Closure properties of CFL

- Find satisfactory answer to the following question; When we perform operations (union, intersection, etc.) with CFLs, do we get as a result a CFL?
- We raised similar questions about RLs, and had easy answers.
- When we ask the same questions about CFLs, we encounter some difficulties.

The Closure Theorems

Theorem: CFLs are closed under **Union**, **Concatenation**, **Kleene Star**, and **Reverse.**

Let $G_1 = (V_1, T_1, S_1, R_1), G_2 = (V_2, T_2, S_2, R_2)$, such that $V_1 \cap V_2 = \{\ \}$.

Union, Proof: We can construct

$G_3 = (V_1 \cup V_2 \cup \{S_3\}, T_1 \cup T_2, S_3, P_3)$, where

$$P_3 = P_1 \cup P_2 \cup \{S_3 \rightarrow S_1 | S_2\}$$

Clearly G_3 is CF, and it is easy to see that $L(G_3) = L(G_1) \cup L(G_2)$.

Concatenation, proof: Similarly we can construct G_3 where

$$P_3 = P_1 \cup P_2 \cup \{S_3 \rightarrow S_1 S_2\}$$

Clearly G_3 is CF and $L(G_3) = L(G_1)L(G_2)$.

The Closure Theorems (cnt.)

Kleene *, Proof: We can construct $G_3 = (V_1 \cup \{S_3\}, T_1, S_3, P_3)$, where

$$P_3 = P_1 \cup \{S_3 \to \epsilon, S_3 \to S_1 S_3\}$$

Clearly G_3 is CF and $L(G_3) = L(G_1)^*$.

Reverse, proof: Given $G = (V, T, P, S)$ in CNF, we can construc $G^R = (V, T, P^R, S)$, where P^R is obtained as follows:

- For every rule in P of the form $X \to AB$, add to P^R the rule $X \to BA$.
- For every rule in P of the form $X \to a$, add to P^R the rule $X \to a$.

It is easy to see that G^R is CF and $L(G^R) = L(G)^R$.

The Closure Theorems (cnt.)

Theorem.: CFLs are not closed under **Intersection**, **Complement**, and **Difference**.

Intersection, proof: If L_1 and L_2 are CF, then $L_1 \cap L_2$ is not necessarily CF.

- Consider two languages CFLs

$$L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\},\ L_2 = \{a^n b^m c^m : n \geq 0, m \geq 0\}$$

- Notice that $L_1 \cap L_2 = \{a^n b^n c^n : n \geq 0, m \geq 0\}$
- But $\{a^n b^n c^n : n \geq 0, m \geq 0\}$ is not CF.
- Using the PL: Let p be a number and $z = a^p b^p c^p \in L_1 \cap L_2$. Considering possible decompositions for $z = uvwxy$, there is no decomposition (obeying the PL reqs.) that would allow to pump v and x and produce a string in the language.

The Closure Theorems (cnt.)

Complement, proof: If $\overline{L}$ always was CF, it would follow that

$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ always would be CF.

However, CFLs are closed under union. If they were closed under complement, then they would necessarily be closed under intersection.

Difference, proof: Given any language L and M,

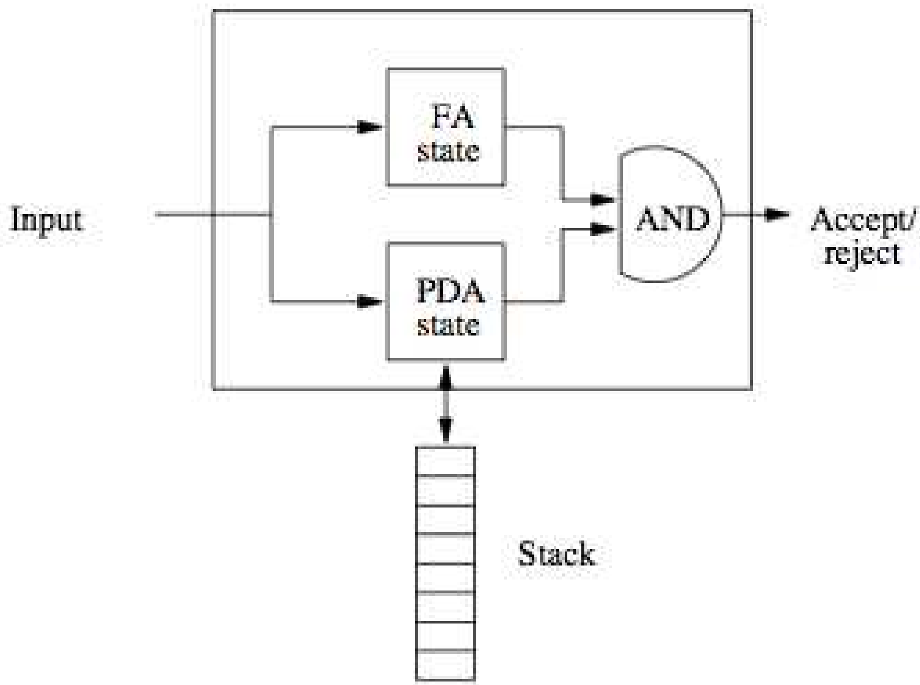
$L - M = L \cap \overline{M}$.

