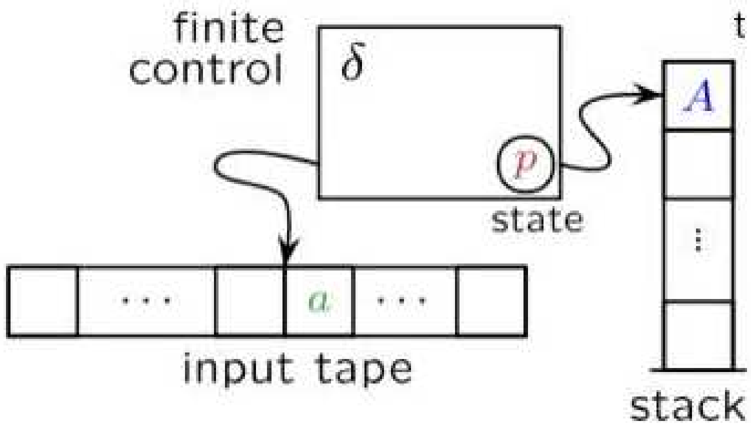
# Procedural descriptors for CFLs

- In our study of regular languages we saw that we can describe such languages “procedurally” using a finite automaton (DFA, NFA, e-NFA), or “declaratively” using a regular expression.
- In our study of context free languages, we first learned how we can use CFGs to “declaratively” describe those languages.
- Today we will learn how we can specify machines called Pushdown Automata that describe CFLs “procedurally”.

# Pushdown Automata

- A pushdown automata (PDA) is essentially an $\epsilon$-NFA with a stack on which it can store symbols.
- PDA can only access information on the stack in a last-in-first out way.
- There are two versions of a PDA, and they differ on how a PDA accepts a string, i.e.,
  - acceptance by accepting state, or
  - acceptance by empty stack.

# PDA informally




- The finite control reads one symbol at a time from the input, and observes the symbol on top of the stack.
- It bases its transitions on its state, the input symbol, and the symbol on top of the stack.
- On a transition, the PDA:
  1. Consumes an input symbol.
  2. Goes to a new state (or stays in the old).
  3. Replaces the top of the stack by a string

# An informal example

- Consider the grammar $P \to aPb \mid \epsilon$, and its language

$$L = \{a^n b^n : n \geq 0\}$$

- Suppose you are a simple machine whose only memory is a stack.
- How would you recognize strings in this language?

# An informal example

Consider the grammar $P \to aPb \mid \epsilon$, and its language

$$L = \{a^n b^n : n \geq 0\}$$

A PDA for $L$ has 3 states and operates as follows:

1. Keeps reading $a$s and pushing them onto the stack until it finds a $b$, which makes it pop an $a$ from the stack and transition to the next state.
2. Keeps reading $b$s and popping as many $a$s; if the "start symbol" is on top of the stack, then accepts.

# PDA, formally

A PDA is a 7-tuple $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where:

- $Q, \Sigma, q_0,$ and $F$, are our old friends;
- $\Gamma$ is the stack alphabet;
- $Z_0$ is the start symbol;
- $\delta$ is the transition function $\delta(q, a, X) = \{(p, \gamma), \cdots\}$, where:
  - $q$ is a state, $a$ is an input symbol, $X$ is a stack symbol
  - $p$ is a state, and $\gamma$ is the string of stack symbols that replaces $X$ on top of the stack. E.g.:

$\delta(q, a, X) = \{(p, \epsilon)\}$ Stack is popped.
$\delta(q, a, X) = \{(p, X)\}$ Stack is unchanged.
$\delta(q, a, X) = \{(p, YZ)\}$ $X$ replaced by $Z$; $Y$ pushed.

# A Graphical notation for PDAs

In the notation used in our textbook, a transition diagram for a PDA consists of the following elements:

- Like in finite automata, nodes represent states, and the start state and accept states are denoted as usual.
- Arcs correspond to transitions of the PDA as follows:


means $(p, \alpha) \in \delta(q, a, X)$

- Conventionally, $Z_0$ or $Z$ denote the start symbol for the stack.

