1

DIGITAL FILTERS

→ Consider the moving average filter:

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

Taking z-transform : $Y(z) = \frac{1}{3}\left(X(z) + z^{-1}X(z) + z^{-2}X(z)\right)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{3}\left(1 + z^{-1} + z^{-2}\right) \text{ or } \frac{z^2+z+1}{3z^2}$$

Calculate magnitude response : $z = e^{j\Omega}$

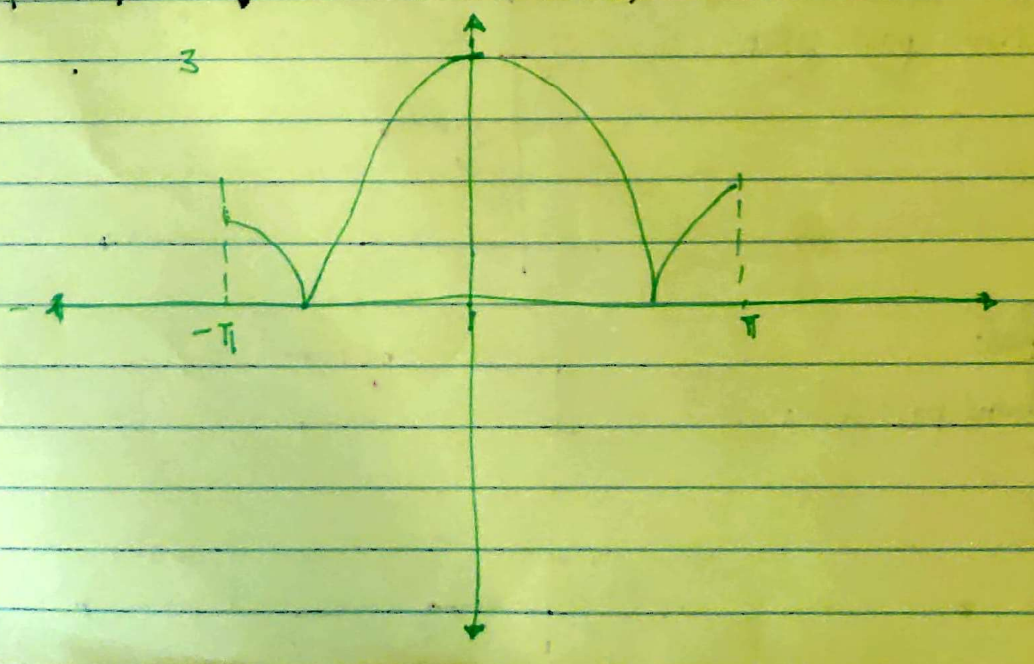
$$H(\Omega) = \frac{1}{3}\left[\left(e^{j\Omega}\right)^{-2} + \left(e^{j\Omega}\right)^{-1} + 1\right]$$

$$= \frac{1}{3}\left[e^{-2j\Omega} + e^{-j\Omega} + 1\right]$$

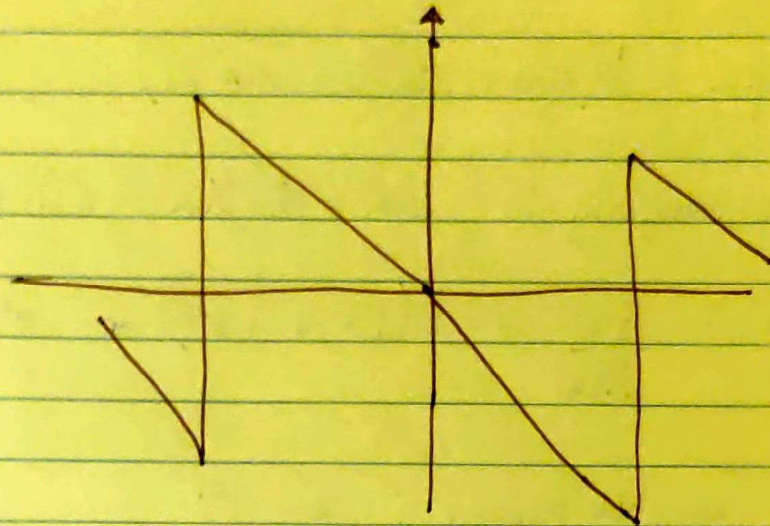
$$\Rightarrow \frac{1}{3}\left[\cos(2\Omega) - j\sin(2\Omega) + \cos(\Omega) - j\sin(\Omega) + 1\right]$$