If L and M are CFLs and they were closed under difference, then they would necessarily be closed under intersection and complement.

CFL $\cap$ Regular is a CFL

Theorem : If L is CF and R is regular, then $L \cap R$ is CF.

Proof: Let L be accepted by PDA $P = (Q_P, \Sigma, \Gamma, \delta_P, q_P, Z_0, F_P)$ by final state, and let R be accepted by DFA $A = (Q_A, \Sigma, \delta_A, q_A, F_A)$
We'll construct a PDA for $L \cap R$ according to the picture



Proof idea

Formally, we define

$$P' = (Q_P \times Q_A, \Sigma, \Gamma, \Delta, (q_P, q_A), Z_0, F_P \times F_A)$$

where

$$\delta((q,p), a, X) = \{((r, \delta_A(p,a)), \gamma) : (r, \gamma) \in \delta_P(q, a, X)\}$$

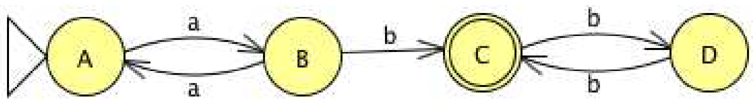
Then carry out a proof by induction on $\overset{*}{\vdash}$ that

$$(q_P, w, Z_0) \underset{P}{\overset{*}{\vdash}} (q, \epsilon, \gamma) \text{ if and only if}$$

$$((q_P, q_A), w, Z_0) \underset{P'}{\overset{*}{\vdash}} ((q, \hat{\delta}(q_A, w)), \epsilon, \gamma)$$

Example

Is $L = \{a^i b^j : \ i, j$ are odd$\} \cap \{a^n b^n : n \geq 1\}$ context free?



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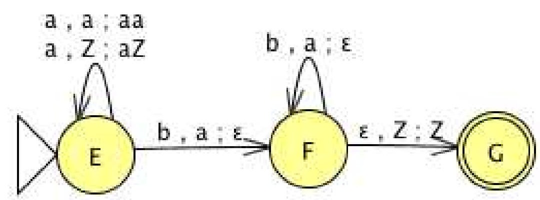
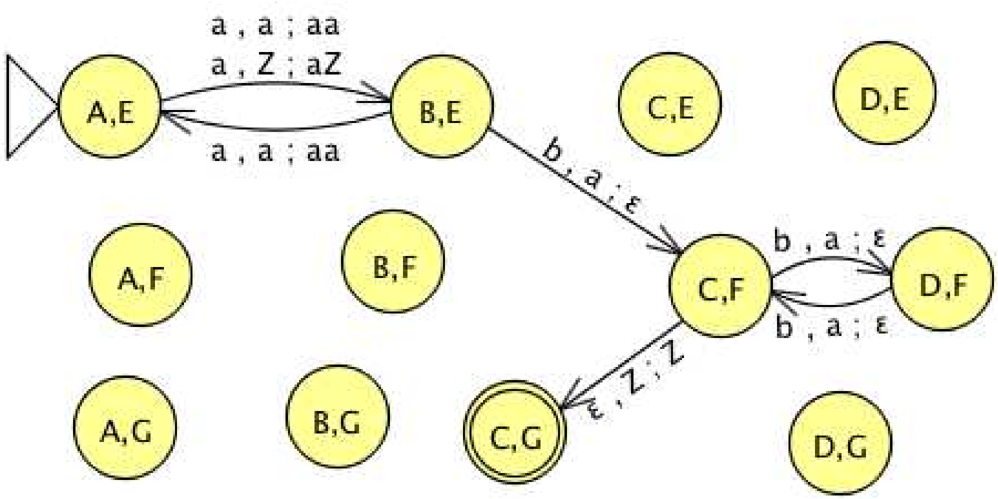