Let's obtain a PDA for the language $\{a^n b^n, n \geq 0\}$ using JFLAP.

# A Nondeterministic PDA

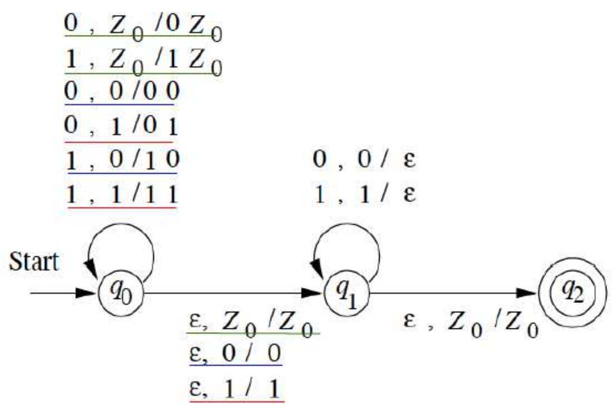
Consider the grammar $P \to 0P0 \mid 1P1 \mid \epsilon$, and its language

$$L = \{ww^R : w \in \{0, 1, \}^*\}$$

A PDA for $L$ has three states and operates as follows:

1. Guesses that it is reading $w$. Stays in state $q_0$, and pushes the input symbol onto the stack.
2. Guesses that it is in the middle of $ww^R$, *i.e.*, $w$ would be on the stack. Goes spontaneously to state $q_1$.
3. It is now reading the head of $w^R$. Compares it to the top of the stack. If they match, pops the stack, and remains in state $q_1$. If they don't match, goes to sleep.
4. If the stack is empty, goes to state $q_2$ and accepts.

# PDA for $L = \{ww^R : w \in \{0, 1, \}^*\}$




$$P = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, \{q_2\}).$$

where $\delta$ is given by the table below

| | $0, Z_0$ | $1, Z_0$ | 0,0 | 0,1 | 1,0 | 1,1 | $\epsilon, Z_0$ | $\epsilon, 0$ | $\epsilon, 1$ |
|---|---|---|---|---|---|---|---|---|---|
| $\rightarrow q_0$ | $q_0, 0Z_0$ | $q_0, 1Z_0$ | $q_0, 00$ | $q_0, 01$ | $q_0, 10$ | $q_0, 11$ | $q_1, Z_0$ | $q_1, 0$ | $q_1, 1$ |
| $q_1$ | | | $q_1, \epsilon$ | | | $q_1, \epsilon$ | $q_2, Z_0$ | | |
| $\star q_2$ | | | | | | | | | |

We assume the PDA accepts a string by consuming it and entering an accepting state.

# PDA: exercise

Design a PDA that recognizes the language

$$\{a^i b^j c^k : i, j, k \geq 1 \text{ and } i = j \text{ or } i = k\}$$

Provide a CFG for the language.

# PDA: exercise

Design a PDA that recognizes the language

$$\{w : w \text{ contains as many 1s as 0s}\}$$

Provide a CFG for the language.

# PDA: exercise

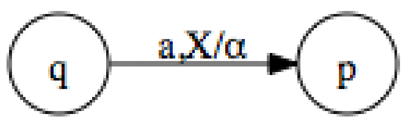
Design a PDA that recognizes the language

$$\{w : w \text{ contains more 1s than 0s}\}$$

Provide a CFG for the language.

