{1}{B(\alpha,\beta)}$$ Propiedad

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1} dx$$

Propiedad

$$B(\alpha+1,\beta) = \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1} dx$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\frac{\alpha}{\alpha+\beta} B(\alpha,\beta)}{B(\alpha,\beta)} = \frac{\alpha}{\alpha+\beta}$$

$$\boxed{E[X] = \frac{\alpha}{\alpha+\beta}}$$

$$B(\alpha+1,\beta) = \frac{\alpha}{\alpha+\beta} B(\alpha,\beta)$$

(propiedad)

$$\left[E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot p(x)\, dx = \int_0^1 x^2 \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx = \right.$$

$$= \int_0^1 x^2 \cdot \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx =$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^2 \cdot x^{\alpha-1} (1\cancel{8}-x)^{\beta-1} dx =$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{(\alpha+2)-1} (1-x)^{\beta-1} dx$$

$$= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)} = \frac{B((\alpha+1)+1,\beta)}{B(\alpha,\beta)} = \frac{\alpha+1}{\alpha+1+\beta} \cdot \frac{B(\alpha+1,\beta)}{\beta(a,b)} =$$

$$B((\alpha+1)+1,\beta) = \frac{\alpha+1}{\alpha+1+\beta} \cdot B(\alpha+1,\beta)$$

$$\left. = \left(\frac{\alpha+1}{\alpha+1+\beta}\right) \cdot \left(\frac{\alpha}{\alpha+\beta}\right) = \frac{\alpha+1}{\alpha+1+\beta} \cdot \frac{\alpha}{\alpha+\beta} \right]$$

Es decir: $\boxed{E[x^2] = \frac{\alpha+1}{\alpha+1+\beta} \cdot \frac{\alpha}{\alpha+\beta}}$

Por lo tanto:

$$\mathrm{var}[X] = E[X^2] - E[X]^2 = \frac{(\alpha+1)\alpha}{(\alpha+1+\beta)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 =$$

$$= \frac{1}{(\alpha+\beta+1)(\alpha+\beta)^2}\left((\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)\right) =$$

$$= \frac{1}{(\alpha+\beta+1)(\alpha+\beta)^2}\left(\cancel{\alpha^3} + \cancel{\alpha^2\beta} + \cancel{\alpha^2} + \alpha\beta - \cancel{\alpha^3} - \cancel{\alpha^2\beta}\right)$$

$$= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$\boxed{\mathrm{var}[X] = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}}$$

Función generadora de momentos de una variable Beta

$$\left[M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx = \right.$$

$$= \int_0^1 e^{tx} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}\, dx =$$

$$\cancel{\frac{1}{B(\alpha,\beta)} = \frac{\Gamma(\alpha)\Gamma(\beta)}{}}$$

$$= \int_0^1 e^{tx} \cdot \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\, dx =$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 e^{tx} \cdot x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

$$e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$$

$$= \frac{1}{B(\alpha,\beta)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 x^k \cdot x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{\int_0^1 x^{\alpha+k-1} (1-x)^{\beta-1}\, dx}_{B(\alpha+k,\beta)} =$$

$$= \cancel{\sum_{k=0}^{\infty} \frac{B(\alpha,\beta)}{}} = \cancel{\sum_k} = \left. \sum_{k=0}^{\infty} \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} \cdot \frac{t^k}{k!} \right]$$