

Quantum Theory of Radiation

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Chapter 1

Quantum Theory of Radiation

1.1 Transverse and Longitudinal Fields

In non-relativistic Quantum Mechanics, the static Electric field is represented by a scalar potential, magnetic fields by the vector potential, and the radiation field also through the vector potential. It will be convenient to keep this separation between the large static atomic Electric field and the radiation fields, however, the equations we have contain the four-vector A_μ with all the fields mixed. When we quantize the field, all E and B fields as well as electromagnetic waves will be made up of photons. It is useful to be able to separate the E fields due to fixed charges from the EM radiation from moving charges. This separation is not Lorentz invariant, but it is still useful. Enrico Fermi showed, in 1930, that A_\parallel together with A_0 give rise to Coulomb interactions between particles, whereas A_\perp gives rise to the EM radiation from moving charges. With this separation, we can maintain the form of our non-relativistic Hamiltonian.

$$\mathbf{H} = \sum_j \frac{1}{2m_j} \left(\vec{p} - \frac{e}{c} \vec{A}_\perp(\vec{x}_j) \right)^2 + \sum_{i>j} \frac{e_i e_j}{4\pi |\vec{x}_i - \vec{x}_j|} + \mathbf{H}_{\text{rad}} \quad (1.1)$$

Where \mathbf{H}_{rad} is purely the Hamiltonian of the radiation (containing only \vec{A}_\perp) and \vec{A}_\perp is the part of the vector potential which satisfies $\nabla \cdot \vec{A}_\perp = 0$. Note that \vec{A}_\parallel and A_0 appear nowhere in the Hamiltonian. Instead, we have the Coulomb potential. This separation allows us to continue with our standard Hydrogen solution and just add radiation. We will not derive this result.

In a region in which there are no source terms,

$$j_\mu = 0 \quad (1.2)$$

we can make a gauge transformation which eliminates A_0 by choosing Λ such that

$$\frac{1}{c} \frac{\partial \Lambda}{\partial t} = A_0 \quad (1.3)$$

Since the fourth component of A_μ is now eliminated, the Lorentz condition now implies that

$$\vec{\nabla} \cdot \vec{A} = 0 \quad (1.4)$$

Again, making one component of a 4-vector zero is not a Lorentz invariant way of working. We have to redo the gauge transformation if we move to another frame.

If $j_\mu \neq 0$, then we cannot eliminate A_0 , since $\boxed{A}_0 = \frac{j_0}{c}$ and we are only allowed to make gauge transformation for which $\boxed{\Lambda} = 0$. In this case we must separate the vector potential into the transverse and longitudinal parts, with

$$\begin{aligned}\vec{A} &= \vec{A}_\perp + \vec{A}_\parallel \\ \vec{\nabla} \cdot \vec{A}_\perp &= 0 \\ \vec{\nabla} \times \vec{A}_\parallel &= 0.\end{aligned}$$

We will now study the radiation field in a region with no sources so that $\vec{\nabla} \cdot \vec{A} = 0$. We will use the equations

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A} \\ \vec{E} &= -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= 0.\end{aligned}$$

1.2 Fourier Decomposition of Radiation Oscillators

Our goal is to write the Hamiltonian for the radiation field in terms of a sum of harmonic oscillators Hamiltonians. The first step is to write the radiation field in a simple way as possible, as a sum of harmonic components. We will work in a cubic volume $V = L^3$ and apply periodic boundary conditions on our electromagnetic waves. We also assume for now that there are no sources inside the region so that we can make a gauge transformation to make $A_0 = 0$ and hence $\vec{\nabla} \cdot \vec{A} = 0$. We decompose the field into its Fourier components at $t = 0$

$$\vec{A}(\vec{x}, t = 0) = \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \hat{\epsilon}^{(\alpha)} \left(c_{k,\alpha}(t=0) e^{i\vec{k} \cdot \vec{x}} + c_{k,\alpha}^*(t=0) e^{-i\vec{k} \cdot \vec{x}} \right) \quad (1.5)$$

