SIGNALS AND SYSTEMS IIB 2017

ELEN 3013

Dr OYEROWDE Olutayo.Oyerinde@wits.ac.za
http://dept.eie.wits.ac.za/~oyerinde/ELEN3013.htm

Course content

→ Discrete Time Signals
- Representations
- Classifications
- Models
- Operations
- Systems

→ Analyse Discrete Time Systems, LTI Systems
- System analysis: Differential equation
- Block diagram representation

→ Z-Transform
- Definition
- Properties
- Inverse transform
- Transfer function
- System realisation

→ Digital Filters
- IIR Filters
- FIR Filters

→ Fourier Transforms
- Discrete-Time Fourier Transform (DTFT)
- Discrete Fourier Transform (DFT)
- Fast Fourier Transform (FFT)
- Applications

28/07/2017

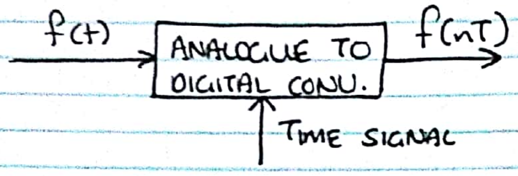
1) DISCRETE TIME SIGNALS

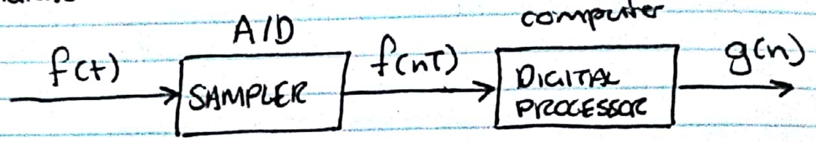
Defn: A <u>discrete time signal</u> is defined only at time instances

A computer processes a continuous-time signal $f(t)$ at regular intervals (Digital Signal Processing DSP) at intervals T gives $f(nT)$ at $n = \ldots -2, -1, 0, 1, 2, \ldots$

The time increment T is called the sampling period

SAMPLING HARDWARE





EXAMPLE 1

A sinosoidal function $g(t) = \cos(2\pi f_0 t)$, is sampled at discrete values $t = nT$

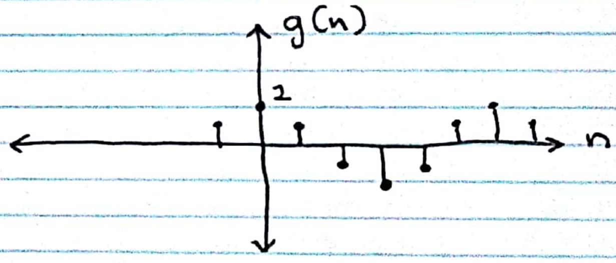
$$g(t) = \cos(2\pi f_0 t)$$

$$g(nT) = g(t)\Big|_{t=nT}$$

$$= \cos(2\pi f_0 nT)$$

$$= \cos(\Omega n)$$

where $\Omega = 2\pi f_0 T$



TYPES OF DISCRETE SIGNALS

A <u>CONTINUOUS - AMPLITUDE SIGNAL</u> is a signal where the amplitude can be any value between $-\infty < x[n] < \infty$

A <u>DISCRETE - AMPLITUDE SIGNAL</u> can only assume certain predefined amplitudes

A DISCRETE-AMPLITUDE, DISCRETE-TIME SIGNAL is also called a <u>DIGITAL SIGNAL</u>.

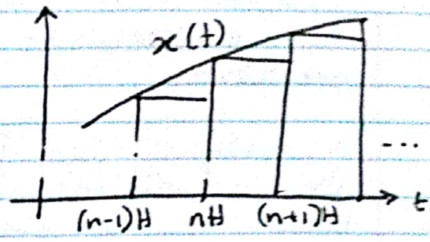
EXAMPLE 2

Integrate a voltage signal $x(t)$ using a digital computer

<u>EULER'S RULE FOR INTEGRATION</u> sums a set of rectangles which fit underneath the curve and calculates their areas

The step size H, the width of the rectangles is called the NUMERICAL-INTEGRATION INCREMENT

$x(t)$ is sampled every H seconds resulting in $x(Hn)$



$$y(t) = \int_0^t x(\tau)\,d\tau$$

$$y(t)\Big|_{t=nH} = \int_0^{nH} x(\tau)\,d\tau$$

$$= \int_0^{(n-1)H} x(\tau)\,d\tau + \int_{(n-1)H}^{nH} x(\tau)\,d\tau$$

$$\approx y[(n-1)H] + Hx[(n-1)H]$$

without the approximation : $y(nH) = y[(1-n)H] + Hx[(n-1)H]$

$$y[n] = y[n-1] + Hx[n-1]$$

This is called a DIFFERENCE EQUATION

An N^{th} order LDE with constant coefficients is given as

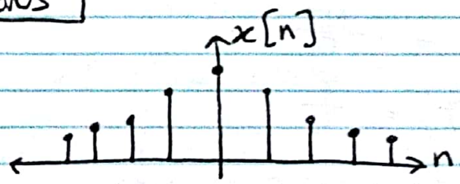
$$y[n] = b_1 y[n-1] + b_2 y[n-2] + \dots + b_N y[n-N] + a_0 x[n] + a_1 x[n-1] + \dots + a_n x[n-N]$$

a_i, b_i are coefficients where $i = 1, 2, \dots, N$ are constants.

The equation is periodic and can also be represented by $n = (n+N)$

DISCRETE SIGNAL REPRESENTATIONS

- GRAPHICAL REPRESENTATION



- FUNCTIONAL REPRESENTATION

$$x[n] = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

- TABULAR REPRESENTATION

n	...	-2	-1	0	1	2	3	4	5 ...
x[n]	...	0	0	0	1	4	1	0	0 ...

- SEQUENTIAL REPRESENTATION

$$x[n] = \{0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0 \dots\}$$

The ↑ represents the time origin

DISCRETE SIGNAL CLASSIFICATIONS

A DETERMINISTIC SIGNAL can have its value determined exactly by its model at a specific time.

A RANDOM or STOCHASTIC SIGNAL cannot have its value predicted.

A signal is PERIODIC if there is a number N for a discrete signal such that $x[n] = x[n+N]$. If there is no such number, the signal is NON-PERIODIC

EXAMPLE 3

Determine the condition in which the signal $x[n]$ obtained by sampling the signal $x(t) = \cos(\omega_0 t)$ every T seconds is periodic.

for $x[n]$ to be periodic:

$$x[n] = x[n+N]$$

$$x(t) = \cos(\omega_0 t)$$

$$x[n] = x(t)\Big|_{t=nT} = \cos(\omega_0 nT)$$

$$= \cos(n\omega_0 T + N\omega_0 T)$$

$$N\omega_0 T = 2\pi k$$
$$N 2\pi f_0 T = 2\pi k$$
$$2\pi k = \frac{2\pi TN}{T_0}$$

$$k = \frac{NT}{T_0}$$

$$\frac{k}{N} = \frac{T}{T_0}$$

The COMPLEX EXPONENTIAL can be represented as

$$x[n] = e^{j\Omega_0 n} = 1\angle\Omega_0 n = \cos(\Omega_0 n) + j\sin(\Omega_0 n)$$

This complex signal is periodic provided

$$x[n] = e^{j\Omega_0 n} = x[n+N] = e^{j(\Omega_0 n + \Omega_0 N)} = e^{j(\Omega_0 n + 2\pi k)}$$

where k is an integer. Thus the complex exponential is periodic if

$$\Omega_0 N = 2\pi k \quad \rightarrow \quad \Omega_0 = \frac{k}{N} 2\pi$$

Hence, the complex exponential signal $e^{j\Omega_0 n}$ is periodic with N samples per period for integer N which satisfies the equation

$$N = \frac{2\pi k}{\Omega_0}$$

where k is the smallest possible integer that satisfies the equation such that N is an integer greater than unity.

* The SUM or PRODUCT of two periodic signals with fundamental periods N_1 and N_2 will have an overall periodicity of the LCM of N_1 and N_2.

EVEN and ODD signals

A signal is EVEN if

$$x_e[n] = x_e[-n]$$

and is ODD if

$$x_o[n] = -x_o[-n]$$

Any discrete time signal can be expressed as the SUM of an EVEN and an ODD signal.

$$x[n] = x_e[n] + x_o[n]$$

Replacing n with $-n$ gives

$$x[-n] = x_e[-n] + x_o[-n]$$
$$= x_e[n] - x_o[n]$$

To extract the EVEN part of $x[n]$ we add $x[-n]$

$$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$$

To extract the ODD part of $x[n]$ we subtract $x[-n]$

$$x_o[n] = \tfrac{1}{2}(x[n] - x[-n])$$

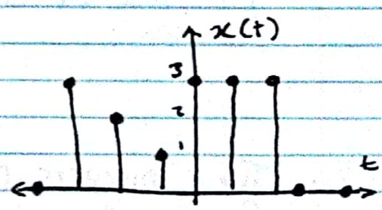
SOME PROPERTIES OF EVEN AND ODD SIGNALS

EVEN + EVEN = EVEN
ODD + ODD = ODD
EVEN + ODD = NEITHER

EVEN × EVEN = EVEN
ODD × ODD = EVEN
EVEN × ODD = ODD

EXAMPLE 4

Find the even and odd parts of the signal $x[n]$

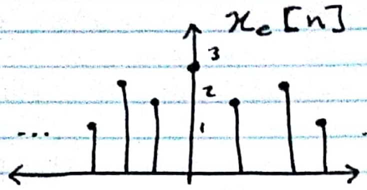


$$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$$

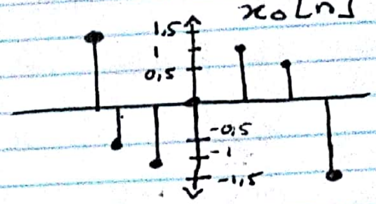
$$x_o[n] = \tfrac{1}{2}(x[n] - x[-n])$$

To analyse this signal, we analyse $x[-n]$, $x[n]$ and then $x_e[n]$, $x_o[n]$

n	$x[n]$	$x[-n]$	$x_e[n]$	$x_o[n]$
-3	3	0	1.5	1.5
-2	2	3	2.5	-0,5
-1	1	3	2	-1
0	3	3	3	0
1	3	1	2	1
2	3	2	2.5	0,5
3	0	3	1.5	-1.5



EVEN component



ODD Component

ENERGY AND POWER SIGNALS

The energy content of a discrete-time signal $x[n]$ is

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

The POWER content of a discrete-time signal $x[n]$ is

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Signals can be classified according to their energy and power as follows

- $x[n]$ is an energy sequence if $0 < E < \infty$ and $P = 0$
- $x[n]$ is a power sequence if $0 < P < \infty$ and ~~E=0~~ $E = \infty$
- A signal can be neither an energy sequence or power sequence

04/07/2017

SIGNAL MODELS

- The UNIT STEP function : $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

The DISCRETE-TIME UNIT STEP function : $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$u[n-n_0] = \begin{cases} 1 & n \geq n_0 \\ 0 & n < n_0 \end{cases}$$

- The DISCRETE-TIME IMPULSE function : $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$

EXAMPLE

Consider $x[n]\delta[n] = x[0]\delta[n]$

Since $\delta[n] = 1$ for $n = 0$ only

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

The unit impulse can be used to find the value of the signal at any point. k.

Consider $x[n]u[n]$, the signal is truncated at $n < 0$

LINEAR CONVOLUTION:

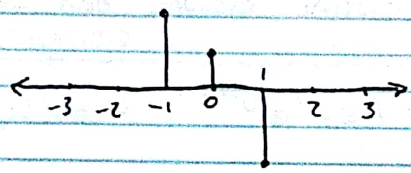
Convolution with an impulse response function shifts the function to where the signal is located

$$x[n] * \delta[n-k] = x[n]\Big|_{n=n-k}$$

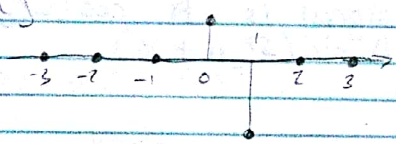
$$= x[n-k]$$

EXAMPLE

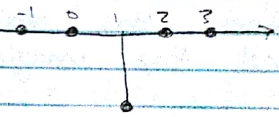
Given the following signal, determine $x[n]u[n]$ and $x[n]u[n-1]$



$x[n]u[n]$

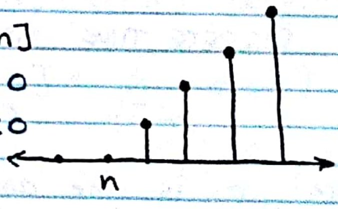


$x[n]u[n-1]$



- The DISCRETE-TIME UNIT RAMP FUNCTION $u_r[n]$

$$u_r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- The DISCRETE-TIME EXPONENTIAL

$$x[n] = a^n \text{, for all } n$$

where

$$a = e^b$$

so that

$$x[n] = e^{bn}, \ \forall n$$

EXAMPLE

Describe a discrete signal $x[n] = 0{,}9^n$ as a discrete time exponential function

$x[n] = a^n = 0{,}9^n \quad \therefore \quad a = 0{,}9 \qquad x[n] = (10^{-1} \cdot 3^2)^n$

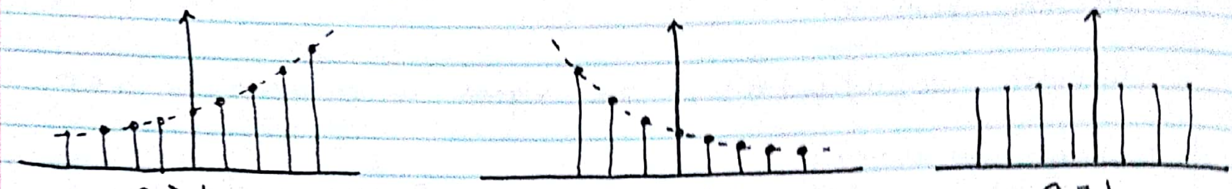
$a = e^b = 0{,}9 = 3^2 \cdot 10^{-1} \quad a = 10^{-1} \cdot 3^2$

The EXPONENTIAL SIGNAL ~~CA~~ can be written as

$$x[n] = c\,a^n$$

where both c and a can be complex.

If C and a are Real



If C and a are COMPLEX

For these systems:

$$C = Ae^{j\theta} = A\angle\phi \qquad a = e^{j\Omega_0}$$

where A, ϕ and Ω_0 are constants.

The COMPLEX EXPONENTIAL SIGNAL can be expressed as follows

$$\boxed{x[n] = Ca^n = Ae^{j\phi}e^{j\Omega_0 n} = A\cos(\Omega_0 n+\phi) + jA\sin(\Omega_0 n+\phi)}$$

Where the real part of the signal is $A\cos(\Omega_0 n+\phi)$ and the imaginary part is $A\sin(\Omega_0 n+\phi)$

Other signal models include:

The DISCRETE TIME SINOSOID

$$r\cos(\Omega n+\theta)$$

where r is the amplitude, Ω is the frequency (rad/sample) and θ is the phase (radians)

The EXPONENTIALLY VARYING DISCRETE-TIME SINOSOID

This is a signal with an exponentially varying amplitude

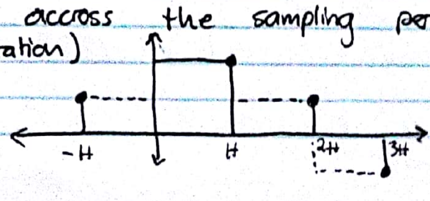
$$e^{j\Omega n}\cos(\Omega n+\theta)$$

BASIC OPERATIONS

EQUIVALENT OPERATIONS: INTEGRATION AND SUMMATION

Integration in continuous time $\approx$ summation in discrete time

(In discrete time, adding a pulse accross the sampling period yeilds area under curve ie integration)



by EULER'S RULE WE SEE:

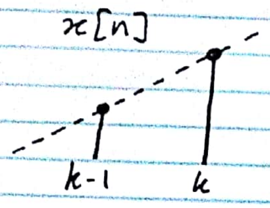
$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow H\sum_{k=-\infty}^{n} x[k]}$$

EQUIVALENT OPERATIONS: FIRST DERIVATIVE AND FIRST DIFFERENCE

The slope of a continuous time signal with samples $x[n]$ can be given as:

$$\boxed{\left.\frac{dx(t)}{dt}\right|_{t=kH} \approx \frac{x[k]-x[k-1]}{H}}$$

In discrete time systems, this is given as



$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow x[n]-x[n-1]}$$

CONTINUOUS TIME OPERATION	DISCRETE TIME OPERATION
$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\sum_{k=-\infty}^{\infty} x[k]$
$\frac{dx(t)}{dt}$	$x[n]-x[n-1]$
$x(t)\delta(t)=x(0)\delta(t)$	$x[n]\delta[n]=x[0]\delta[n]$
$\delta(t)=\frac{du(t)}{dt}$	$\delta[n]=u[n]-u[n-1]$
$u(t)=\int_{-\infty}^{t}\delta(\tau)\,d\tau$	$u[n]=\sum_{k=-\infty}^{n}\delta[k]$

TIME TRANSFORMATIONS

TIME ~~TRANSFORM~~ SHIFTING

$$y[n] = x[m]\Big|_{m=n-n_0} = x[n-n_0]$$

EXAMPLE

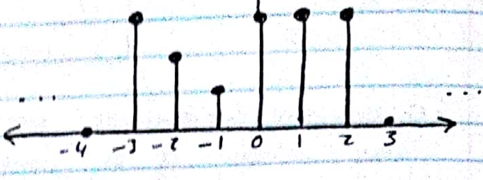
Given $x[m] = a^m \cos\left(\frac{\pi m}{4}\right)u[m]$, find the time-shifted signal $x[n-3]$

$$x[n-3] = x[m]\Big|_{m=n-3}$$

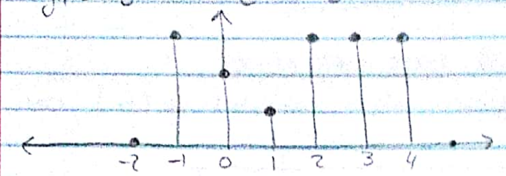
$$= a^{n-3}\cos\left(\frac{\pi}{4}(n-3)\right)u[n-3]$$

EXAMPLE

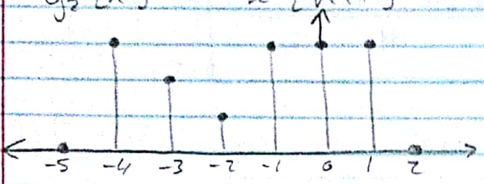
Given a discrete time signal $x[m]$ as shown, find the time-shifted versions $y_1[n] = x[n-2]$ and $y_2[n] = x[n+1]$



$y_1[n] = x[n-2]$



$y_2[n] = x[n+1]$



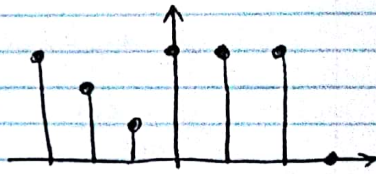
TIME REVERSAL

$$y[n] = x[m]\Big|_{m=-n} = x[-n]$$

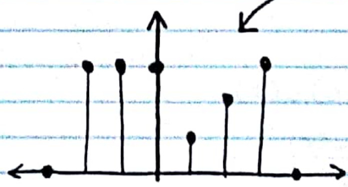
We create a mirror image of the signal about the vertical axis.

EXAMPLE

Given a discrete signal $x[m]$ as shown, obtain signal



$y[n] = x[-n]$



$y[n] = \{3,3,3,1,2,3\}$

TIME SCALING

Given a signal $x[m]$ in discrete time, time scaling refers to the operation that gives us the signal $x[an]$ obtained as:

$$y[n] = x[m]\Big|_{m=an} = x[an]$$

We have achieved:

i) Linearly stressed version of the original $x[n]$ if $|a| < 1$

ii) Linearly compressed version of the original if ~~$a < |a| < 1$~~ $|a| > 1$

$a < 0$?

iii) Reversed in time version of the original signal if $|a| < 0$

EXAMPLE

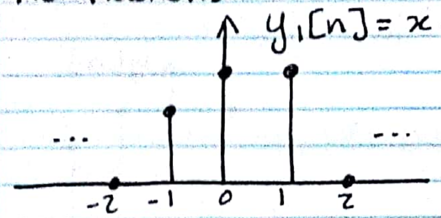
Given a discrete signal $x[m]$, obtain the time-scaled versions $y_1[n] = x[2n]$; $y_2[n] = x[n/3]$.

