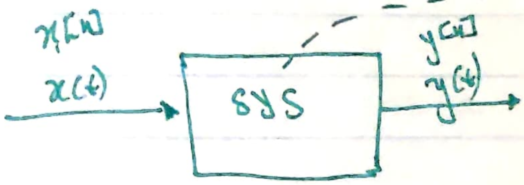
~~Lecture Note~~ Lecture 2

Defn: System

- System is an abstraction of a physical process that relates an input signal to the output signal

in general



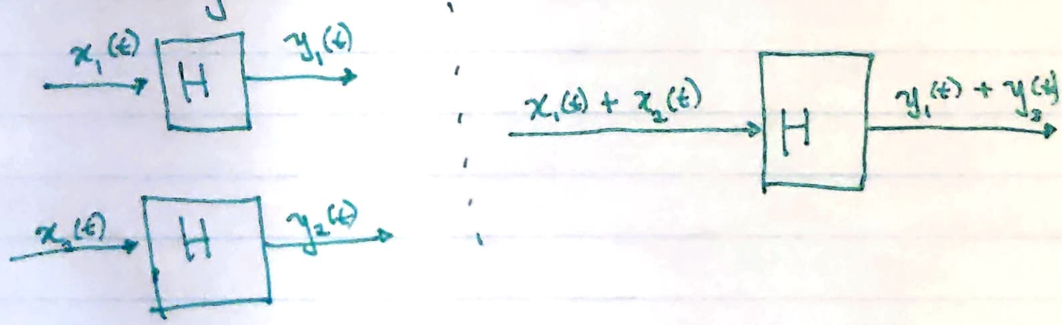
[this representation can be anything]

Properties for system

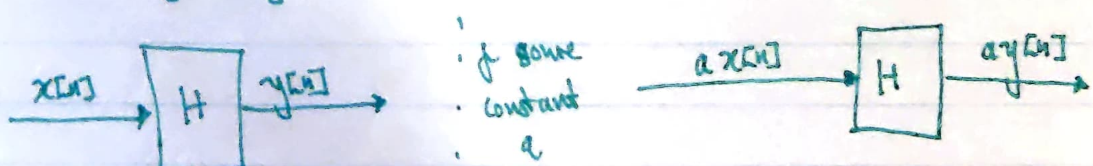
Linearity

- consider a system H; ~~we sa~~ H is linear if it meets the following conditions:

① Additivity:



② Homogeneity:



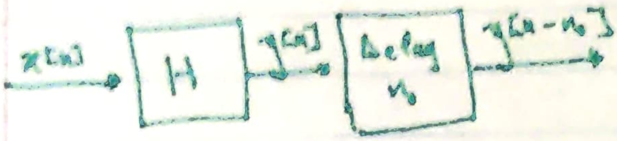
of some constant a

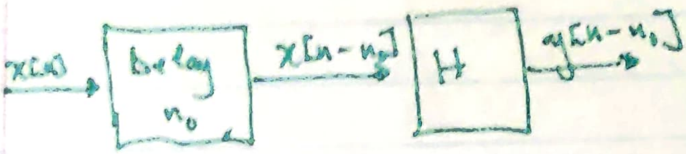
$$a_1 x_1[n] + a_2 x_2[n] \xrightarrow{H} a_1 y_1[n] + a_2 y_2[n]$$

principle of superposition

Time - invariance

if $x[n] \xrightarrow{H} y[n]$ then for some n_0 $x[n-n_0] \xrightarrow{H} y[n-n_0]$

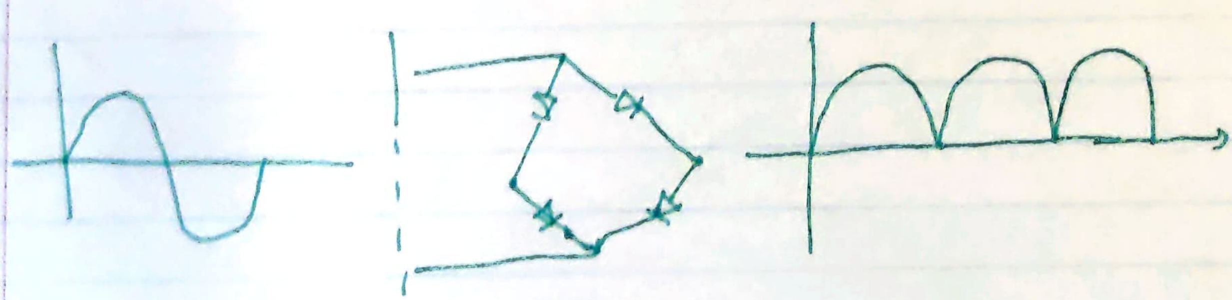




~~Inverse~~ Invertibility

→ a sys H is said to be invertible if the input can be determined from its output

