Primal-Circular Substitutions

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Abstract. There are two ongoing tensions in the creation of new musical systems that traditional innovation procedures in the domain of harmony have acknowledged yet left under-explored in the systematic sense from the composer's perspective. Learned terminological constraints and reactionary creative practices often limit the realization of underlying creative logic that connects past influences to present innovations, and creative procedures on one structural unit of a particular musical element to those on another. The following paper will proceed by example from theory toward a compositional end, employing algebraic techniques to create a system of chord substitution which serves as an exemplary solution of the aforementioned issues. Primal-circular substitution has a basis in western tertian harmony, shows compatibility with neo-Riemannian local transformation sequences, and re-envisions substitutions as the realization of globally applicable systems

Keywords: chord substitution, re-harmonization system, composition practices.

1 Introduction

Two major themes have long pervaded music composition across cultural, chronological, and creative boundaries. One is the tendency of creative progress to be of a reactionary rather than systematic nature, and involve a dichotomous choice between slightly editing or wholly rejecting a past creative system. Initially, a composer learns the traditions of the nearby and recent, as a social being learns a language in preparation for the task of being effective socially. Then he or she reacts to these in the process of incorporating them into the worldview that contextualizes the compositional products. A decision is made about how effective or not these traditional approaches are, and action is taken over time to increase the efficacy of the compositional tools and transmit the composer's message.

The second theme at play in the compositional process is consideration of the interplay between local and global musical structures. Focusing on either, the composer must decide which combinations of one or more musical elements (e.g. melody, harmony, timing, timbre, etc…) most strongly serve the overall purpose of the work (or some other large formal construct), and which large scale musical forms and techniques will constrain the treatment of these elements. The orientation described in the first theme gives rise to particular tendencies in this decision making process and becomes the starting point for implementation. The composer's palette is governed by the evolution of a language and its implied "conditions of possibility" [6]. Over the course of composing a work, making a hierarchy of decisions of the aforementioned nature, the composer intuitively creates a system of patterns of which he or she is only partially conscious in the moment. When one level of patterned abstraction is being acted upon creatively, another is unfolding logically without agency.

In what follows, a method of chord substitution will be introduced in attempt to provide an example of thinking about harmony and altering traditional chord progressions that overcomes the epistemological limitations previously described. The approach, called primal-circular substitution, makes use of cyclical quotient groups in Z_{12} to explain mutually dependent interval circularities and the harmonies they relate for use in re-harmonization procedures.

2 Aesthetic and Philosophical Concerns

While aesthetic judgments in the absolute sense may be left to the individual and the situational, in the case of employing mathematics to compositional ends, a few words are needed in explanation and defense of the approach. Without this, we are left with the trivial case that any finite collection of objects, musical or not, can be placed in an organizational and axiomatic framework that appears mathematically sound because it was initially designed to be so. While not sufficient to justify a new system, it is certainly necessary to view this extreme not as a potential derailment of the theory, but as an indication that the astounding variety of organizational structures that mathematics explains make mathematical thinking central to human intuition [3]. In the case of primal-circular substitution, this defense then has two parts.

The first is that the theory is not a freely generative approach to harmony, but one with existing musical anchors. It is a theory of substitution which uses prime number patterns to catalogue similarities between chords and allow substitution of part or all of a known chord progression. Regarding commonly accepted features of tonality, the example of the process soon to be introduced is centered on circularity and western tertian notions of consonance, maintains chord shapes and distances [7], and introduces a new scope of underlying structural similarity that re-envisions sonic relationships known to work aesthetically for audiences of the past and present. This maintains the composer's opportunity to manipulate emotions rather than just logically connected patterns.