$$\Rightarrow \frac{1}{3}\left[\left(1 + \cos(\Omega) + \cos(2\Omega)\right) - j\left(\sin(\Omega) + \sin(2\Omega)\right)\right]$$

$$|H(\Omega)| = \sqrt{\left(1 + \cos(\Omega) + \cos(2\Omega)\right)^2 + \left(\sin(\Omega) + \sin(2\Omega)\right)^2}$$



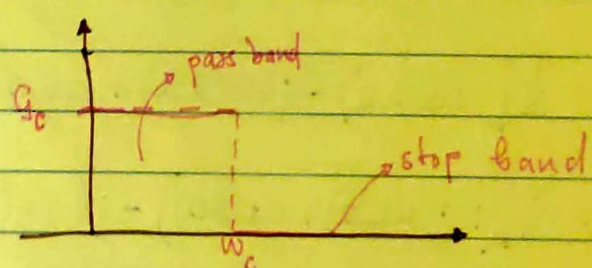
$$\measuredangle H(\Omega) = \tan^{-1}\left[\frac{-\left(\sin(\Omega) + \sin(2\Omega)\right)}{\left(1 + \cos(\Omega) + \cos(2\Omega)\right)}\right]$$



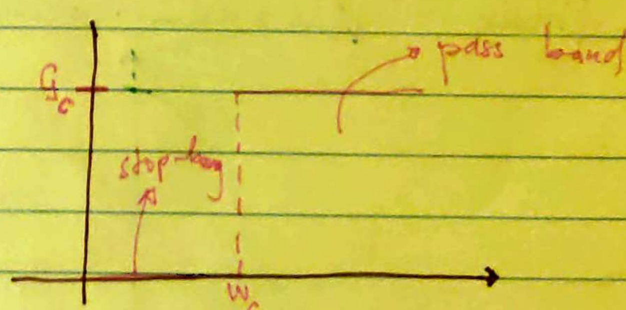
RECAP FROM SIGNALS II B

Types of filters

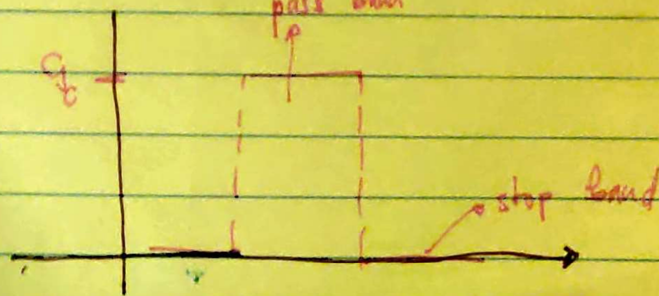
Low - pass filte



High pass filter

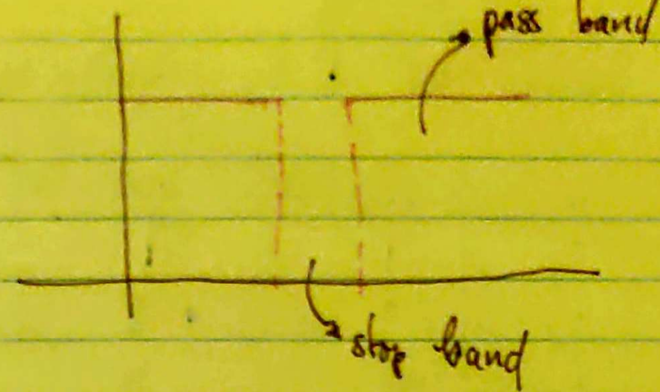


~~Pass~~
Band pass filter



2

Band stop filter :

