Relativistic Quantum Waves (Klein-Gordon Equation

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Contents

Chapter 1

Deriving the KG Equation

1.1 double deriving

Definition 1.1.1: Relativity: the mass shell

$$p \cdot p = (mc)^2 \rightarrow (mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2$$

Definition 1.1.2: Quantum: energy and momentum operators

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \text{ so } \left(\frac{E}{c}\right)^2 \text{ becomes } -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}.$$

$$\hat{p} = -i\hbar\nabla, \text{so } -p_x^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial x^2}.$$

$$\text{likewise, } -p_y^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial y^2} \text{ and } -p_z^2 \text{ becomes } \hbar^2 \frac{\partial^2}{\partial z^2}.$$

1.2 Plugging in the new values

we can now plugin these into the original equation:

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_y^2$$

$$(mc)^2 = -\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2\frac{\partial^2}{\partial y^2} + \hbar^2\frac{\partial^2}{\partial z^2}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0.$$

1.3 this is the Klein-Gordon Equation!

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2 = 0 \tag{1.1}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.2}$$

1.4 Replacing with Laplacian

We know that $-\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2} - \dfrac{\partial^2}{\partial z^2} = \nabla^2$ -otherwise known as a Laplacian

So the function becomes

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right] \psi = 0 \tag{1.3}$$

1.5 d'Alembertian

$\square$ = d'Alembertian
We can also rewrite $\left(\dfrac{mc}{\hbar} \right)^2$ as μ^2

Note:-

We could also write $\left(\dfrac{mc}{\hbar} \right)$ as just m as in this universe it would become $\left(\frac{m \cdot 1}{1} \right)$ which is just the mass but this is the correct way to write it.

So our final equation becomes

$$\left[\square + \mu^2 \right] \psi = 0 \tag{1.4}$$