The second point worthy of mention is that mathematics provides a language for relating formal components of different types of inquiry about the world, and is not itself responsible for cases in which the user does not choose to do so. Again, this is not an argument restrictively in favor of a particular teleological position on the use of mathematics in music. It is safe to say though, that representationalism has been a key mode of creative endeavor in music from inspirations acknowledged to form and content decided upon by composers, so it provides one solid base for defending the widespread relevance of the quantitative approach. With mathematics being largely a study of structures and processes independent of content that may conform to its particular models, its use in connecting music to other human endeavors and thought patterns knows few boundaries. Beyond its representational potential by way of mapping relations, it also facilitates specific instances of the reductionist thought that has been pervasive across different types of scientific inquiry at least sense the Age of Enlightenment [3]. From Chemillier's "ethnomathematics" to Xenakis' inspiration drawn from architecture and Cage's interests in chance and choice, many exemplary connections between thought structure and medium-independent creative output can be found which invite mathematical scrutiny as described.

Now that the long surviving relevance of mathematics to time and efficacy tested aesthetic practices has been summarized, let us move back to the plan to introduce the primal-circular substitution framework. In the next section, we will begin by defining the basic tools for creating one version of the system from the starting point of accepted tonal principles: interval cycles (in all such systems) and major/minor chord structures (this version). Following that, we will advance our conceptualization to the codependency of intervallic cycles, populating the chord substitution charts in the process so they are ready for compositional use. In doing this, mathematical generalizations will become apparent and proofs of those will be offered. Finally we will show the application of the system in an especially restrictive formal context: the mapping of one Hamiltonian progression (typically generated by P, L, and R operations) to another, connecting the local (neo-Riemannian, chord to chord) to the global (primal-circular substitution systems, chord/progression to superstructure) with mathematical continuity.

3 Primal-Circular Substitutions

We will begin the example construction of primal-circular systems by noting some behaviors of single intervals, as chord types as defined in the western European tradition contain predefined subsets of the intervallic possibilities. The reader is assumed to be familiar with determining basic cycling of intervals, so this part need not be directly illustrated (numerical results are given exhaustively in the table at the top of page 4). Interval cycling is isomorphic to dividing a whole number of equal temperament octaves into equal parts, or multiplying an interval out until it finishes one complete cycle through distinct pitch classes. So to go from the culturally dependent notion of interval names to something that can be extended free of terminological needs, we proceed to a mathematical explanation. In this case, to allow both relative and absolute interpretation of the system, we will move from consideration of the interval to consideration of the pitch class numbers that arise. The table below shows the behavior of each pitch class n when cycled multiplicatively by the interval M and adjusted modulo 12 for labeling.

The generating formula for the chart below is the familiar:

$$Pitch\ \mathrm{Category} = Mn \bmod 12$$

Table 1. Catalogue of mod cycles: Note numbers *n* are presented horizontally and multipliers *M* are catalogued vertically, with the value of the above equation in each interior box

n➔	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	0	2	4	6	8	10	0
3	3	6	9	0	3	6	9	0	3	6	9	0
4	4	8	0	4	8	0	4	8	0	4	8	0
5	5	10	3	8	1	6	11	4	9	2	7	0
6	6	0	6	0	6	0	6	0	6	0	6	0
7	7	2	9	4	11	6	1	8	3	10	5	0
8	8	4	0	8	4	0	8	4	0	8	4	0
9	9	6	3	0	9	6	3	0	9	6	3	0
10	10	8	6	4	2	0	10	8	6	4	2	0
11	11	10	9	8	7	6	5	4	3	2	1	0

In the body of the table above, one can observe the mod cycles for a given multiplier horizontally. Tertian chords then, correspond to groupings of three elements.

For a root pitch p, the following sets represent major and minor chords:

$$Major\ chord = (p,\ p{+}4,\ p{+}7)$$

$$Minor\ chord = (p,\ p{+}3,\ p{+}7)$$

Multiplication example:

$$4\ x\ \mathrm{C}\ major = 4\ x\ (1, 5, 8)\ \mathrm{mod}\ 12 = (4, 8, 8)$$

$$8\ x\ \mathrm{C}\ major = 8\ x\ (1, 5, 8)\ \mathrm{mod}\ 12 = (4, 4, 8)$$

So C Major is type 448/488 then.