where $\hat{\epsilon}^{(\alpha)}$ are real unit vectors, and $c_{k,\alpha}$ is the coefficient of the wave with wave vector \vec{k} and polarization vector $\hat{\epsilon}^{(\alpha)}$. Once the wave vector is chose, the two polarization vectors must be picked so that $\hat{\epsilon}^{(1)}$, $\hat{\epsilon}^{(2)}$, and \vec{k} form a right handed orthogonal system. The components of the wave vector must satisfy

$$k_i = \frac{2\pi n_i}{L} \quad (1.6)$$

due to the periodic boundary conditions. The factor out front is set to normalize the states nicely since

$$\frac{1}{V} \int d^3x e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k}' \cdot \vec{x}} = \delta_{\vec{k}\vec{k}'}, \quad (1.7)$$

and

$$\hat{\epsilon}^{(\alpha)} \cdot \hat{\epsilon}^{(\alpha')} = \delta_{\alpha\alpha'} \quad (1.8)$$

We know the time dependence of the waves from Maxwell's equation,

$$c_{k,\alpha}(t) = c_{k,\alpha}(0) e^{-i\omega t} \quad (1.9)$$

where $\omega = kc$. We can now write the vector potential as a function of position and time.

$$\vec{A}(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \hat{\epsilon}^{(\alpha)} \left(c_{k,\alpha}(t) e^{i\vec{k} \cdot \vec{x}} + c_{k,\alpha}^*(t) e^{-i\vec{k} \cdot \vec{x}} \right) \quad (1.10)$$

We may need to write this solution in several different ways, and use the best one for the calculation being performed. One nice way to write this is in terms 4-vector k_μ , the wave number,

$$k_\mu = \frac{p_\mu}{\hbar} = (k_x, k_y, k_z, ik) = \left(k_x, k_y, k_z, i \frac{\omega}{c} \right) \quad (1.11)$$

so that

$$k_p x_p = k \cdot x = \vec{k} \cdot \vec{x} - \omega t \quad (1.12)$$

We can then write the radiation field in a more covariant way

$$\vec{A}(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \hat{\epsilon}^{(\alpha)} \left(c_{k,\alpha}(0) e^{ik_p x_p} + c_{k,\alpha}^*(0) e^{-ik_p x_p} \right) \quad (1.13)$$

A convenient shorthand for calculations is possible by noticing that the second term is just the complex conjugate of the first.

$$\begin{aligned} \vec{A}(\vec{x}, t) &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \hat{\epsilon}^{(\alpha)} \left(c_{k,\alpha}(0) e^{ik_p x_p} + c.c. \right) \\ \vec{A}(\vec{x}, t) &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \hat{\epsilon}^{(\alpha)} c_{k,\alpha}(0) e^{ik_p x_p} + c.c. \end{aligned}$$

Note again that we have made this a transverse field by construction. The unit vectors $\hat{\epsilon}^{(\alpha)}$ are transverse to the direction of propagation. Also note that we are working in a gauge with $A_4 = 0$, so this can also represent

the 4-vector form of the potential. The Fourier Decomposition of the radiation field can be written very simply.

$$A_\mu = \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} c_{k,\alpha}(0) e^{ik_p x_p} + c.c. \quad (1.14)$$

This choice of gauge makes switching between 4-vector and 3-vector expressions for the potential trivial. Let's verify that this decomposition of the radiation field satisfies the Maxwell equation, just for some practice. It's most convenient to use the covariant form of the equation and field.

$$\begin{aligned} \square A_\mu &= 0 \\ \square \left(\frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} c_{k,\alpha}(0) e^{ik_p x_p} + c.c. \right) &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} c_{k,\alpha}(0) \square e^{ik_p x_p} + c.c. \\ &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} c_{k,\alpha}(0) (-k_\nu k_\nu) e^{ik_p x_p} + c.c. = 0 \end{aligned}$$

The result is zero since $k_\nu k_\nu = k^2 - k^2 = 0$.

