

Exercise 05

1. Projector representation of isotropic fourth-order tensor

Problem

- A projector decomposition of an arbitrary symmetric tensor of fourth-order is of the form

$$\mathbb{C} = \sum_{\alpha=1}^N \lambda_{\alpha} \mathbb{P}_{\alpha} .$$

Let N be the number of distinct eigenvalues λ_{α} of \mathbb{C} . Then the projectors \mathbb{P}_{α} fulfill subsequent projector rules:

$$\begin{aligned} \text{idempotence: } \mathbb{P}_{\alpha} \mathbb{P}_{\alpha} &= \mathbb{P}_{\alpha} & \forall \alpha = \{1, \dots, N\} \\ \text{orthogonality: } \mathbb{P}_{\alpha} \mathbb{P}_{\beta} &= \mathbb{0} & \forall \alpha \neq \beta \\ \text{completeness: } \sum \mathbb{P}_{\alpha} &= \mathbb{I}^{\text{sym}} \end{aligned}$$

If there are only simple eigenvalues, the projector decomposition coincides with the spectral form of the tensor. Determine the projector decomposition of an isotropic \mathbb{C} .

Solution

- The fourth-order identity tensor is given with respect to an ortonormal basis $\{\mathbf{e}_i\}$.

$$\mathbb{I} = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j$$

- The fourth-order transposer is given as follows.

$$\mathbb{T} = \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_j$$

- The fourth-order symmetrizer is given as

$$\mathbb{I}^{\text{sym}} = \frac{1}{2} (\mathbb{I} + \mathbb{T})$$

while the fourth-order anti(sym)metrizer is

$$\mathbb{I}^{\text{skw}} = \frac{1}{2} (\mathbb{I} - \mathbb{T}) .$$

The isotropic fourth-order tensor is given, while we additionally apply above introduced abbreviations

$$\begin{aligned} \mathbb{C} &= a \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j + b \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j + c \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i \\ &= a \mathbf{I} \otimes \mathbf{I} + (b+c) \mathbb{I}^{\text{sym}} + (b-c) \mathbb{I}^{\text{skw}} \\ &= (3a+b+c) \mathbb{P}_1 + (b+c) \mathbb{P}_2 + (b-c) \mathbb{P}_3 \end{aligned}$$

with $a, b, c \in \mathcal{R}$ (set of real numbers) and

$$\mathbb{P}_1 = \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \qquad \mathbb{P}_2 = \mathbb{I}^{\text{sym}} - \mathbb{P}_1 \qquad \mathbb{P}_3 = \mathbb{I}^{\text{skw}} .$$

These tensors fulfill the projector rules. Thus, the eigenvalues of \mathbb{C} are as follows.

$$\lambda_1 = 3a + b + c \qquad \lambda_2 = b + c \qquad \lambda_3 = b - c$$



2. Value ranges of material parameters

Problem

- Determine the constraints on the value range for Young's modulus and Poisson's ratio, which arise from the requirement of positive definiteness of the stiffness tetrad of the isotropic St. Venant-Kirchhoff material law

$$\mathbb{C} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I}^{\text{sym}}$$

with λ and μ being Lamé parameters.

Solution

- The spectral representation of this law is given as

$$\mathbb{C} = 3K\mathbb{P}_1 + 2G\mathbb{P}_2$$

with bulk modulus $K = \lambda + 2/3\mu$ and shear modulus $G = \mu$. \mathbb{P}_α are eigenspace projectors:

$$\mathbb{P}_1 = \frac{1}{3} \mathbf{I} \otimes \mathbf{I}$$

$$\mathbb{P}_2 = \mathbb{I}^{\text{sym}} - \mathbb{P}_1$$

- The projector rules are as follows:

$$\text{idempotence: } \mathbb{P}_\alpha \mathbb{P}_\alpha = \mathbb{P}_\alpha$$

$$\forall \alpha = \{1, 2\}$$

$$\text{orthogonality: } \mathbb{P}_1 \mathbb{P}_2 = \mathbb{P}_2 \mathbb{P}_1 = \mathbb{O}$$

$$\text{completeness: } \mathbb{P}_1 + \mathbb{P}_2 = \mathbb{I}^{\text{sym}}$$

- The projector representation implies that $3K$ and $2G$ are the onefold and the fivefold eigenvalue of \mathbb{C} . Since \mathbb{C} is (major) symmetric and positive definite, these eigenvalues are also positive.

$$K > 0$$

$$G > 0$$

- The relation between K and G as well as Young's modulus Y and Poisson's ratio ν is analogous to the Hooke's law.

$$3K = \frac{Y}{1 - 2\nu}$$

$$2G = \frac{Y}{1 + \nu}$$

- From $K > 0$ and $G > 0$ follows

$$0 < \frac{1}{9K} + \frac{1}{3G} = \frac{1}{Y}.$$

This indicates the $Y > 0$ for Young's modulus. We can transform the relations from two equations above

$$\nu_- = \frac{1}{2} \frac{Y}{G} - 1$$

$$\nu_+ = -\frac{1}{6} \frac{Y}{K} + \frac{1}{2}$$

and set the bounds $Y/G = Y/K = 0$ to determine the range of Poisson's ratio: $-1 < \nu < 1/2$