# Instantaneous Descriptions (IDs)

- $(q, w, \gamma)$ is an Instantaneous Description of a PDA configuration.
  - $q$ is the current state
  - $w$ is the remaining input
  - $\gamma$ is the contents the stack
- Using IDs to represent a computation step by a PDA:

  suppose $(p, \alpha) \in \delta(q, a, X)$, i.e., 

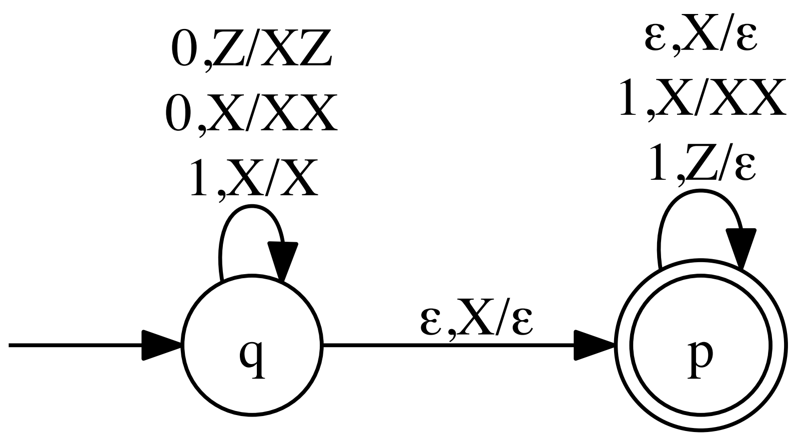
then for all strings $w$,

$$(q, aw, X\beta) \vdash (p, w, \alpha\beta)$$

- $\overset{*}{\vdash}$ is the closure of $\vdash$, meaning "zero or more computation steps (moves) of the PDA".

# Computation as a sequence of IDs

Example: Starting from the ID $(q, 010, Z)$, show all the reachable ID's for the PDA below.



# The languages of a PDA

- For a DFA $D$, we saw that $L(D) = \{w : \hat{\delta}(q_0, w) \in F\}$, where $F$ is the set of accepting states.
- For PDAs, there are two equivalent approaches to define the languages they accept:
  - Acceptance by final state
  - Acceptance by empty stack

# Acceptance by final state

- Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA.
- The language accepted by $P$ by final state is

$$L_f(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \alpha), q \in F\}.$$

Notice:

- $P$ consumes $w$ completely.
- $P$ halts in an accepting state.
- The stack might not be emptied.

# Acceptance by empty stack

- Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA.
- The language accepted by $P$ by empty stack is

$$L_e(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \epsilon)\}$$

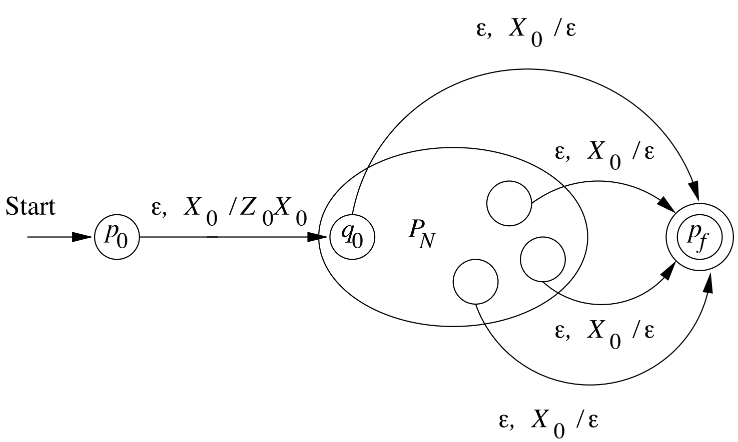
Notice:

- $q$ can be any state
- $P$ at the same time consumes $w$ and empties the stack
- $F$ is redundant in the definition of the PDA and is usually left off in the representation.

# From Empty Stack to Final State

**Theorem**: If $P_N = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ accepts by empty stack, then $\exists$ PDA $P_F$ that accepts by final state such that $L_e(P_N) = L_f(P_F)$ .

- The idea behind the proof of this theorem is to build a PDA $P_F$ that simulates $P_N$ and accepts if $P_N$ empties the stack.




- $\delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$
- Keep all state transitions of $P_N$.
- Add transitions $\delta_F(q, \epsilon, X_0) = \{(p_f, \epsilon)\}$ for every state $q \in Q$

$$(p_0, w, X_0) \underset{P_F}{\vdash} (q_0, w, Z_0X_0) \overset{*}{\underset{P_N}{\vdash}} (q, \epsilon, X_0) \underset{P_F}{\vdash} (p_f, \epsilon, \epsilon)$$

# From Final State to Empty Stack

**Theorem**: If $P_F = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ accepts by final state, then $\exists$ PDA $P_N$ that accepts by empty stack such that $L_f(P_F) = L_e(P_N)$.

