Relativistic Quantum Waves (Klein-Gordon Equation

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Chapter 1

Deriving the KG Equation

1.1 double deriving

Definition 1.1.1: Relativity: the mass shell

$p \cdot p = (mc)^2 \longrightarrow (mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2$

Definition 1.1.2: Quantum: energy and momentum operators

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \text{ so } \left(\frac{E}{c}\right)^2 \text{ becomes } -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}.$$

$$\hat{p} = -i\hbar\nabla, \text{so } -p_x^2 \text{ becomes } \hbar^2\frac{\partial^2}{\partial x^2}.$$

$$\text{likewise, } -p_y^2 \text{ becomes } \hbar^2\frac{\partial^2}{\partial y^2} \text{ and } -p_z^2 \text{ becomes } \hbar^2\frac{\partial^2}{\partial z^2}.$$

1.2 Plugging in the new values

we can now plugin these into the original equation:

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_y^2$$

$$(mc)^2 = -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2\frac{\partial^2}{\partial y^2} + \hbar^2\frac{\partial^2}{\partial z^2}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0.$$

1.3 this is the Klein-Gordon Equation!

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0 \tag{1.1}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.2}$$

1.4 Replacing with Laplacian

We know that $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \nabla^2$ -otherwise known as a Laplacian

So the function becomes

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.3}$$

1.5 d'Alembertian

$\Box =$ d'Alembertian
We can also rewrite $\left(\frac{mc}{\hbar}\right)^2$ as μ^2

Note:-

We could also write $\left(\frac{mc}{\hbar}\right)$ as just m as in this universe it would become $\left(\frac{m \cdot 1}{1}\right)$ which is just the mass but this is the correct way to write it.

So our final equation becomes

$$\left[\Box + \mu^2\right]\psi = 0 \tag{1.4}$$

Chapter 2

Four-momentum Eigenstates

2.1 Klein-Gordon Plane Wave

Definition 2.1.1: Klein-Gorden Plane Wave function

$\psi = A \text{ exp } \left(-\frac{i}{\hbar} p \cdot x\right)$

$$p = \left[E/c, \vec{p}\right], \qquad x = \left[ct, \vec{x}\right]$$

$$A \in \mathbb{C}, \qquad p^0 = \frac{E}{c} 0 \pm \sqrt{\left|\vec{p}\right|^2 + m^2 c^2}$$

we can rewrite the original Equation as:

$$\psi = A \text{ exp } \left(\frac{i}{\hbar} \left(\vec{p} \cdot \vec{x} - Et\right)\right)$$
$$\psi = A \text{ exp } \left(\frac{i}{\hbar} \left(\vec{p} \cdot \vec{x} \pm c\sqrt{\left|\vec{p}\right|^2 + m^2 c^2 t}\right)\right)$$

2.2 Proof that plane waves $\psi = A \text{ exp } \left[-ip \cdot x/\hbar\right]$ satisfy K.G.

rewrite K.G.: $\left[\square + \mu^3\right] \psi = 0 \rightarrow \square\psi = -\mu^2\psi$

Does d'Alembertian ddo the same thing as multiplying by $-\mu^2$?

$$\square = \square \left[\text{exp} \left[-\frac{i}{\hbar} p \cdot x\right]\right] = \square \left[\text{exp} \left[\frac{i}{\hbar} \left(\vec{p} \cdot \vec{x} - Et\right)\right]\right] \tag{2.1}$$