Let's also verify that $\vec{\nabla} \cdot \vec{A} = 0$

$$\begin{aligned} \vec{\nabla} \cdot \left(\frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \hat{\epsilon}^{(\alpha)} c_{k,\alpha}(t) e^{i\vec{k} \cdot \vec{x}} + c.c. \right) &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 c_{k,\alpha}(t) \hat{\epsilon}^{(\alpha)} \cdot \vec{\nabla} e^{i\vec{k} \cdot \vec{x}} + c.c. \\ &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 c_{k,\alpha}(t) \hat{\epsilon}^{(\alpha)} \vec{k} e^{i\vec{k} \cdot \vec{x}} + c.c. = 0. \end{aligned}$$

The result here is zero because $\hat{\epsilon}^{(\alpha)} \cdot \vec{k} = 0$

1.3 The Hamiltonian for the Radiation Field

We now wish to compute the Hamiltonian in terms of the coefficients $c_{k,\alpha}(t)$. This is an important calculation because we will use the Hamiltonian formalism to do the quantization of the field. We will do the calculation using the covariant notation (while Sakurai outlines an alternate calculation using 3-vectors). We have already calculated the Hamiltonian density of a classical EM field.

$$\mathcal{H} = F_{\mu 4} \frac{\partial A_\mu}{\partial x_4} + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \quad (1.15)$$

$$\begin{aligned} \mathcal{H} &= \left(\frac{\partial A_4}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_4} \right) \frac{\partial A_\mu}{\partial x_4} + \frac{1}{4} \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) \\ \mathcal{H} &= -\frac{\partial A_4}{\partial x_\mu} \frac{\partial A_\mu}{\partial x_4} + \frac{1}{2} \left(\frac{\partial A_\nu}{\partial x_4} \frac{\partial A_\nu}{\partial x_4} - \frac{\partial A_\nu}{\partial x_\nu} \frac{\partial A_\mu}{\partial x_\nu} \right). \end{aligned}$$

Now lets compute the basic element of the above formula for our decomposed radiation field.

$$\begin{aligned}
A_\mu &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} \left(c_{k,\alpha}(0) e^{ik_p x_p} + c_{k,\alpha}^*(0) e^{-ik_p x_p} \right) \\
\frac{\partial A_\mu}{\partial x_\nu} &= \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} \left(c_{k,\alpha}(0) (ik_\nu) e^{ik_p x_p} + c_{k,\alpha}^*(0) (-ik_\nu) e^{-ik_p x_p} \right) \\
\frac{\partial A_\mu}{\partial x_\nu} &= i \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} \frac{\omega}{c} \left(c_{k,\alpha}(0) e^{ik_p x_p} - c_{k,\alpha}^*(0) e^{-ik_p x_p} \right) \\
\frac{\partial A_\mu}{\partial x_4} &= -\frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} \frac{\omega}{c} \left(c_{k,\alpha}(0) e^{ik_p x_p} - c_{k,\alpha}^*(0) e^{-ik_p x_p} \right).
\end{aligned}$$

We have all the elements to finish the calculation of the Hamiltonian. Before pulling this all together in a brute force way, it's good to realize that almost all the terms will give zero. We see that the derivative of A_μ is proportional to a 4-vector, say k_ν and to a polarization vector, say $\epsilon_\mu^{(\alpha)}$. The dot products of the 4-vectors, either k with itself, or k with ϵ are zero. Going back to our expression for the Hamiltonian density, we can eliminate some terms.

