# Computational Physics: PS 9

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## 1 Discussion

My GitHub repo is `https://github.com/marcusHoskinsNYU/phys-ga2000`. For images that are blurry here, please see the ps-9 folder for individual image files.

### 1.1 Problem 1

#### 1.1.1 Part (a)

The second order equation we are given is that of a simple harmonic oscillator:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x.$$

Thus, we can construct two coupled first-order equations given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2 x$$

which are equivalent to the second order equation. Then, the numerical implementation of solving these equations is found in the repo, where I used fourth-order Runge-Kutta. The appropriate graph is found in figure 1.

#### 1.1.2 Part (b)

If we increase the amplitude of the oscillations now to $x_0 = 2$, we get the orange plot seen in figure 2. Clearly from the overlap of the plots we can see that the period of the oscillations stays roughly the same if we increase the amplitude of the oscillations.

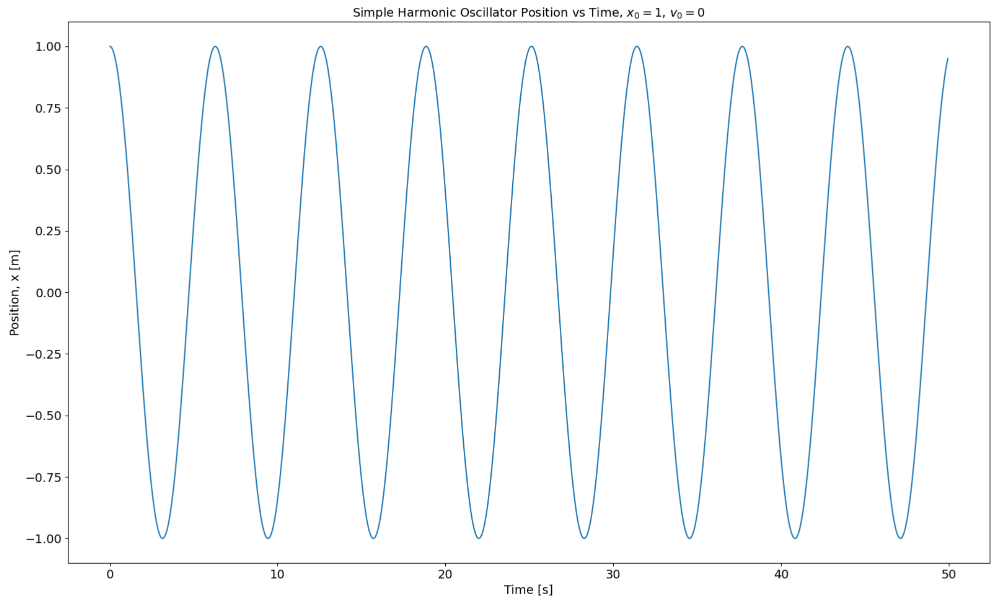


Figure 1: Plot of the solution to the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$.