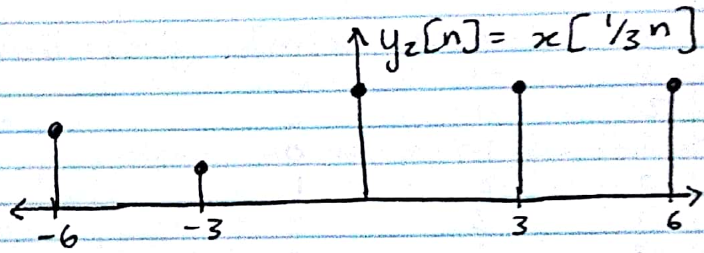
$y_1[n] = x[m]\big|_{m=2n} \rightarrow n = \frac{m}{2}$

$y_2[n] = x[m]\big|_{m=\frac{1}{3}n} \rightarrow n = 3m$

m	$x[m]$	$n = m/2$	$y_1[n]$	$n = 3m$	$y_2[n]$
3	0	—	—	9	0
2	3	1	3	6	3
1	3	—	—	3	3
0	3	0	3	0	3
−1	1	—	—	−3	1
−2	2	−1	2	−6	2
−3	3	—	—	−9	3
−4	0	−2	0	−12	0

Discrete time values cannot be ~~ne~~ fractions

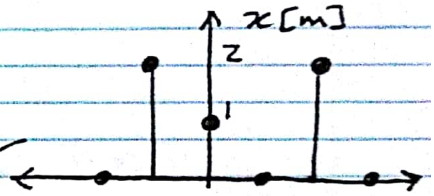


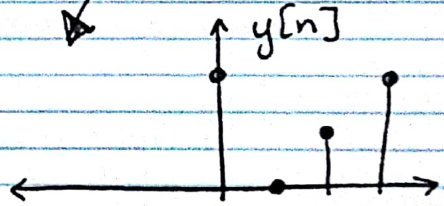


EXAMPLE

Consider a discrete time signal $x[m]$. Plot, the transformation $m = 2 - n$.

m	$x[m]$	$n = 2 - m$	$y[n]$
−1	2	3	2
0	1	2	1
1	0	1	0
2	2	0	2





$$y[n] = \{\underset{\uparrow}{2}, 0, 1, 2\}$$

AMPLITUDE TRANSFORM

The three transformations of amplitude follow the form

$$y[n] = Ax[n] + B$$

where A and B are constants

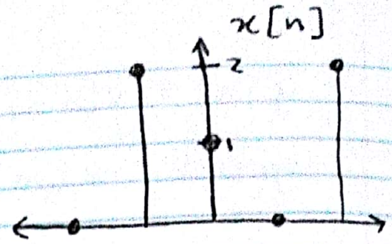
EXAMPLE

A discrete signal $y[n]$ is obtained from $x[n]$ as

$y[n] = -3.1\, x[n] - 5.75$

$A = -3.1$ gives the amplitude scaling and reversal

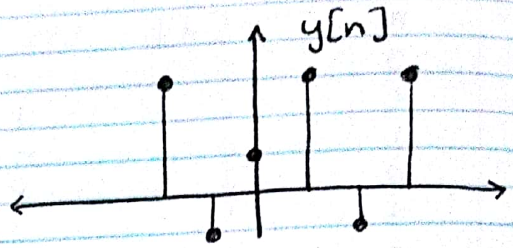
$B = -5.75$ gives the amplitude shifting and changes the DC level of the signal



EXAMPLE

Consider a discrete signal $x[n]$. Plot the time-transformed signal

$y[n] = 3 - 2x[n]$



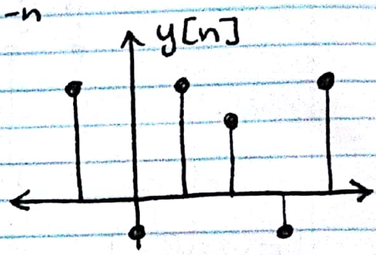
m	$x[m]$	$y[n] = 3 - 2x[n]$
3	0	3
2	2	-1
1	1	~~3~~ 1
0	0	3
-1	2	-1
-2	0	3

EXAMPLE

Consider the general example $x[n] \rightarrow y[n] = 3 - 2x[2-n]$

$m = 2 - n \rightarrow n = 2 - m$

m	$2-m=n$	$x[m]$	$y[n] = 3 - 2x[m]\big\|_{m=2-n}$
3	-1	0	3
2	0	2	-1
1	1	0	3
0	2	1	1
-1	3	2	-1
-2	4	0	3
-3			



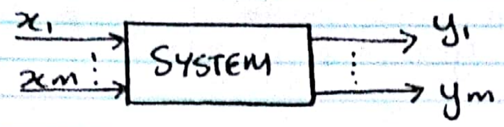
Scanned by CamScanner

DISCRETE TIME SYSTEMS

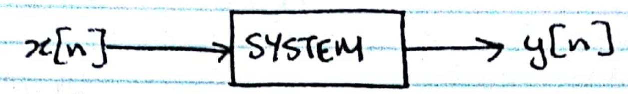
A **SYSTEM** is a mathematical model of a physical process that relates the input (or EXCITATION) signal to the output (or RESPONSE) signal.

$$Y = T x$$

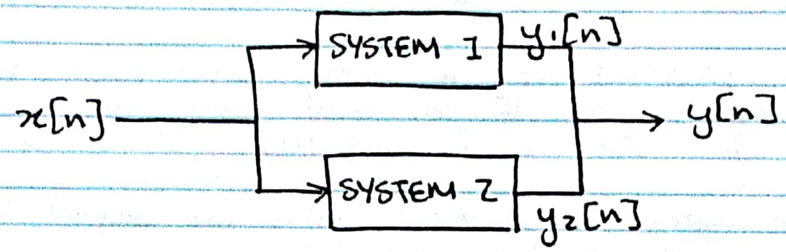
T is the operator representing some well-defined rule whereby x is transformed into Y.



A **DISCRETE TIME SYSTEM** ~~/signal~~ is one which has discrete time signals as inputs and outputs

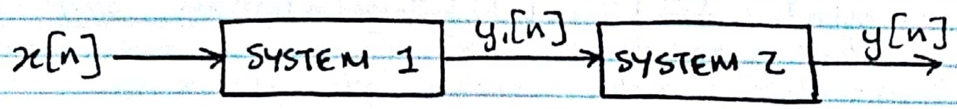


A **PARALLEL CONNECTION** of systems is represented as follows:



$$y[n] = y_1[n] + y_2[n] = T_1 x[n] + T_2 x[n] = T x[n]$$

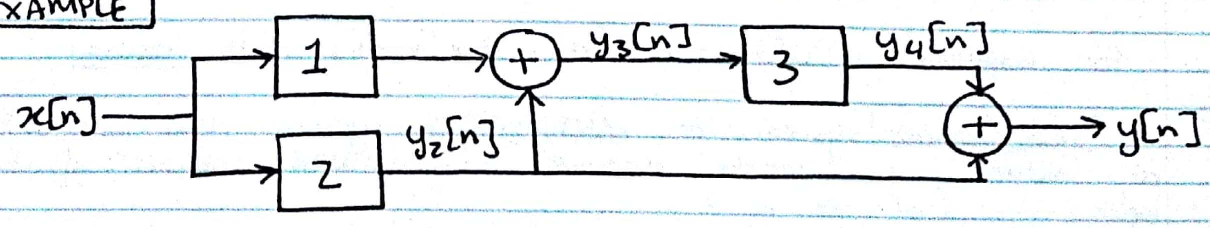
A **SERIES OR CASCADE CONNECTED SYSTEM** is given as follows:



$$y_1[n] = T_1 x[n]$$

$$y[n] = T_2 (T_1 (x[n])) = T x[n]$$

EXAMPLE



$$y[n] = T_2 x[n] + T_3 (T_1 x[n] + T_2 x[n])$$

$$\simeq T_2 * x[n] + T_3 * (T_1 * x[n] + T_2 * x[n])$$

PROPERTIES OF DISCRETE TIME SYSTEMS

SYSTEMS WITH MEMORY

A system has memory if its output at time n_0, $y[n_0]$ depends on the input of values other than $x[n_0]$. Otherwise, the system is memory-less

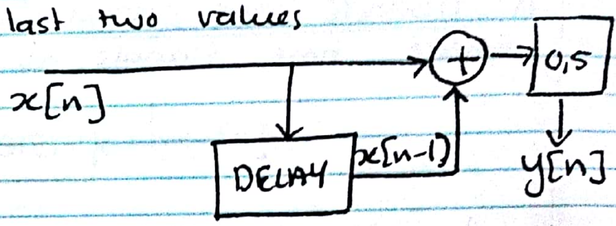
A <u>MEMORYLESS SYSTEM</u> is also called a <u>STATIC SYSTEM</u>

A system with memory is also called a <u>DYNAMIC SYSTEM</u>
for example:

$$y[n] = H \sum_{k=-\infty}^{n-1} x[n]$$

EXAMPLE

Output gives average of the last two values



$$y[n] = \frac{1}{2}\left[x[n] + x[n-1]\right]$$

INVERTABILITY

A system T is <u>INVERTABLE</u> if if every output signal corresponds to a unique input signal. If so, there exists an inverse system T^{-1}.

When the inverse system is cascaded with the original system, the output is equal to the input.

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n] \longrightarrow \boxed{T^{-1}} \longrightarrow x[n]$$

Thus

$$y[n] = T^{-1}\left[T(x[n])\right] = x[n]$$

CAUSALITY

A <u>CAUSAL</u> or <u>NON-ANTICIPATIVE</u> SYSTEM is a system whoose output at instant n depends on the value of the input at the same instant n and on past values of the input. but <u>not</u> on <u>FUTURE VALUES</u> of the input.

For EXAMPLE: $y[n] = \alpha x[n] + \beta x[n-1]$ is <u>causal</u>

A <u>NON-CAUSAL</u> system or <u>ANTICIPATIVE</u> system has an output at instant n which depends on future VALUES of the input.

for EXAMPLE: $y[n] = \alpha x[n] + \beta x[n+1]$

STABILITY

A STABLE SYSTEM is one where the output does not diverge as long as the input does not diverge

A bounded input produces a bounded output, which gives the name BOUNDED INPUT-BOUNDED OUTPUT (BIBO)

$$|x[n]| \leq M \qquad \text{for all } n$$

Is the definition of a bounded signal. Hence, a system is BIBO stable if

$$|y[n]| \leq R \qquad \text{for all } n$$

for all $x[n]$ so that $|x[n]| \leq M$.

To determine BIBO stability, R (normally a function of M), must be found so $|y[n]| \leq R$.

An UNSTABLE SYSTEM has an output which grows without limit from a bounded input.

TIME INVARIENCE

A system is TIME INVARIENT if a time shift in the input results only in the same time shift in the output.

$$x[n] \longrightarrow y[n]$$

$$x[n-n_0] \longrightarrow y[n-n_0]$$

TIME VARYING

A TIME VARYING system is one which if $y[n]$ is the output when the input $x[n]$ is applied, then $y[n-n_0]$ is not neccessarilly the output, when $x[n-n_0]$ is applied.

EXAMPLE

$$y[n] = n\,x[n]$$

$$x_2[n] = x_1[n-n_0] : \quad y_1[n] = n\,x_2[n]$$

$$= n\,x_1[n-n_0]$$

This system is not time-invarient but time-varying.

$$\neq (n-n_0)\,x[n-n_0]$$

$$= y[n-n_0].$$

LINEARITY

A linear system is one that has the property of superposition.

ADDITIVE

if $x_1[n] \rightarrow y_1[n]$ and $x_2[n] \rightarrow y_2[n]$

then $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$

HOMOGENEITY

if $x[n] \rightarrow y[n]$

then $a x[n] \rightarrow a y[n]$

$$a_1 x_1[n] + a_2 x_2[n] \rightarrow a_1 y_1[n] + a_2 y_2[n]$$

EXAMPLES

LINEAR : $y[n] = k x[n]$

NON LINEAR : $y[n] = x^2[n]$

EXAMPLE

Examine the given system for its properties

$$y = \left[\frac{n+2.5}{n+1.5}\right]^2 x[n]$$

~~for n=0~~ i) Memoryless → Only function of present time

ii) Invertable → since we can solve for $x[n]$

$$x[n] = \left[\frac{n+1.5}{n+2.5}\right]^2 y[n]$$

iii) CAUSALITY → Causal, the output does not depend on the input at a future time.

iv) STABLE → for $|x[n]| \leq M$, $|y[n]| \leq R = 9M$

Let $y[n] = k[n]x[n]$, $k[n] = \left[\frac{n+2.5}{n+1.5}\right]^2$

n	2	1	0	-1	-2	-3
$k[n]$	1.65	1.96	1.67	9.0	1	0,11

As $n \rightarrow \infty$, $k[n] \rightarrow 1$

$k[n]$ is max for $n = -1$

$$|y[n]| = 9|x[n]|$$

v) <u>TIME VARYING</u> → the applied input at $n=0$ produces a different one from that when a signal is applied at $n=1$

vi) <u>LINEARITY</u> → Linear, by superposition

$$\left[\frac{n+2.5}{n+1.5}\right]^2 (a_1 x_1[n] + a_2 x_2[n]) = a_1 \left[\frac{n+2.5}{n+1.5}\right]^2 x_1[n] + a_2 \left[\frac{n+2.5}{n+1.5}\right]^2 x_2[n]$$

$$= a_1 y_1[n] + a_2 y_2[n]$$

<u>SUMMARY</u> of <u>DISCRETE TIME SIGNALS</u>

- Definitions and need for discrete time signals
- Representations of discrete time signals. (Graphical, tabular, functional, sequential)
- Classifications of discrete time signals (Deterministic, or random, periodic and non-periodic, even and odd, energy and power signals)
- Discrete signal models (Discrete-time functions)
- Basic operations (Equivalent, Time transformations, Amplitude transformations)
- Discrete time systems.

2) ANALYSIS OF DISCRETE TIME SYSTEMS: LINEAR INVARIENT

An LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

LINEAR TIME INVARIENT SYSTEMS

CONVOLUTION SUM

The convolution sum representation of an LTI system is given in the following equation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

and can also be represented as

$$y[n] = x[n] * h[n]$$

Convolution is commutative, as shown:

$$y = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = x[n]*h[k]$$
$$= h[k] * x[k]$$

and associative:

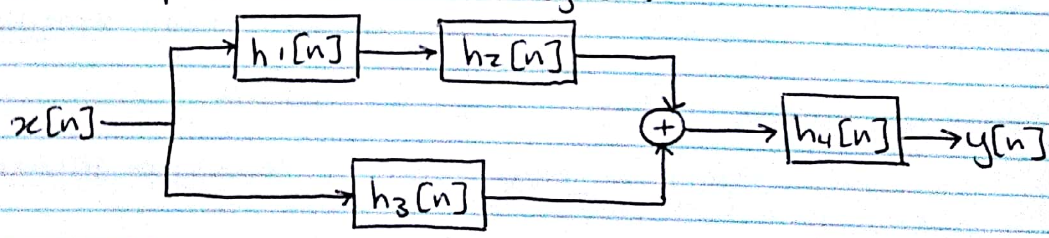
$$(f[n]*g[n])*h[n] = f[n]*(g[n]*h[n]) = (h[n]*f[n])*g[n]$$

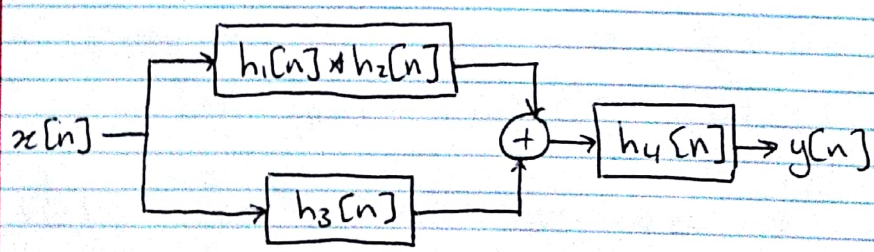
and distributive:

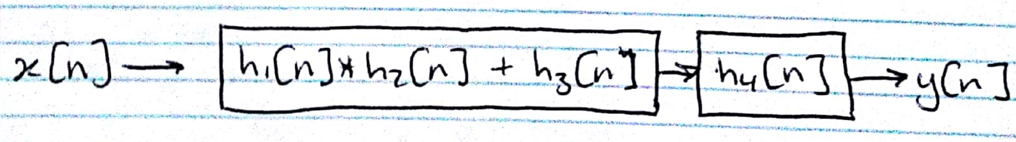
$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n] + x[n]*h_2[n]$$

EXAMPLE

Determine the impulse response of the following system in terms of impulse responses in the subsystems







$$x[n] \rightarrow \boxed{(h_1[n] * h_2[n] + h_3[n]) * h_4[n]} \rightarrow y[n]$$

Recall : Convolution sum gives the output signal in terms of the input signal as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

- **SYSTEMS WITH MEMORY**

A memoryless (static) system has an output $y[n]$ which only depends on the current value of $x[n]$

$$\boxed{h[n] = k\,\delta[n]}$$

AKA, all functions are functions of $x[n]$ not $n-k$ or $n+k$.

A MEMORYLESS system is therefore a <u>PURE GAIN</u> system

$$y[n] = k\,x[n]$$

- **INVERTABILITY**

A system is INVERTABLE if its input can be determined from its <u>OUTPUT</u>

$$x[n] \rightarrow \boxed{h[n]} \xrightarrow{y[n]} \boxed{h_i[n]} \rightarrow y_i[n] = x[n]$$

where $h[n] * h_i[n] = I\,\delta[n]$

Easiest way to prove invertability is to sub in an n value for $x[n]$ and solve for y and vice versa and see if they correspond. If not, then they are not invertable.

- **CAUSALITY**

A discrete time signal is CAUSAL if the current value of the output depends only on current or past values of the input.

$$\boxed{h[n] = 0 \text{ for } n < 0}$$

Sub in $n=0$. If x is a function of any positive numbers, it is NON CAUSAL.

- **STABILITY**

A system is BIBO stable if the output remains stable for any bounded input. The boundedness of any output can be expressed as

$$|x[n]| \leq M$$

Where M is a real constant Therefore

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[n-k]\,h[k]\right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]\,h[k]|$$

$$= \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq \sum_{k=-\infty}^{\infty} M\,|h[k]|$$

$$= M \sum_{k=-\infty}^{\infty} |h[k]|$$

Since M is finite, it is sufficient to prove

$$|h[k]| < \infty$$

If this is satisfied, $h[k]$ is called "ABSOLUTELY SUMMABLE"

$\boxed{\text{EXAMPLE}}$

Check the following systems for their properties (also find a better way to phrase this question your parents would be ashamed of you)

a) $h[n] = \left(\frac{1}{2}\right)^n u[n]$

i) Memoryless? Nope $\rightarrow h[n] \neq 0$ when $n = 0$

ii) Causal? Yes $\rightarrow h[n] = 0$ for $n < 0$

iii) Stable? Yes $\rightarrow$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |h[k]| = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

Recall power series:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1$$

$$\therefore \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

$$0 < 2 < \infty$$

$\therefore$ System is stable.

b) $h[n] = 2^n u[n]$

i) Memoryless? Nope $\rightarrow h[n] = 0$ for $n \neq 0$

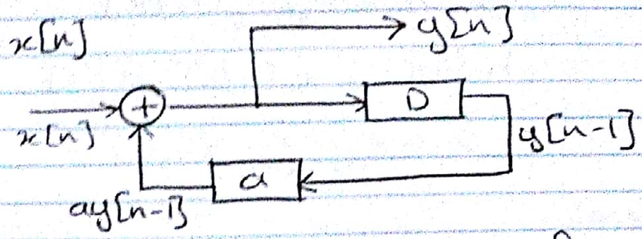
ii) Causal? Yep $\rightarrow h[n] = 0$ for $n < 0$

iii) Stable? No, the function diverges

IMPULSE RESPONSE

Consider the following system

$$y[n] = ay[n-1] + x[n]$$



The system's ~~unit~~ UNIT IMPULSE RESPONSE is obtained from this equation by applying a UNIT IMPULSE FUNCTION with $x[0] = \delta[0] = 1$

$$y[0] = ay[-1] + x[0] = a(0) + 1 = 1$$
$$y[1] = ay[0] + x[1] = a(1) + 0 = a$$
$$y[2] = ay[1] + x[2] = a(a) + 0 = a^2$$
$$y[3] = ay[2] + x[3] = a(a^2) + 0 = a^3$$

The unit impulse response for this function is

$$\underline{h[n]} = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} = a^n u[n]$$

Note, the system is causal so there is nothing outputted when $n < 0$.