Example: Full-wave rectifier



Causality

→ a system H is causal if the output @ $y[n]$ is dependent on the input @ time $x[n]$ & in the past $x[n-1]$, $x[n-2]$, . . .

Example: derivative - finite difference

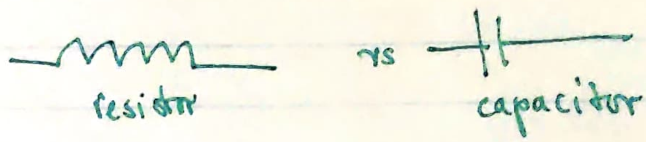
$y[n] = x[n$

3

Memory

→ A system H has memory if the output $y[n_0]$ depends on input values ~~@ n_0; $x[n_0]$~~ @ other than n_0

Example:

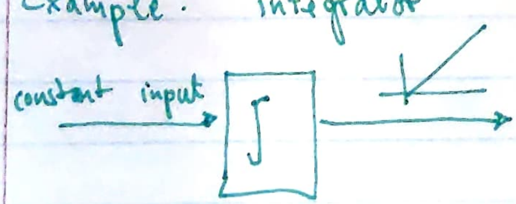


Stability

→ lots of versions of stability we will focus on bounded input bounded output (BIBO)

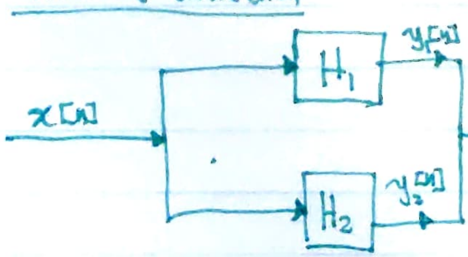
→ A system H is said to BIBO ~~stability~~ stable if $\exists$ a M such that $|x[n]| \leq M \ \forall n$ then $\exists$ a number R such that $|y[n]| \leq R$ for $\forall n$ where $x[n] \longrightarrow y[n]$

Example: integrator



SYSTEM INTERCONNECTION

Parallel connection

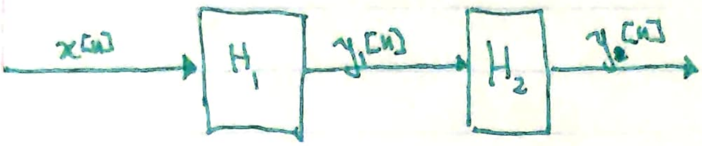


$y[n] = y_1[n] + y_2[n]$

$y[n] = H_1(x_1[n]) + H_2(x_2[n])$

(A)

Cascade connection



$$y[n] = H_2(H_1(x_1[n])) = (H_2 \circ H_1)(x[n])$$

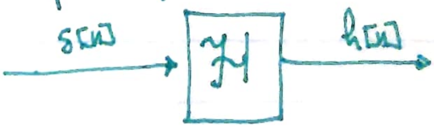
SYSTEM REPRESENTATION

→ Impulse response

→ recall any signal can be represented as a sum weighted of impulse functions

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

→ ~~define the system~~ consider a sys $\mathcal{H}$ we will denote its response to an impulse function as $h[n]$



→ for a signal

$$\boxed{\text{for input } x[n] \text{ \& system with impulse response } h[n] \text{ will have output } y[n] = \sum x[k]\,h[n-k]}$$

Determining system properties from the impulse response.

