Relativistic Quantum Waves (Klein-Gordon Equation

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Contents

Chapter 1

Deriving the KG Equation

1.1 double deriving

Definition 1.1.1: Relativity: the mass shell

$p \cdot p = (mc)^2 \longrightarrow (mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2$

Definition 1.1.2: Quantum: energy and momentum operators

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \text{ so } \left(\frac{E}{c}\right)^2 \text{ becomes } -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}.$$

$$\hat{p} = -i\hbar\nabla, \text{so } -p_x^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial x^2}.$$

$$\text{likewise, } -p_y^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial y^2} \text{ and } -p_z^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial z^2}.$$

1.2 Plugging in the new values

we can now plugin these into the original equation:

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_y^2$$

$$(mc)^2 = -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2 \frac{\partial^2}{\partial x^2} + \hbar^2 \frac{\partial^2}{\partial y^2} + \hbar^2 \frac{\partial^2}{\partial z^2}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0.$$

1.3 this is the Klein-Gordon Equation!

$$\frac{2}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0 \tag{1.1}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.2}$$

1.4 Replacing with Laplacian

We know that $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \nabla^2$ -otherwise known as a Laplacian

So the function becomes

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.3}$$

1.5 d'Alembertian

$\Box =$ d'Alembertian
We can also rewrite $\left(\frac{mc}{\hbar}\right)^2$ as μ^2

Note:-

We could also write $\left(\frac{mc}{\hbar}\right)$ as just m as in this universe it would become $\left(\frac{m \cdot 1}{1}\right)$ which is just the mass but this is the correct way to write it.

So our final equation becomes

$$\left[\Box + \mu^2\right]\psi = 0 \tag{1.4}$$

Chapter 2

Four-momentum Eigenstates

2.1 Klein-Gordon Plane Wave

Definition 2.1.1: Klein-Gorden Plane Wave function

$\psi = A \text{ exp } \left(-\frac{i}{\hbar} p \cdot x\right)$

$$p = \left[E/c, \vec{p}\right], \qquad x = \left[ct, \vec{x}\right]$$
$$A \in \mathbb{C}, \qquad p^0 = \frac{E}{c}0 \pm \sqrt{\left|\vec{p}\right|^2 + m^2c^2}$$

we can rewrite the original Equation as:

$$\psi = A \text{ exp } \left(\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} - Et\right)\right)$$
$$\psi = A \text{ exp } \left(\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} \pm c\sqrt{\left|\vec{p}\right|^2 + m^2c^2t}\right)\right)$$

2.2 Proof that plane waves $\psi = A \text{ exp } \left[-ip \cdot x/\hbar\right]$ satisfy K.G.

rewrite K.G.: $\left[\Box + \mu^3\right]\psi = 0 \to \Box\psi = -\mu^2\psi$

Does d'Alembertian ddo the same thing as multiplying by $-\mu^2$?

$$\Box = \Box\left[\text{exp}\left[-\frac{i}{\hbar}p \cdot x\right]\right] = \Box\left[\text{exp}\left[\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} - Et\right)\right]\right] \tag{2.1}$$

Definition of d'Alembertian: $\Box = \dfrac{1}{c^2}\dfrac{\partial}{\partial t^2} - \nabla^2$

$$\Box = \left[-\left(\frac{E}{c\hbar}\right) + \left(\frac{\left|\vec{p}\right|^2}{\hbar}\right)\right]\text{exp}\left[\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} - Et\right)\right] = -\left(\frac{mc}{\hbar}\right)^2\psi \tag{2.2}$$

...