$$\begin{aligned}
\mathcal{H} &= -\frac{\partial A_\mu}{\partial x_4} \frac{\partial A_\mu}{\partial x_4} + \frac{1}{2} \left(\frac{\partial A_\nu}{\partial x_\mu} \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\nu}{\partial x_\mu} \frac{\partial A_\mu}{\partial x_\nu} \right) \\
\mathcal{H} &= -\frac{\partial A_\nu}{\partial x_4} \frac{\partial A_\mu}{\partial x_4} + \frac{1}{2} (0 - 0) \\
\mathcal{H} &= -\frac{\partial A_\mu}{\partial x_4} \frac{\partial A_\mu}{\partial x_4}.
\end{aligned}$$

The remaining term has a dot product between polarization vectors which will be nonzero if the polarization vectors are the same. (Note that this simplification is possible because we have assumed no sources in the region.) The total Hamiltonian we are aiming at, is the integral of the Hamiltonian density.

$$H = \int d^3x \mathcal{H} \quad (1.16)$$

When we integrate over the volume only products like $e^{ik_p x_p} e^{-ik_p' x_p}$ will give a nonzero result. So when we multiply one sum over k by another, only the terms with the same k will contribute to the integral, basically because the waves with different wave number are orthogonal.

$$\frac{1}{V} \int d^3x e^{ik_p x_p} e^{-ik_p' x_p} = \delta_{kk'} \quad (1.17)$$

$$\begin{aligned}
H &= \int d^3x \mathcal{H} \\
\mathcal{H} &= -\frac{\partial A_\mu}{\partial x_4} \frac{\partial A_\mu}{\partial x_4} \\
\frac{\partial A_\mu}{\partial x_4} &= -\frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1}^2 \epsilon_\mu^{(\alpha)} \left(c_{k,\alpha}(0) \frac{\omega}{c} e^{ik_p x_p} - c_{k,\alpha}^*(0) \frac{\omega}{c} e^{-ik_p x_p} \right) \\
H &= -\int d^3x \frac{\partial A_\mu}{\partial x_4} \frac{\partial A_\mu}{\partial x_4} \\
H &= -\int d^3x \frac{1}{V} \sum_k \sum_{\alpha}^2 \left(c_{k,\alpha}(0) \frac{\omega}{c} e^{ik_p x_p} - c_{k,\alpha}^*(0) e^{-ik_p x_p} \right) \\
H &= -\sum_k \sum_{\alpha=1}^2 \left(\frac{\omega}{c} \right)^2 \left[-c_{k,\alpha}(t) c_{k,\alpha}^*(t) - c_{k,\alpha}^*(t) c_{k,\alpha}(t) \right] \\
H &= \sum_k \sum_{\alpha}^2 \left(\frac{\omega}{c} \right)^2 \left[c_{k,\alpha}(t) c_{k,\alpha}^*(t) + c_{k,\alpha}^*(t) c_{k,\alpha}(t) \right] \\
H &= \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 \left[c_{k,\alpha}(t) c_{k,\alpha}^*(t) + c_{k,\alpha}^*(t) c_{k,\alpha}(t) \right].
\end{aligned}$$

This is the result we will use to quantize the field. We have been careful not to commute C and C^* here in anticipation of the fact that they do not commute.

It should not be a surprise that the terms that made up the Lagrangian gave a zero contribution because $\mathcal{L} = \frac{1}{2}(E^2 - B^2)$ and we know that E and B have the same magnitude in radiation field. (There is one wrinkle we have glossed over; terms with $\vec{k}' = -\vec{k}$.)

1.4 Canonical Coordinates and Momenta

We now have the Hamiltonian for the radiation field

$$\boxed{H = \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 \left[c_{k,\alpha}(t) c_{k,\alpha}^*(t) + c_{k,\alpha}^*(t) c_{k,\alpha}(t) \right]} \quad (1.18)$$

It was with the Hamiltonian that we first quantized the non-relativistic motion of particles. The position and momentum became operators which did not commute. Lets define $c_{k,\alpha}$ to be the time dependent Fourier coefficient.

$$\ddot{c}_{k,\alpha} = -\omega^2 c_{k,\alpha} \quad (1.19)$$

We can then simplify our notation a bit

$$H = \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 \left[c_{k,\alpha} c_{k,\alpha}^* + c_{k,\alpha}^* c_{k,\alpha} \right] \quad (1.20)$$

This now clearly looks like the Hamiltonian for a collection of uncoupled oscillators; one oscillator for each wave vector and polariation.