UNIT STEP RESPONSE

The system UNIT STEP RESPONSE can be calculated directly from the impulse response.

If the input to the above system is a unit step ~~u(t)~~, $u[n]$, the unit step response $s[n]$ can be written as:

$$\boxed{s[n] = \sum_{k=-\infty}^{\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k]}$$

The impulse response can be calculated in the same way as follows:

$$s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n]$$

EXAMPLE

Consider a causal, stable system with memory which has impulse response $h[n] = (0{,}6)^n u[n]$

i) Find the unit step response
ii) Re-determine the impulse response from the unit step.

i) Impulse response $\longrightarrow$ Unit step response

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$h[n] = (0,6)^n u[k]$$

$$= \sum_{k=0}^{n} (0,6)^k$$

Recall:
$$\sum_{k=0}^{n} a^k = \frac{1+a^{n+1}}{1-a}$$

$$s[n] = \frac{1-(0,6)^{n+1}}{1-0,6} u[n]$$

$$= 2.5\left(1-0,6\right)^{n+1} u[n]$$

$$s[n] = 0 \quad \text{for} \quad n<0$$

ii) $h[n] = s[n] - s[n-1]$

$$= 2.5(1-0,6^{n+1})\,u[n]$$

$$= 2.5\left(1-0,6^{n}\right) u[n-1]$$

$u[n]=1$ for $n \geq 0$
$u[n-1]=1$ for $n \geq 1$

for $n=0$, $u[n]=1$, $u[n-1]=0$

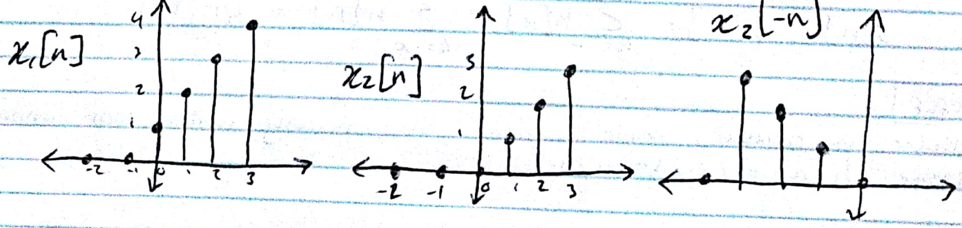
$h[n] = 2.5(1-0,6) = 1$ for $n=0$
for $n \geq 1$, $u[n]=1$, $u[n-1]=1$

$$h[n] = 2.5(1-0,6^{n+1} - 1 + 0,6^{n+1})$$
$$= 2.5(0,6^n)(1-0,6)$$
$$= [0,6^n]u[n]$$

$\boxed{\text{EXAMPLE}}$

Given signals $x_1[n]$ and $x_2[n]$ shown below, compute their convolutional sum ~~to~~ $x_1[n] * x_2[n]$

$\cancel{x_1[n] * x_2[n]}$:



$x_1[n] = [\underset{\uparrow}{1}, 2, 3, 4]$ $\quad$ $x_2[n] = [\underset{\uparrow}{0}, 1, 2, 3]$

$x_2[-n] = [3, 2, 1, 0]$ $\quad$ $y[0] = \sum x_1[n] x_2[0-n]$

DISCRETE CONVOLUTION ROCKS! ↴

$y[0]$: 1 2 3 4 × 3 2 1 0 = 0

0 + 0 + 0 + 0 + 0 + 0 + 0

$y[1]$: 1 2 3 4 × 3 2 1 0 = 1

0 + 0 + 1 + 0 + 0 + 0

$y[2]$: 1 2 3 4 × 3 2 1 0 = 4

0 + 2 + 2 + 0 + 0

$y[3]$: 1 2 3 4 × 3 2 1 0 = 10

3 + 4 + 3 + 0

$y[4]$: 1 2 3 4 × 3 2 1 0 = 16

0 + 6 + 6 + 4 + 0

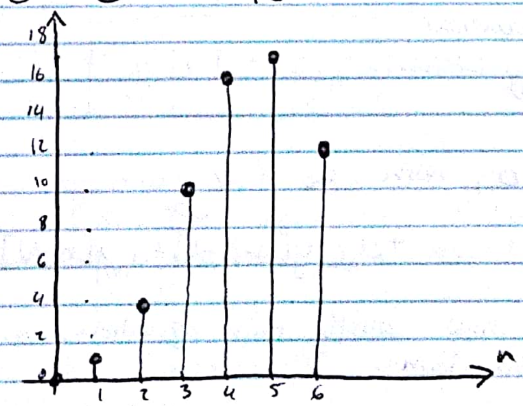
$y[5]$: 1 2 3 4 × 3 2 1 0 = 17

0 + 0 + 9 + 8 + 0 + 0

$y[6]$ 1 2 3 4 × 3 2 0 = 12

0 + 0 + 0 + 12 + 0 + 0

$\therefore y[n]$:



SYSTEM ANALYSIS : THE DIFFERENCE EQUATION

Signals and systems can be described by differential equations in one of two ways:

DELAY TERMS :

$y[n-1], y[n-2], x[n-1], x[n-2], \ldots$

ADVANCE TERMS

$y[n+1], y[n+2], \ldots$

The GENERAL FORM of a difference equation:

$$a_0 y[n] + a_1 y[n-1] + \ldots a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots$$
$$\ldots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

or:

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]}$$

We can solve these equations with the CLASSICAL or RECURSIVE methods

THE CLASSICAL METHOD

The classical method solves the differential equation as the sum of two functions.

$$\boxed{y[n] = y_c[n] + y_p[n]}$$

$y_c[n]$ = "The COMPLIMENTARY FUNCTION"
= "THE NATURAL RESPONSE"
= "THE HOMOGENIUS EQUATION"

$y_p[n]$ = "THE PARTICULAR SOLUTION"
= "THE ~~NATURAL~~ FORCED RESPONSE"

THE NATURAL RESPONSE

Let the LHS equal 0, with $a_0 \neq 0$

$$a_0 y[n] + a_1 y[n-1] + \ldots + a_{N-1} y[n-N+1] + a_N y[n-N] = 0$$

The natural response must satisfy this equation, and is assumed to be of the form:

$$\boxed{y_c[n] = C z^n}$$

If this is true, then:

$$y_c[n-1] = C z^{n-1} = C z^{-1} z^n$$

$$y[n-2] = Cz^{n-2} = Cz^{-2}z^n$$

$$\vdots$$

$$y_c[n-N] = Cz^{n-N} = Cz^{-N}z^n$$

Each of the points z^n, z^{n-1}, z^{n-2} ... z^{n-N} are called the <u>MODES</u> of the D.E.

if we apply this form to our condition:

$$(a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{-(N-1)} + a_N z^{-N}) C z^n$$

$$=$$

$$(a_0 z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N) C z^{-N} z^n = 0$$

In order for this to be true, we have the following.

$$(a_0 z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N) = 0$$

This is called the <u>CHARACTERISTIC EQUATION</u> or the <u>AUXILIARY EQUATION</u>. (a polynomial in z set to 0). This polynomial may be factored as follows:

$$a_0 z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N = a_0(z-z_1)(z-z_2)\dots(z-z_N) = 0$$

Hence, the N values of z, denoted z_i ; $i = 1, 2, 3 \dots N$ satisfy the equation.

For the case of NO REPEATED ROOTS, the solution to the complimentary function is:

$$\boxed{y_c[n] = C_1 z_1^n + C_2 z_2^n + \dots + C_N z_N^n}$$

... since the equation is linear. It contains N unknown coefficients C_1, C_2, ... C_N. The unknown constant C can only be found when the complete solution has been found.

EXAMPLE

Given the first order DE: $y[n] - 0{,}6y[n-1] = x[n]$, find the natural response.

$$y[n]: \quad z - 0{,}6 = 0$$

$$z = 0{,}6$$

$$y_c[n] = Cz^n = C(0{,}6)^n$$

~~THE~~

THE NATURAL RESPONSE FOR REPEATED ROOTS

The characteristic equation for the complimentary formula no longer holds when the roots are repeated.

EXAMPLE

$$z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = (z - z_1)^3 (z - z_4)$$

The above is given as the characteristic equation for a forth order system.

We can no longer use the standard characteristic equation. It is instead given as the following

$$\boxed{y_c[n] = (C_1 + C_2 n + C_3 n^2) z_1^n + C_4 z_4^n}$$

the remainder of the procedure stays the same.

For the general case of an r^{th} order root:

$$\boxed{\text{term} = (C_1 + C_2 n + C_3 n^2 + \dots + C_r n^{r-1}) z_1^n}$$

THE ~~NA~~ FORCED RESPONSE OF THE DE

In the same general equation for a D.E. we can extract the forced or particular response $y_p[n]$

$$\boxed{\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x[n-k]}$$

The forced response is any response which satisfies the DE.

We have 2 ways of finding the forced response. But he only gave us one? WTF?

METHOD OF ~~UNDETER~~ UNDETERMINATE COEFFICIENTS.

The solution is the sum of the excitation $x[n]$ and all the delayed reactions of $x[n-k]$. that are different from $x[n]$.

This method doesn't work for an infinite number of terms.

EXAMPLES

If $x[n] = n^3$, the delayed excitation $x[n-k]$ is:

$$x[n-k] = (n-k)^3$$

Which expands into the functional forms n^3, n^2, n^1, n^0

For this case, the forced response is chosen as:

$$y_p[n] = P_1 + P_2 n + P_3 n^2 + P_4 n^3$$

where all the P coefficients are unknowns.

$\boxed{\text{EXAMPLE}}$

If $x[n] = \cos 2n$, the delayed excitation can be expressed as the sum of a cos function and a shifted cos function or the sum of a cos function and a sine function.

The response is chosen as:

$$y_p[n] = P_1 \cos 2n + P_2 \sin 2n$$

The unknown coefficients are evaluated by substituting them into the original D.E. Equate the coefficients of terms of the same degree.

$\boxed{\text{EXAMPLE}}$

Again, consider: $y[n] - 0{,}6\,y[n-1] = x[n]$. It is now given that $x[n] = 4u[n]$. If the initial conditions are, 0, what is the total solution?

Total Solution using CLASSICAL METHOD.

$y_p[n] = P$ $\qquad y_p[n-1] = y_p[n-2]$

$$y_p[n] - 0{,}6\,y_p[n-1] = 4u[n]$$

$$P - 0{,}6P = 4$$

$$0{,}4P = 4$$

$$P = 10$$

$$y_p[n] = 10$$

$$y[n] = y_c[n] + y_p[n]$$

$$= C(0{,}6)^n + 10$$

$@\, n=0 \rightarrow y[0] = 0$

$$y[0] = 0 = C(0{,}6)^0 + 10$$

$$C = -10$$

$$y[n] = 10\left(1-(0{,}6)^n\right)u[n]$$

THE ITERATIVE METHOD

If we consider a simple linear DE:

$$y[n] + ay[n-1] = bx[n]$$

by solving the equation recursively, it is posssible to generate an expression for the complete solution $y[n]$ in terms of the initial conditions and the input $x[n]$

$n=0 \rightarrow \quad y[0] = -ay[-1] + bx(0)$

$n=1 \rightarrow \quad y[1] = -a[y[0]] + bx[1] = a^2y[-1] - abx[0] + bx[1]$

$n=2 \rightarrow \quad y[2] = -ay[n1] + bx[2] = -a^3y[-1] + a^2bx[0] - abx[1] + bx[2]$

EXAMPLE

Verify the classical method with the recursive method. Reconsider the function:

$$y[n] - 0{,}6y[n-1] = x[n] \qquad x[n] = 4u[n]$$

Solve for the total solution with the classical method and the recursive method.

CLASSICAL METHOD:

$y[n] = 10(1-(0{,}6)^n)u[n]$

$n=0 \rightarrow 10(1-(0{,}6)^0]\times 1$
$10(1-1)\times 1$
$=0$

$n=1 \rightarrow y[1] = 10[1-(0{,}6)^1]\times 1$
$= 10[1-0{,}6]$
$= 4$

$n=2 \rightarrow y[2] = 10[1-(0{,}6)^2]$
$= 10[1-(0{,}6)^2]$
$= 0{,}4$

$n=3 \rightarrow y[3] = 10[1-(0{,}6)^3]$
$= 7{,}84$

RECURSIVE METHOD

$y[n] = 0{,}6y[n-1] + 4u[n]$

$n=0 \rightarrow \quad y[0] = 0{,}6y[-1] + 4 \quad = 0$

$n=1 \rightarrow \quad y[1] = 0{,}6y[0] + 4$
$= 0{,}6(0) + 4$
$= 4$

$n=2 \rightarrow \quad y[2] = 0{,}6(y[1]) + 4$
$= 0{,}6(4) + 4$
$= 0{,}4$

$n=3 \rightarrow \quad y[3] = 0{,}6y[2] + 4$
$= 0{,}6(0{,}4) + 4$
$= 7{,}84$

STEADY STATE OUTPUT $\quad y_{ss}[n]$

Consider $\quad y[n] = 10[1-(0{,}6)^n]u[n]$

$$y[n] = 10[1-0{,}6^{n}]u[n] \longrightarrow \lim_{n\to\infty} y[n] = y_{ss}[n] = 10$$

$$y[n-1] = 10[1-0{,}6^{n-1}]u[n-1] \rightarrow \lim_{n\to\infty} y[n-1] \rightarrow y_{ss}[n]$$

Therefore, the given differen difference equation

$$y[n] - 0{,}6y[n-1] = 4u[n]$$

can be given as

$$y_{ss}[n] - 0{,}6y_{ss}[n] = 4$$

$$0{,}4\,y_{ss}[n] = 4$$

$$y_{ss}[n] = 10$$

Therefore the steady state value has been confirmed.

The NATURAL RESPONSE

For a linear difference equation, the natural response part of the system is given as follows

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad a_0 \neq 0$$

The NATURAL RESPONSE is independent of the forcing function $x[n]$ and is dependent only on the structure of the system.

In the previous example:

$$y_c[n] = c\,(0.6)^n$$

It can be seen that this function is independent of any input to the system.

A NOTE

Forced Response = Zero State Response = Steady State Response
Zero State → Zero Initial Conditions.

The forced response is a function of the input and the system structure.

TERMS IN THE NATURAL RESPONSE $y_c[n]$

The mathematical forms of the terms of the NATURAL Response are given by the roots of the characteristic equation.

$$a_0 z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N = a_0 (z - z_1)(z - z_2)\ldots(z - z_N) = 0$$

When the roots of the equation are distinct, the natural response is given as

$$y_c[n] = C_1 z_1^n + C_2 z_2^n + \ldots + C_N z_N^n$$

The general terms of the Natural response is called the SYSTEM MODES which are multiplied by a constant

$$\underbrace{c_i z_i^n}_{\text{SYSTEM MODE}}$$

The root may be Real or Complex.

$\boxed{\text{STABILITY}}$

The magnitude of the NATURAL RESPONSE TERMS are given as

$$|c_i||z_i|^n$$

$|z_i| < 1 \rightarrow$ Term magnitude $\rightarrow 0$ as $n \rightarrow \infty$

$|z_i| > 1 \rightarrow$ Term magnitude $\rightarrow \infty$ as $n \rightarrow \infty$

If

$$y[n] = y_c[n] + y_p[n]$$

If ALL ~~TERMS~~ ROOTS SATISFY $|z_i| < 1$, then each term is also unbounded.

A ~~BIBO~~ CAUSAL DISCRETE TIME LTI SYSTEM IS BIBO STABLE HAS

$$|z_i| < 1 \quad \forall i \in N.$$

$\boxed{\text{EXAMPLE}}$

Consider the causal system.

$$y[n] - 1.25y[n-1] + 0{,}375y[n-2] = x[n]$$

Determine if the system is stable.

N^{th} order difference equation: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad a_0 \neq 0$

a ~~fac~~ polynomial may be factored as:

$$a_0 z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N = a_0 (z - z_1)(z - z_2) \dots (z - z_N) = 0$$

We can describe the transfer function as:

$$h[n] = \frac{\cancel{x[n]}\ y[n]}{x[n]} \qquad \cancel{z = y[n-1]}$$

$\therefore \quad z^2 - 1.25z + 0{,}375 = 0$ is our characteristic equation

$\rightarrow (z - 0{,}75)(z - 0{,}5) = 0$

This system is stable as the magnitudes of the roots is unity. The natural response is given by

$$y_c[n] = C_1(0.75)^n + C_2(0.5)^n$$

The function approaches 0 as $n \to \infty$

EXAMPLE

Is the system stable?

$$y[n] - 2.5y[n-1] + y[n-2] = x[n]$$

$$z^2 - 2.5z + 1 = 0$$

$$(z-2)(z-0.5) = 0$$

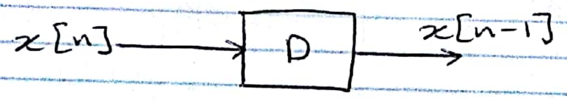
Unstable as one root is larger than unity

$$y_c[n] = C_1(2)^n + C_2(0.5)^n$$

The natural response tends to ∞ as $n \to \infty$

BLOCK DIAGRAM REPRESENTATION OF LTI SYSTEMS

The Ideal Delay:



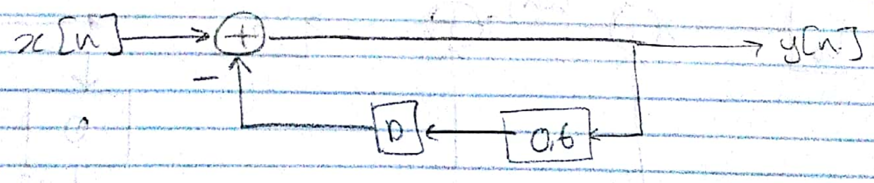
$$x[n] \to y[n] = x[n-1]$$

EXAMPLE

Find the block diagram for:

$$y[n] - 0.6y[n-1] = x[n]$$

* Check with Voss + Ani



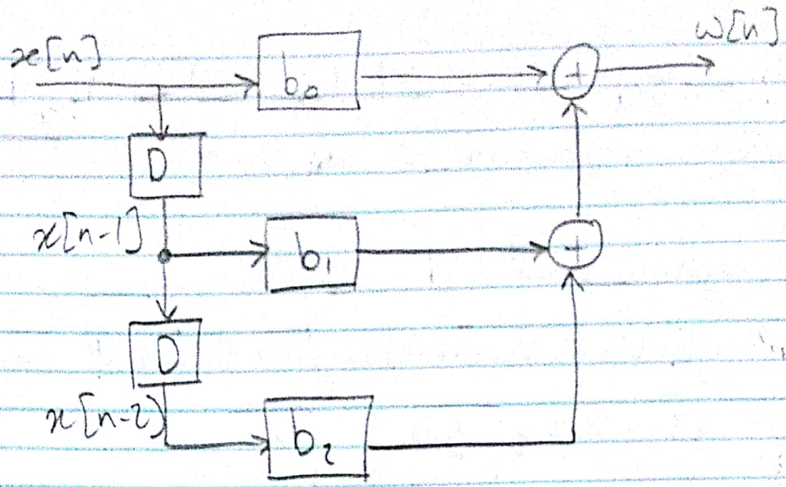
STANDARD BLOCK DIAGRAM FORMS

for our standard form: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad a_0 \neq 0$