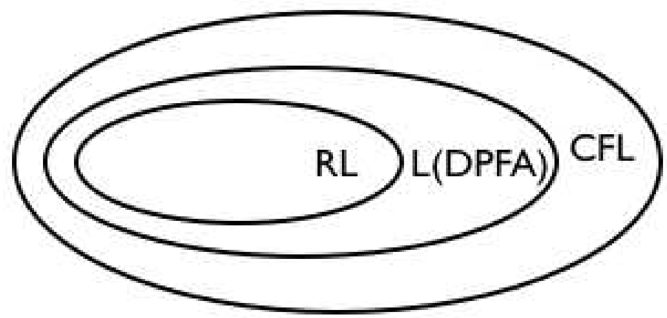
For clarity sake, I have not included transitions that lead to "trap states."



Our recent studies

- So far, we have been studying simple classes of languages (problems related to searching text, analysis of protocols, and parsing).

- We saw that PDAs are more powerful than finite automata.

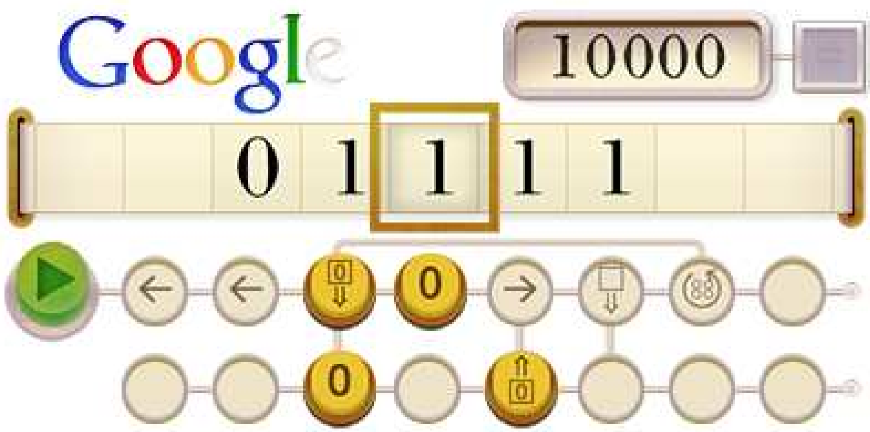


- We saw that CFL, while fundamental to the study of programming languages, are limited in scope; e.g., $a^nb^nc^n$ and ww although quite simple, are not CF.
- What is then beyond CFL? And how can we define new language families that include these examples?

In search for a more powerful automaton

- **FA vs PDA**: the nature of the temporary storage captures the difference between them.
  - if there is no storage, we have an FA; if the storage is a stack, then we have the more powerful PDA
- By extrapolation, we can expect to discover even more powerful languages if we give the automaton more flexible storage, e.g., two stacks, three stacks, a queue or some other storage device, etc.
- What can we say of the most powerful automaton and the limits of computation?
- The Turing Thesis maintains that the Turing Machine is such automaton.

Turing Machines



From Google's "Turing Machine doodle" (June 23, 2012), in commemoration of Alan Turing's 100th Birthday.

Motivation for studying this stuff

- To provide guidance to programmers on what they might or might not be able to accomplish through programming.
- How do Turing Machines fit in this context?
  - It is a tool that will allow us to prove everyday questions undecidable or intractable (those problems that are decidable but require large amounts of time to solve them).
- To prove a problem undecidable, we'll reduce it to a problem that the TM cannot decide.

The Turing Machine (1936)

- Has a finite-state control unit, like all automata.
- One infinite read-write tape serves as both input and unbounded storage device.

- The tape:
  - divided into cells;
  - each cell holds one symbol from the tape alphabet;
  - the head marks the *current* cell, which is the only cell that can influence the move of the TM.
  - Initially, tape holds $a_1a_2\cdots a_nBB\cdots$, where $a_1a_2\cdots a_n$ is the input, chosen from an input alphabet (subset of the tape alphabet) and B is the “blank” symbol.

A move of a TM

- A move of a Turing machine (TM) is a function of the state of the finite control and the tape symbol just scanned.
- In one move, the TM will:
  1. Change state.
  2. Write a tape symbol in the cell scanned.
  3. Move the tape head left or right.