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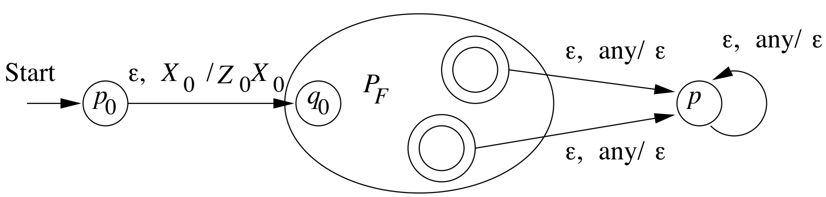



- $\delta_N(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$
- Keep all state transitions of $P_F$.
- Add transitions $\delta_N(q, \epsilon, X) = \{(p_f, \epsilon)\}$ for every state $q \in F$ and symbol $X \in \Gamma \cup \{X_0\}$.

# Equivalence of PDAs and CFGs

- We shall see that PDAs and CFGs are equivalent in power: both represent the same class of languages.
- We will do that by showing how to obtain a PDA from a CFG, and vice-versa.
- Our objective is to prove the following theorem:

  *A language is context-free* *iff* *some PDA recognizes it.*

  Therefore, two lemmas:

**L1**: If a language is context-free, then some PDA recognizes it.

**L2**: If a PDA recognizes some language, then it is context-free.

# Proving L1

- Proof idea: show how to convert a CFG $G$ to a PDA $P$.
- We will design $P$ to simulate leftmost derivations of strings according to $G$.
- The stack will contain the strings of the derivations.

The derivation of a string, according to a given grammar, gives us a hint on the operation of the PDA.

$$A \to aAb \mid \epsilon$$

$$A \Rightarrow aAb \Rightarrow aaAbb \Rightarrow aaaAbbb \Rightarrow aaabbb$$

# Proof idea of L1 (cnt.)

This is how a PDA would accept a string based on the grammar rules:

- Place the start variable on the stack.
- Repeat the following steps forever:
  a) If the top of the stack is a variable $A$, nondeterministically select a rule $A \to \gamma$ and substitute $A$ on the stack by $\gamma$.
  b) If the top of the stack is a terminal symbol $a$, read the next symbol from the input and compare it to $a$. If they match, pop the stack, repeat (b). If they don't match, reject this branch of the nondeterminism.
  c) If the top of the stack is the start symbol, pop it without reading a symbol from input.

# Example: $A \rightarrow aAb|\epsilon$

# Proof idea of L2

- Our task: given a PDA $P$, design a CFG that generates the strings that $P$ accepts.
- Fundamental event in a PDA's history: the popping of a symbol from the stack while consuming input.
- We design a grammar that will have variables named $A_{pXq}$ that generate all strings $w$ such that $(p, w, X\alpha) \overset{*}{\vdash} (q, \epsilon, \alpha), \forall p, q, X$.
- But to facilitate this task, we must make sure $P$ has the following features:
  - It accepts by empty stack.
  - Each transition either pushes a symbol onto the stack or pops one off the stack, but not both at the same time.

# Describing $G$'s rules.

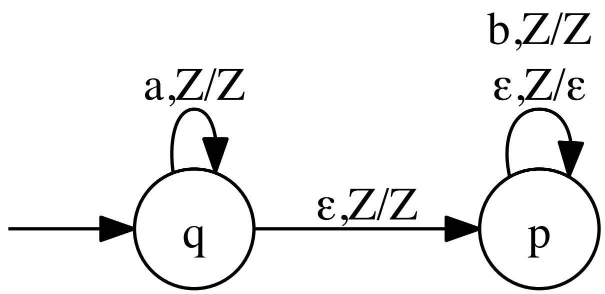
Given $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$, let's construct $G = (V, \Sigma, R, S)$:

1. $V$ consists of the start symbol $S$ and all symbols $A_{pXq}$, where $p, q \in Q$ and $X \in \Gamma$.
2. The production rules in $R$ are:
   (a) For all states $p$, $G$ has the production $S \rightarrow A_{q_0 Z_0 p}$. Therefore, these productions say: $S$ will generate all strings $w$ that cause $P$ to empty its stack.
   (b) Let $\delta(q, a, X) = \{(r, Y_1 Y_2 \cdots Y_k), \cdots\}$, where $a$ is either a symbol of $\Sigma$ or $\epsilon$, and $k \geq 0$.
      - if $k = 0$, i.e., $\delta(q, a, X) = \{(r, \epsilon), \cdots\}$, then $A_{qXr} \rightarrow \epsilon$.
      - if $k > 0$, then for all lists of states $r_1 r_2 \cdots r_k$, $G$ has the production

$$A_{qX\underline{r_k}} \rightarrow a \quad \underbrace{A_{rY_1\underline{r_1}}}_{\text{pops } Y_1} \quad \underbrace{A_{\underline{r_1}Y_2\underline{r_2}}}_{\text{pops } Y_2} \quad \cdots \quad \underbrace{A_{\underline{r_{k-1}}Y_k\underline{r_k}}}_{\text{pops } Y_k}$$

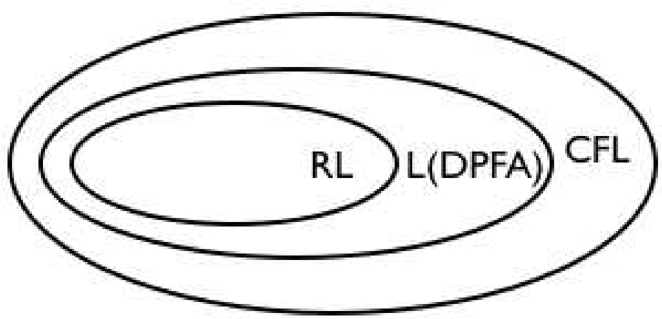
# Example

Obtain a CFG for the following PDA:



# Deterministic PDA

- Recall that DFAs and NFAs are equivalent in language recognition power.
- The same does not happen with PDAs.
- NPDAs are more powerful then DPDAs.
- DPDAs accept a class of languages that is between RLs and CFLs.
- Let's start by defining what is a DPDA. Then show that DPDA languages include all RLs.