We wish to write the Hamiltonian in terms of a coordinate for each oscillator and the conjugate momenta. The coordinate should be real so it can be represented by a Hermitian operator and have a physical meaning. The simplest choice for a real coordinate is $c + c^*$. With a little effort we can identify the coordinate

$$Q_{k,\alpha} = \frac{1}{c} \left(c_{k,\alpha} + c_{k,\alpha}^* \right) \quad (1.21)$$

and its conjugate momentum for each oscillator,

$$P_{k,\alpha} = -\frac{i\omega}{c} \left(c_{k,\alpha} - c_{k,\alpha}^* \right) \quad (1.22)$$

The Hamiltonian can be written in terms of these

$$\begin{aligned} H &= \frac{1}{2} \sum_{k,\alpha} \left[P_{k,\alpha}^2 + \omega^2 Q_{k,\alpha}^2 \right] \\ &= \frac{1}{2} \sum_{k,\alpha} \left[-\left(\frac{\omega}{c} \right)^2 \left(c_{k,\alpha} - c_{k,\alpha}^* \right)^2 + \left(\frac{\omega}{c} \right)^2 \left(c_{k,\alpha} + c_{k,\alpha}^* \right)^2 \right] \\ &= \frac{1}{2} \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 \left[-\left(c_{k,\alpha} - c_{k,\alpha}^* \right)^2 + \left(c_{k,\alpha} + c_{k,\alpha}^* \right)^2 \right] \\ &= \frac{1}{2} \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 2 \left[c_{k,\alpha} c_{k,\alpha}^* + c_{k,\alpha}^* c_{k,\alpha} \right] \\ &= \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 \left[c_{k,\alpha} c_{k,\alpha}^* + c_{k,\alpha}^* c_{k,\alpha} \right]. \end{aligned}$$

This verifies that this choice gives the right Hamiltonian. We should also check that this choice of coordinates and momenta satisfy Hamiltonian's equations to identify them as the canonical coordinates. The first equation is

$$\begin{aligned} \frac{\partial H}{\partial Q_{k,\alpha}} &= -\dot{P}_{k,\alpha} \\ \omega^2 Q_{k,\alpha} &= \frac{i\omega}{c} \left(\dot{c}_{k,\alpha} - \dot{c}_{k,\alpha}^* \right) \\ \frac{\omega^2}{c} \left(c_{k,\alpha} + c_{k,\alpha}^* \right) &= \frac{i\omega}{c} \left(-i\omega c_{k,\alpha} - i\omega c_{k,\alpha}^* \right) \\ \frac{\omega^2}{c} \left(c_{k,\alpha} + c_{k,\alpha}^* \right) &= \frac{\omega^2}{c} \left(c_{k,\alpha} + c_{k,\alpha}^* \right). \end{aligned}$$

This one checks out OK.

The other equation of Hamiltonian is

$$\begin{aligned}\frac{\partial H}{\partial P_{k,\alpha}} &= Q_{k,\alpha} \\ P_{k,\alpha} &= \frac{1}{c} \left(\dot{c}_{k,\alpha} + \dot{c}_{k,\alpha}^* \right) \\ -\frac{i\omega}{c} \left(c_{k,\alpha} - c_{k,\alpha}^* \right) &= \frac{1}{c} \left(-i\omega c_{k,\alpha} + i\omega c_{k,\alpha}^* \right) \\ \frac{-i\omega}{c} \left(c_{k,\alpha} - c_{k,\alpha}^* \right) &= -\frac{i\omega}{c} \left(c_{k,\alpha} - c_{k,\alpha}^* \right).\end{aligned}$$

This also checks out, so we have identified the canonical coordinates and momenta of our oscillators.

