1.

DISCRETE TIME FOURIER TRANSFORM (DTFT)

DEFN :

The DTFT is defined by:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}$$

& the Inverse DTFT (IDTFT) is defined by:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{jn\Omega} d\Omega$$

EXAMPLE :

$x[n] = a^n u[n]$. FIND $X(\Omega)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} = \sum_{n=0}^{\infty} a^n e^{-jn\Omega}$$

$\rightarrow$ geometric series

common ratio $= ae^{-j\Omega}$

$$=$$

$$\therefore X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

for $|ae^{-j\Omega}| < 1$

$\Rightarrow |a| < 1 \longrightarrow$ region of convergence.

RELATIONSHIP BETWEEN z-Transform & DTFF

Recall z-transform of $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad \left(X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \right)$$

we can evaluate the DTFT from the z-transform

$$X(\Omega) = X(z) \Big|_{z = e^{-j\Omega}}$$

DTFT PROPERTIES

① $X(\Omega)$ is a continuous function of Ω

② $X(\Omega)$ is periodic in Ω with period 2π

note $X(\Omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn(\Omega + 2\pi k)}$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \cdot \underbrace{e^{-jn2\pi k}}_{1}$$

$$= X(\Omega)$$

3

(3) ~~Pt~~ Linearity : ~~FT~~

$$a x_1[n] + b x_2[n] \longleftrightarrow a X_1(\Omega) + b X_2(\Omega)$$

(4) (a) Convolution in time :

$$x[n] * y[n] \xrightarrow{\mathcal{F}} X(\Omega) Y(\Omega)$$

(b) Convolution in frequency :

$$x[n] y[n] \longleftrightarrow \cancel{\frac{1}{2\pi} \int_{2\pi} X(\Omega) Y(\Omega}$$

$$x[n] y[n] \longleftrightarrow \underbrace{\frac{1}{2\pi} \int_{2\pi} X(\theta) Y(\Omega - \theta) \, d\theta}_{\text{circular convolution}}$$

$$\frac{1}{2\pi} X(\Omega) \circledast Y(\Omega)$$

Discrete Fourier Transform (DFT)

→ The DFT is an approximation of the DTFT.

→ Consider $x_N[n]$ to be an N-length sequence the DFT is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j 2\pi k n / N} \qquad k = 0, 1, 2, \ldots, N-1$$

→ The inverse DFT (IDF) is defined by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N} \qquad n = 0, 1, 2, .., N-1$$

NOTATION (SHORTHAND): for brevity we can write these definitions as

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1$$

$$W_N = e^{-j2\pi/N}$$

COMMENT

→ for a N-length sequence $x[n]$ the DFT is an N-length sequence of complex numbers $X[k]$

→ the DFT is a sampled version of the DTFT; the sampling period is the frequency resolution. This is equal to:

$$\Delta\Omega = \frac{2\pi}{N}$$

→ To improve the frequency resolution we need to increase N (number of samples). An easy way to do this is appending zero; this is termed zero padding.

3.

FFT (Fast Fourier Transform)

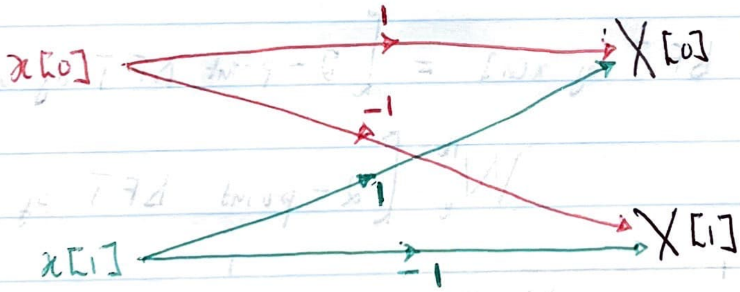
→ This is an efficient algorithm for calculating DFT (straight-forward evaluation of DFT has N^2 complexity)

Decomposition In Time (N - point radix - 2 FFT)
$N \rightarrow 2^m$

2 - point DFT

$$X[k] = \sum_{n=0}^{1} x[n] W_2^{kn} = x[0] W_2^{k0} + x[1] W_2^{k1}$$

$$X[0] = x[0] W_2^{00} + x[1] W_2^{01} = x[0] + x[1]$$
$$X[1] = x[0] W_2^{10} + x[1] W_2^{11} = x[0] - x[1]$$



Butterfly diagram
Signal flow.

4 - point D.FT

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{kn} = x[0] W_4^{k0} + x[1] W_4^{k1} + x[2] W_4^{k2} + x[3] W_4^{k3}$$

$$W_4^{k0} = \left(e^{-j\pi/2}\right)^{k0} = 1$$

$$W_4^{k1} = \left(e^{-j\pi/2}\right)^{k} = (-j)^k$$

$$W_4^{k2} = \left(e^{-j\pi}\right)^{k} = (-1)^k$$

$$W_4^{k3} = \left(e^{-j\frac{3\pi}{2}}\right)^{k} = \left(e^{-j\pi}\right)^k \cdot \left(e^{-j\pi/2}\right)^k = (-1)^k \cdot W_4^{1k}$$

$$X[k] = \Big[\underbrace{\overbrace{x[0] + x[2](-1)^k}^{X_e[k]}}_{\text{2-point DFT}}\Big] + \Big[\underbrace{\overbrace{x[1] + x[3](-1)^k}^{X_o[k]}}_{\text{2-point DFT}}\Big] W_4^{1k}$$

$$X_e[0] = x[0] + x[2]$$
$$X_e[1] = x[0] - x[2]$$
$$X_e[2] = x[0] + x[2] = X_e[0]$$
$$X_e[3] = x[0] - x[2] = X_e[1]$$

$$X_o[0] = x[1] \cancel{+} x[3]$$
$$X_o[1] = x[1] - x[3]$$
$$X_o[2] = x[1] + x[3] = X_o[0]$$
$$X_o[3] = x[1] - x[3] = X_1[0]$$

$$X[0] = X_e[0] + X_o[0]\, W_4^{10}$$
$$X[1] = X_e[1] + X[\bullet]\, W_4^{11}$$
$$X[2] = X_e[0] + X_o[0]\, W_4^{12}$$
$$X[3] = X_e[1] + X_o[1]\, W_4^{13}$$

$$W_4^{10} = 1$$
$$W_4^{\bullet} = e^{-j\frac{2\pi}{4}} =$$
$$W_4^{12} = e^{-(j\frac{2\pi}{4})2} = -1$$
$$W_4^{13} = (-1)^k W_4^{1}$$

essentially : 4-point DFT of $x[n]$ = [2-point DFT of $x_e[n]$] + $W_k^{1k} \cdot$ [2-point DFT of $x_o[n]$]