Categorizations can then be visualized in the chart above, but for ease of use in compositional practices or experimentation with the theory, they will be organized in a table below with by category labels and chord names before laying out the mathematical reasoning explicitly and discussing the codependency of the three intervals present in a tertian chord. Before we do this, note the following two necessary abstractions:

1. Because this is a theory of chord substitution, which is an operation done on chord progressions rather than on specific voice leadings, permutation of voices is abstracted away by always ordering pitch class numbers from least to greatest after each operation in *modulo 12*. And…
2. Because the substitution approach is based on comparisons of prime factors of M with prime factors of 12, considering that the quotient groups Z_{12} / MZ_{12} and $Z_{12} / (12\text{-}M)Z_{12}$ are isomorphic, we categorize chords according to results of multiplication by M and $12\text{-}M$ together, since the residue classes are re-orderings, which we said above we abstract away.

The names of the chord types in the charts below correspond to the positions of their individual tones in the mod cycles for M and 12-M. Four steps are now given, explaining how to use the chord charts:

1. Write a traditional western tonal chord progression, e.g. I-vi-IV-V-I in C major:
   C Am F G C
2. Choose the 3x8 chart or the 4x6 chart (or other examples you generate via the mathematics) below. (for this example we will use the 3x8 chart)
3. Find the category (column) of each chord by locating it in the chart:
   C Am F G C = 448/488, 448/488, 044/088, 004/008, 448/488.
4. Write a new progression using chords from each corresponding category.
   Example: Cm Eb Ab Fm C.

Table 2. 3 Substitution Categories of 8 Chords Each: M = 4, 12-M = 8.

Type 448/488	**Type 004/008**	**Type 044/088**
C Major	C#/Db Major	C#/Db Minor
C Minor	D Minor	D Major
D#/Eb Major	E Major	E Minor
D#/Eb Minor	F Minor	F Major
F#/Gb Major	G Major	G Minor
F#/Gb Minor	G#/Ab Minor	G#/Ab Major
A Major	A#/Bb Major	A#/Bb Minor
A Minor	B Minor	B Major

Table 3. 4 substitution categories of 6 chords each: M = 3, 12-M = 9.

C Major	C#/Db Major	C#/Db Minor	C Minor
D#/Eb Minor	D Minor	D Major	D#/Eb Major
E Major	F Major	F Minor	E Minor
G Minor	F#/Gb Minor	F#/Gb Major	G Major
G#/Ab Major	A Major	A Minor	G#/Ab Minor
B Minor	A#/Bb Minor	Bb Major	B Major

Any values of M on the interval [1, 12] can be used to generate systems of harmony around points of symmetry dividing the octave as this procedure does. The particular choices for the examples above were made because of their connection to common rhythmic/metric divisions of 3 and 4 as well as observations by the author of their relationship to neo-Riemannian transformations. This in part serves the intent expressed earlier to show that this system aligns with certain traditional features of western music and common creative treatments of other musical elements. Neo-Riemannian transformations were originally devised as an explanation for movement from individual chord to chord, the local [5]. Primal-circular substitution theory connects them to the global. This will be demonstrated following some further mathematical explanations of how the chord charts above are determined.

Briefly outlined earlier, chords are a set of fixed intervallic relationships that we then map onto two of the M systems in the table on page 4. The M system takes each circularity, i.e. that of the root, third, and fifth of the chord, and cycles them through the remainder classes. Depending upon the value of M and the position of each chord tone relative to the points of symmetry defined by cycling M itself starting from pitch class 0 and ending at the first pitch p such that the product of M and p is congruent to 0 mod 12, two of the three chord tones with distinct behaviors will define the category. One of these tones is always the root, simply because that is how the positions are defined and the chord is named. This leaves either the 3rd or the 5th of the chord irrelevant to the categorization depending on the mapping (with p as root) of p+3 (for the third of minor chords), or p+4 (for major chords) and p+7 (the fifth) to the corresponding positions in the cycle of Mp mod 12. A proof of this is given below.