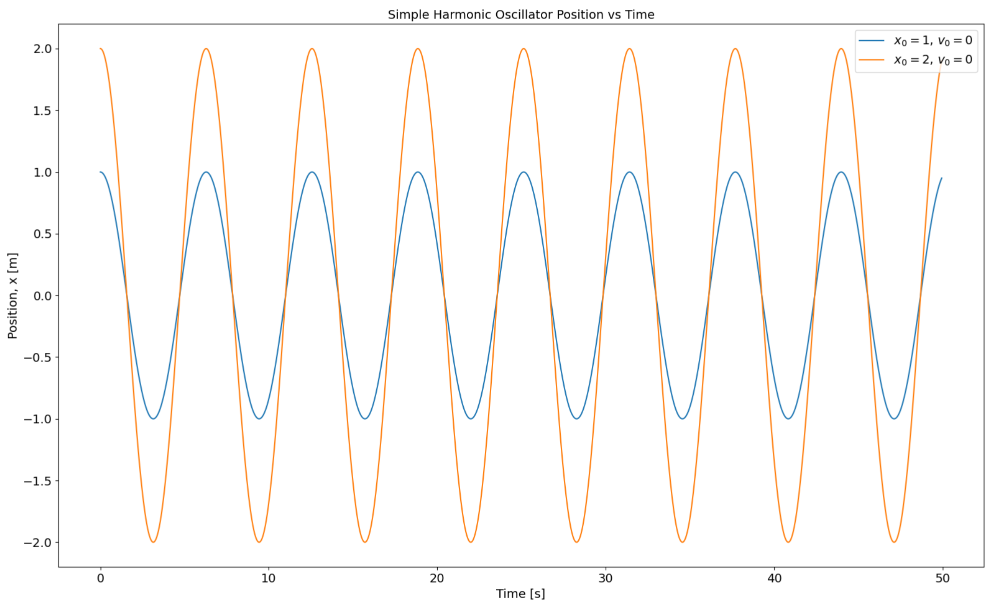


Figure 2: Plots of the solution to the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the blue plot and $x_0 = 2$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the orange. Visually they have roughly the same periods.

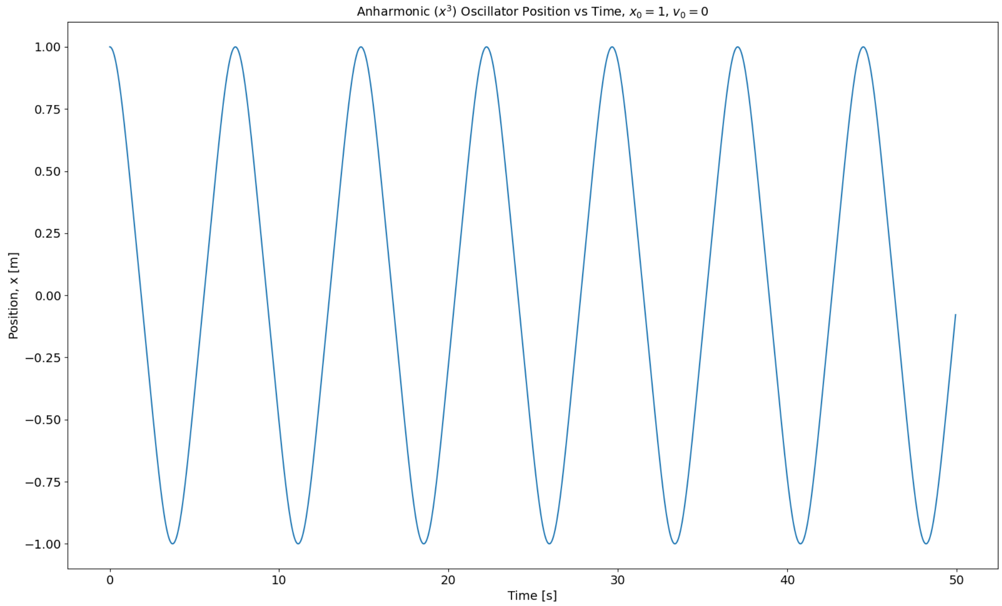


Figure 3: Plot of the solution to the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2x^3$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$.

### 1.1.3 Part (c)

Now, suppose, we wish to solve the equation of motion for the anharmonic oscillator:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2x^3.$$

Thus, we can construct two coupled first-order equations given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2x^3$$

which are equivalent to the second order equation. Then, the plot of the motion of the oscillator is seen in figure 3. And, when we change the amplitude of oscillation, the period of the oscillator also changes, as seen in figure 4.

### 1.1.4 Part (d)

The modification is seen in the repo. And, the plots of the phase space of the anharmonic oscillator are seen in figures 5 and 6. And, the plot of the phase space of the simple harmonic oscillator is seen in figure 7.

