# Some Class
# Random Examples

## Marcus Denslow

2025-10-15

# Contents

# Chapter 1

## 1.1 Random Examples

---

**Definition 1.1.1: Limit of Sequence in $\mathbb{R}$**

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say
$$\lim_{n\to\infty} s_n = s$$
where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number $N$ such that for $n > N$
$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

---

**Question 1**

Is the set $x-\text{axis}\backslash\{\text{Origin}\}$ a closed set

---

*Solution:* We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

---

**Note:-**

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space
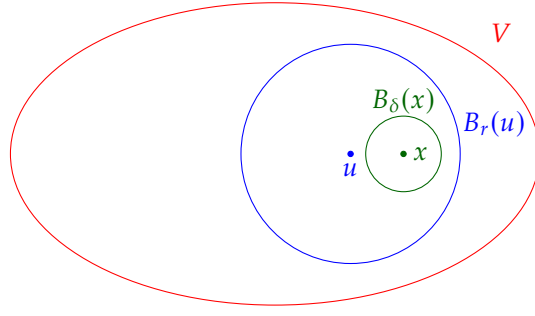
---

**Claim 1.1.1** Topology

Topology is cool

---

**Example 1.1.1** (Open Set and Close Set)

Open Set:
- $\phi$
- $\bigcup_{x\in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

Closed Set:
- $X, \phi$
- $\overline{B_r(x)}$

$x-\text{axis} \cup y-\text{axis}$

---

**Theorem 1.1.1**

If $x \in$ open set $V$ then $\exists \delta > 0$ such that $B_\delta(x) \subset V$

---

*Proof:* By openness of $V$, $x \in B_r(u) \subset V$

Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u,x)$. Choose $\delta$ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u,y) < r$ but

$$d(u,y) \leqslant d(u,x) + d(x,y) < d + \delta < r$$

☺

---

**Corollary 1.1.1**

By the result of the proof, we can then show...

---

**Lemma 1.1.1**

Suppose $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

---

**Proposition 1.1.1**

$1 + 1 = 2$.

---

## 1.2 Random

---

**Definition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$**

Let $V$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on $V$ is function $\|\cdot\| \, V \to \mathbb{R}_{\geqslant 0}$ satisfying

① $\|x\| = 0 \iff x = 0 \; \forall \, x \in V$

② $\|\lambda x\| = |\lambda| \|x\| \; \forall \, \lambda \in \mathbb{R}(\text{or } \mathbb{C}), \; x \in V$

③ $\|x + y\| \leqslant \|x\| + \|y\| \; \forall \, x, y \in V$ (Triangle Inequality/Subadditivity)

And $V$ is called a normed linear space.
● Same definition works with $V$ a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geqslant 0}$) where ② becomes $\|\lambda x\| = |\lambda| \|x\|$
$\forall \, \lambda \in \mathbb{C}, \; x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

---

**Example 1.2.1** ($p$−Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left( |x_1|^p + |x_2|^p + \cdots + |x_m|^p \right)^{\frac{1}{p}}$$

(In school $p = 2$)

---

**Special Case $p = 1$**: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
**Special Case $p \to \infty$** ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p$−norms are nothing but $|x|$. Now exercise

3

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$ ?)

*Solution:* **For Property ③ for norm-2**

**When field is $\mathbb{R}$ :**

We have to show

$$\sum_i (x_i + y_i)^2 \leqslant \left( \sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2$$

$$\implies \sum_i (x_i^2 + 2x_i y_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_i y_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

> **Note:-**
>
> - $\|x\|^2 = \langle x, x \rangle$
>
> - $\langle x, y \rangle = \langle y, x \rangle$
>
> - $\langle \cdot, \cdot \rangle$ is $\mathbb{R}-$linear in each slot i.e.
>
> $$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$
>
> Here in $\langle x, y \rangle$ $x$ is in first slot and $y$ is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable $\lambda$

$$\langle x - \lambda y, x - \lambda y \rangle = \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle$$
$$= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle$$
$$= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

**When field is $\mathbb{C}$ :**

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geqslant 0$

## 1.3 Algorithms

---

**Algorithm 1:** what

**Input:** This is some input
**Output:** This is some output
/* This is a comment */

```
1  some code here;
2  x ← 0;
3  y ← 0;
4  if x > 5 then
5  |   x is greater than 5 ;                                    // This is also a comment
6  else
7  |   x is less than or equal to 5;
8  end
9  foreach y in 0..5 do
10 |   y ← y + 1;
11 end
12 for y in 0..5 do
13 |   y ← y − 1;
14 end
15 while x > 5 do
16 |   x ← x − 1;
17 end
18 return Return something here;
```

---