Memory

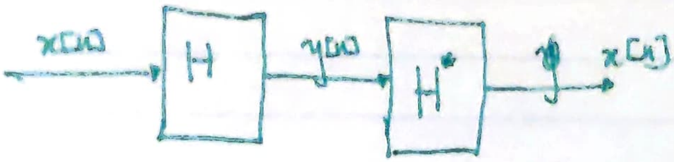
~~$y[n] = \dots + x[n+2] + x[n+1] + x$~~

$y[n] = \dots + \cancel{x[-2]h[n+}$

$y[n] = \dots + x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]\cdot h[n-2]$

\# ~~for memory~~ to be memoryless $h[n] = 0$ for $n \neq 0$

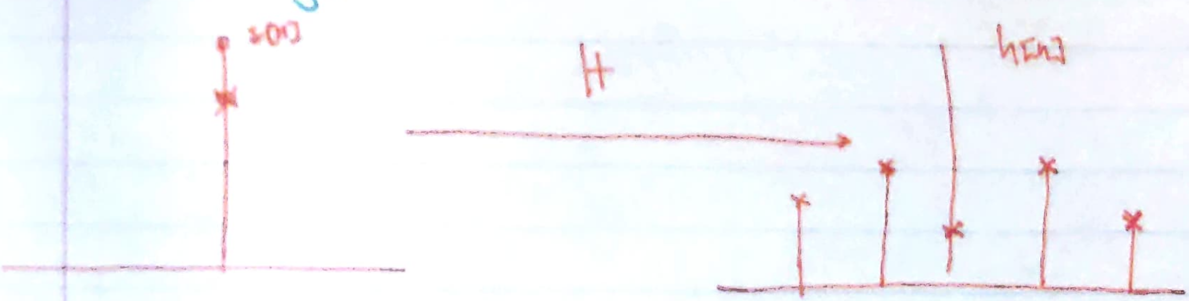
Invertibility



a system is invertible if $\exists$ a function $h^*[n]$ such that

$$h[n] * h^*[n] = \delta[n]$$

Causality



for causality $h[n] = 0$ for $n < 0$

6

BIBO Stability

consider a system H defined by impulse response $h[n]$. For input $x[n]$ we have output $y[n]$

$$\therefore \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

assume $|x[n]| \leq M$; for all n

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \leq \cancel{\sum_{k=-\infty}^{\infty} |x[k]\,h[n-k]|}$$

$$\leq \sum_{k=-\infty}^{\infty} |x[k]h[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |x[k]|\,|h[n-k]| \leq \sum_{k=-\infty}^{\infty} M\,|h[n-k]|$$

$$= M\sum_{k=-\infty}^{\infty} |h[n-k]|$$

for $|y[n]| \leq R$ we need $\sum_{k=-\infty}^{\infty} |h[n-k]| < \infty$

If $h[n]$ is absolutely summable then the system is BIBO stable

Proving necessity is left as an exercize

Properties of the convolution sum

① Commutative property

$$x[n] * h[n] = h[n] * x[n]$$

PROOF

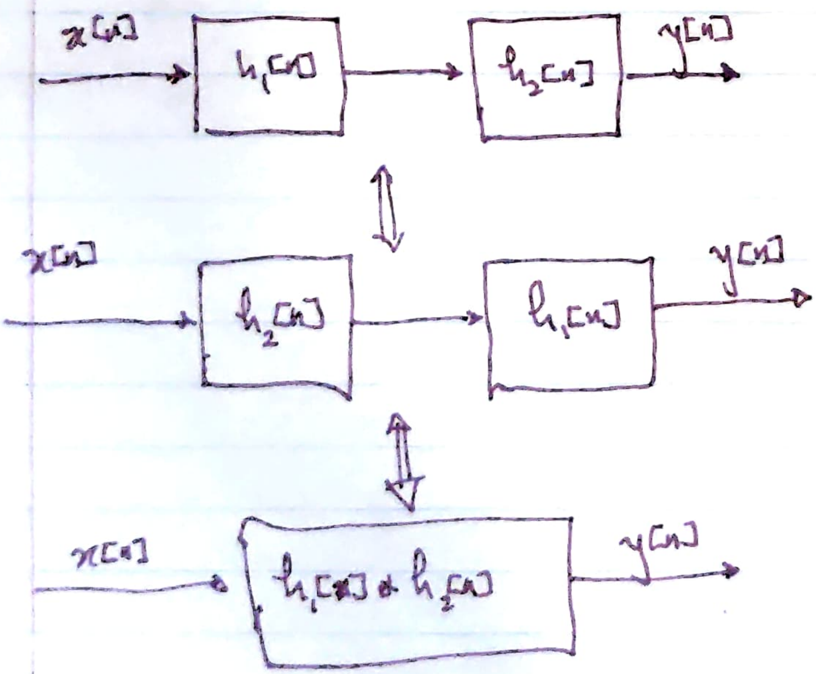
$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