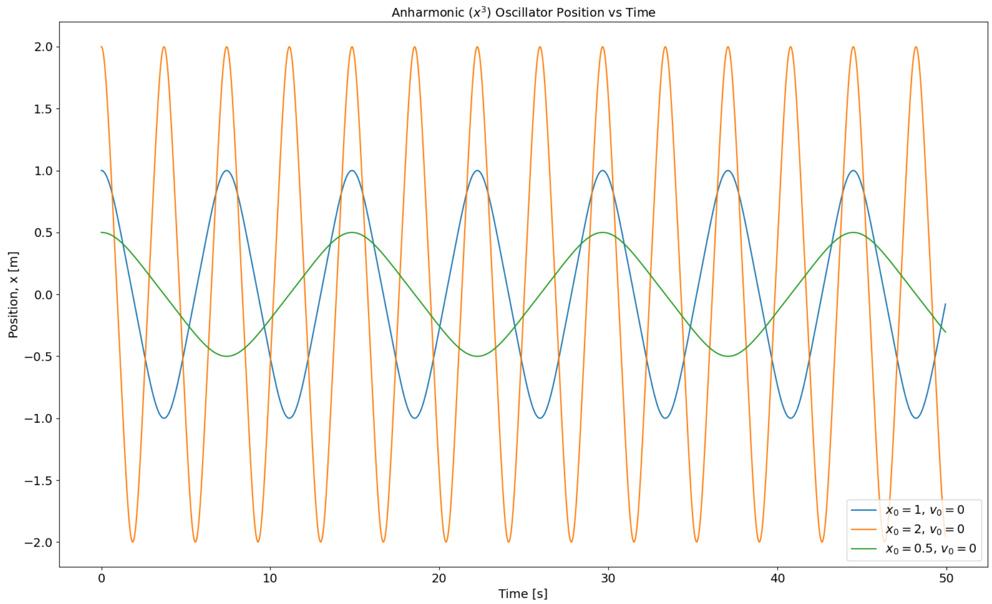


Figure 4: Plots of the solution to the equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x^3$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the blue plot, $x_0 = 2$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the orange plot, and $x_0 = 0.5$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the green. Visually they clearly have different periods.

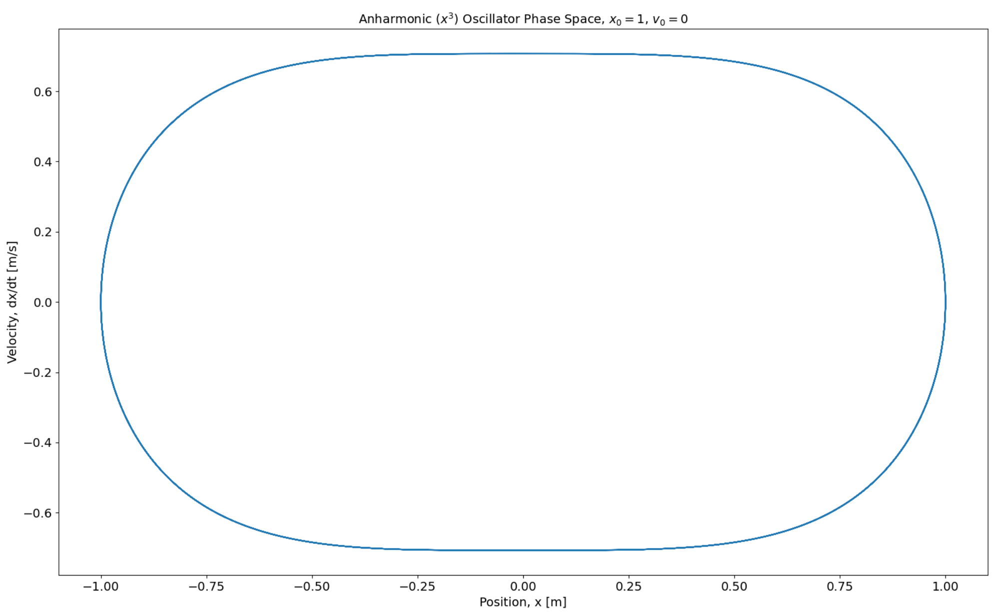


Figure 5: Plot of the phase space of the equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x^3$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$.