TMs: formally

A TM is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$, where:

- Q, Σ, δ, q_0 and F are our old friends.
- Γ is the set of tape symbols, $\Sigma \subset \Gamma$.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$, e.g.: $\delta(q, X) = (p, Y, \mathcal{D})$, where q is a state, X is a tape symbol
  - p is the next state,
  - Y is the symbol written in the cell being scanned, replacing whatever was there, and
  - $\mathcal{D}$ is the direction, either L or R, of the next move.
- $B \in \Gamma$ is the blank symbol; $B \notin \Sigma$

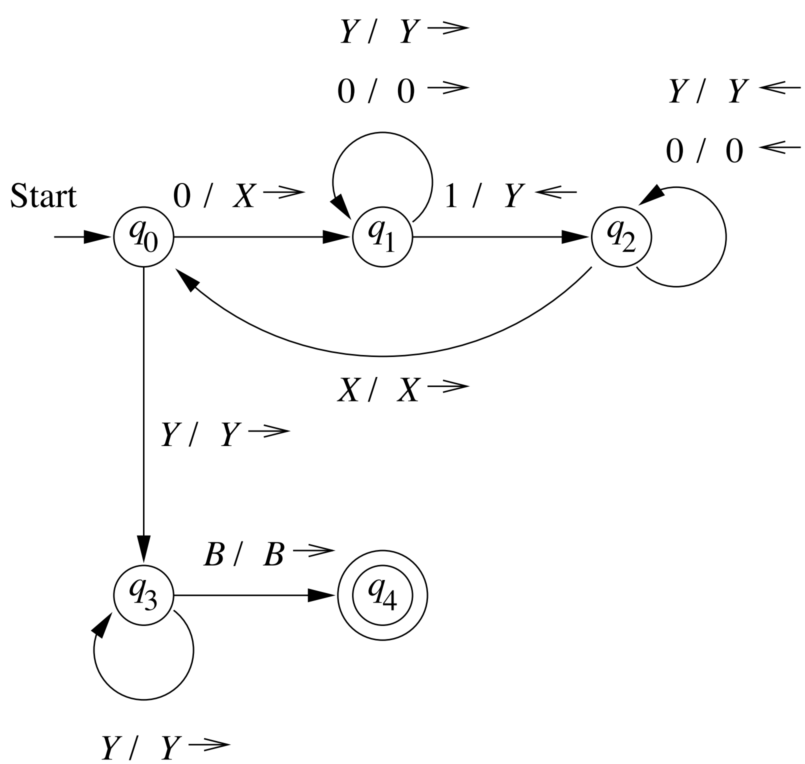
Instantaneous description for TMs

- An ID, denoted $\alpha q \beta$, represents what is going on at any moment with a TM:
  - α is the tape contents to the left of the head.
  - q is the state the TM is at the moment.
  - β is the nonblank tape contents at or to the right of the tape head.
- E.g.: XXq_3Y1BB
- as before, $\vdash$ denotes one move; $\overset{*}{\vdash}$ denotes zero, one, or more moves.

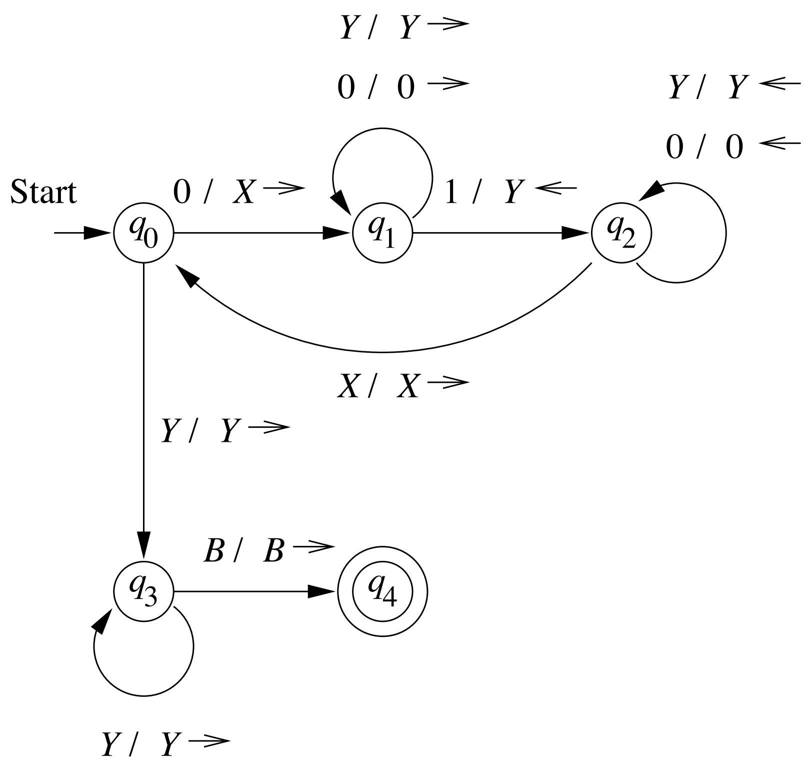
Example: a TM for $\{0^n1^n : n \geq 1\}$