substitute $k' = n-k$

$$= \sum_{k'=-\infty}^{\infty} x[n-k']\, h[k'] = h[n] * x[n]$$

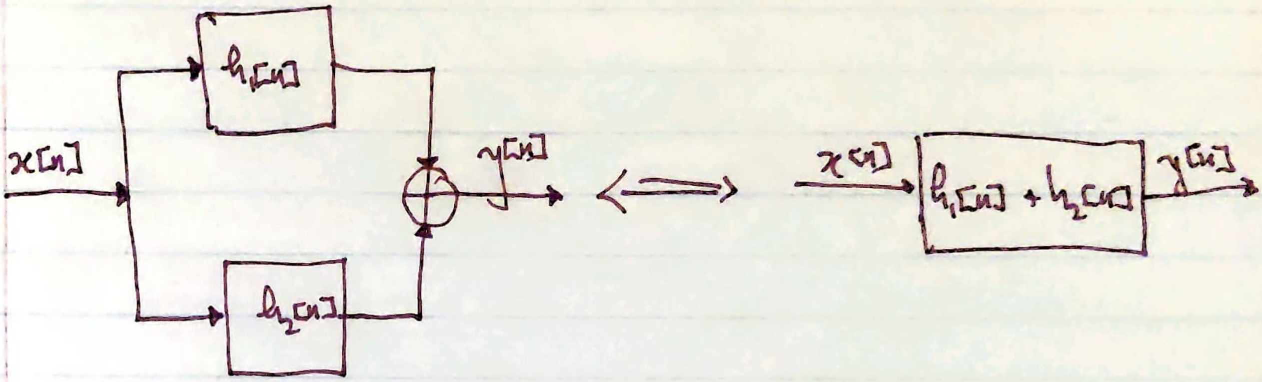
② Associative property

$$(f[n] * g[n]) * h[n] = f[n] * (g[n] * h[n]) = \cancel{f[n]*} \, (h[n] * f[n]) * g[n]$$



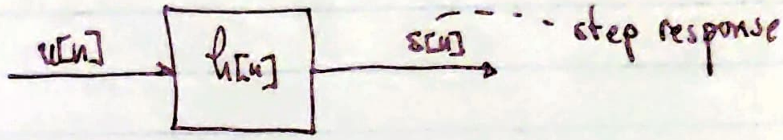
Distributive property

$$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])$$



Step Response

Consider a system with impulse response $h[n]$



$$s[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$

$$= \sum_{k=-\infty}^{n} h[k]$$

consider

$$s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k]$$

$$= h[n]$$

We can get the impulse response from the step response.

10

DIFFERENCE EQUATION REPRESENTATION

→ We can represent a discrete LTI system with linear difference equations with constant coefficient.

* We can represent DE using either:
  (i) delay terms $y[n-1], y[n-2], x[n-1], \ldots$
  (ii) advance terms $y[n+1], y[n+2], x[n+1]$

we will use delay notation

DE General form

An N^{th} order linear difference equation with constant coefficients is given as:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] =$$

$$b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

where $a_0, \ldots, a_N$ & $b_0, \ldots, b_M$ are constant real values $a_0 \neq 0$

more compactly

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{k'=0}^{M} b_{k'} x[n-k']$$

11.

SOLUTION OF D.E using CLASSICAL METHOD

consider the general N^{th} of D.E.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \longrightarrow (1)$$

The general solution to (1) can be expressed as

$$y[n] = y_c[n] + y_p[n]$$

$y_c[n]$: complementary (c) function, natural response

$y_p[n]$: particular function, forced response.

Natural response

(1) zero the R.H.S of (1)

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = 0 \longrightarrow (2)$$

$y_c[n]$ is a solution to equation (2)

assume $y_c[n]$ has the general form Cz^n ; C & z are constants

$$y_c[n] = Cz^n$$

(12).

$$y_0[n] = Cz^n$$
$$y_0[n-1] = Cz^{n-1} = Cz^n \cdot z^{-1}$$
$$y_0[n-2] = Cz^{n-2} = Cz^n \cdot z^{-2}$$
$$\vdots$$
$$y_0[n-N] = Cz^{n-N} = Cz^n \cdot z^{-N}$$

substituting into (2):

$$a_0 \cdot Cz^n + a_1 \cdot Cz^n \cdot z^{-1} + \cdots + a_{N-1} \cdot Cz^n \cdot z^{-N+1} + a_N \cdot C \cdot z^n \cdot z^{-N} = 0$$

$$\Rightarrow \left(a_0 + a_1 z^{-1} + \cdots + a_{N-1} \cdot z^{-N+1} + a_N \cdot z^{-N}\right) Cz^n = 0$$