First, note that:

# of substitution categories = length of mod cycle = 12 / GCF(12, M)

This follows from the concept of circularity. Because the relative positions p, p+ (3 or 4), and p+7 are fixed, as soon as one of the chord tones lands on an Mp such that Mp mod 12 = 0, the cycle is complete. Referring to the table on page 4, we see the following regarding the particular substitution sets we will continue to discuss:

Table 4. Number of categories for various M values.

M Value	**# of Categories (12/GCF(12, M)**
3	4
4	3
5	3
6	4

Table 5. Number of chords per category for various M values.

M Value	**# of Categories**	**# of Chords in Category**
3	4	6
4	3	8
8	3	6
9	4	8

Now we are prepared to prove that either the third or the fifth of the chord is irrelevant to its categorization. Cases for multiple M values will be presented as the point applies to each.

Consider again the general major and minor chord structures:

$$Major: [p, p{+}4, p{+}7] \qquad Minor: [p, p{+}3, p{+}7]$$

Given the property:

$$(x + y)\ mod\ M = x\ mod\ M + y\ mod\ M$$

Proofs that the 3rd or the 5th of any chord is irrelevant for categorization purposes:

$$([p, p+4, p+7] \;\|\; [p, p+3, p+7]) \; mod \; (Size\ of\ Category: [2, 3, 4, 6, or\ 12])$$

Now in order to map them onto the mod cycles for different M values, consider the following:

Mod 2 Example (corresponding to M = 6):

$$(p+3) \; mod \; 2 = (p+7) \; mod \; 2 \; (p + (3 \,\|\, 7)\;) \; mod \; 2 = ((p \; mod \; 2) + 1) \; mod \; 2$$

So for a major or minor chord in root position, you have 3 tones with only 2 distinct results modulo 2

$$3 \; mod \; 2 = 7 \; mod \; 2$$

$$4 \; mod \; 2 = 0 \; mod \; 2 ==>$$

Depending on the root, either 3rd = 5th or Root = 3rd.

Mod 3 Example (corresponding to M = 4 || M = 8).

$$(p+3) \; mod \; 3 = p \; mod \; 3$$

$$(p+7) \; mod \; 3 = p \; mod \; 3 + 1$$

$$(p+4) \; mod \; 3 = p \; mod \; 3 + 1$$

$$(p+4) \; mod \; 3 = (p+7) \; mod \; 3$$

Again for a major or minor chord in root position, you have 3 tones with only two distinct results modulo 3.

$$3 \; mod \; 3 = 0$$

$$4 \; mod \; 3 = 7 \; mod \; 3 ==>$$

Depending on the root, either 3rd = 5th or root = 3rd

Mod 4 Example (corresponding to M = 3 || M = 9)

$$(p+3)\ mod\ 4 = p\ mod\ 4 + 3$$

$$(p+7)\ mod\ 4 = p\ mod\ 4 + 3$$

$$(p+4)\ mod\ 4 = p\ mod\ 4$$

$$(p+3)\ mod\ 4 = (p+7)\ mod\ 4$$

So for a major or minor chord in root position, you have 3 tones with only 2 distinct results modulo 4.

$$4\ mod\ 4 = 0\ mod\ 4$$

$$3\ mod\ 4 = 7\ mod\ 4 ==>$$

Depending on the root, either 3rd = 5th or root = 3rd.

Now that the system has been explained mathematically and the construction has been shown, we conclude by showing musical examples on the next page, each of which satisfies the intentions described at the outset. Here, primal-circular substitution is used to alter local harmonies while satisfying the global choice to preserve a particularly restrictive form. For this purpose, the progression chosen for re-harmonization is Moreno Andreatta's Hamiltonian progression entitled “Aprile”, from which new Hamiltonian progressions are generated.

Note: “Aprile” was composed using P, L, and R transformations. Primal-circular substitution systems can be understood locally in this way as detailed in **Appendix B**.