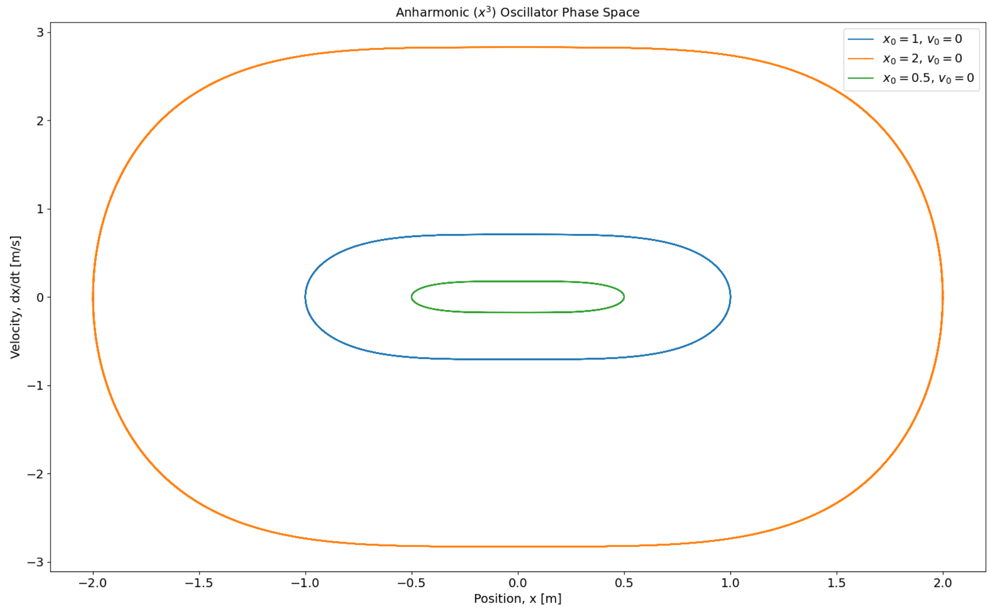


Figure 6: Plots of the phase space of the equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x^3$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the blue plot, $x_0 = 2$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the orange plot, and $x_0 = 0.5$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for the green.

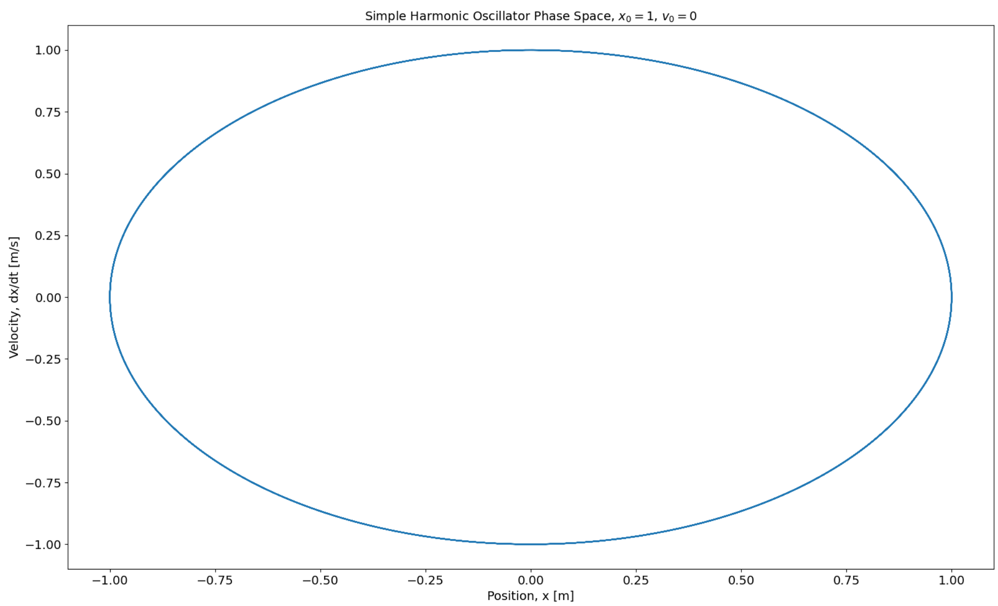


Figure 7: Plot of the phase space of the equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$.