We have a collection of uncoupled oscillators with identified canonical coordinate and momentum. The next step is to quantize the oscillators.

1.5 Quantization of the Oscillators

To summarize the result of the calculations of the last section we have the Hamiltonian for the radiation field.

$\mathbf{H} = \sum_{k,\alpha} \left(\frac{\omega}{c} \right)^2 \left[c_{k,\alpha} c_{k,\alpha}^* + c_{k,\alpha}^* c_{k,\alpha} \right]$	(1.23)
$Q_{k,\alpha} = \frac{1}{c} \left(c_{k,\alpha} + c_{k,\alpha}^* \right)$	
$P_{k,\alpha} = -\frac{i\omega}{c} \left(c_{k,\alpha} - c_{k,\alpha}^* \right)$	
$H = \frac{1}{2} \sum_{k,\alpha} \left[P_{k,\alpha}^2 + \omega^2 Q_{k,\alpha}^2 \right]$	

Soon after the development of non-relativistic quantum mechanics, Dirac proposed that the canonical variables of the radiation oscillators be treated like p and x in the quantum mechanics we know. The place to start is with the commutators. The coordinate and its corresponding momentum do not commute. For example $[p_x, x] = \frac{\hbar}{i}$. Coordinates and momenta that do not correspond, do not commute. For example $[p_y, x] = 0$. Different coordinates commute with each other as do different momenta. We will impose the same rules here.

$$\begin{aligned}[Q_{k,\alpha}, P_{k',\alpha'}] &= i\hbar \delta_{kk'} \delta_{\alpha\alpha'} \\ [Q_{k,\alpha}, Q_{k',\alpha'}] &= 0 \\ [P_{k,\alpha}, P_{k',\alpha'}] &= 0.\end{aligned}$$

By now we know that if the Q and P do not commute, neither do the c and c^* so we should continue to avoid commuting them.

Since we are dealing with harmonic oscillators, we want to find the analog of the raising and lowering operators. We developed the raising and lowering operators by trying to write the Hamiltonian as $H = A^\dagger A \hbar \omega$. Following the same idea, we get

$$\begin{aligned}
a_{k,\alpha} &= \frac{1}{\sqrt{2\hbar\omega}} (\omega Q_{k,\alpha} + iP_{k,\alpha}) \\
a_{k,\alpha}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} (\omega Q_{k,\alpha} - iP_{k,\alpha}) \\
a_{k,\alpha}^\dagger a_{k,\alpha} &= \frac{1}{2\hbar\omega} (\omega Q_{k,\alpha} - iP_{k,\alpha}) (\omega Q_{k,\alpha} + iP_{k,\alpha}) \\
&= \frac{1}{2\hbar\omega} \left(\omega^2 Q_{k,\alpha}^2 + P_{k,\alpha}^2 + i\omega Q_{k,\alpha} P_{k,\alpha} - i\omega P_{k,\alpha} Q_{k,\alpha} \right) \\
&= \frac{1}{2\hbar\omega} \left(\omega^2 Q_{k,\alpha}^2 + P_{k,\alpha}^2 + i\omega Q_{k,\alpha} P_{k,\alpha} - i\omega \left(Q_{k,\alpha} P_{k,\alpha} + \frac{\hbar}{i} \right) \right) \\
&= \frac{1}{2\hbar\omega} \left(\omega^2 Q_{k,\alpha}^2 + P_{k,\alpha}^2 - \hbar\omega \right) \\
a_{k,\alpha}^\dagger a_{k,\alpha} + \frac{1}{2} &= \frac{1}{2\hbar\omega} \left(\omega^2 Q_{k,\alpha}^2 + P_{k,\alpha}^2 \right).
\end{aligned}$$