3x8 (M = 4 or 8) random selection re-harmonization of “Aprile”:

Random Chord Selections = 1 8 3 7 6 5 2 4 3 7 1 8 5 6 1 4 5 2 4 3 6 8 2 7 *1

Progression (category, chord #) = (2,1) (3,8) (1,3) (1,7) (3,6) (2,5) (2,2**)** (3,4) (2,3) (3,7) (1,1) (1,8) (1,5) (1,6) (3,1) (2,4) (3,5) (1,2) (1,4) (3,3) (2,6) (2,8) (3,2) (2,7) *(2,1)

Progression (chord names) = Db major, B major, Eb major, A major, Ab major, G major**,** D minor**,** F major**,** E major**,** Bb minor, C major, A minor**,** Gb major**,** F# minor, C# minor, F minor, G minor, C minor, Eb minor, E minor, G# minor, B minor, D major**,** Bb major, *Db major

4x6 (M = 3 or 9) random selection.

Random Chord Selections = 6 3 6 1 5 3 4 5 5 6 1 3 2 3 1 2 6 4 5 2 1 2 4 4 6

Progression (category, chord) = (3,6), (1,3), (4,6), (1,1), (4,5), (4,3), (1,4), (3,5), (2,5), (2,6), (3,1), (2,3), (2,2), (3,3), (2,1), (3,2), (1,6), (4,4), (1,5), (4,2), (4,1), (1,2), (3,4), (2,4), *(3,6)

Progression (chord names) = Bb major, E major, B major, C major, G# minor, E minor, G minor, A minor, A major, Bb minor, C# minor, F major, D minor, F minor, Db major, D major, B minor, G major, Ab major, Eb major, C minor, D# minor, Gb major, F# minor, *Bb major

With an extreme example of primal-circular substitution (for M = 3 and M = 4) in western harmonic context shown above, further experimentation on shorter and more conventional progressions is now left to the readers and composers. This approach to harmonic substitution can now be performed on aesthetically grounded materials according to the user, to novel creative ends serving his or her particular purpose at any harmonic structural level.

Appendix A: Primal-Circular Substitution Charts (Reproductions)

Table 6. M = 4, 12-M = 8.

Type 448/488	**Type 004/008**	**Type 044/088**
C Major	C#/Db Major	C#/Db Minor
C Minor	D Minor	D Major
D#/Eb Major	E Major	E Minor
D#/Eb Minor	F Minor	F Major
F#/Gb Major	G Major	G Minor
F#/Gb Minor	G#/Ab Minor	G#/Ab Major
A Major	A#/Bb Major	A#/Bb Minor
A Minor	B Minor	B Major

Table 7. M = 3, 12-M = 9.

C Major	C#/Db Major	C#/Db Minor	C Minor
D#/Eb Minor	D Minor	D Major	D#/Eb Major
E Major	F Major	F Minor	E Minor
G Minor	F#/Gb Minor	F#/Gb Major	G Major
G#/Ab Major	A Major	A Minor	G#/Ab Minor
B Minor	A#/Bb Minor	Bb Major	B Major

Appendix B: Neo-Riemannian Analogues of Primal-Circular Substitutions

Assumption: Use each transformation (P, L, R) either 0 or 1 times.

Note: The term "**preserves"** is used below to indicate that a particular transformation on a chord in a particular category yields another chord in that category, whereas other transformations would yield chords outside the category.

Summary of Neo-Riemannian Results:

- The category **448/488** preserves: **P || R in any order and place, no L.**
- The category **004/008** preserves: **P & R consecutively in any order and place, L alone.**
- The category **044/088** preserves **P || R first && P || R last.**
- The category **033/099** preserves **P || L first && P || L last.**
- The category **366/669** preserves **L & P consecutively in any order/place, R alone.**
- The category **336/699** preserves **P || L first && P || L last.**
- The category **003/009** preserves **L & P consecutively in any order/place, R alone.**

Examples for clarification (use charts to verify):

Ex. 1: Transformation of a chord in 448/488 by PR or RP yields another chord in 44/.488.

Ex. 2: Transformation of a chord in 004/008 by L alone yields another chord in 004/008. So does transformation of a chord in 004/008 by PR or RP.

Ex. 3: Transformation of a chord in 044/088 by PR, RP, PLR, or RLP yields another chord in 044/088.

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