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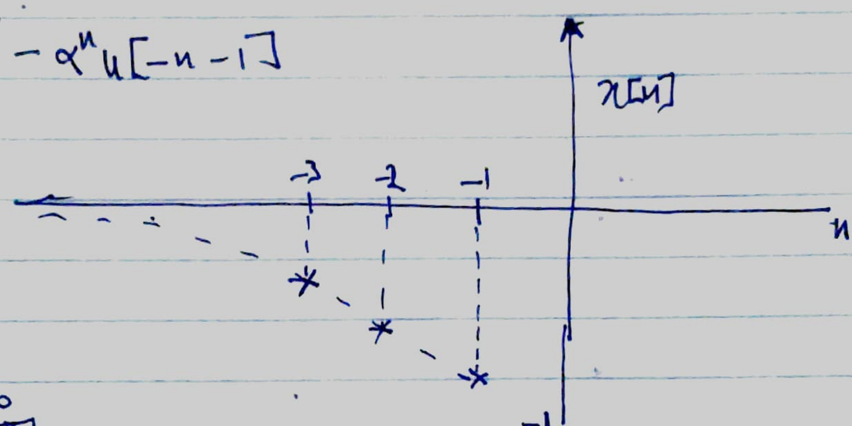
REGIONS OF CONVERGENCE

Consider the left-hand sided exponential

$$x[n] = -\alpha^n u[-n-1]$$



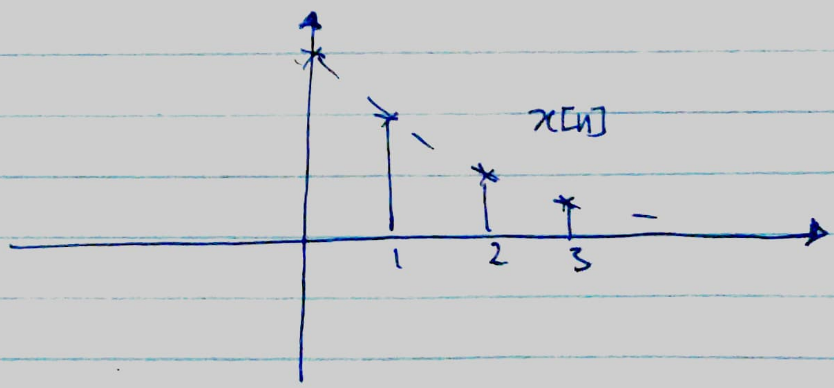
$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1]\, z^{-n} = \sum_{-\infty}^{-1} -\alpha^n z^{-n}$$

$$k = -n \quad X(z) = \sum_{k=1}^{\infty} -\alpha^{-k} z^k = -\sum_{k=1}^{\infty} (\alpha^{-1} z)^k$$

$$X(z) = \frac{\alpha^{-1} z}{1 - \alpha^{-1} z} \Rightarrow \frac{1}{1 - \alpha z^{-1}}$$

$$R.O.C = |\alpha^{-1} z| < 1 \Rightarrow \cancel{|z| < |\alpha|}$$
$$|z| < |\alpha|$$

Consider : $x[n] = \alpha^n u[n]$



$$X(z) = \sum_{k=-\infty}^{\infty} \underbrace{\alpha^n}_{u[n]} z^{-n} = \sum_{k=0}^{\infty} (\alpha z^{-1})^n$$

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad |\alpha z^{-1}| < 1$$

$$\Rightarrow |z| > |\alpha|$$

$x[n] = -\alpha^n u[-n-1] \qquad\qquad x[n] = \alpha^n u[n]$

$$\Big\updownarrow \mathcal{Z} \qquad\qquad\qquad\qquad \Big\updownarrow$$

$\dfrac{1}{1 - \alpha z^{-1}}$; $|z| < |\alpha| \qquad\qquad \dfrac{1}{1 - a z^{-1}}$ $|z| > |\alpha|$

why R.O.C is important

PROPERTIES of R.O.C

① The R.O.C does not include any poles of the z-transform, follows from definitions

② for right handed sequences ($x[n]$ such that $x[n] = 0$ for $n < n_0$;) the R.O.C has the form $|z| > z_0$

③ for left handed sequences ($x[n]$ such that $x[n] = 0$ for $n > n_0$) the R.O.C has the form $|z| < z_0$

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→ For double-sided sequences ($x[n]$ extends $-\infty \leftrightarrow \infty$)

the R.O.C has the form $z_1 < |z| < z_2$ (annulus)

~~PROOF~~

→ For a finite length sequence ($x[n] = 0$ for $n > n_0$ & $n < n_1$)

the R.O.C is the entire z-plane.

