Relativistic Quantum Waves (Klein-Gordon Equation

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Chapter 1

Deriving the KG Equation

1.1 double deriving

Definition 1.1.1: Relativity: the mass shell

$p \cdot p = (mc)^2 \longrightarrow (mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2$

Definition 1.1.2: Quantum: energy and momentum operators

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \text{ so } \left(\frac{E}{c}\right)^2 \text{ becomes } -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}.$$

$$\hat{p} = -i\hbar\nabla, \text{so } -p_x^2 \text{ becomes } \hbar^2\frac{\partial^2}{\partial x^2}.$$

$$\text{likewise, } -p_y^2 \text{ becomes } \hbar^2\frac{\partial^2}{\partial y^2} \text{ and } -p_z^2 \text{ becomes } \hbar^2\frac{\partial^2}{\partial z^2}.$$

1.2 Plugging in the new values

we can now plugin these into the original equation:

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_y^2$$

$$(mc)^2 = -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2\frac{\partial^2}{\partial y^2} + \hbar^2\frac{\partial^2}{\partial z^2}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0.$$

1.3 this is the Klein-Gordon Equation!

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0 \tag{1.1}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.2}$$

1.4 Replacing with Laplacian

We know that $-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}=\nabla^2$ -otherwise known as a Laplacian

So the function becomes

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\left(\frac{mc}{\hbar}\right)^2\right]\psi=0 \tag{1.3}$$

1.5 d'Alembertian

$\square =$ d'Alembertian

We can also rewrite $\left(\frac{mc}{\hbar}\right)^2$ as μ^2

Note:-

We could also write $\left(\frac{mc}{\hbar}\right)$ as just m as in this universe it would become $\left(\frac{m\cdot 1}{1}\right)$ which is just the mass but this is the correct way to write it.

So our final equation becomes

$$\left[\square+\mu^2\right]\psi=0 \tag{1.4}$$

Chapter 2

Four-momentum Eigenstates

2.1 Klein-Gordon Plane Wave

Definition 2.1.1: Klein-Gorden Plane Wave function

$\psi = A \text{ exp } \left(-\frac{i}{\hbar} p \cdot x\right)$

$$p = \left[E/c, \vec{p}\right], \qquad x = \left[ct, \vec{x}\right]$$

$$A \in \mathbb{C}, \qquad p^0 = \frac{E}{c}0 \pm \sqrt{\left|\vec{p}\right|^2 + m^2c^2}$$

we can rewrite the original Equation as:

$$\psi = A \text{ exp } \left(\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} - Et\right)\right)$$
$$\psi = A \text{ exp } \left(\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} \pm c\sqrt{\left|\vec{p}\right|^2 + m^2c^2t}\right)\right)$$

2.2 Proof that plane waves $\psi = A \text{ exp } \left[-ip \cdot x/\hbar\right]$ satisfy K.G.

rewrite K.G.: $\qquad \left[\Box + \mu^3\right]\psi = 0 \longrightarrow \Box\psi = -\mu^2\psi$

Does d'Alembertian ddo the same thing as multiplying by $-\mu^2$?

$$\Box = \Box\left[\text{exp}\left[-\frac{i}{\hbar}p \cdot x\right]\right] = \Box\left[\text{exp}\left[\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} - Et\right)\right]\right] \tag{2.1}$$

Definition of d'Alembertian: $\qquad \Box = \dfrac{1}{c^2}\dfrac{\partial}{\partial t^2} - \nabla^2$

$$\Box = \left[-\left(\frac{E}{c\hbar}\right) + \left(\frac{\left|\vec{p}\right|^2}{\hbar}\right)\right]\text{exp}\left[\frac{i}{\hbar}\left(\vec{p} \cdot \vec{x} - Et\right)\right] = -\left(\frac{mc}{\hbar}\right)^2\psi \tag{2.2}$$

...

Chapter 3

Superposition

3.1 You can add together many states that solve K.G., and the resulting sum also solves K.G.

Another important concept we have to know about when working with the Klein-Gordon Equation is that any superposition of wave functions that satisfy the Klein-Gordon Equation, also satisfy the Klein-Gordon Equation. That lets us create a complex landscape starting with the simple basis set of functions

Let's say that you have two functions that satisfy the Klein-Gordon Equation, call them ψ_1 and ψ_2

$$\left[\Box + \mu^2\right] \psi_1 = 0 \tag{3.1}$$

$$\left[\Box + \mu^2\right] \psi_2 = 0 \tag{3.2}$$

Let's call their sum ψ_3 so

$$\psi_3 = \psi_1 + \psi_2.$$