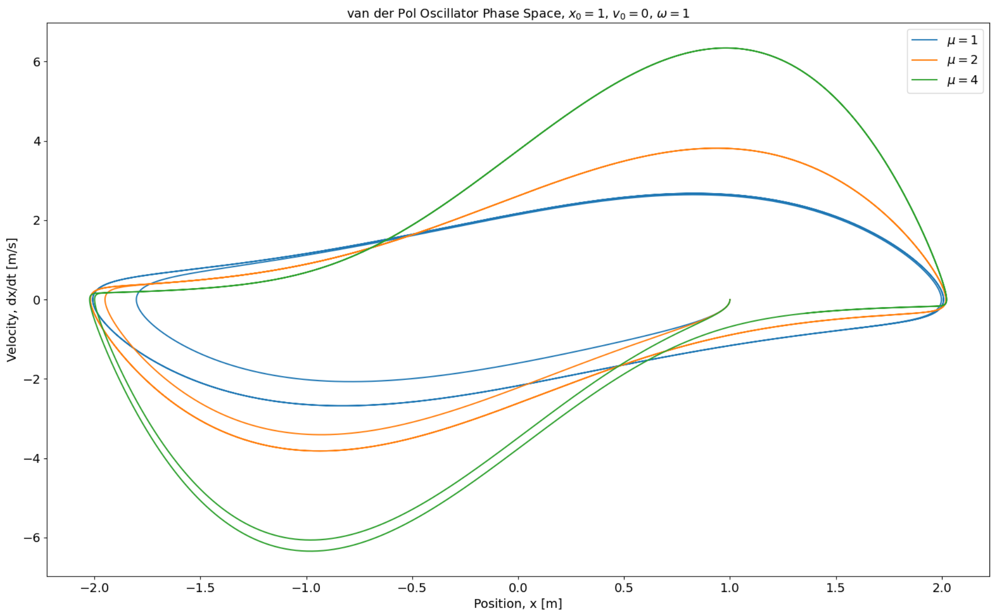


Figure 8: Plots of the phase space of the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \mu(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0$, where here $\omega = 1$, and we have the initial conditions $x_0 = 1$, $\frac{\mathrm{d}x_0}{\mathrm{d}t} = 0$ for all three plots, and $\mu = 1$ for the blue plot, $\mu = 2$ for the orange, and $\mu = 4$ for the green.

#### 1.1.5 Part (e)

Now, suppose, we wish to solve the equation of motion for the van der Pol oscillator:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \mu(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0.$$

Thus, we can construct two coupled first-order equations given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \mu(1-x^2)v - \omega^2 x$$

which are equivalent to the second order equation. Then, the phase space plot for various $\mu$ values is seen in figure 8.

### 1.2 Problem 2

#### 1.2.1 Part (a)

Suppose a cannonball is fired in the positive x and y directions at some velocity at an angle $\theta$ above the horizontal. Starting from Newton's second law, in the x-direction the cannonball has only the drag force acting on it, meaning:

$$F_x = ma_x \implies m\ddot{x} = \left(-\frac{1}{2}\pi R^2 \rho C v^2\right)_x = -\frac{1}{2}\pi R^2 \rho C \left(v^2\right)_x = -\frac{1}{2}\pi R^2 \rho C v^2 \cos\theta.$$

But, as $v_x = \dot{x} = v\cos\theta$, and $v = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$, we can write this as:

$$\ddot{x} = -\frac{\pi R^2 \rho C}{2m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2},$$

as $v^2\cos\theta = v \cdot v\cos\theta = v \cdot v_x$.

And, in the y-direction, the cannonball has two forces acting on it. Gravity always points downward, and on the way up in its motion, the drag force points downward, while on the way down in its motion, the drag force points upward. The reason for this is that the drag force always opposes the direction of motion. So, Newton's second law in the y-direction reads:

$$F_y = ma_y \implies m\ddot{y} = -mg + \left(-\frac{1}{2}\pi R^2 \rho C v^2\right)_y = -mg - \frac{1}{2}\pi R^2 \rho C \left(v^2\right)_y$$

$$= -mg - \frac{1}{2}\pi R^2 \rho C v^2 \sin\theta.$$

Then, for the same reason as in the x-direction, we can write:

$$\ddot{y} = -g - \frac{\pi R^2 \rho C}{2m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2},$$

where here we have defined $\theta$ as the instantaneous angle between the velocity vector and the horizontal, and thus automatically takes into account the sign of the y-velocity, changing the sign of the drag term as the cannonball is moving back down in its trajectory.