PROOF

$x[n]$: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ in R.O.C $|X(z)| < \infty$

general signal

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| |z^{-n}|$$

$$= \underbrace{\sum_{n=-\infty}^{-1} |x[n]| |z^{-n}|}_{N(z)} + \underbrace{\sum_{n=0}^{\infty} |x[n]| |z^{-n}|}_{P(z)}$$

for $X(z)$ to be fine $|x[n]|$ must be bounded

→ for right handed signal $N(z) = 0$

$$|X(z)| \le \sum_{n=0}^{\infty} |x[n]| |z^{-n}|$$

$$< \sum_{n=0}^{\infty} r_0^n |z^{-n}|$$ for convergence $|z^{-1} r_0| < 1$

$\Rightarrow |z| > r_0$

→ for left sided $x[n]$ $\Rightarrow$ $P(z) = 0$

$$X(z) \leq \sum_{n=-\infty}^{\infty} |x[n]| |z^{-n}|$$

$$\leq \sum_{n=-\infty}^{-\infty} |r_0^n| |z^{-n}| \qquad k = -n$$

$$\leq \sum_{k=1}^{\infty} |r_0^{-k}| |z^k|$$

↳ converges if $|z r_0^{-1}| < 1$

$|z| < |r_0|$

→ Double sided is just superposition.

Example.

$$X(z) = \frac{z}{2z^2 - 3z + 1} \; ; \quad |z| < \frac{1}{2}$$

from R.O.C $\Rightarrow$ $x[n]$ is a left sided signal

do long devision.

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$$\cancel{\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} - \frac{11}{8}z^{-3} \ldots}$$

$$2z^2 - 3z + 1 \overline{\big) z}$$

$$z - \frac{3}{2} + \frac{1}{2}z^{-1}$$

$$\frac{3}{2} - \frac{1}{2}z^{-1}$$

$$+\frac{3}{2} + \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}$$

$$-\frac{11}{4}z^{-1} - \frac{3}{4}z^{-2}$$

$$-\frac{11}{4}z^{-1} + \frac{33}{8}z^{-2} - \frac{11}{8}z^{-3}$$

$$-\frac{39}{8}z^{-2} + \frac{11}{8}z^{-3}$$

$$\cancel{X(z) = +\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} - \frac{11}{8}z^{-3} \ldots}$$

$$\sum_{n=-\infty}^{-1} x[n]z^{-n} \longrightarrow \sum_{k=1}^{\infty} x[-k]z^{k}$$

$$z + 3z^2 + 7z^3 + 15z^4 + \ldots$$

$$\cancel{1 - 3z + 1} \qquad 1 - 3z + 2z^2 \overline{\big) z}$$

$$z - 3z^2 + 2z^3$$

$$3z^2 - 2z^3$$

$$3z^2 - 9z^3 + 6z^4$$

$$7z^3 - 6z^4$$

$$7z^3 - 21z^4 + 14z^5$$

$$\sum_{k=1}^{\infty} x[-k]z^k = \cancel{x[-1] + x[-2]z + x[-3]z}$$

$$x[-1]z + x[-2]z^2 + x[-3]z^3$$

$x[-1] = 1$

$x[-2] = 3$

$x[-3] = 7$

$x[-4] = 15$

⋮

(b) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$ $|z| > 1$

$\downarrow$

right sided

$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

$$\begin{array}{r} \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \dots \\ 2z^2 - 3z + 1 \,\overline{)\, z \qquad\qquad\qquad\qquad} \\ \underline{z - \frac{3}{2} - \frac{1}{2}z^{-1}} \\ \frac{3}{2} - \frac{1}{2}z^{-1} \\ \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\ \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \end{array}$$

$X(z) = \underbrace{\frac{1}{2}z^{-1}}_{x[1]} + \underbrace{\frac{3}{4}z^{-2}}_{x[2]} + \underbrace{\frac{7}{8}z^{-3}}_{x[3]} \dots$

[alternatively can use tables but be careful to make sure the R.O.C is correct

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TRANSFER/SYSTEM FUNCTION

→ recall we showed that

$$y[n] = x[n] * h[n] \xrightarrow{\mathcal{Z}} Y(z) = X(z)\,H(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} \quad \{\text{transfer function}\}$$

(i) Causality :

→ for a system $h[n]$ is right-sided i.e $h[n] = 0$ for all $n < 0$.

∴ R.O.C of a causal system has the form

$$|z| > r_{max}$$

$|z| < r_{max}$ is R.O.C for anti-causal sys

(ii) a system is stable if all the poles of $H(z)$ lie inside the unit-circle of the z-plane

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Big\downarrow \mathcal{Z}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \longrightarrow$$ ~~transformation~~ transfer function

start with transfer function

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

$H(z) \rightarrow$ rational divide by z^N

$M \leq N$

$$\rightarrow \frac{Y(z)}{X(z)} = \frac{b_0 z^{M-N} + b_1 z^{M-N-1} + \cdots + b_{M-1} z^{-N+1} + b_M z^{-N}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1} + a_N z^{-N}}$$

$$Y(z)\left[a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1} + a_N z^{-N}\right] = X(z)\left[b_0 z^{M-N} + b_1 z^{M-N-1} + \cdots + b_{M-1} z^{-N+1} + b_M z^{-N}\right]$$