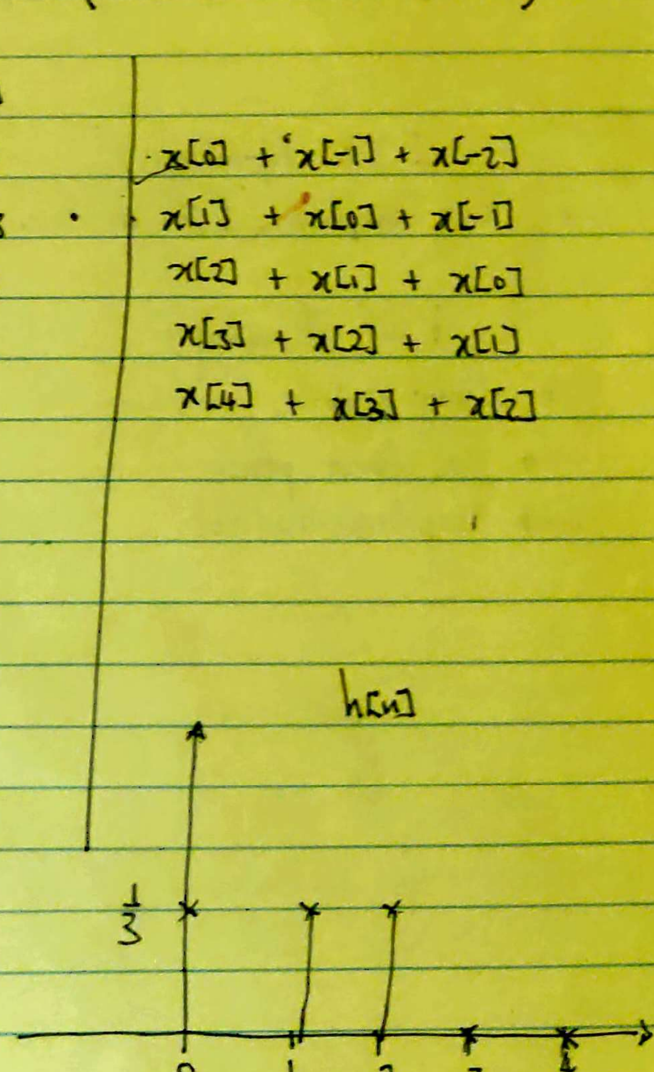


→ For digital filters we have 2-types of filter

consider the moving average :

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2] \right)$$

n	$x[n] = \delta[n]$	$y[n]$	
0	1	1/3	$x[0] + x[-1] + x[-2]$
1	0	1/3	$x[1] + x[0] + x[-1]$
2	0	1/3	$x[2] + x[1] + x[0]$
3	0	0	$x[3] + x[2] + x[1]$
4	0	0	$x[4] + x[3] + x[2]$
5	0	.	
6	0	.	
7	0	.	
8	0	.	
9	0		
10	0		



Consider the D.E. $y[n] - \frac{1}{2}y[n-1] = x[n]$

n	$x[n] = \delta[n]$	$y[n] = h[n]$
0	1	1
1	0	+ 1/2
2	0	1/4
3	0	1/8
4	0	⋮
5	0	⋮
6	0	⋮
7	0	⋮
		$\frac{1}{2^n}$

$y[0] = \frac{1}{2}y[-1] + x[0]$

$y[1] = \frac{1}{2}y[0] + x[1]$

$y[2] = \frac{1}{2}y[1] + x[2]$

$y[3] = \frac{1}{2}y[2] + x[2]$

RECURSIVE COMPUTATION

$h[n]$ is an infinite sequence

IIR Advantages

→ less complex (D.E are less complex compared to FIR

FIR Advantages

→ always stable

→ has linear phase

→ implementation

3

FILTER REALIZATION (SIMULINK)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