### 1.2.2 Rescaling

Given a typical timescale $T$ of our system, we have a set of parameters $R$, $\rho$, $C$, $m$, and $g$, which in combination with $T$ can form a unitless parameter:

$$Q = \frac{R^2 \rho C g T^2}{m}.$$

Then, rescaling $t$ to $t' = t/T$, and further (since $g$ is the only natural acceleration in our system) $x$ to $x' = x/gT^2$ and $y$ to $y' = y/gT^2$, we can rewrite our previously found equations of motion:

$$\ddot{x} = -\frac{\pi R^2 \rho C}{2m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\ddot{y} = -g - \frac{\pi R^2 \rho C}{2m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}.$$

To do this, we use the rescalings $x = x'gT^2$, $y = y'gT^2$, $t = t'T$, and the observations $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}(x'gT^2)}{\mathrm{d}(t'T)} = gT\frac{\mathrm{d}x'}{\mathrm{d}t'} = gT\dot{x}'$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2(x'gT^2)}{\mathrm{d}(t'T)^2} = g\frac{\mathrm{d}^2x'}{\mathrm{d}(t')^2} = g\ddot{x}'$ (with similar equations holding for $\dot{y}$ and $\ddot{y}$), we can write these equations as:

$$g\ddot{x}' = -\frac{\pi R^2\rho C}{2m}gT\dot{x}'\sqrt{g^2T^2\dot{x}'^2 + g^2T^2\dot{y}'^2}$$

$$\implies \quad \ddot{x}' = -\frac{\pi R^2\rho C}{2m}gT^2\dot{x}'\sqrt{\dot{x}'^2 + \dot{y}'^2}$$

and

$$g\ddot{y}' = -g - \frac{\pi R^2\rho C}{2m}gT\dot{y}'\sqrt{g^2T^2\dot{x}'^2 + g^2T^2\dot{y}'^2}$$

$$\implies \quad \ddot{y}' = -1 - \frac{\pi R^2\rho C}{2m}gT^2\dot{y}'\sqrt{\dot{x}'^2 + \dot{y}'^2}.$$

Or, in terms of the unitless parameter $Q$ :

$$\ddot{x}' = -\frac{\pi}{2}Q\dot{x}'\sqrt{\dot{x}'^2 + \dot{y}'^2}$$

$$\ddot{y}' = -1 - \frac{\pi}{2}Q\dot{y}'\sqrt{\dot{x}'^2 + \dot{y}'^2}.$$

Thus, as expected, we see that we have a unitless set of equations (as $x'$, $y'$, and $t'$ are unitless) with one free (unitless) parameter, $Q$.

### 1.2.3 Part (b)

Changing the two second-order equations into four first-order equations, we see that the x-equations become:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{\pi R^2\rho C}{2m}v_x\sqrt{v_x^2 + v_y^2},$$

while the y-equations become:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_y$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -g - \frac{\pi R^2\rho C}{2m}v_y\sqrt{v_x^2 + v_y^2}.$$

Then, implementing fourth-order Runge-Kutta to numerically solve these equations, using the initial conditions given, we get the trajectory given in Figure 9.