[~~non-trivial solution~~: $\cancel{\underbrace{a_0 + a_1 z^{-1} + \cdots + a_{N-1} \cdot z^{-N+1} + a_N \cdot z^{-N} = 0}_{\text{characterie equation}}}$]

$$\Rightarrow \left(a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} \cdot z^{-1} + a_N\right) \cdot Cz^{-N} \cdot z^n = 0$$

non-trivial solution: $\underbrace{a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} \cdot z^{-1} + a_N = 0}_{\text{characteristic/auxilary equation.}}$

→ this is a polynomial in z which we can factorise:

$$a_0 (z - z_1)(z - z_2) \cdots (z - z_N) = 0$$

$z_1, z_2, \ldots, z_N$ are solutions to the characteristic equation.

13

no repeated roots

$$y_c[n] = C_1 z_1^n + C_2 z_2^n + \dots + C_N \cdot z_N^n \quad (C_1, \dots, C_N \rightarrow \text{unknown coefficients})$$

repeated roots

→ if a root has multiplicity > 1 the repeated solution must be multiplied by each n from 0 to r-1 (r = multiplicity)

i.e if the root z_i has multiplicity

$$\text{term} = (C_1 + C_2 n + C_3 n^2 + \dots + C_r n^{r-1}) \cdot z_i^n$$

complex roots

conjugate if z_1 is complex the it's conjugate is also a root

$$z_i \longleftrightarrow z_i^*$$

∴ we have 2 terms $C_i \cdot z_i^n + C_i^* \cdot (z_i^*)^n$

let $C_i = |C_i| e^{j\beta_i}$ & $C_i^* = |C_i| e^{-j\beta_i}$

$$z_i = |z_i| e^{j\theta_i} \qquad z_i^* = |z_i| e^{-j\theta_i}$$

$$C_i \cdot z_i^n + C_i^* (z_i^*)^n = |C_i||z_i|^n e^{jn\theta_i} \cdot e^{j\beta_i} + |C_1||z_1|^n e^{-jn\theta_1} \cdot e^{-j\beta_1}$$

$$|C_1||z_1|^n \left[e^{j(n\theta_1 + \beta_1)} + e^{-j(n\theta_1 + \beta_1)} \right]$$

$$= |C_1||z_1|^n \cos(n\theta_1 + \beta_1)$$

(14).

For particular solution

if we know what kind of input $x[n]$ is exciting the system we can use the method of undetermined coefficients

~~Input $x[n]$ | $y_p[n]$~~

→ the particular solution $y_p[n]$ should satisfy:

$$\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

known

→ In the method of undetermined co-efficients we assume that $y_p[n]$ is the sum of functions ~~with~~ of the mathematical form of $x[n]$ & delayed versions of $x[n]$

# Input $x[n]$	Particular solution
A (constant)	K (constant
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \dots + K_M$
$A^n n^M$	$A^n(K_0 n^M + K_1 n^{M-1} + \dots + K_M)$
$A\cos(w_0 n)$, $A\sin(w_0 n)$	$K_1 \cos(w_0 n) + K_2 \sin(w_0 n)$

15

$$y[n] = y_c[n] + y_p[n]$$

$$y_c[n] \left\{ \cancel{C_1 z_1^n + C_2 z_2^n + \cdots + C_{N-r}} \right.$$

$$y_c[n] = \left(C_1 + C_2 n + C_3 n^2 + \cdots + C_r n^{r-1}\right) \cdot z_{N-r+1}^n + \sum_{k=1}^{N-r} C_k \cdot z_k^n$$

$$+ \; y_p[n]$$

Note

(i) the roots of the characteristic equation determine the behavior of the system when its unforced!

→ if $|z_i| < 1$ then $y_c[n] \longrightarrow 0$

if $|z_i| > 1$ then $y_c[n] \longrightarrow \infty$

(ii) for the ~~cal~~ case where $|z_i| < 1$ then the particular solution $y_p[n]$ dominates the system behaviour as $n \to \infty$