

Relativistic Quantum Waves (Klein-Gordon Equation

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Chapter 1

Deriving the KG Equation

1.1 double deriving

Definition 1.1.1: Relativity: the mass shell

$$p \cdot p = (mc)^2 \rightarrow (mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2$$

Definition 1.1.2: Quantum: energy and momentum operators

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \text{ so } \left(\frac{E}{c}\right)^2 \text{ becomes } -\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2}.$$

$$\hat{p} = -i\hbar \nabla, \text{ so } -p_x^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial x^2}.$$

$$\text{likewise, } -p_y^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial y^2} \text{ and } -p_z^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial z^2}.$$

1.2 Plugging in the new values

we can now plugin these into the original equation:

$$\begin{aligned} (mc)^2 &= \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 \\ (mc)^2 &= -\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \hbar^2 \frac{\partial^2}{\partial x^2} + \hbar^2 \frac{\partial^2}{\partial y^2} + \hbar^2 \frac{\partial^2}{\partial z^2} \\ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 &= 0. \end{aligned}$$

1.3 this is the Klein-Gordon Equation!

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0 \quad (1.1)$$

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 \right] \psi = 0 \quad (1.2)$$

1.4 Replacing with Laplacian

We know that $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \nabla^2$ -otherwise known as a Laplacian

So the function becomes

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right] \psi = 0 \quad (1.3)$$

1.5 d'Alembertian

$\square =$ d'Alembertian

We can also rewrite $\left(\frac{mc}{\hbar} \right)^2$ as μ^2

Note:-

We could also write $\left(\frac{mc}{\hbar} \right)$ as just m as in this universe it would become $\left(\frac{m \cdot 1}{1} \right)$ which is just the mass but this is the correct way to write it.

So our final equation becomes

$$[\square + \mu^2] \psi = 0 \quad (1.4)$$

Chapter 2

Four-momentum Eigenstates

2.1 Klein-Gordon Plane Wave

Definition 2.1.1: Klein-Gordon Plane Wave function

$$\psi = A \exp \left(-\frac{i}{\hbar} p \cdot x \right)$$

$$p = [E/c, \vec{p}], \quad x = [ct, \vec{x}]$$

$$A \in \mathbb{C}, \quad p^0 = \frac{E}{c} = 0 \pm \sqrt{|\vec{p}|^2 + m^2 c^2}$$

we can rewrite the original Equation as:

$$\psi = A \exp \left(\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right)$$

$$\psi = A \exp \left(\frac{i}{\hbar} \left(\vec{p} \cdot \vec{x} \pm c \sqrt{|\vec{p}|^2 + m^2 c^2} t \right) \right)$$

2.2 Proof that plane waves $\psi = A \exp [-ip \cdot x/\hbar]$ satisfy K.G.

rewrite K.G.: $[\square + \mu^3] \psi = 0 \rightarrow \square \psi = -\mu^2 \psi$

Does d'Alembertian do the same thing as multiplying by $-\mu^2$?

$$\square = \square \left[\exp \left[-\frac{i}{\hbar} p \cdot x \right] \right] = \square \left[\exp \left[\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right] \right] \quad (2.1)$$

Definition of d'Alembertian: $\square = \frac{1}{c^2} \frac{\partial}{\partial t^2} - \nabla^2$

$$\square = \left[-\left(\frac{E}{c\hbar} \right) + \left(\frac{|\vec{p}|^2}{\hbar} \right) \right] \exp \left[\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right] = -\left(\frac{mc}{\hbar} \right)^2 \psi \quad (2.2)$$