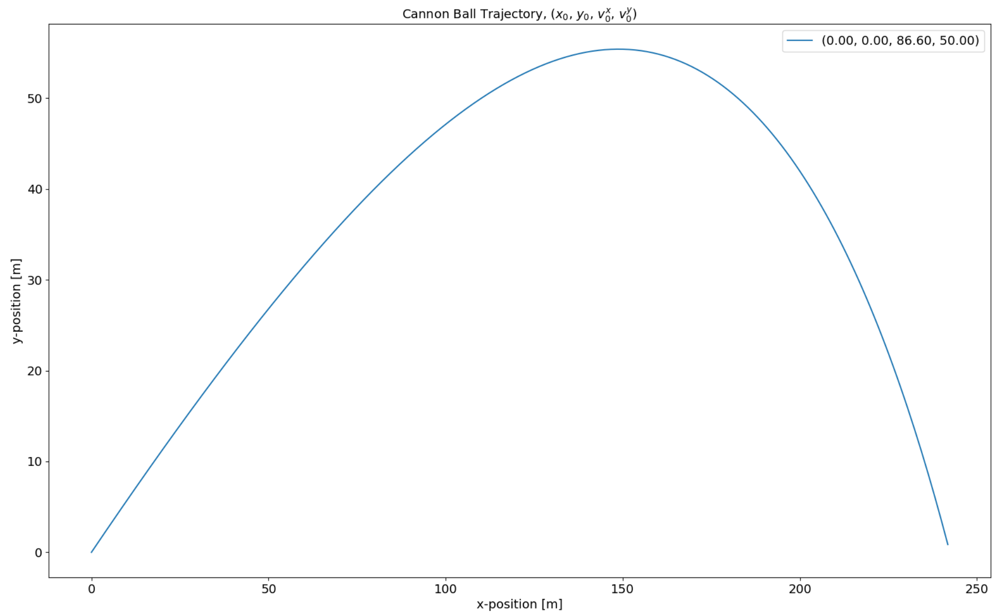


Figure 9: This is the trajectory of a cannonball that experiences drag, with initial conditions $v_0 = 100\text{ms}^{-1}$, $\theta = 30°$, and parameters $m = 1\text{kg}$, $R = 0.08\text{m}$, $\rho_{\text{air}} = 1.22\text{kgm}^{-3}$, and coefficient of drag $C = 0.47$.

#### 1.2.4 Part (c)

By varying the mass of the cannonball, we can change the ball's trajectory. A collection of such trajectories is seen in Figure 10. As we can see, the larger the mass, the farther the cannonball goes. And, as seen in Figure 11 (for the smaller mass values, see Figure 12), we see that the total x-distance traveled flattens out to a maximum value as $m$ increases. The reason that the x-distance traveled flattens out for sufficiently large masses is because, in this limit, the y-direction equation of motion becomes:

$$\ddot{y} = -g,$$

and the x-direction equation of motion becomes:

$$\ddot{x} = 0,$$

which together are the equations of motion for a cannonball that doesn't experience any drag force. That is, as $m \to \infty$, we approach the usual drag-free limit.

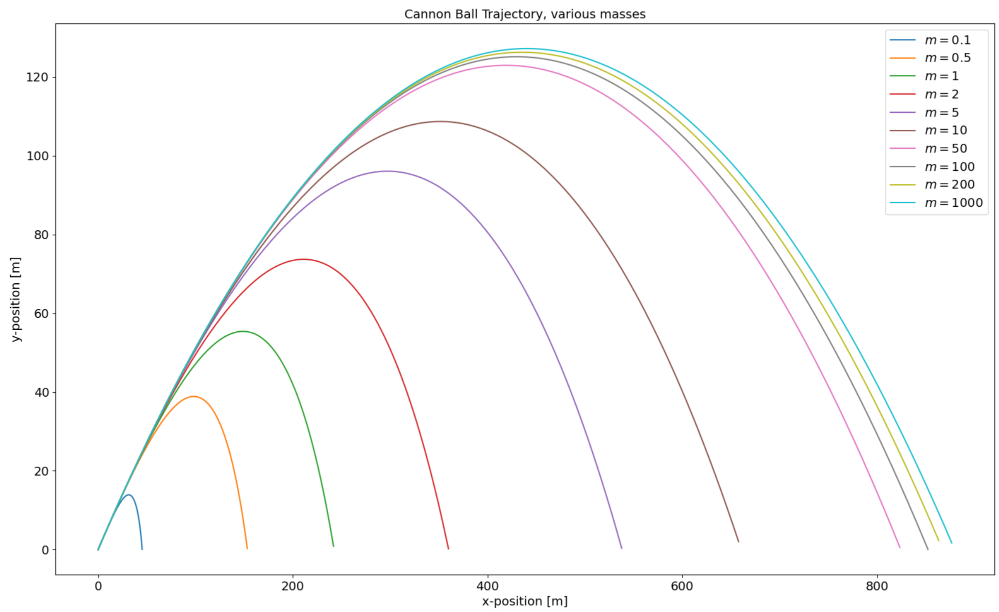


Figure 10: These are the trajectories of a cannonball at various mass values, with the same initial conditions and other parameter values as described in Figure 9.