introduce intermediate ~~var~~ F variable $F(z)$

$$X(z) = \frac{1}{z^3 + a_2 z^2 + a_1 z + a_0} \cdot F(z)$$

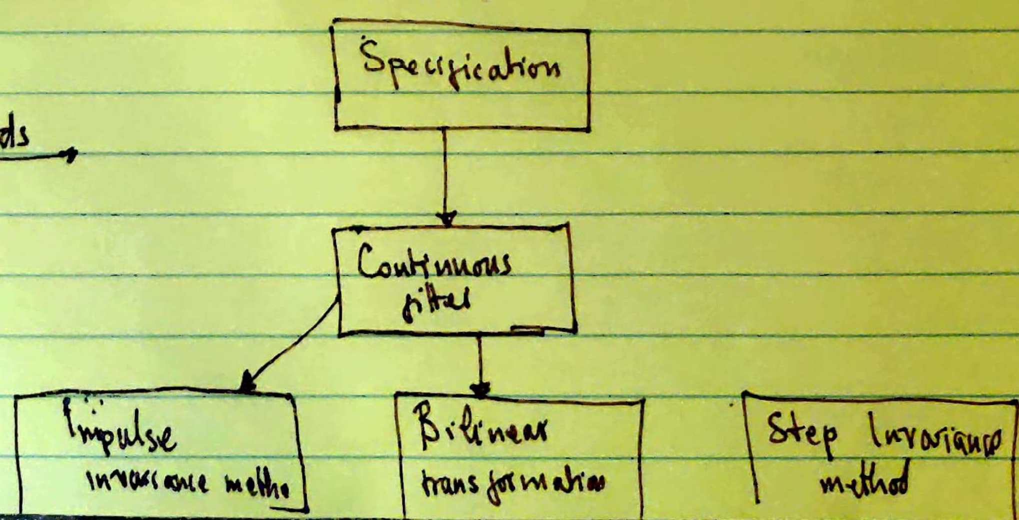
~~look @ filter realization for i~~

DIGITAL FILTER DESIGN

→ the general approach: (a) desired specification (sketch & highlight critical points)
(b) Design continuous filter
(c) use transformation / approximation to derive discrete filter from continuous.