Chapter 3

Superposition

3.1 You can add together many states that solve K.G., and the resulting sum also solves K.G.

Another important concept we have to know about when working with the Klein-Gordon Equation is that any superposition of wave functions that satisfy the Klein-Gordon Equation, also satisfy the Klein-Gordon Equation. That lets us create a complex landscape starting with the simple basis set of functions

Let's say that you have two functions that satisfy the Klein-Gordon Equation, call them ψ_1 and ψ_2

$$\left[\Box + \mu^2\right] \psi_1 = 0 \tag{3.1}$$

$$\left[\Box + \mu^2\right] \psi_2 = 0 \tag{3.2}$$

Let's call their sum ψ_3 so

$$\psi_3 = \psi_1 + \psi_2.$$

The question then arises: does ψ_3 satisfy the Klein-Gordon Equation?
Well, yes it does. We can show this with:

$$\begin{aligned}\left[\Box + \mu^2\right] \psi_3 &= \left[\Box + \mu^2\right] (\psi_1 + \psi_3) \\ &= \left[\Box + \mu^2\right] \psi_1 + \left[\Box + \mu^2\right] \psi_2 = 0 + 0 = 0 \text{ ... also satisfies K.G..}\end{aligned}$$

The following reasoning does not just apply to the sum of two wave functions, but also scaling up a wave function or taking the arbitrary sum over many wave functions. Or even more usefully if you have a basis set of functions that satisfy the Klein-Gordon Equation, then you can make wave functions of that basis set by summing over them with some series of coefficients

Say we write a wavefunction ψ as a linear combination of $\left\{\psi_n\right\}$

$$\psi = \sum_n C_n \psi_n, \qquad C_n \in \mathbb{C} \tag{3.3}$$

Since each basis function in $\psi_n \in \left\{\psi_n\right\}$ satisfies K.G....

$$\left[\Box + \mu^2\right] \psi_n = 0 \rightarrow \left[\Box + \mu^2\right] \sum_n C_n \psi_n = 0 \tag{3.4}$$

And that's how we actually use energy and momentum in Eigen-states most of the time.

Chapter 4

Group Velocity, and c Speed limit

4.1 calculating the group velocity of a Klein-Gordon Wave packet

In order to calculate the group velocity of a Klein-Gordon Wave packet, we first have to calculate something called the **DispersionRelation**.

The **DispersionRelation** is when **angularfrequency**ω is written as a function of **angularwavenumber**.

General form of plane wave:

$$\psi = A\ \exp\ \left(i\left(\vec{k}\cdot\vec{x}-\omega t\right)\right) \tag{4.1}$$

Klein-Gordon plane wave:

$$\psi = A\ \exp\ \left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}\pm c\sqrt{\left|\vec{p}\right|^2+m^2c^2t}\right)\right) \tag{4.2}$$

$$\omega = \frac{c}{\hbar}\sqrt{\left|\vec{p}\right|^2+m^2c^2}, \qquad k = \left|\vec{p}\right|/\hbar$$

$$\left|\vec{p}\right|^2 = \hbar^2k^2, \qquad \rightarrow \omega = \sqrt{(kc)^2+\left(\frac{mc^2}{\hbar}\right)^2}$$

4.2 Finding the group relation

Now we have the Dispersion Relation: $\omega = \sqrt{(kc)^2+\left(\frac{mc^2}{\hbar}\right)^2}$

The group velocity is then: $v_g = \frac{d\omega}{dk}$

$$v_g = \frac{d}{dk}\left[\sqrt{(kc)^2+(mc^2/\hbar)^2}\right] \tag{4.3}$$

with $k = \left|\vec{p}\right|/\hbar$ in mind:

$$v_g = \frac{kc}{\sqrt{k^2+(mc/\hbar)^2}} \rightarrow v_g = \frac{c\left|\vec{p}\right|}{\sqrt{\left|\vec{p}\right|^2+(mc)^2}} \tag{4.4}$$

Playing around with the Klein-Gordon equation, we have found a speed limit: the speed of light!

Massive particle: can only approach c as momentum increases, can never reach c.
Massless particle: always travels at c, regardless of momentum. – Light always travels at c in all reference frames!

Chapter 5

Fourier transform and antimatter

5.1 Complex Coefficients on the Mass Shell

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) \exp\left[-\frac{i}{\hbar} p \cdot x\right] dp \tag{5.1}$$

- wavefunction in spacetime
- Normalization constant
- $\int_{-\infty}^{\infty} dp$ Add up eigenstate for each p, weighted $\phi(p)$
- $\phi(p)$ Momentum-space wavefunction
- $\exp\left[-\frac{i}{\hbar} p \cdot x\right]$ p-eigenstate