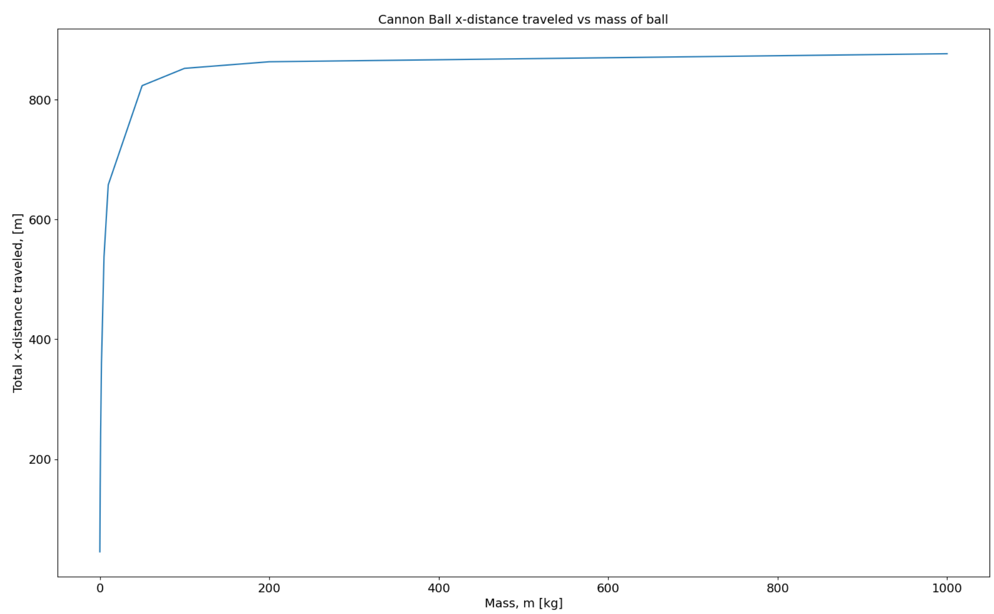


Figure 11: These are the total x distances traveled by the cannonball as a function of mass. As we can see, as mass increases, so too does the distance traveled in the x direction, where for sufficiently large masses the x distance traveled flattens out. A zoomed in Figure for the smaller masses are seen in Figure 12.

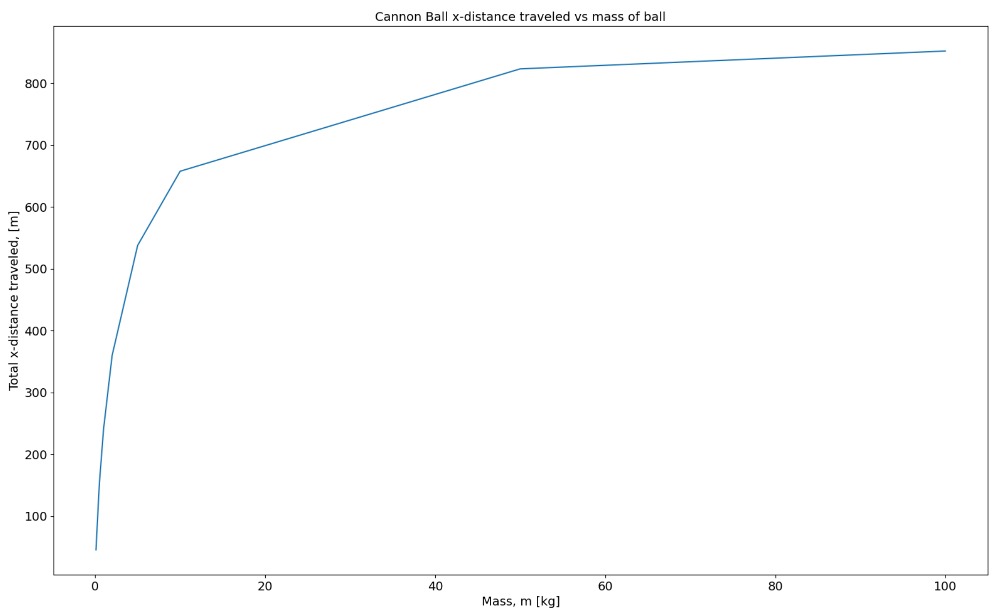


Figure 12: These are the total x distances traveled by the cannonball as a function of mass. This is a zoomed in version of figure 11.