IIR Filter Designs

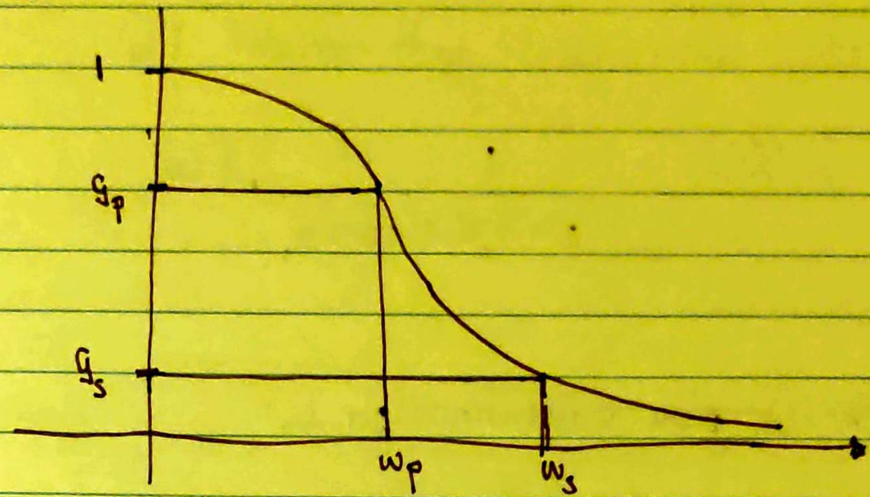
Methods

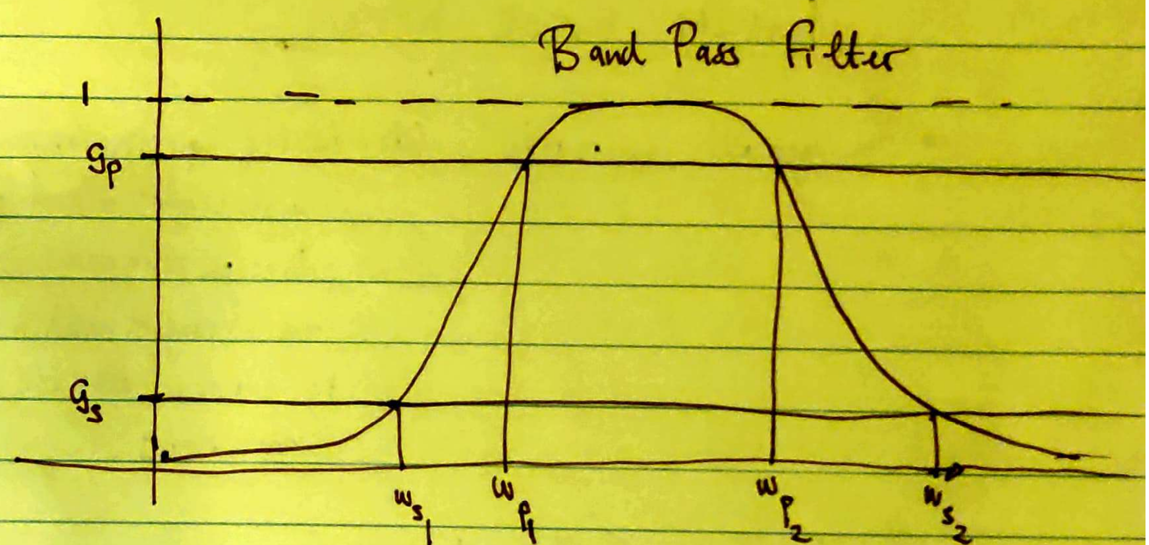


Impulse Invariance Method

RECAP : CONTINOUOS FILTER DESIGN

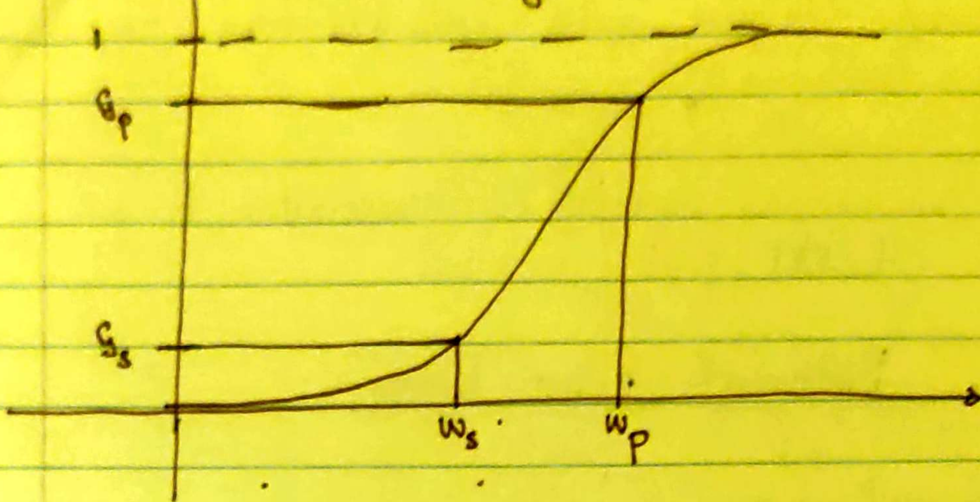
LPF



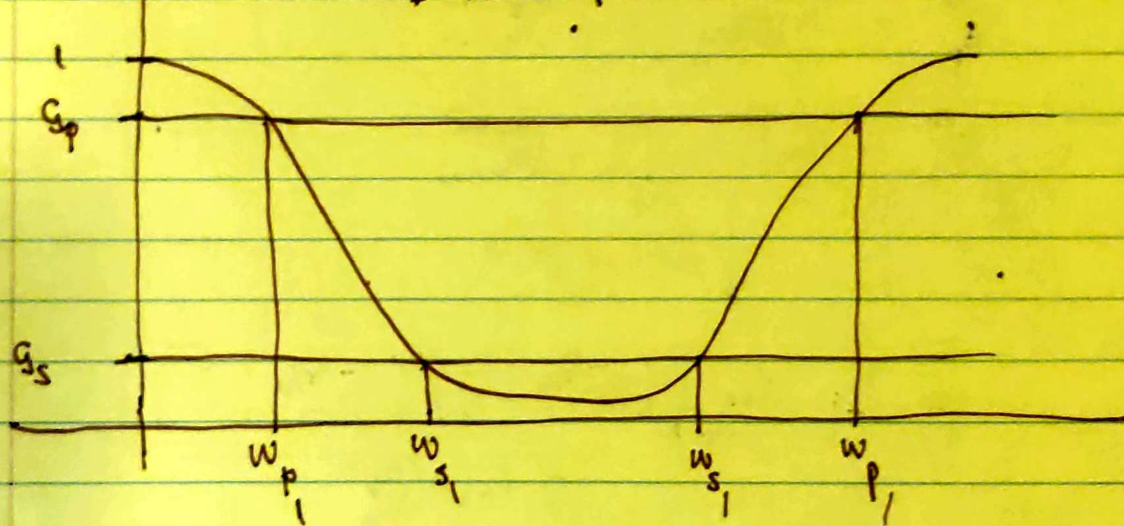


(4)

High Pass Filter



Band-stop filter



→ Practical filters have transitions i.e stop band → pass band takes some finite frequency intervals

→ Gain can't be zero ~~for~~ over a finite band therefore stop band is defined as the interval where gain $< G_s$ where G_s is small

→ Same for pass band.