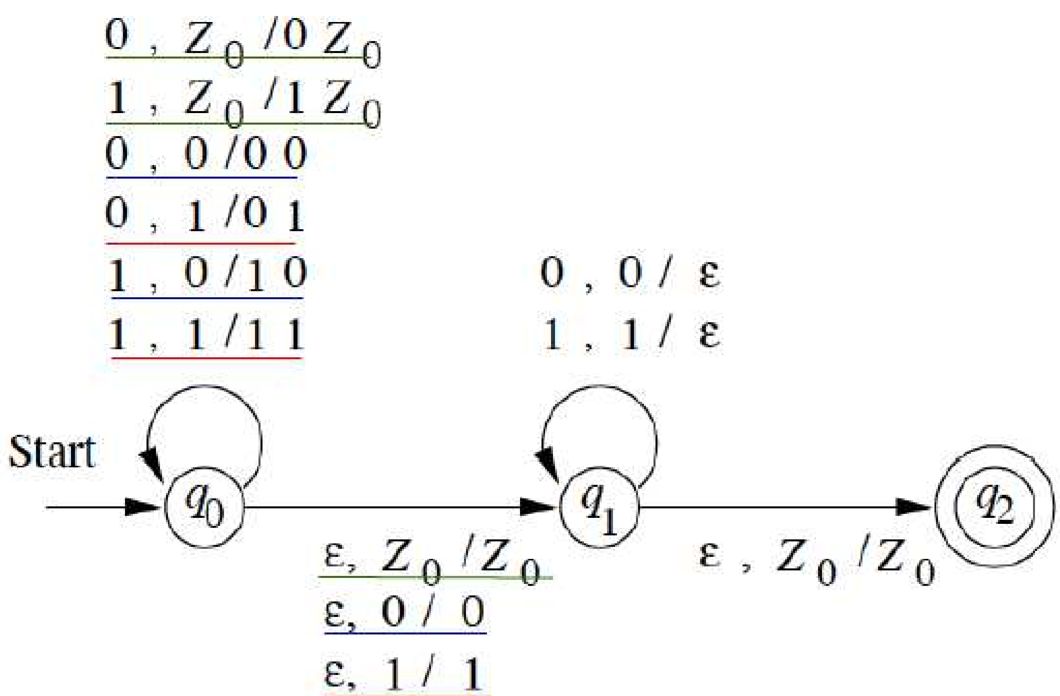


# Deterministic PDA (DPDA)

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is deterministic iff:

- $\delta(q, a, X)$ is always empty or a singleton.
- If $\delta(q, a, X)$ is nonempty, then $\delta(q, \epsilon, X)$ must be empty.

But before analyzing a DPDA, let's see the source of nondeterminism in the PDA for $\{ww^R : w \in \{0,1\}^*\}$

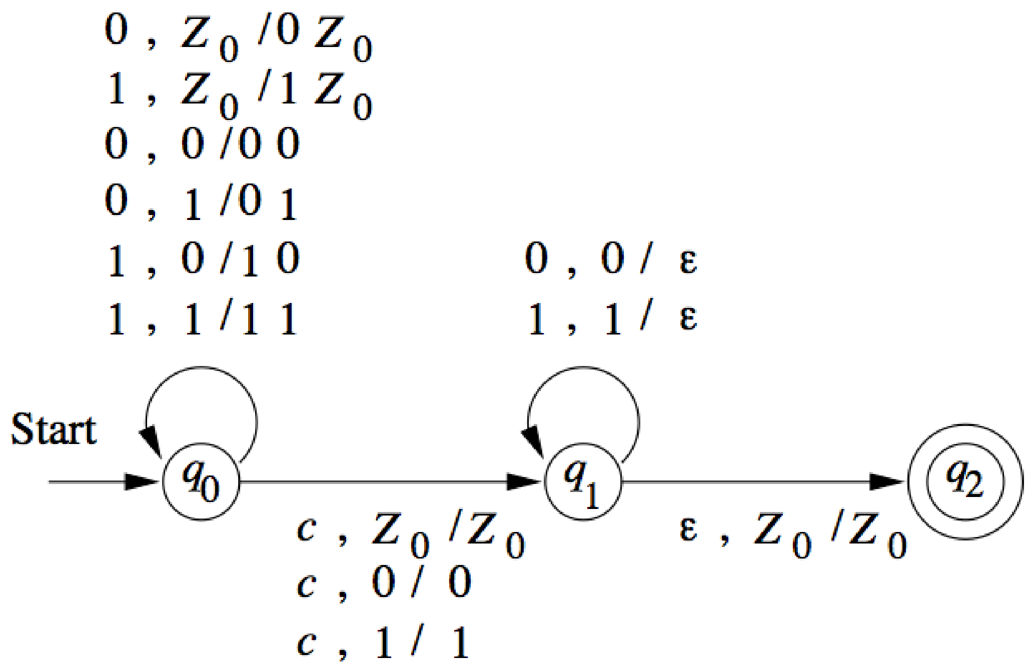


# Example of a DPDA

Let us define the following language

$L_{wcwr} = \{wcw^R : w \in \{0,1\}^*\}$

Then $L_{wcwr}$ is recognized by the following DPDA




Unlike in the previous example, all move choices for this PDA are deterministic.

$pda.1$

# Exercise

Provide a DPDA for the following languages:

$$L = \{0^n 1^m : n \leq m\}$$

$$L = \{0^n 1^m : n \geq m\}$$