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

The Right hand side of the equation is called $w[n]$ and we can represent it as follows:

$$w[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$



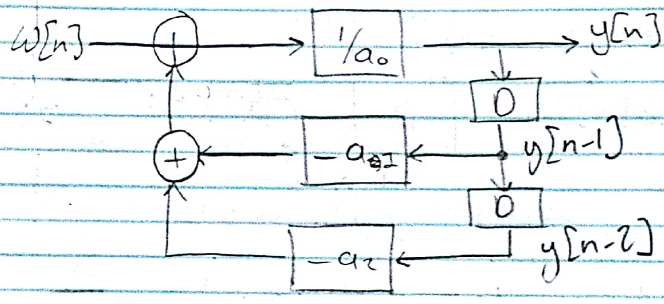
The Left hand side can be rewritten as

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] = \omega[n]$$

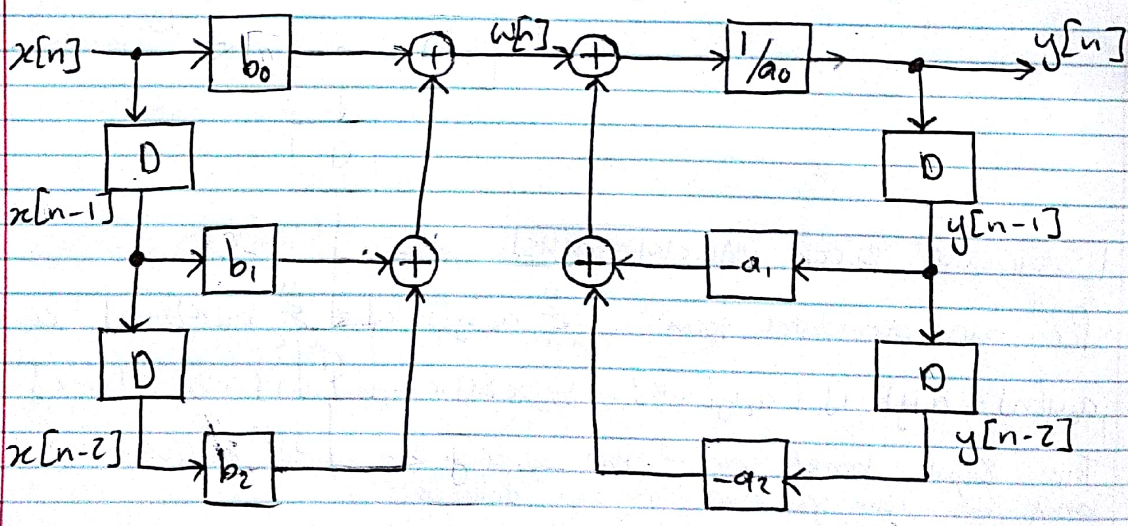
we solve for $y[n]$:

$$y[n] = \frac{1}{a_0}\left(w[n] - a_1 y[n-1] - a_2 y[n-2]\right) \qquad a_0 \neq 0$$

Which can be represented as:

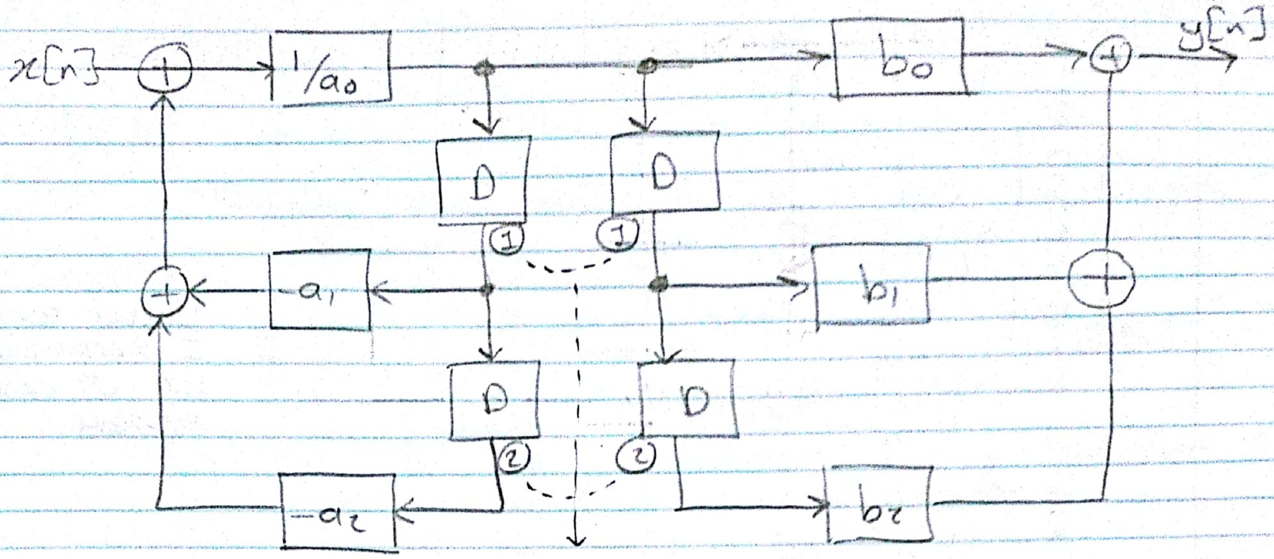


DIRECT FORM I REALISATION

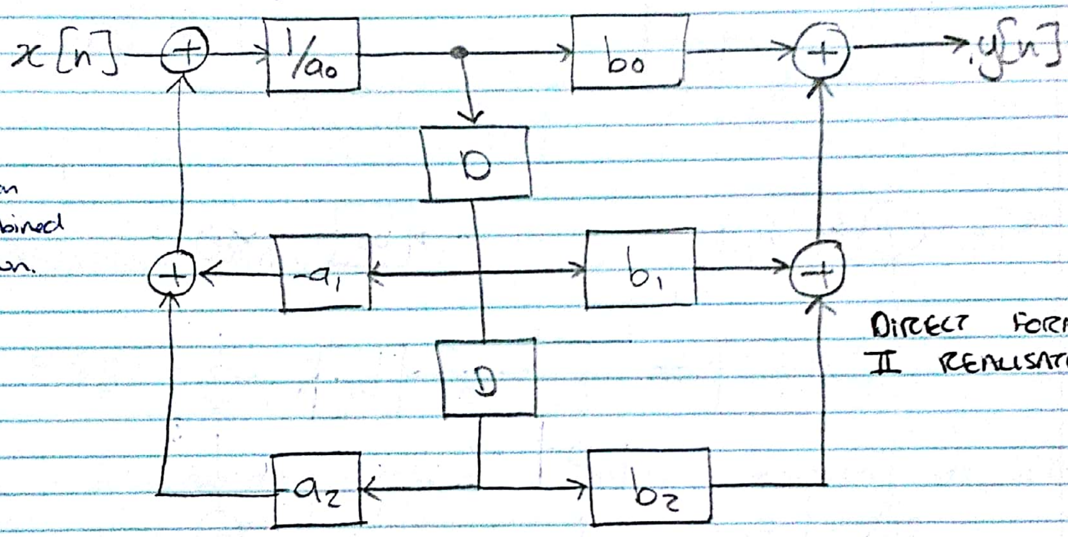


DIRECT FORM II REALISATION

We manipulate direct form I to achieve direct form II



each of these delays affects the same signal



∴ They can be combined as shown.

DIRECT FORM II REALISATION

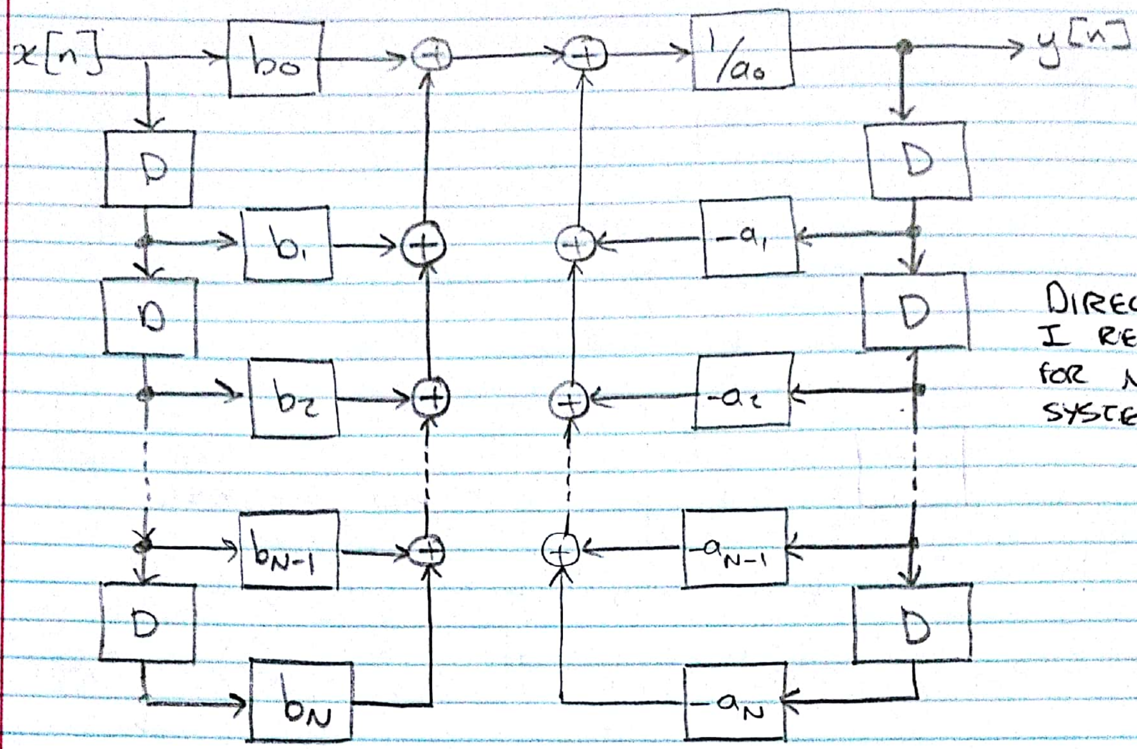
For an LTI system: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

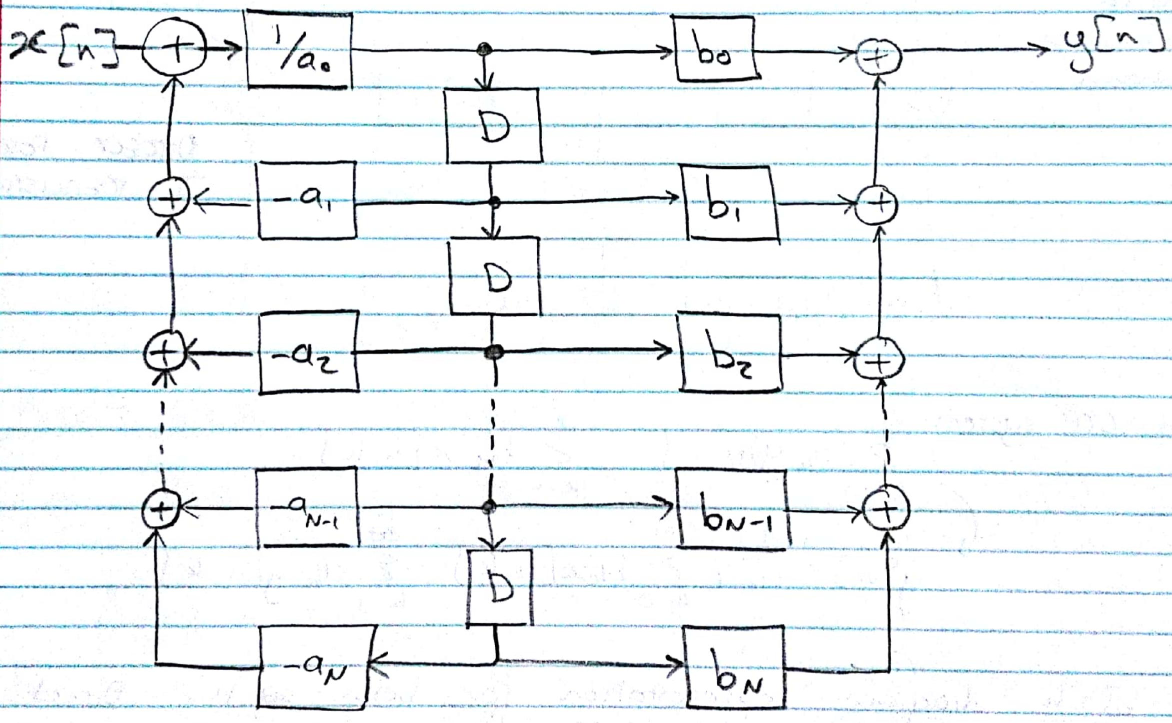
The block diagram representation for both ~~sy~~ the Direct I form and the direct II form are given on the following page for an N^{th} order system.

SUMMARY OF DISCRETE TIME SYSTEMS

- Linear time invariant systems and their properties.
- Systems Analysis based on the Difference Equation
  - → Classical Method
  - → Iterative Method.
- Block diagram representation of LTI systems.



DIRECT FORM I REALISATION FOR N^{th} ORDER SYSTEM



3 The Z-TRANSFORM

~~DEFINITION~~ !

DTFT : frequency domain representation of DISCRETE time signals

Generalisation :

$$\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}} \quad \text{DTFT}$$

this shit leads to the z transform. The z-transform doesn't exist if the ~~DE~~ DTFT does not exist.

DEFINITION: Bilateral Z-TRANSFORM

~~F[z]~~ $F[z]$ of a discrete-time function $f[n]$

$$\boxed{F[z] = \mathcal{Z}\{f[n]\} = \sum_{n=-\infty}^{\infty} f[n] z^{-n}}$$

Its expanded form:

$$\boxed{F[z] = \ldots + f[-2]z^2 + f[-1]z + f[0] + f[1]z^{-1}}$$

z is a complex-valued number expressed as follows

$$\boxed{z = re^{i\Omega}}$$

The z transform here is the Bilateral Transform. (Two sided)

<u>Unilateral (or single-sided) z-transform</u>

→ A z transform which is restricted to <u>CAUSAL</u> <u>INPUTS</u> → $n \geq 0$

→ The unilateral z-transform is the same as the bilateral z-transform except the limit sum is from 0 to ∞

→ Defined by the POWER SERIES

$$\boxed{F[z] = \mathcal{Z}\{f[n]\} = \sum_{n=0}^{\infty} f[n] z^{-n}}$$

LAPLACE TX → Z-TRANSFORM

A Z-Transform is the DISCRETE-TIME counterpart of the Laplace Transform.

Laplace Transform is continuous:

$$\boxed{X(s) = \int_0^{\infty} x(t)e^{-st}\,dt}$$

~~If we SAMPLE the LAPLACE TRANSFORM definition~~:

Sampling Formula:

$$\boxed{x_s[t] = x(t)\delta_{T_s}(t) = \sum_{m=-\infty}^{\infty} x(mT_s)\,\delta(t-mT_s)}$$

if $x(t)=0$, if $m<0$

$$\boxed{x_s[t] = \sum_{m=0}^{\infty} x(mT_s)\,\delta(t-mT_s)}$$

taking Laplace.

$$X(s) = \int_0^{\infty} x_s(t)e^{-st}\,dt$$

$$= \int_0^{\infty}\left(\sum_{m=0}^{\infty} x(mT_s)\delta(t-mT_s)\right)e^{-st}\,dt$$

$$= \sum_{m=0}^{\infty} x(mT_s)\int_0^{\infty}\delta(t-mT_s)e^{-st}\,dt$$

$$\mathcal{L}\{\delta(t-mT_s) = \int_0^{\infty}(\delta(t-mT_s))e^{-st}\,dt = e^{-st}\Big|_{t=mT_s}$$

$$= e^{-smT_s}$$

$$\therefore \quad \boxed{X(s) = \sum_{m=0}^{\infty} x(mT_s)e^{-smT_s}}$$

if we let $z = e^{s}$, $T_s = 1$

$$X(z) = \sum_{m=0}^{\infty} x(m)z^{-m}$$

So for a one sided z transform and double sided respectively:

$$\boxed{X(z) = \sum_{m=0}^{\infty} x(m)z^{-m}} \qquad \boxed{X(z) = \sum_{m=-\infty}^{\infty} x(m)z^{-m}}$$

INVERSE Z-TRANSFORM

The inverse z-transform can be found according to:

$$\boxed{f[n] = \frac{1}{2\pi j}\oint F[z]\, z^{n-1}\, dz}$$

$\oint$ represents integration in a z plane closed path

REGION OF CONVERGENCE

The infinite sums in the z-transform definitions may not exist if the series does not converge

EXAMPLE

For the causal sequence $x[n] = a^n u[n]$, determine the region of convergence.

$$X[z] = Z\{x[n]\} = \sum_{n=0}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

POWER SERIES: (or something) $\displaystyle\sum_{n=0}^{\infty} k^n = \frac{1}{1-k} \qquad |k|<1$

Hence,

$$X[z] = \frac{1}{1-az^{-1}} \quad \text{if} \quad |az^{-1}|<1$$

$$= \frac{z}{z-a} \quad \text{if} \quad |a|<|z|$$

It could be said that $X[z]$ only exists if $|a|<|z|$ as, if the sum goes to infinity, it doesn't exist.

There is 1 x ZERO at ~~z=0~~ (roots of numerator)

$$z=0$$

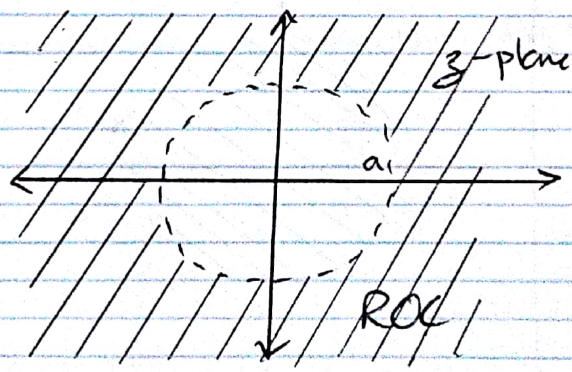
There is 1 x POLE at (roots of denominator)

$$z=a$$

REGION OF CONVERGENCE

$$|z|>|a|$$

(Not defined for $|z|=|a|$)



Example 2

Solve for the z-transform and ROC of $x[n] = -a^n u[-n-1]$ (anti-causal)

$$X[z] = Z\{x[n]\}$$
$$= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$u[-n-1] = 1$ when $-n-1 \geq 0$, $n+1 \leq 0$, $n \leq -1$

$u[-n-1] = 0$ when $n > -1$

So $X[z] = -\sum_{n=-\infty}^{-1} (az^{-1})^n$

→ IF YOU WANT TO SWAP THE LIMITS IN A SUMMATION, RAISE SUM TERM TO POWER OF −1

→ TAKE A −VE OUT OF SUM TERM AND MULTIPLY WITH BOTH LIMITS TO CANCEL IT

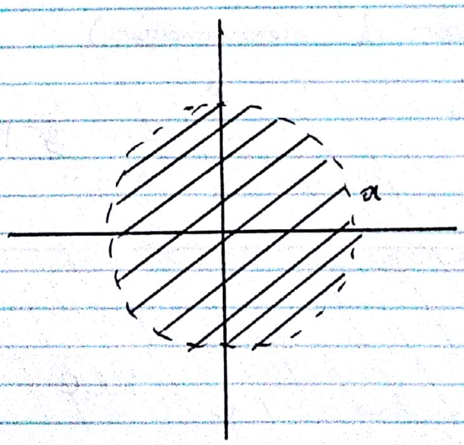
$$X[z] = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$
$$= 1 - (a^{-1}z)^0 - \sum_{n=1}^{\infty} (a^{-1}z)^n$$
$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$\sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-za^{-1}} = \frac{a}{a-z} \qquad |za^{-1}| < 1$$

$$\therefore X\{z\} = 1 - \frac{a}{a-z} = \frac{z}{z-a}$$

$|za^{-1}| < 1$

$|z| < |a|$



PROPERTIES OF THE z-TRANSFORM

- Linearity
- Delay (Left / Right shift)
- Convolution
- Differentiation (xn, scaling in z-domain)
- Initial + Final value theorems

$$f[n] \Longleftrightarrow F[z]$$

LINEARITY

if $f_1[n] \Longleftrightarrow F_1[z]$

and $f_2[n] \Longleftrightarrow F_2[z]$

then

$$a_1 f_1[n] + a_2 f_2[n] \Longleftrightarrow a_1 F_1[z] + a_2 F_2[z]$$

DELAY

if $f[n]\,u[n] \Longleftrightarrow F[z]$

then RIGHT SHIFT : $f[n-n_0]\,u[n-n_0] \Longleftrightarrow z^{-n_0} F[z]$

$$\boxed{f[n-n_0]\,u[n] \Longleftrightarrow z^{-n_0}\left[F[z] + \sum_{n=0}^{n_0-1} f[-n]\,z^{n}\right]}$$

then LEFT SHIFT

$$\boxed{f[n+n_0]\,u[n] \Longleftrightarrow z^{n_0}\left[F[z] - \sum_{n=0}^{n_0-1} f[n]\,z^{-n}\right]}$$

CONVOLUTION (whyyyyyy...)

if $f_1[n] \Longleftrightarrow F_1[z]$ and $f_2[n] \Longleftrightarrow F_2[z]$

then

$$f_1[n] * f_2[n] = F_1[z]\,F_2[z]$$

(Discrete time conv)

DIFFERENTIATION

if $f[n]\,u[n] \Longleftrightarrow F[z]$ then

$$n f[n]\,u[n] \Longleftrightarrow -z \frac{d}{dz} F[z]$$

INITIAL VALUE THEOREM

find the initial value of $f[0]$ from the z transform. $F[z]$

$$F[z] = f[0] + f[1]\,z^{-1} + f[2]\,z^{-2} + \dots$$

$$f[0] = \lim_{z\to\infty} F[z] = \lim_{z\to\infty}\left[f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} \dots\right]$$

$$\boxed{f[0] = \lim_{z\to\infty} F[z]}$$

FINAL VALUE THEOREM

Find final (steady-state) value of a function from $F[z]$

$$\boxed{F[\infty] = \lim_{n\to\infty} f[n] = \lim_{z\to 1} (z-1)F[z]}$$

Obviously, this is only if this limit exists. It won't exist if the transform has a final steady state value, ie, the system is unstable → has ~~no~~ poles ~~inside~~ ~~on~~ outside circle of convergence.