G_p → minimum pass-band gain

G_s → maximum stop-band gain.

→ Design low-pass filter with pass-band → ω_p
stop-band → ω_s

→ Then use the appropriate transformation to get HPF | BPF | BSF

Butterworth Filter Design

→ General form : $|H(j\omega)| = \dfrac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$

$$G_p^{dB} = -10\log_{10}\left[1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n}\right]$$

$$G_s^{dB} = -10\log_{10}\left[1 + \left(\frac{\omega_s}{\omega_c}\right)^{2n}\right]$$

↓ solving for n

$$n \geq \frac{\log_{10}\left[\left(10^{-G_s^{\prime}/10} - 1\right) / \left(10^{-G_p^{\prime}/10} - 1\right)\right]}{2\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}$$

$$\omega_c = \frac{\omega_p}{\left[10^{-G_p^{\prime}/10} - 1\right]^{1/2n}}$$

or

$$\omega_c = \frac{\omega_s}{\left[10^{-G_s/10} - 1\right]^{1/2n}}$$

5

→ From n we can calculate the poles

$$s_k = \cos\left(\frac{\pi(2k+n-1)}{2n}\right) + j\sin\left(\frac{\pi(2k+n-1)}{2n}\right).$$

$$k = 1, 2, 3, \ldots, n$$

get normalized transfer function:

$$H_a(s) = \prod_{k=1}^{n} \frac{1}{s - s_k}$$

→ $s \longrightarrow s/w_c$ (L.P.F

$s \longrightarrow w_p/s$. (HPF)

$$s \longrightarrow \frac{s^2 + w_{p_1} w_{p_2}}{(w_{p_2} - w_{p_1}) \cdot s} \quad \text{BPF}$$

$$s \longrightarrow \frac{(w_{p_2} - w_{p_1})s}{s^2 + w_{p_1} w_{p_2}} \quad \text{BSF}$$

Procedure different for ~~Chebsh~~ Chebyshev filter
or Elliptical filter
Cauer, filter
⋮
e.t.c

IIR Design by Impulse Invariance

impulse invariance condition $h[n] = T h_a(t)\big|_{t=nT} = T h_a(nT)$

$n = 1, 2, 3, \ldots$

→ T is the sampling period.

Assume we have designed the continuous filter $H_a(s)$;

$$H_a(s) = \frac{\sum_{k=0}^{N} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

$$h_a(t) = \mathcal{L}^{-1}\left[H_a(s)\right] = \mathcal{L}^{-1}\left[\sum_{k=1}^{N} \frac{c_k}{s-p_k}\right] =$$

$$\sum_{k=1}^{N} \mathcal{L}^{-1}\left[\frac{c_k}{s-p_k}\right] = \sum_{k=1}^{N} c_k e^{p_k t}$$

$$h[n] = T h_a(nT) = T \sum_{k=1}^{N} c_k e^{p_k nT}, \quad n = 1, 2, 3$$

→ infinite sequence.

$$H(z) \doteq \mathcal{Z}\{h[n]\} = T \sum_{n=0}^{\infty} \sum_{k=1}^{N} c_k e^{p_k nT} \cdot z^{-n}$$

$$= T \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^n$$

⑥

→ sum to infinity

$$H(z) = T \sum_{k=1}^{N} c_k \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$= \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}}$$

$$\left[\text{compare with } H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \right]$$

→ Choice of T ; T has to be small enough to avoid aliasing

→ Recall Nyquist's Theorem : $\omega_s > 2\omega_B$ → bandwidth

→ in our case $H_a(s)$ is not band limited, so

we define ω_B as $|H_a(j\omega_B)| = \underbrace{0.01}_{\text{our choice}} \cdot |H_a(j\omega)|_{max}$

$$T = \frac{\pi}{\omega_B}$$

this requirement means we can't use the impulse invariance method for high-pass and stop-band filters.