$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{0, 1\}, \{0, 1, X, Y, B\}, \delta, q_0, B, \{q_4\})$
where δ is given by the following table:

	0	1	X	Y	B
$\rightarrow q_0$	(q_1, X, R)			(q_3, Y, R)	
q_1	$(q_1, 0, R)$	(q_2, Y, L)		(q_1, Y, R)	
q_2	$(q_2, 0, L)$		(q_0, X, R)	(q_2, Y, L)	
q_3				(q_3, Y, R)	(q_4, B, R)
$\star\ q_4$					

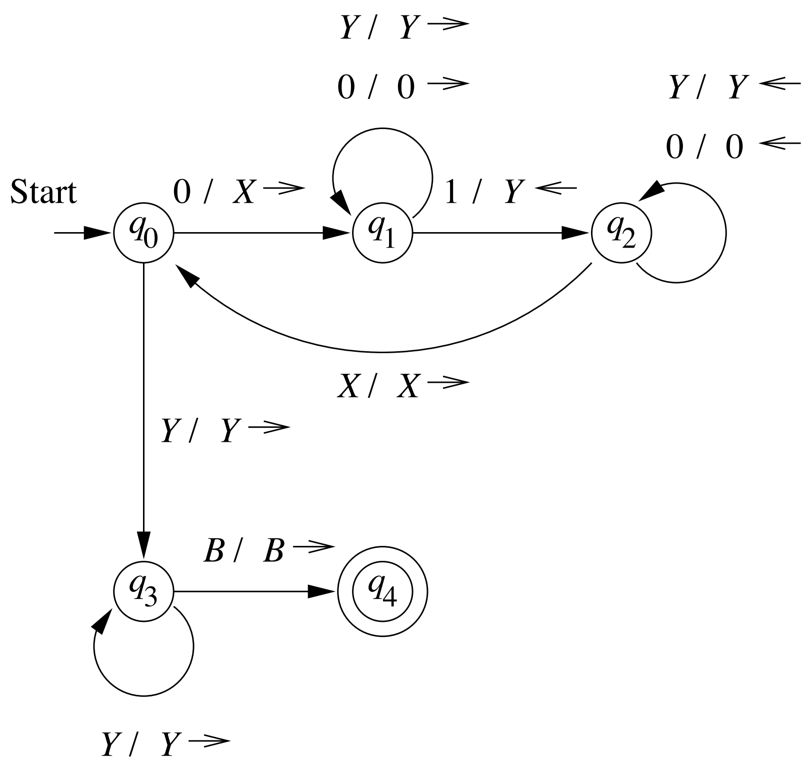


An accepting computation by M



$$q_0 0011 \vdash Xq_1 011 \vdash X0q_1 11 \vdash Xq_2 0Y1 \vdash q_2 X0Y1$$
$$Xq_0 0Y1 \vdash XXq_1 Y1 \vdash XXYq_1 1 \vdash XXq_2 YY \vdash Xq_2 XYY \vdash$$
$$XXq_0 YY \vdash XXYq_3 Y \vdash XXYYq_3 B \vdash XXYYBq_4 B$$

A non-accepting computation by M



$$q_0 0010 \vdash X q_1 010 \vdash X 0 q_1 10 \vdash X q_2 0 Y 0 \vdash q_2 X 0 Y 0$$
$$X q_0 0 Y 0 \vdash X X q_1 Y 0 \vdash X X Y q_1 0 \vdash X X Y 0 q_1 B$$

The language of a TM

- Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a TM. The language accepted by M is defined as

  $L(M) = \{w : w \in \Sigma^*, q_0w \overset{*}{\vdash} \alpha p \beta\}$, where $p \in F$ and any tape strings α and β.

  - A language is called recursively enumerable (*Turing-recognizable*) if a TM accepts it.

- But what if a TM does not halt (loops) on a given input? (it may be hard to determine whether it is actually looping or just taking too long to compute).

TM: acceptance by halting

- A Turing machine halts if it enters a state q, scanning a tape symbol X, and there is no move in this situation, i.e., $\delta(q, X)$ is undefined.
- A TM that halts on all inputs is called a decider, as it decides to accept an input or not.
  - A language is called decidable if a TM *decides* it.

Exercise

Provide a TM that decides the language

$$L = \{w\#w : |w| \geq 0, w \in \{0,1\}^*\}$$