EXAMPLE

Find the initial value and final value of the unit step function:

$$F[z] = \frac{z}{z-1}$$

Initial value : $f[0] = \lim_{z\to\infty} \frac{z}{z-1} = \lim_{z\to\infty} \frac{1}{1-z^{-1}} = 1$

Final value : $f[\infty] = \lim_{z\to 1} (z-1)F[z] = \lim_{z\to 1} z = 1$

INVERSE Z-TRANSFORM

There are 4 procedures:

- Contour Integration Method (I mean you took Math III ...)
- Inspection Method
- Power series expansion
- Partial Fraction Expansion

1) CONTOUR INTEGRATION METHOD

$$x[n] = \frac{1}{2\pi j} \oint X[z]\, z^{n-1}\, dz$$

Evaluate using Math 3 principles
Your skills will finally be useful ...

2) INSPECTION METHOD

"Do you have a table of Z transforms"
If "yes" then →

EXAMPLE

$$X[z] = \frac{z}{z-b}$$

$$x[n] = b^n u[n]$$

difficulty

3) POWER SERIES EXPANSION

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \dots x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} \dots$$

You can determine each term in the ~~sequency~~ sequence by solving for the coefficients.

Great if you have time and the question is super short and doesn't have too many terms.

EXAMPLE

$$X[z] = \frac{z^2 (7z-2)}{(z-0,2)(z-0,5)(z-1)} \quad \text{use Power series:}$$

$$X[z] = \frac{7z^3 - 2z^2}{z^3 - 1.7z^2 + 0,8z - 0,1}$$

So... you need to actually do the division...

$$\begin{array}{r|l} & 7 + 9,9z^{-1} + 11,23z^{-2} + 11,87z^{-3} \\ \hline z^3 - 1.7z + 0,8z - 0,1 & 7z^3 - 2z^2 \\ & 7z^3 - 11,9z^2 + 5,6z - 0,7 \\ \hline & 9,9z^2 - 5,6z + 0,7 \\ & \quad 11,23z - 0,1 \end{array}$$

#long division.

$$\therefore X[z] = 7 + 9,9z^{-1} + 11,23z^{-2} + 11,87z^{-3} + \dots$$

$x[0] = 7$, $x[1] = 9,9$ $x[2] = 11,23$ $x[3] = 11,87$

EXAMPLE

$$X[z] = \left(\frac{0,1z}{z-0,9}\right)\left(\frac{z}{z-1}\right)$$

$$= \frac{0,1z^2}{z^2 - 1.9z + 0,9}$$

$$\begin{array}{r|l} & 0,1 + 0,19z^{-1} + 0,271z^{-2} + \dots \\ \hline z^2 - 1.9z + 0,9 & 0,1z^2 \\ & 0,1z^2 - 0,19z + 0,09 \\ \hline & 0,19z - 0,09 \\ & 0,19z - 0,361 + 0,171z^{-1} \\ \hline & 0,271 - 0,171z^{-1} \\ & \vdots \end{array}$$

$x[0] = 0,1$, $x[1] = 0,19$, $x[2] = 0,271$

n = zeros
m = poles

4) PARTIAL FRACTION

Notes: Just use this one

→ ONLY SIMPLE POLES + (MORE DEN TERMS THAN NUM TERMS)

$$\frac{X[z]}{z} = \frac{C_0}{z} + \frac{C_1}{z-p_1} + \dots + \frac{C_n}{z-p_n} = \frac{C_0}{z} + \sum_{k=1}^{n} \frac{C_k}{z-p_k}$$

$$\boxed{C_0 = X[z]\Big|_{z=0}} \quad \boxed{C_k = (z-p_k)\frac{X[z]}{z}\Big|_{z=p_k}}$$

→ ONLY SIMPLE TERMS + (MORE NUM TERMS THAN DEN TERMS)

Need to compensate for the question being stupid by adding an extra term.

$$X[z] = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} C_k \frac{z}{z-p_k}$$

You can solve for b_q by yet again conducting long division with numerator and denominator

→ MULTIPLE / REPEATED POLES

$$\frac{\lambda_m}{z-p_i} + \frac{\lambda_{m-1}}{(z-p_i)^2} + \dots + \frac{\lambda_1}{(z-p_i)^m}$$

Method 1

$$\lambda_k = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}\left[(z-p_i)^m \frac{X[z]}{z}\right]\Bigg|_{z=p_i}$$

$(k=m)$

METHOD 2

Lean on them partial fractions

$$\frac{X[z]}{z} = \sum_{k=1}^{n} \sum_{j=1}^{m} \frac{C_{kj}}{(z-p_k)^j}$$

→ COMPLEX POLE

$$\frac{X[z]}{z} = \frac{c}{(z-p)} + \frac{c^*}{(z-p^*)}$$

$$\boxed{x_{complex}[n] = 2|c||p|^n \cos(\theta_p n + \theta_c)}$$

C^* complex conjugate of c $\qquad \theta_c = \measuredangle c$

p^* complex conjugate of p $\qquad \theta_p = \measuredangle p$

EXAMPLE : 1st order real pole

$$X[z] = \frac{8z - 19}{(z-2)(z-3)}$$

$$X[z] = \frac{c_1}{(z-2)} + \frac{c_2}{z-3}$$

$$c_1 = (z-2)X[z]\Big|_{z=2} = 3$$

$$c_2 = (z-3)X[z]\Big|_{z=3} = 5$$

$$X[z] = \frac{3}{(z-2)} + \frac{5}{z-3}$$

$$x[n] = [3(2)^{n-1} + 5(3)^{n-1}]u[n-1]$$

No

~~no~~ one no one wants to see this

So we literally diten it by dividing through by z before.

Since $[n > m]$

$$\frac{X[z]}{z} = \frac{8z - 19}{z(z-2)(z-3)} = \frac{c_0}{z} + \frac{c_1}{z-2} + \frac{c_2}{z-3}$$

$$c_0 = X[z]\Big|_{z=0} = -\frac{19}{6} \quad c_1 = (x-2)\frac{X[z]}{z}\Big|_{z=2} = \frac{3}{2} \quad c_1 = (z-3)\frac{X[z]}{z}\Big|_{z=3} = \frac{5}{3}$$

$$\frac{X[z]}{z} = \frac{-19/6}{z} + \frac{3/2}{z-2} + \frac{5-3}{z-3}$$

$$X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

$$x[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}(2)^n + \frac{5}{3}(3)^n\right]u[n]$$

It's the same. But it's prettier. So we hire this one.

EXAMPLE : Repeated Poles

$$X[z] = \frac{z^2}{(z-1)(z-0,5)^2}$$

$$\frac{X[z]}{z} = \frac{z^2}{z(z-1)(z-0,5)^2} = \frac{z}{(z-1)(z-0,5)^2}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-0,5)} + \frac{C}{(z-0,5)^2}$$

$$A: (z-1)\frac{X[z]}{z}\bigg|_{z=1} = 4$$

METHOD 1

$$\gamma_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-p_i)^m \frac{X[z]}{z}\right]\bigg|_{z=p_i}$$

$$B = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-0,5)^2\frac{X[z]}{z}\right] = -4$$

$$C = \frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left[(z-0,5)^2\frac{X[z]}{z}\right] = -1$$

$$\frac{X[z]}{z} = \frac{4}{z-1} - \frac{4}{z-0,5} - \frac{1}{(z-0,5)^2}$$

$$x[n] = \left[4 - 4(0,5)^n - 2n(0,5)^n\right]u[n]$$

The partial fractions way (method 2) works too.

EXAMPLE

$$X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$$

$$\frac{X[z]}{z} = \frac{C_0}{(z-1)} + \frac{C_1}{(z-3-j4)} + \frac{C_2}{(z-3+j4)}$$

$$C_0 = (z-1)\frac{X[z]}{z}\bigg|_{z=1} = 2$$

$$C_1 = (z-3-j4)\frac{X[z]}{z}\bigg|_{z=3+j4} = 1{,}6e^{-j2,246}$$

$$X[z] = 2\frac{z}{z-1} + \left(1.6e^{-j2,246}\right)\frac{z}{(z-3-j4)} + \left(1.6e^{j2,246}\right)\frac{z}{(z-3+j4)}$$

$$|c| = 1.6 \qquad \theta_c = -2,246 \qquad |p| = \sqrt{3^2+4^2} = 5$$

$$\theta_p = \tan^{-1}(4/3) = 0,927$$

$$x[n] = 2u[n] + x_{complex}[n] = \left[2 + 3.2(5)^n\cos(0,927n - 2.246)\right]u[n]$$

SOLUTION OF LINEAR DIFFERENCE EQUATION

Z transform converts D.E into algebraic equations. finding the inverse will provide the solution in the time domain

EXAMPLE:

Solve $y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$

$y[-1] = 11/6$; $y[-2] = 37/36$; $x[n] = 2^{-n} u[n]$

To take advantage of the initial conditions, we shift everything

$y[n] \to y[n-2]$

$\therefore y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$

all of this is as if it is multiplied with $u[n]$. as we are at initial conditions

$$y[n-1]u[n] \Rightarrow z^{-1} Y[z] + y[-1] = z^{-1} Y[z] + 11/6$$

$$y[n-2]u[n] \Rightarrow z^{-2} Y[z] + z^{-1} y[-1] + y[-2] =$$

$$z^{-2} Y[z] + z^{-1} \tfrac{11}{6} + \tfrac{37}{36}$$

$$\therefore 2^{-n} u[n] = (2^{-1})^n u[n] = (0{,}5)^n u[n] = \frac{z}{z-0{,}5}$$

Buuuut, if we want to delay it $n \to n-2$

we need to use the time shifting property

$$x[n-r] = z^{-r} X[z] + z^{-r} \sum_{n=1}^{r} x[-n]$$

IF THE INPUT IS CAUSAL: $x[-1] = x[-2] = x[-3] = \dots = x[-n] = 0$

$$\therefore \boxed{x[n-r] = z^{-r} X[z]}$$

$$3x[n-1] = z^{-1} \frac{3z}{z-0{,}5} = \frac{3}{z-0{,}5} \qquad 5x[n-2] = z^{-2} \frac{5z}{z-0{,}5} = \frac{5}{z(z-0{,}5)}$$

$$Y[z] - 5\left[z^{-1} Y[z] + \frac{11}{6}\right] + 6\left[z^{-2} Y[z] + z^{-1}\frac{11}{6} + \frac{37}{36}\right] = \frac{3}{z-0{,}5} + \frac{5}{z(z-0{,}5)}$$

Much_algebra

$$Y[z] = \frac{z(3z^2 - 9.5z + 10.5)}{(z-0{,}5)(z^2 - 5z + 6)}$$

$$y[n] = \left[\frac{26}{15}(0{,}5)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n\right] u[n]$$

TRANSFER FUNCTION AND SYSTEM REALISATION

Ideal: linear TF with constant coefficients

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad a_0 \neq 0$$

In general, if $M=N$

$$a_0 y[n] + a_1 y[n-1] + \dots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

z transform

$$\sum_{k=0}^{N} a_k z^{-k} Y[z] = \sum_{k=0}^{M} b_k z^{-k} X[z]$$

In general:

$$[a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{-N+1} + a_N z^{-N}] Y[z] = [b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-M+1} + b_M z^{-M}] X[z]$$

$$H(z) = \frac{Y[z]}{X[z]} = \frac{b_0 z^N + b_1 z^{N-1} \dots + b_{N-1} z + b_N}{a_0 z^N + a_1 z^{N-1} \dots + a_{N-1} z + a_N}$$

SYSTEM REALISATION

We want to generally keep to using a UNIT DELAY device: $\frac{1}{z}$

Consider a 3rd order system

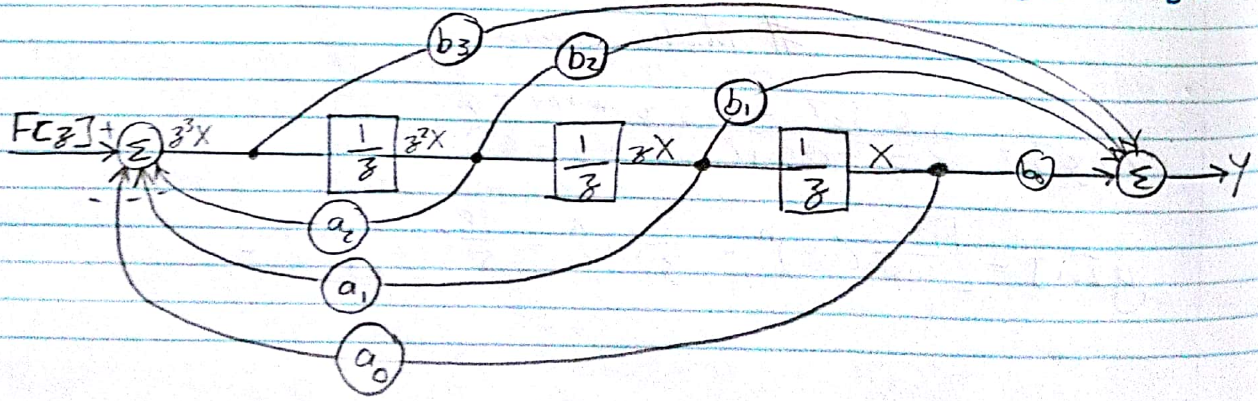
$$H(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z^1 + b_0}{a_3 z^3 + a_2 z^2 + a_1 z^1 + a_0}$$

Which we break into $X[z]$ and $Y[z]$

$$X[z] = \frac{1}{z^3 + a_2 z^2 + a_1 z + a_0} F[z] \qquad Y[z] = X[z]\left(b_3 z^3 + b_2 z^2 + b_1 z + b_0\right)$$

Which can be written as

$$z^3 X[z] = -a_2 z^2 X[z] - a_1 z X[z] - a_0 X[z] + F[z]$$



EXAMPLE : Find the transfer function of an α-filter

$$y[n] - (1-\alpha)\, y[n-1] = \alpha x[n] \quad \text{A} \quad (\alpha = 0{,}1)$$

$$y[n] - 0{,}9\, y[n-1] = \cancel{\alpha x}\ 0{,}1\, x[n]$$

$\# \ \mathcal{Z}\{\}$

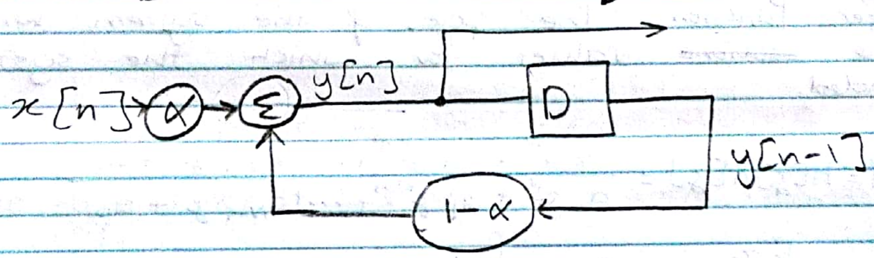
$$Y[z] - 0{,}9 z^{-1} Y[z] = 0{,}1\, X[z]$$

$$(1 - 0{,}9 z^{-1})\, Y[z] = 0{,}1\, X[z] \quad \text{B}$$

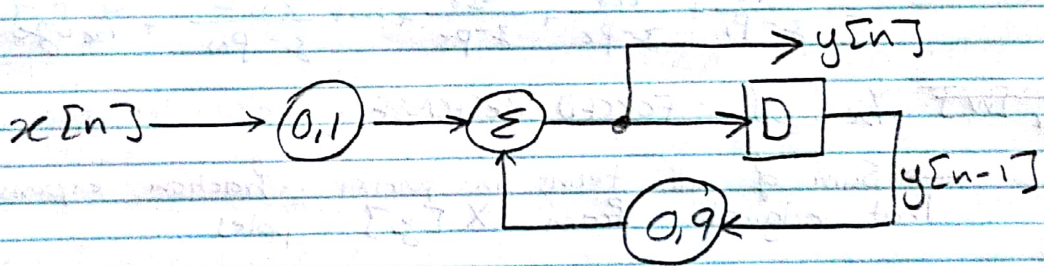
$$H[z] = \frac{0{,}1}{1 - 0{,}9 z^{-1}}$$

$$= \frac{0{,}1 z}{z - 0{,}9} \quad \text{C}$$

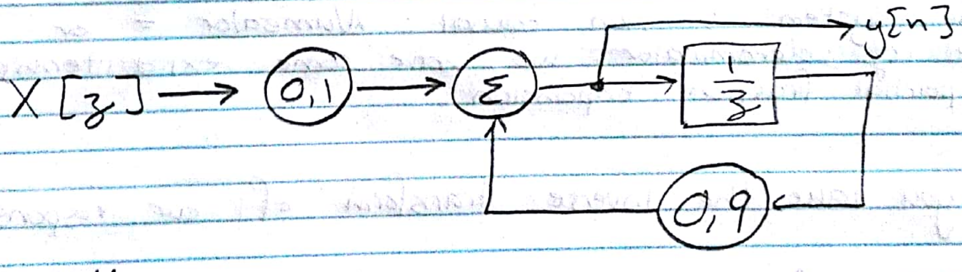
A



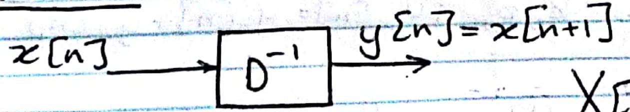
B



C



CAUSALITY



This system is called a
UNIT ADVANCE

N cascaded advance systems will have a transfer function of

$$Y[z] = z^{N} X[z]$$

A UNIT ADVANCE is <u>NOT CAUSAL</u>

<u>EXAMPLE</u>

$$H(z) = \frac{z^2 + 0{,}4\,z + 0{,}9}{z - 0{,}6}$$

$$= z + \frac{z + 0{,}9}{z - 0{,}6}$$

System is non causal. Represented as a realisable system in // with a unit advance

Numerator is of higher degree than the denominator

<u>STABILITY</u>

BIBO Stable: Output remains bounded for a bounded input

In a transfer function, the poles of the system will represent the ~~assume~~ values for which the system is unbounded.

$$Y[z] = H[z]X[z] = \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_{N-1} z + b_0}{a_0 z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_0}$$

$$= \frac{k_1 z}{z - p_1} + \frac{k_2 z}{z - p_2} + \frac{k_3 z}{z - p_3} + \ldots + \frac{k_N z}{z - p_N} + Y_x[z]$$

~~$Y_x[n]$~~ $Y_x[z]$: FORCED RESPONSE

→ Sum of all terms in partial fraction expansion that originate from $X[z]$ poles

If the system is non causal: Numerator = or ~~mo~~ higer degree of denominator, we have some extra terms in the partial fraction expansion

If you take the inverse transform of our response:

$$y[n] = k_1 p_1^n + k_2 p_2^n + \ldots + k_N p_N^n + y_x[n] = y_n[n] + y_x[n]$$

$y_N[n]$: NATURAL RESPONSE : arises from the partial fraction expansion of the TF terms.

p_i^n : Mode of the system

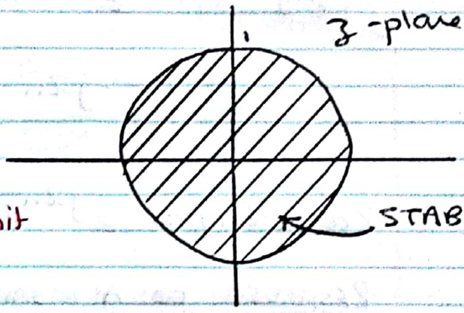
$Y_x[n]$ has the same poles as $X[n] \rightarrow$ if $x[n]$ is bounded, then $Y_x[n]$ is bounded

$\therefore$ Only $y_n[n]$ can be responsible for an unbounded output.

UNBOUNDEDNESS WILL OCCUR IF ANY OF THE POLE MAGNITUDES $|p_i|$ IS LARGER THAN 1

"An LTI discrete-time ~~signal~~ causal system is BIBO stable if all poles of the system transfer function lie inside the unit circle in the z-plane"

Causal systems have a ROC everywhere outside the unit circle



An LTI system (causal) is BIBO stable provided the region of convergence of its transfer function includes the unit circle.

System characteristic equation: denominator = 0

Stable systems have roots of characteristic equation in the unit circle.

EXAMPLE

Is the system stable? $$H(z) = \frac{2z^2 - 1.6z - 0{,}9}{z^3 - 2.5z^2 + 1.96z - 0{,}48}$$

$$z^3 - 2.5z^2 + 1.96z - 0{,}48 = 0$$

$$(z-1.2)(z-0{,}8)(z-0{,}5) = 0$$

Unstable system: $z = 1.2$ is outside unit circle.

$$\lim_{n\to\infty} (1.2)^n \rightarrow \infty$$

INVERTABILITY

$$X[z] \rightarrow \boxed{H[z]} \rightarrow \boxed{H_i[z]} \rightarrow X[z]$$

$$H_i = \frac{1}{H[z]}$$

Summary

- Bilateral and unilateral z-transform
- z-transform from Laplace transform
- Inverse z-transform (Procedures for finding the inverse)
- Properties of the z-transform
- ~~Procedures~~ Transfer function + system realisation
- Stability, causality, invertability of the z-transform

4) DIGITAL FILTERS

Filtering characteristics of a system are defined by its frequency response

The response of a digital filter to a complex exponential

z^n is also a complex exponential $H[z] = z^n$

$$y[n] = z^n H[z]$$

EXAMPLE

$z = e^{j\Omega}$: $e^{j\Omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\Omega})\, e^{j\Omega n}$

$e^{-j\Omega n} \longrightarrow H(e^{-j\Omega})\, e^{-j\Omega n}$

$2\cos(\Omega n) = e^{j\Omega n} + e^{-j\Omega n} \longrightarrow H[e^{j\Omega}]e^{j\Omega n} + H[e^{-j\Omega}]e^{-j\Omega n}$

RESPONSE TO A SAMPLED SINOSOID:

$x(t) = \cos(\omega t)$

$x[n] = z\, x(t)\big|_{t=nT_s} = \cos(n\omega T_s)$

Let $\Omega = \omega T_s$

EXAMPLE

Frequency response of $y[n+1] - 0{,}8\,y[n] = x[n+1]$, w/ zero initial conditions

z transform: $zY[z] - 0{,}8\,Y[z] = zX[z]$

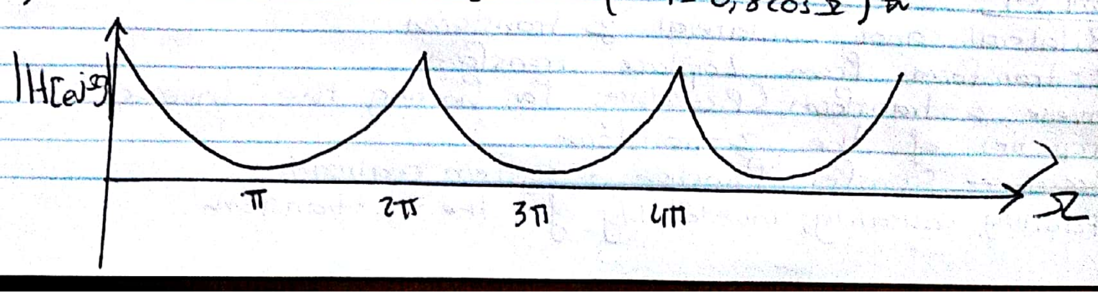
$$H[z] = \frac{z}{z - 0{,}8} = \frac{1}{1 - 0{,}8z^{-1}}$$

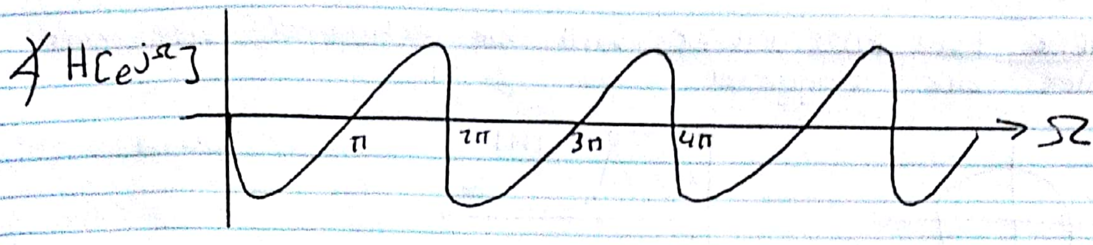
$$H[e^{j\Omega}] = \frac{1}{1 - 0{,}8\,e^{-j\Omega}} = \frac{1}{(1 - 0{,}8\cos\Omega) + j0{,}8\sin\Omega}$$

$$|H[e^{j\Omega}]| = \frac{1}{\sqrt{(1 - 0{,}8\cos\Omega)^2 + (0{,}8\sin\Omega)^2}}$$

$$= \frac{1}{\sqrt{1{,}64 - 1{,}64\cos\Omega}}$$

$$\measuredangle H[e^{j\Omega}] = 0 - \tan^{-1}\left(\frac{0{,}8\sin\Omega}{1 - 0{,}8\cos\Omega}\right)$$

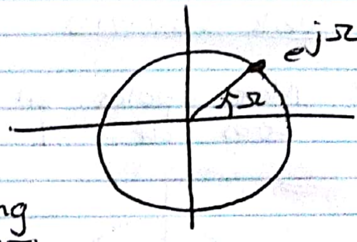




FREQUENCY RESPONSE FROM POLE-ZERO LOCATION

Evaluate $H[e^{j\Omega}]$ at $z = e^{j\Omega}$

$|z| = 1 \qquad \measuredangle z = \Omega$



Create a vector sketch by connecting $z = e^{j\Omega}$ with every pole and zero

We can represent $H[e^{j\Omega}]$ as using the properties of these vectors

$$\boxed{|H[e^{j\Omega}]| = b_N \frac{r_1 r_2 \ldots r_n}{d_1 d_2 \ldots d_n}}$$

r: pole vector magnitude
d: zero vector magnitude

$$\boxed{\measuredangle H[e^{j\Omega}] = (\phi_1 + \phi_2 + \ldots + \phi_N) - (\theta_1 + \theta_2 + \ldots \theta_3)}$$

ϕ : Pole angle $\qquad \theta$: zero angle

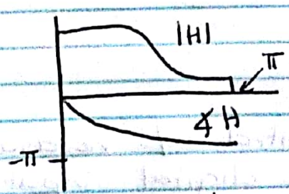
$$\boxed{H[e^{j\Omega}] = b_N \frac{r_1 r_2 \ldots r_N}{d_1 d_2 \ldots d_N} e^{j[(\phi_1 + \phi_2 + \ldots + \phi_N) - (\theta_1 + \theta_2 + \ldots + \theta_N)]}}$$

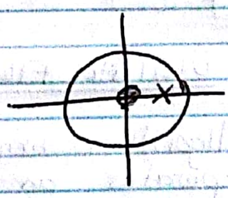
TYPES OF FILTERS

- → Low Pass Filters
- → High Pass Filters
- → Band Pass Filters
- → Band Stop Filters
- → All Pass Filters

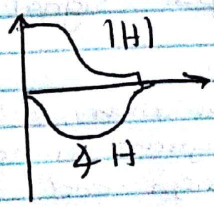
POLES NEAR UNIT CIRCLE: PASSBANDS
ZEROS NEAR UNIT CIRCLE: STOP BANDS

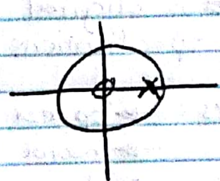
LOWPASS
Pole near $z = e^{j\Omega} = 1$



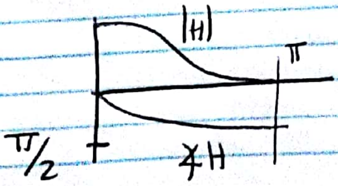


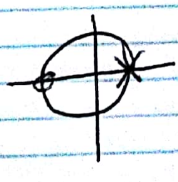
Zero ~~Pole near~~ @ origin
Only phase is changed





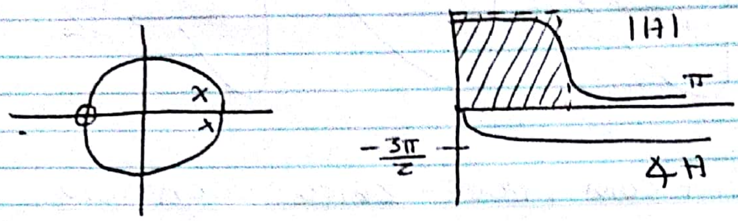
Zero @ -1





amplitude + phase changed

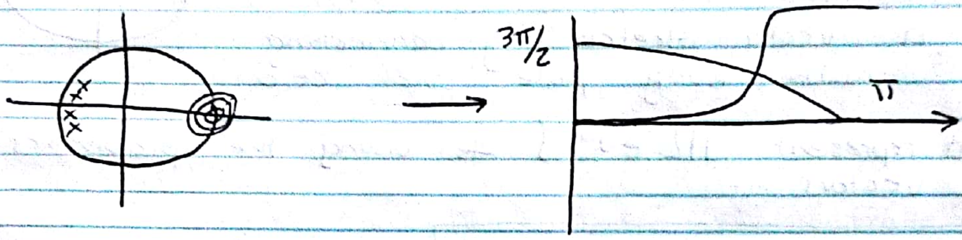
IDEAL LOW PASS FILTER will be achieved as more poles are staggered near $z = 1$



/// : ideal

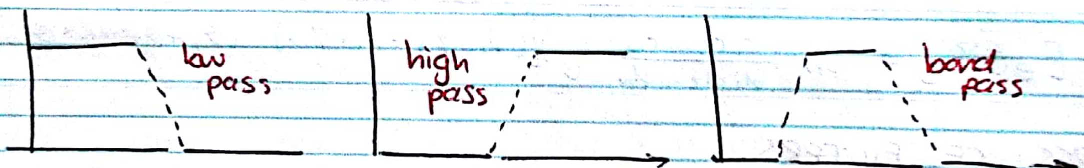
IDEAL LOW PASS

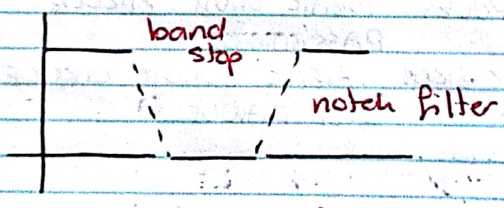
More poles at $z = -1$ → GAIN @ π highest
Zeros at 1 → LOWERS gain @ low frequency



SELECTIVE FREQUENCY FILTERS

Select frequencies, reject others
The parts of the frequency band that has response 1 is pass bands. 0's are stop bands
In between these regions exists the transition band





DIGITAL FILTER

Algorithm implemented w/ hardware or software which operates on a digital input signal to produce digital output within specifications

Disadvantages:
- → Speed limitation
- → Word lengths finite
- → long design and development times

RECURSIVE FILTERS

Feedback from output to input
Propogates forever

$$y[n] = (-a_2 y[n-1] - a_1 y[n-2] - a_0 y[n-3]) + (b_3 x[n] + b_2 x[n-1] + b_1 x[n-2])$$

eg) Infinite impulse response (IIR) filters

NON RECURSIVE FILTERS

No feedback, output is indipendent of any past outputs

if recursive coefficients are set to 0 (a_0, a_1, a_2)

$$H[z] = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3}$$

$$y[n] = b_3 x[n] + b_2 x[n-1] + b_1 x[n-2] + b_0 x[n-3] \ldots$$

eg) Finite impulse response (FIR) filters

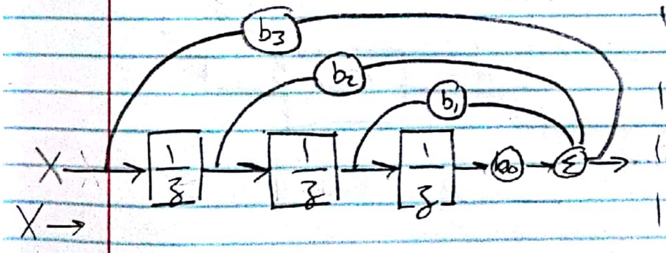
FIR vs IIR

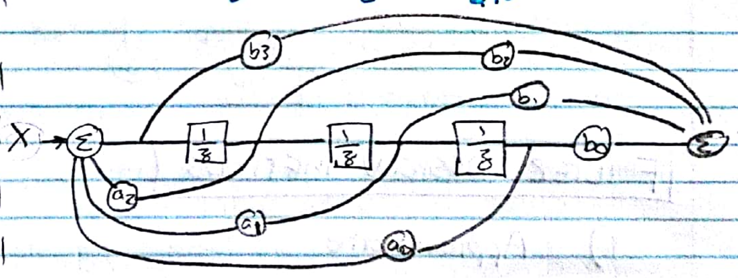
FIR → Always stable
→ Linear phase
→ Several design methods
→ Convenient implementation

IIR → Less complex
→ Way more expensive
→ Convert Butterworth / Chebyshev to discrete time domain

$$H = b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0$$

$$H = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$



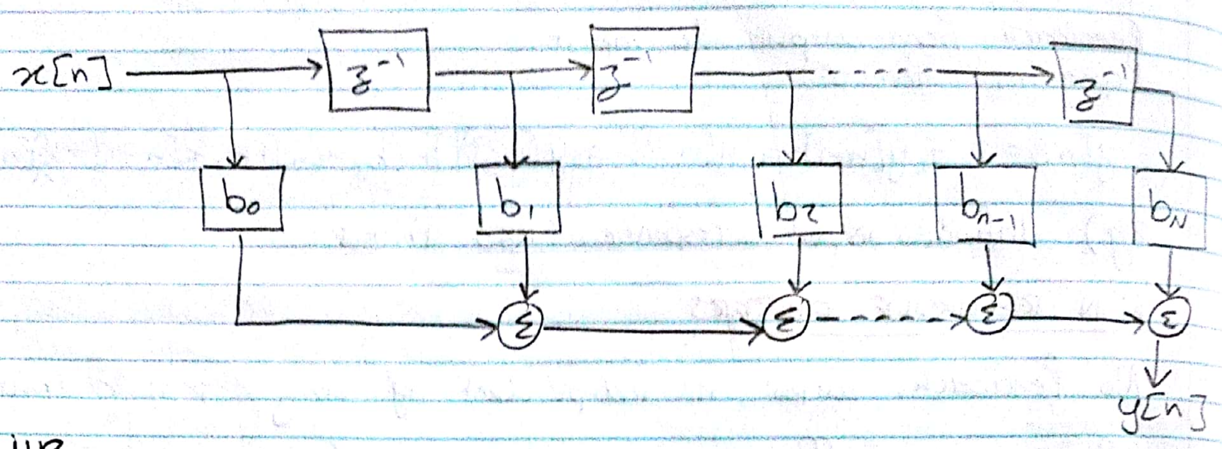


$$y[n] = \sum_{k=0}^{N} b_k \cdot x[n-k]$$

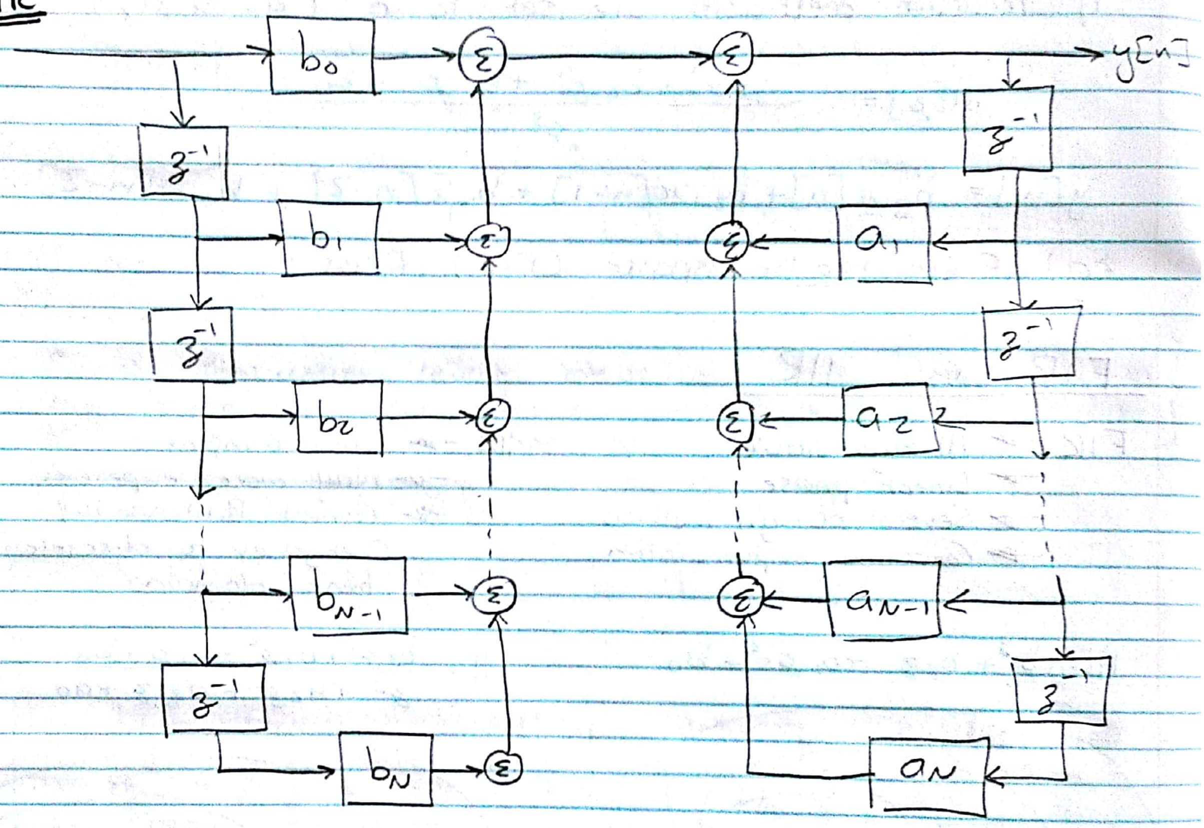
~~$y[n] = \sum_{k=0}^{N} b_k \cdot x$~~

$$y[n] = \sum_{k=0}^{N} b_k \cdot x[n-k] + \sum_{m=-1}^{M} -a_m \cdot y[n-m]$$

FIR



IIR



FILTER DESIGN METHODS

1) Approximate

2) Realization

Approximation

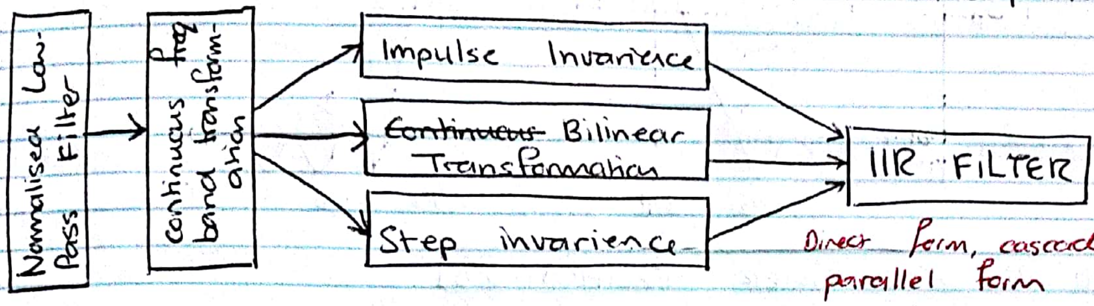
- Desired frequency domain response
- Filter class chosen (FIR or IIR)
- Design criteria
- Algorithm to design TF.

Realization

- Set of structures
- Criteria for comparing implementations
- Best structure chosen, parameters calculated from TF
- Structure implemented in software / hardware

IIR FILTER DESIGN

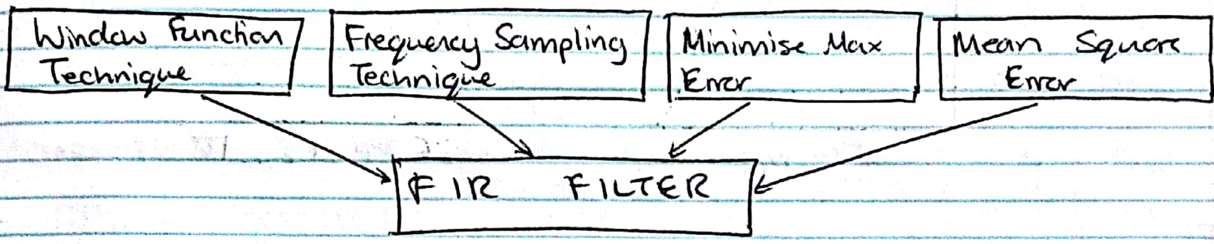
3 Methods : Impulse Invarience, Bilinear Transformation, Step invarience



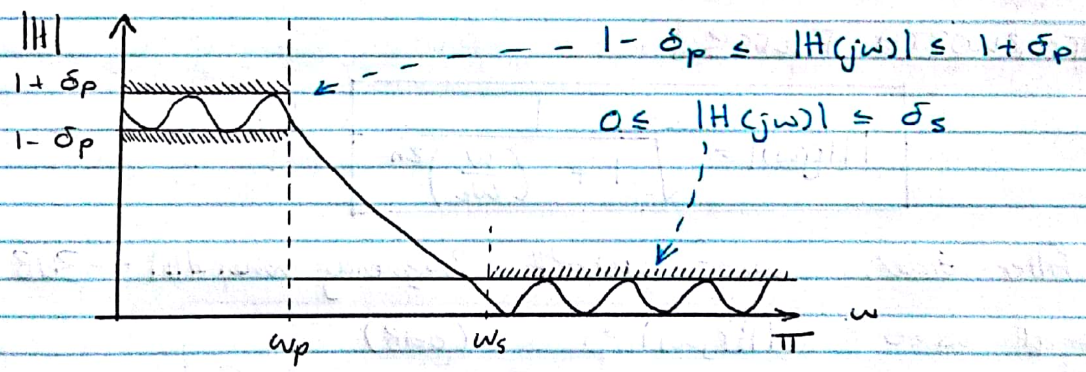
Direct form, cascade form, parallel form

FIR FILTER DESIGN

Methods: Window function, Frequency sampling, minimise max error, MSE



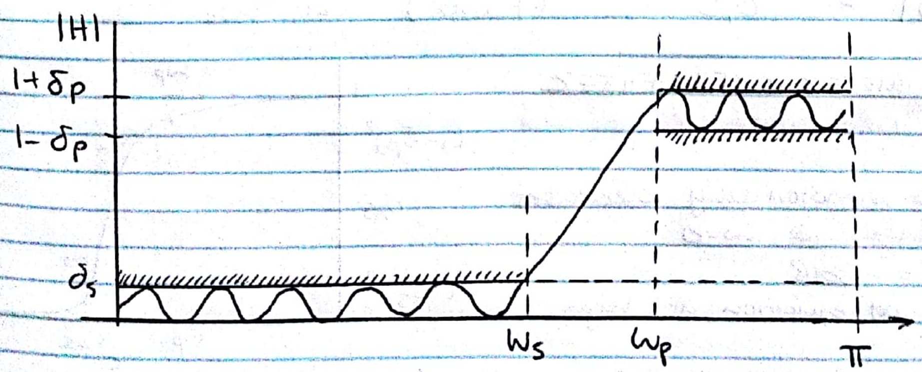
LOW PASS FILTER SPECIFICATION



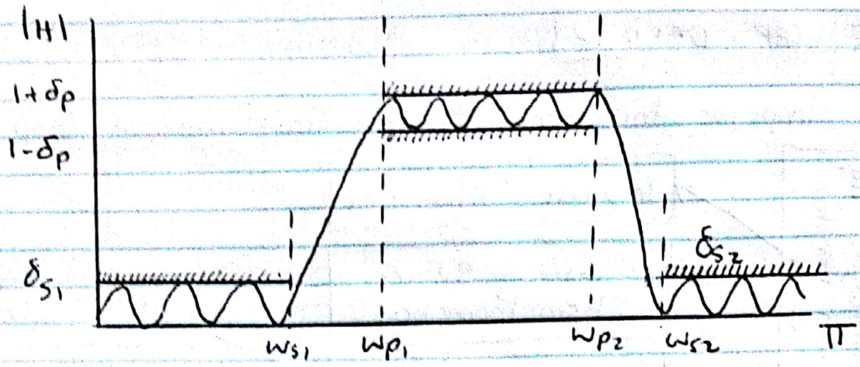
Pass band ripple: $20\log_{10}(1+\delta_p)$ dB

Stop band attenuation: $-20\log_{10}(\delta_s)$ dB

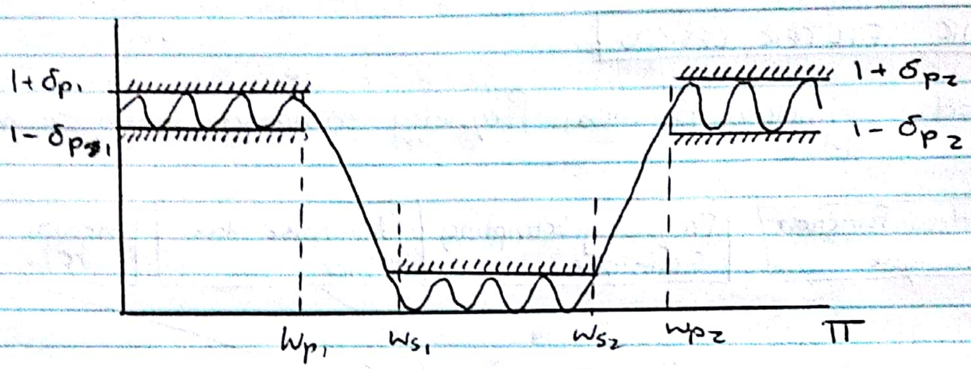
HIGH PASS FILTER SPECIFICATION



BAND PASS FILTER



BAND ~~PAS~~ STOP FILTER SPECIFICATIONS



{ We normally just design the LOW PASS filter and then use frequency transformations to obtain the other filters }

BUTTERWORTH FILTER

$$\boxed{|H(jw)| = \frac{1}{\sqrt{1 + \left(\frac{w}{w_c}\right)^{2n}}}}$$

n: filter order, w_c = cutoff frequency when $|H| = -3dB$

→ at $w=0$: $|H(jw)| = 1$ (0dB)

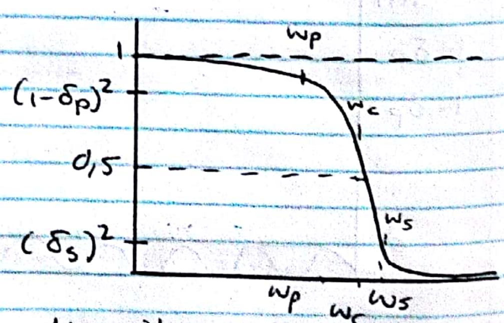
→ at $w=w_c$: $|H(jw)| = \frac{1}{\sqrt{2}}$ (−3 dB)

w_c is called the half power frequency because

$|H(jw)|^2 = 0{,}5$ (Power) at w_c

LOW PASS BUTTERWORTH FILTER

Properties:
- → Magnitude monotonically decreases
- → Max gain = 1 @ $w=0$
- → $|H(jw_c)| = -3dB$
- → Asymptotic attenuation at high frequency
- → Maximally flat at DC



Magnitude Squared Response

NORMALISED BUTTERWORTH DESIGN PROCEDURE

- Normalised: $\omega_c = 1$

$$\boxed{|H(j\omega)| = \frac{1}{\sqrt{1+\omega_n^{2n}}}}$$

- Solve for $H(s)$

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega) = \frac{1}{1+\omega^{2n}}$$

$$|H(s)|^2 = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$

- Find poles, where $k = 1, 2, \ldots n$

$$S_k = e^{\frac{j\pi}{2n}(2k+n-1)} = \cos\left(\frac{\pi}{2n}(2k+n-1)\right) + \sin\left(\frac{\pi}{2n}(2k+n-1)\right)$$

- Desired TF can be found by replacing s w/ s/ω_c

DESIGN PROCEDURE

1)
$$\boxed{\begin{aligned} G_p' &= \cancel{20}\,{-10}\log_{10}\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right) \\ G_s' &= \cancel{20}\,{-10}\log_{10}\left(1+\left(\frac{\omega_s}{\omega_c}\right)^{2n}\right) \end{aligned}}$$

$G_p = 1 - \delta_p$

$G_s = \delta_s$

$\omega_p' = \omega_p T$

($G_p' = 20\log_{10} G_p$ @ ω_p $\quad G_s' = 20\log_{10} G_s$) @ ω_s $\quad \omega_s' = \omega_s T$

2) Compute n → Round UP to nearest integer

$$\boxed{n \geq \frac{\log\left[(10^{-G_s'/10}-1)/(10^{-G_p'/10}-1)\right]}{2\log_{10}(\omega_s'/\omega_p')}}$$

3) ω_c : -3dB cutoff frequency

$$\boxed{\omega_c = \frac{\omega_p}{[10^{-G_p/10}-1]^{1/2n}}}$$

exactly satisfies G_p'
oversatisfies G_s'

OR

$$\boxed{\omega_c = \frac{\omega_s}{[10^{-G_s/10}-1]^{1/2n}}}$$

Exactley satisfies G_s', oversatisfies G_p'

Choose accordingly

4) Find the Poles S_k

$$\boxed{S_k = \cos\left(\frac{\pi}{2n}(2k+n-1)\right) + j\sin\left(\frac{\pi}{2n}(2k+n-1)\right)}$$

5) Compute Normalised Transfer Function $\bar{H}_a(s)$

$$\boxed{\bar{H}_a(s) = \prod_{k=1}^{n} \frac{1}{s - s_k}}$$

6) Find unnormalised transfer function $H_a(s)$

$$s \to s/\omega_c$$

HIGH PASS FILTER VERSION

→ Design prototype low pass filter w/

$$\omega_p = 1 \qquad \omega_s = \frac{\omega_p'}{\omega_s'}$$

→ when solving for ω_c' use $\omega_s = \frac{\omega_p'}{\omega_s'}$

ω_p' and ω_s' are PREWARPED FREQUENCY SPECS

→ In the final transfer function, replace s w/ $\frac{\omega_p}{s}$

BAND PASS FILTER VERSION

$\omega_p = 1$ and use whichever of the following is smallest

$$\boxed{\omega_s = \frac{\omega_{p_1}' \omega_{p_2}' - (\omega_{s_1}')^2}{\omega_{s_1}' (\omega_{p_2}' - \omega_{p_1}')} \quad \text{or} \quad \omega_s = \frac{(\omega_{s_2}')^2 - (\omega_{p_1}' \omega_{p_2}')}{\omega_{s_2}' (\omega_{p_2}' - \omega_{p_1}')}}$$

ω_{p_1}', ω_{p_2}', ω_{s_1}', ω_{s_2}' are PREWARPED

→ In the final TF, replace

$$\boxed{s \to \frac{s^2 + \omega_{p_1}' \omega_{p_2}'}{(\omega_{p_2}' - \omega_{p_1}') s}}$$

(ω_c found as per low pass)

BAND STOP FILTER VERSION

$\omega_p = 1$ and use whichever of the following is smallest

$$\omega_s = \frac{(\omega_{p_2}' - \omega_{p_1}') \omega_{s_1}'}{\omega_{p_1}' \omega_{p_2}' - (\omega_{s_1}')^2} \quad \text{or} \quad \omega_s = \frac{(\omega_{p_2}' - \omega_{p_1}') \omega_{s_2}'}{(\omega_{s_2}')^2 - \omega_{p_1}' \omega_{p_2}'}$$

ω_{p_1}', ω_{p_2}', ω_{s_1}', ω_{s_2}' are PREWARPED

→ In the final TF, replace

$$\boxed{s \to \frac{(\omega_{p_2}' - \omega_{p_1}') s}{s^2 + \omega_{p_1}' \omega_{p_2}'}}$$

CHEBYSHEV FILTER

$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(\omega)}}$$

$C_n(\omega)$ is the "Chebyshev Polynomial"

$$C_n = \cos(n\cos^{-1}(\omega)) = \cosh(n\cosh^{-1}(\omega))$$

ε controls the height of the ripples.
r : ratio of max ripple height over min ripple height

$$r = \sqrt{1+\varepsilon^2}$$

$$r'(dB) = 20\log\sqrt{1+\varepsilon^2} = 10\log(1+\varepsilon^2)$$

$$\varepsilon^2 = 10^{\frac{r'}{10}} - 1$$

NORMALISED CHEBYSHEV DESIGN PROCEDURE

1) Gains :

$$G(s) = G_s = |H(j\omega_s)| \quad @ \ \omega_s$$

$$G_s = -10\log\left[1+\varepsilon^2 C_n^2(\omega_s)\right]$$

$$\therefore \ \varepsilon^2 C_n^2(\omega_s) = 10^{-\frac{G_s}{10}} - 1$$

For Chebyshev filters, we evaluate r' instead of $-G_p'$

$r' = -G_p$ is a thing

2) Compute n and round up

$$n \geq \frac{1}{\cosh^{-1}(\omega_s)} \cosh^{-1}\left[\frac{10^{-\frac{G_s}{10}} - 1}{10^{r'/10} - 1}\right]^{0.5}$$

(For not normalised filters $\omega_s \to \frac{\omega_s}{\omega_p}$)

3) Compute Normalised Transfer Function $\overline{H}_a(s)$

$$\overline{H}_a(s) = \frac{K_n}{C_n'(s)} = \frac{K}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

$$K = \begin{cases} a_0 & n, \text{ odd} \\ \dfrac{a_0}{\sqrt{1+\varepsilon^2}} = \dfrac{a_0}{10^{r'/20}} & n, \text{ even} \end{cases}$$

Tables exist w/ ready made coefficients which make this whole process easier.

If the table doesn't have r' or n you want,

a) $\boxed{\varepsilon^2 = 10^{\frac{r'}{10}} - 1}$

b) Poles :

$$\boxed{S_k = -\sin\left[\frac{(2k-1)\pi}{2n}\right]\sinh x + j\cos\left[\frac{(2k-1)\pi}{2n}\right]\cosh x}$$

$$\boxed{x = \frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)} \qquad k = 1, 2, 3 \ldots n$$

c) Normalised TF $\bar{H}_a(s)$

$$\boxed{\bar{H}_a(s) = \prod_{k=1}^{n} \frac{k_n}{s - s_k}}$$

5) Get $H_a(s)$: $s \to \frac{s}{\omega_{pc}}$

HIGH PASS FILTER VERSION

Design low pass w/ $\omega_p = 1$, $\omega_s = \frac{\omega_p'}{\omega_s'}$

$$s \to \frac{\omega_p'}{s}$$

BAND PASS FILTER VERSION

Design $\omega_p = 1$, ω_s the smaller of :

$$\boxed{\omega_s = \frac{\omega_{p1}'\omega_{p2}' - (\omega_{s1}')^2}{\omega_{s1}'(\omega_{p2}' - \omega_{p1}')}} \qquad \boxed{\omega_s = \frac{(\omega_{s2}')^2 - \omega_{p1}'\omega_{p2}'}{\omega_{s2}'(\omega_{p2}' - \omega_{p1}')}}$$

$$\boxed{s \to \frac{s^2 + \omega_{p1}'\omega_{p2}'}{(\omega_{p2}' - \omega_{p1}')s}}$$

BAND STOP FILTER VERSION

Design $\omega_p = 1$ and ω_s the smaller of :

$$\boxed{\omega_s = \frac{(\omega_{p2}' - \omega_{p1}')\omega_{s1}'}{\omega_{p1}'\omega_{p2}' - (\omega_{s1}')^2}} \quad \text{or} \quad \boxed{\omega_s = \frac{(\omega_{p2}' - \omega_{p1}')\omega_{s2}'}{(\omega_{s2}')^2 - \omega_{p1}'\omega_{p2}'}}$$

DISCRETE IIR FILTERS

IMPULSE INVARIENCE DESIGN METHOD

Impulse response of the digital filter is chosen proportionally to sampled continuous filter

$$\boxed{h[n] = T h_a(t)\big|_{t=nT} = T h_a(nT)}$$

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$ ← Partial fraction form

Inverse Laplace :

$$h_a(t) = \mathcal{L}^{-1}[H_a(s)] = \mathcal{L}^{-1}\left[\sum_{k=1}^{N} \frac{c_k}{s-p_k}\right] = \sum_{k=1}^{N} c_k e^{p_k t}$$

but then to digital:

$$h[n] = T h_a(nT) = T \sum_{k=1}^{N} c_k e^{p_k n T}$$

to Z domain, Z transform

$$H[z] = Z\{h[n]\} = T \sum_{n=0}^{\infty} \sum_{k=1}^{N} c_k e^{p_k n T} z^{-n}$$

$$= T \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n$$

$$= T \sum_{k=1}^{N} c_k \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$= \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}}$$

If we skip all this:

$$\boxed{H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \longrightarrow H[z] = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}}$$

$$\boxed{\frac{c_k}{s - p_k} \longrightarrow \frac{c_k}{1 - e^{p_k T} z^{-1}}}$$

ALIASING EFFECTS AND THE CHOICE OF SAMPLING INTERVAL

If for some frequency w_0, $|H_a(jw)| = 0{,}01\, |H_a(jw)|_{max}$

then: $T = \dfrac{\pi}{w_0}$

This method is ~~fine~~ NOT FINE for High pass filters, or stop band filters

Not really in use.

EXAMPLE 1

Design a low pass discrete time filter using impulse invariance method for Butterworth filter to satisfy passband specification if $T_s = 1$

Specs: $0,89125 \leq |H(e^{jwT})| \leq 1 \qquad 0 \leq |wT| \leq 0,2\pi$

$|H(e^{jwT})| \leq 0,17783 \qquad 0,3\pi \leq |wT| \leq \pi$

$G_p' = 20 \log G_p = \cancel{0,8}\ 20 \log(0,891250) = -1$

$G_s' = 20 \log G_s = 20 \log(0,17783) = -14,999$

$w_p' = T \times w_p = 1 \times 0,2\pi = 0,2\pi$

$w_s' = T \times w_s = 1 \times 0,3\pi = 0,3\pi$

$$n \geq \frac{\log[(10^{-G_s/10} - 1)/(10^{-G_p/10} - 1)]}{2 \log(w_s/w_p)}$$

$$\geq \frac{\log[(10^{1.4999} - 1)/(10^{0,1} - 1)]}{2 \log(0,3/0,2)}$$

$\geq 5,8858$

$= \underline{6}$

$$w_c = \frac{w_p'}{[10^{-G_p/10} - 1]^{1/2n}}$$

$$= \frac{0,2\pi}{(10^{0,1} - 1)^{1/12}} = 0,7032$$

$$s_k = \cos\left(\frac{\pi}{2n}(2k+n-1)\right) + j\sin\left(\frac{\pi}{2n}(2k+n-1)\right)$$

$s_1 = -0,2588 + j0,9659$

$s_2 = -0,7071 + j0,7071$

$s_3 = \cancel{-0,9656}\ -0,9659 + j0,2588$

$s_4 = -0,9659 - j0,2588$

$s_5 = -0,7071 - j0,7071$

$s_6 = -0,2588 - j0,9659$

Scanned by CamScanner

$$\overline{H}a(s) = \prod_{k=0}^{N-1} \frac{1}{s - s_k}$$

$$= \frac{1}{(s^2 + 0{,}51765s + 1)(s^2 + 1{,}4142s + 1)(s^2 + 1.9318s + 1)}$$

$$Ha(s) = \overline{H}a(s)\Big|_{s = \frac{s}{0{,}7032}}$$

$$= \frac{0{,}12093}{(s^2 + 0{,}3640s + 0{,}4945)(s^2 + 0{,}9945s + 0{,}4945)(s^2 + 1.3585s + 0{,}4945)}$$

$$Ha(s) = \sum_{k=1}^{6} \frac{C_k}{s - p_k} \qquad C_k = Ha(s)(s - p_k)\Big|_{s = p_k}$$

$$h_a(t) = \sum_{k=1}^{6} C_k e^{p_k t}$$

$$h[n] = \sum_{k=1}^{6} C_k e^{p_k n}$$

$$H[z] = \sum_{k=1}^{6} \frac{C_k}{1 - e^{p_k} z^{-1}}$$

Using the summation formula for geometric series ...

$$H[z] = \frac{0{,}2871 - 0{,}4466z^{-1}}{1 - 1.2971z^{-1} + 0{,}6949z^{-2}} + \frac{-2{,}1428 + 1.2455z^{-1}}{1 - 1{,}0691z^{-1} + 0{,}3699z^{-2}}$$

$$+ \frac{1{,}8597 - 0{,}6303z^{-1}}{1 - 0{,}9972z^{-1} + 0{,}2570z^{-2}}$$

EXAMPLE 2

$$Ha(s) = \frac{0{,}5(s+4)}{(s+1)(s+2)}$$

$$h_a(t) = \mathcal{L}^{-1}\{Ha(s)\} = \mathcal{L}^{-1}\left\{\frac{1.5}{s+1} + \frac{-1}{s+2}\right\} = 1.5e^{-t} - e^{-2t}$$

~~h[n] = Z{h~~ $h[n] = h_a(nT) = 1.5e^{-nT} - e^{-2nT}$

$$H[z] = Z\{h[n]\} = \sum_{n=0}^{\infty} (1.5e^{-nT} - e^{-2nT}) z^{-n}$$

$$\cancel{= 1.5\sum_{n=0}^{\infty}(e^{-T})^n - \sum_{n=0}^{\infty}(e^{-2T}z)^n}$$

$$= 1.5\sum_{n=0}^{\infty}(e^{-T}z^{-1})^n - \sum_{n=0}^{\infty}(e^{-2T}z^{-1})^n$$

$$= \frac{1.5}{1 - e^{-T}z^{-1}} + \frac{1}{1 - e^{-2T}z^{-1}}$$

Scanned by CamScanner

BILINEAR TRANSFORMATION METHOD

Use this one, it is the most widely used and also avoids the aliasing problems of the impulse invariance method

$$\boxed{H[z] = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}}$$

SPECIAL CASE: BAND PASS FILTER

When re-warping the TF by replacing s w/ $s = \frac{s^2 + w_{p1}' w_{p2}'}{(w_{p2}' - w_{p1}')s}$, the next step would be to apply the bilinear transformation. We can combine these to form the super-special hidden jutsu:

BILINEAR BAND PASS TRANSFORMATION

$$H[z] = H_a(s)\Big|_{s = T_{dp}[z]}$$

$$\boxed{T_{dp} = \frac{z^2 + 2az + 1}{b(z^2-1)}} \qquad \boxed{a = \frac{w_{p1}' w_{p2}' - 1}{w_{p1}' w_{p2}' + 1}}$$

$$\boxed{b = \frac{w_{p2}' - w_{p1}'}{w_{p1}' w_{p2}' + 1}}$$

SPECIAL CASE: BAND STOP FILTER.

Same story:

$$H[z] = H_a(s)\Big|_{s = T_{bs}[z]}$$

$$\boxed{T_{bs} = \frac{b(z^2-1)}{z^2 + 2az + 1}} \qquad \boxed{a = \frac{w_{p1}' w_{p2}' - 1}{w_{p1}' w_{p2}' + 1}} \qquad \boxed{b = \frac{w_{p2}' - w_{p1}'}{w_{p1}' w_{p2}' + 1}}$$

BILINEAR TX METHOD

- Stable analogue → Stable digital filter
- Max and min of analogue response are preserved
  - Passband ripple, minimum stop band attenuation, all preserved.

THE WARPING EFFECT

ω : ~~Fa~~ Analogue frequency variable
Ω : Digital frequency variable

$\therefore$
$$H[z] = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$H[e^{j\Omega T}] = H_a(j\omega)$$

iff

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \longrightarrow j\omega = \frac{2}{T}\left(\frac{e^{j\omega T}-1}{e^{j\omega T}+1}\right) \approx \frac{2}{T}\left(\frac{e^{j\Omega}-1}{e^{j\Omega}+1}\right)$$

$$\omega = \omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right) = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right)$$

$$\boxed{\omega = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right)} \qquad \boxed{\Omega = \omega_d T}$$

~~As~~ at low frequencies: $\omega = \omega_a < \frac{0.3}{T}$

$$\omega = \omega_a \approx \omega_d$$

$\therefore$ The digital filter has the same frequency response as the analogue filter over this range

At high frequencies, DISTORTION is introduced in the frequency ~~domain~~ scale of the digital filter relative to that of the analogue filter $\longrightarrow$ Non linear relationship

$$\boxed{\text{WARPING EFFECT}}$$

BILINEAR TRANSFORMATION WITH PREWARPING EFFECT

$$\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right)$$

frequency ω corresponds to frequency Ω in digital filter

for each of the critical frequencies perform the following calculations:

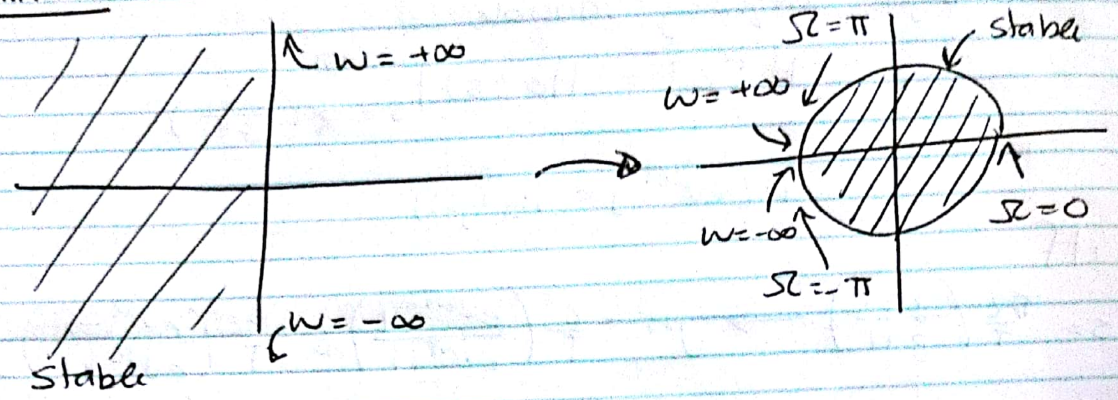
$$\Omega_i = \frac{2}{T}\tan^{-1}\left(\frac{\omega_i T}{2}\right) \qquad i = 1, 2 \ldots$$

Therefore, if you want to achieve the Ω specs calculated here, the analogue filter must be PREWARPED.

$$\omega_i = \frac{2}{T}\tan\left(\frac{\omega_i' T}{2}\right) = \frac{2}{T}\tan\left(\frac{\Omega_i}{2}\right)$$

$\frac{2}{T}$ is a scaling factor which can be omitted.

MAPPING



CHOICE of T

f_H F is the highest frequency to be processed

To avoid aliasing

$$\boxed{T = \frac{1}{F_s} \leq \frac{1}{2F_h}}$$

EXAMPLE

$$H_a = \frac{w_c}{s + w_c} \quad ; \quad w_c = 10^5 \quad ; \quad T = \frac{\pi}{10 w_c}$$

$$H[z] = H_a(s) \Big|_{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{w_c}{\frac{2}{T}\left(\frac{z-1}{z+1}\right) + w_c}$$

$$= \frac{w_c T (z+1)}{(2 + w_c T) z - (2 - w_c T)}$$

$$= 0{,}1357 \left(\frac{z+1}{z - 0{,}7284}\right)$$

$$H[e^{jwT}] = 0{,}1357 \left(\frac{e^{jwT} + 1}{e^{jwT} - 0{,}7284}\right)$$

$$= \frac{0{,}1357 (\cos wT + 1 + j \sin wT)}{\cos(wT) - 0{,}7284 + j \sin wT}$$

$$|H[e^{jwT}]| = \left[\frac{0{,}024[1 + \cos wT)}{(1 - 0{,}9518 \cos wT)}\right]^{0{,}5}$$

$$\measuredangle H[e^{jwT}] = \tan^{-1}\left(\frac{\sin wT}{1 + \cos wT}\right) - \tan^{-1}\left(\frac{\sin wT}{\cos wT - 0{,}7285}\right)$$

EXAMPLE

Design a low pass filter

$$|H[e^{j\omega T}]| = 1 \ (0dB) \quad @ \ DC \quad \omega = 0$$

$$\geq -2 dB \quad 0 \leq \omega \leq 8$$

$$\leq -11 dB \quad \omega \geq 15$$

highest frequency to be processed = 35 rad/s

Use Bilinear Tx an Butterworth filter prototype

$$\omega_p' = 8 ; \quad \omega_s' = 15 ; \quad G_p' = -2dB ; \quad G_s' = -11dB$$

Choose ω_c to satisfy stopband.

$$T \leq \frac{1}{2F_h} = \cancel{\frac{1}{70}} \ \frac{1}{2\left(\frac{35}{2\pi}\right)} = \frac{\pi}{35}$$

$$\omega_p = \frac{2}{T} \tan\left(\frac{\omega_p' T}{2}\right) = \frac{70}{\pi} \tan\left(\frac{4\pi}{35}\right) = 8{,}3623$$

$$\omega_s = \frac{2}{T} \tan\left(\frac{\omega_s' T}{2}\right) = \frac{70}{\pi} \tan\left(\frac{15\pi}{35}\right) = 17{,}7696$$

$$n \geq \frac{\log[(10^{-\frac{G_s'}{10}} - 1)/(10^{-\frac{G_p'}{10}} - 1)]}{2 \log\{\omega_s/\omega_p\}}$$

$$\geq \frac{\log[(10^{1.1} - 1)/(10^{0.2} - 1)]}{2 \log(17{,}7696/8{,}3623)} = 1.9405 \approx 2$$

ω_c to satisfy ~~stob~~ stopband: $\omega_c = \dfrac{\omega_s}{[10^{-G_s/10} - 1]^{1/2n}}$

$$= \frac{17{,}7696}{(10^{1.1} - 1)^{1/4}} = 9{,}6308$$

$$S_k = \cos\left(\frac{\pi}{2n}(2k + n - 1)\right) + j\sin\left(\frac{\pi}{2n}(2k + n - 1)\right)$$

$$s_1 = -0{,}7071 + j\,0{,}7071$$

$$s_2 = -0{,}7071 - j\,0{,}7071$$

$$H_a(s) = \frac{1}{s^2 + 1.4142 s + 1} \qquad s \to \frac{s}{\omega_c}$$

$$H_a(s) = \frac{1}{\left(\frac{s}{9{,}6308}\right)^2 + 1.4142\left(\frac{s}{9{,}6308}\right) + 1}$$

$$= \frac{92{,}7529}{s^2 + 13{,}62 s + 92{,}7529}$$

$$H[z] = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{0{,}1039\,(z+1)^2}{z^2 - 0{,}9045z + 0{,}3201}$$

$z = e^{j\omega t}$

$$H[e^{j\omega T}] = \frac{0{,}1039\,(e^{j\omega T}+1)^2}{e^{j2\omega T} - 0{,}9045\,e^{j\omega T} + 0{,}3201}$$

DISCRETE FIR FILTERS

Always stable + simple to implement
Can be designed to have linear phase.
Windowing technique → simplist method for design

THE WINDOWING TECHNIQUE

Design begins by specifying the ideal zero phase frequency response $H_a(j\omega)$

The corresponding impulse response $h_a(t)$ is centred at $t=0$ and has infinite duration.

The corresponding discrete time filter impulse response $h[n]$ is then made finite and causal (beginning at zero)

The corresponding frequency response is determined from the zero phase frequency response of the specified ideal filter.

1) DETERMINATION OF DISCRETE FILTER IMPULSE RESPONSE

$$\cancel{h[n]} \quad h[n] = T h_a(t)\Big|_{t=nT} = T h_a(nT)$$

$$h_a(t) = \mathcal{L}^{-1}\,H_a(s) = \mathcal{F}^{-1}\{H_a(j\omega)\}$$

$$h_a(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} H_a(j\omega)\,e^{j\omega t}\,d\omega$$

$$\boxed{h[n] = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} H_a(j\omega)\,e^{j\omega nT}\,d\omega}$$

2) WINDOWING

The $h[n]$ we calculated has an infinite duration. So we truncate it w/ a window

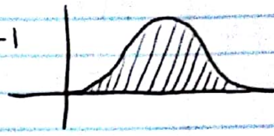
Truncation = pre multiplication of $h[n]$ by window

Some windows

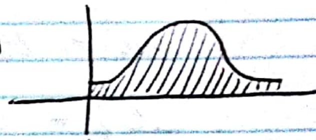
RECTANGULAR $w[n] = \begin{cases} 1 & -M \le n \le M \\ 0 \end{cases}$

BARLETT WINDOW $w[n] = \begin{cases} 2n/M & 0 \le n \le M/2 \\ 2 - 2n/M & M/2 \le n \le M \\ 0 \end{cases}$

HANNING WINDOW $w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos \frac{2\pi n}{N_0 - 1}\right] & 0 \le n \le N_0 - 1 \\ 0 \end{cases}$

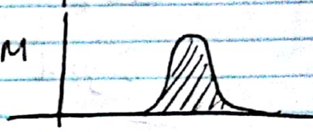


HAMMING WINDOW $w[n] = \begin{cases} 0{,}56 - 0{,}46 \cos \frac{2\pi n}{N_0 - 1} & 0 \le n \le N_0 - 1 \\ 0 \end{cases}$

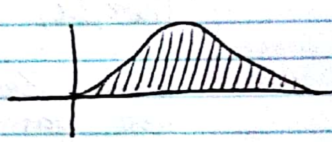


KAISER'S WINDOW $w[n] = \dfrac{I_0\left[\alpha \sqrt{1 - 4\left(\frac{n}{N_0 - 1}\right)^2}\,\right]}{I_0[\alpha \xi]}$ $-M \le n \le M$

WTF



BLACKMAN WINDOW $w[n] = \begin{cases} 0{,}42 + 0{,}5 \cos \frac{2\pi n}{N_0 - 1} + 0{,}08 \cos \frac{4\pi n}{N_0 - 1} \\ 0 \end{cases}$ $-M \le n \le M$



	Main lobe width	Rolloff	Peak sidelobe
Rectangular	$4\pi/N_0$	-6	-13.3
Barlett	$8\pi/N_0$	-12	$-26{,}5$
Hanning	$8\pi/N_0$	-18	$-31{,}5$
Hamming	$8\pi/N_0$	-6	$-42{,}\cancel{5}7$
Kaiser	$12\pi/N_0$	-6	~~-58,1~~ $-59{,}9$
Blackman	$11.2\pi/N_0$	-18	$-58{,}1$

3) DETERMINE FREQUENCY RESPONSE

You should know $h[0], h[1] \ldots h[n]$

$$H[z] = \sum_{k=0}^{\infty} h[k] z^{-k} = h[0] + \frac{h[1]}{z} + \ldots \frac{h[n]}{z^n}$$

$$H[e^{j\omega T}] = H[z]\Big|_{z = e^{j\omega t}}$$

EXAMPLE

Ideal low pass filter w/ $\omega_c = 20kHz$. 6^{th} order NON RECURSIVE using rectangular + hamming windows

$F_H = 40kHz$

$n = 6 \rightarrow \boxed{N_o - 1 = n}$ $\quad N_o = 6 + 1 = 7$

$$M = \frac{N_o - 1}{2} = \frac{7-1}{2} = 3$$

$$T = \frac{1}{2F_H} = 12{,}5 \times 10^{-6}$$

$$\omega_c = 2\pi F_c = 2\pi(20000) = 40\,000\pi = \frac{\pi}{2T}$$

$$\Omega_c = \omega_c T = 40000\pi \times 12{,}5 \times 10^{-6} = \frac{\pi}{2}$$

Digital frequency range is from $-\pi$ to π only

Therefore, filter: $\frac{2\pi}{T}$ on ω scale and 2π on Ω scale

Ideal zero phase frequency response of the ideal low pass filter

$$H_a(\omega) = \text{rect}\left(\frac{\omega}{2\omega_c}\right) \rightarrow \text{rect}\left(\frac{\omega T}{\pi}\right)$$

Corresponding impulse response:

$$h_a(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \text{rect}\left(\frac{\omega T}{\pi}\right) e^{j\omega t}\, d\omega$$

IF YOU ARE GIVEN $H_d[\Omega]$ at start, use IDFFT to get $h_d[n]$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\Omega) e^{jn\Omega}\, d\Omega$$

$$\frac{\omega_c}{\pi}\,\text{sinc}(\omega_c t) \Leftrightarrow \text{rect}\left(\frac{\omega}{2\omega_c}\right)$$

$$h_a(t) = \frac{\omega_c}{\pi}\,\text{sinc}(\omega_c t)$$

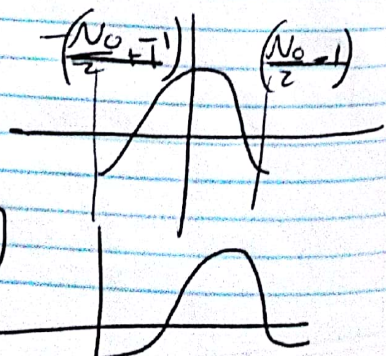
$$= \frac{1}{2T}\,\text{sinc}\left(\frac{\pi t}{2T}\right)$$

$$h[n] = T h_a[nT] = \frac{1}{2}\,\text{sinc}\left(\frac{\pi n}{2}\right)$$

WINDOWING:

Rectangular: $w[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \end{cases}$

delayed by $\frac{N_o - 1}{2}$



$$h_e[n] = h[n-3] = \frac{1}{2}\,\text{sinc}\left(\frac{\pi\{n-3\}}{2}\right)$$

$$H[z] = z^{-3}\left(-\frac{1}{3\pi}z^{3} + \frac{1}{\pi}z^{1} + \frac{1}{2} + \frac{1}{\pi}z^{-1} - \frac{1}{3\pi}z^{-3}\right)$$

~~$H[e^{-j3w}$~~ $$H[e^{j\omega T}] = e^{-j3\omega T}\left[\frac{1}{2} + \frac{2}{\pi}\cos\left(\frac{w}{8000}\right) - \frac{2}{3\pi}\cos\left(\frac{3w}{8000}\right)\right]$$

HAMMING WINDOW

$$w_H[n] = \begin{cases} 0{,}54 + 0{,}46\cos\left(\frac{2\pi n}{N_0-1}\right) & -M \le n \le M \\ 0 \end{cases}$$

$N_0 = 7$, $M = \frac{N_0-1}{2}$ $\quad w_H[n] = 0{,}54 + 0{,}46\cos\left(\frac{2\pi n}{6}\right)$

$-3 \le n \le 3$

$$H[e^{j\omega T}] = e^{-j3\omega T}\left[\frac{1}{2} + 0{,}49\cos\omega T - 0{,}01696\cos 3\omega T\right]$$

GIBBS PHENOMENON

Observing $H[e^{j\omega T}]$, there is a persistant oscilliatory nature which will remain there no matter the filter order, which decays slowly over the stop band.

The Oscillation is called the Gibs Phenomenon

Caused by truncating the impulse response of a causal FIR filter. so ruthlessly with rectangle

SOLUTION Use of a tapper function like a Hamming window. → Eliminates oscilliatory behavior

Cost ~~decrease~~ increase of transition band.

SUMMARY

- → Response to complex signals
- → Response to discrete time sinosoid
- → Response to sampled sinosoid
- → Frequency response of digital filters
- → Types of filters
- → Classifications of filters (Recursive (IIR), Non recursive (FIR))
- → Filter design Methods (IIR + FIR methods)