Digital Filters: Impulse Invariance Design Method for IIR Filter

IIR Filter's Design by Impulse Invariance Method contd

- **Example 1:** Design a lowpass discrete-time filter by applying impulse invariance to an Butterworth continuous-time filter **to satisfy the passband specification**, assuming that the sampling period $T = 1$. The specifications for the discrete-time filter are:

$$0.89125 \le \left|H\left(e^{j\omega T}\right)\right| \le 1 \qquad 0 \le |\omega T| \le 0.2\pi$$

$$\left|H\left(e^{j\omega T}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega T| \le \pi$$

- **Solution:**

$$G_p = 20log10(0.891250) = -1.0$$

$$G_s = 20log10(0.17783) = -14.9999$$

$$\Omega_p = \omega_p' = \omega_p T = 0.2\pi, \qquad \Omega_s = \omega_s' = \omega_s T = 0.3\pi$$

The Butterworth filter's order, n, is obtained using (12.12)

$$n \ge \frac{\log_{10}\left[\left(10^{-G_s/10} - 1\right)/\left(10^{-G_p/10} - 1\right)\right]}{2\log_{10}\left(\Omega_s/\Omega_p\right)} \ge \frac{\log_{10}[(10^{1.49999} - 1)/(10^{0.1} - 1)]}{2\log_{10}\left(\frac{0.3\pi}{0.2\pi}\right)} \ge 5.8858 = 6$$

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❑ IIR Filter's Design by Impulse Invariance Method contd

➢ Solution contd:

The cut off frequency, ω_c is obtained using (12.13a) that satisfies the pass band specification as :

$$\omega_c = \frac{\omega_p'}{\left[10^{-G_p/10} - 1\right]^{1/2n}} = \frac{0.2\pi}{(10^{0.1} - 1)^{1/12}} = 0.7032$$

The poles s_k for **normalized** Butterworth filter are obtained using (12.14):

$$s_k = cos\left(\frac{\pi(2k+n-1)}{2n}\right) + jsin\left(\frac{\pi(2k+n-1)}{2n}\right), \qquad k = 1, 2, 3, \ldots, n \ldots (12.14)$$

$$s_1 = cos\left(\frac{\pi(2+6-1)}{12}\right) + jsin\left(\frac{\pi(2+6-1)}{12}\right) = -0.2588 + j0.9659$$

$$s_2 = cos\left(\frac{\pi(4+6-1)}{12}\right) + jsin\left(\frac{\pi(4+6-1)}{12}\right) = -0.7071 + j0.7071$$

$$s_3 = cos\left(\frac{\pi(6+6-1)}{12}\right) + jsin\left(\frac{\pi(6+6-1)}{12}\right) = -0.9659 + j0.2588$$

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❑ IIR Filter's Design by Impulse Invariance Method contd

➢ **Solution contd:**

$$s_4 = cos\left(\frac{\pi(8+6-1)}{12}\right) + jsin\left(\frac{\pi(8+6-1)}{12}\right) = -0.9659 \; - j\;0.2588$$

$$s_5 = cos\left(\frac{\pi(10+6-1)}{12}\right) + jsin\left(\frac{\pi(10+6-1)}{12}\right) = -0.7071 \; - j0.7071$$

$$s_6 = cos\left(\frac{\pi(12+6-1)}{12}\right) + jsin\left(\frac{\pi(12+6-1)}{12}\right) = -0.2588 \; - j0.9659$$

$$\mathcal{H}_a(s) = \prod_{k=0}^{N-1} \frac{1}{s - s_k} = \frac{1}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}$$

$$\mathcal{H}_a(s) = \frac{1}{(s+0.2588 \; - j0.9659)(s+0.2588+j0.9659)(s+0.7071 \; - j0.7071)(s+0.7071+j0.7071)(s+0.9659 \; - j0.2588)(s+0.9659+j0.2588)}$$

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❑ IIR Filter's Design by Impulse Invariance Method contd

➢ **Solution contd:**

$$\mathcal{H}_a(s) = \frac{1}{(s^2 + 0.51765s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9318s + 1)}$$

➢ $H_a(s)$ is obtained by replacing s with s/ω_c in $\mathcal{H}_a(s)$:

$$H_a(s) = \mathcal{H}_a(s)|_{s=\frac{s}{0.7032}}$$

$$= \frac{1}{\left((\frac{s}{0.7032})^2+0.51765(\frac{s}{0.7032}) + 1\right)\left((\frac{s}{0.7032})^2+1.4142(\frac{s}{0.7032}) + 1\right)\left((\frac{s}{0.7032})^2+1.9318(\frac{s}{0.7032}) + 1\right)}$$

$$H_a(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

The partial fraction expansion for $H_a(s)$ could be written using (12.28) as:

$$H_a(s) = \sum_{k=1}^{6} \frac{c_k}{s - p_k}, \qquad where \qquad c_k = H_a(s)(s - p_k)|_{s=p_k}$$

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❑ IIR Filter's Design by Impulse Invariance Method contd

➢ Solution contd:

Using inverse Laplace, we have:

$$h_a(t) = L^{-1}[H_a(s)] = L^{-1}\left[\sum_{k=1}^{6} \frac{c_k}{s - p_k}\right] = \sum_{k=1}^{6} L^{-1}\left[\frac{c_k}{s - p_k}\right] = \sum_{k=1}^{6} c_k e^{p_k t}$$

Hence, with $T = 1$,

$$h[n] = T h_a(nT) = \sum_{k=1}^{6} c_k e^{p_k n}, \qquad n = 1, 2, 3, \ldots$$

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❑ IIR Filter's Design by Impulse Invariance Method contd

➢ **Solution contd:**

❖ $H[z]$ is obtained by taking the z-transform of $h[n]$ as:

$$H[z] = \mathcal{Z}\{h[n]\} = \sum_{k=1}^{6} \frac{c_k z}{z - e^{p_k}} = \sum_{k=1}^{6} \frac{c_k}{1 - e^{p_k} z^{-1}}, \qquad where \quad c_k = H_a(s)(s - p_k)|_{s=p_k}$$

A bit of algebra, including using the summation formula for geometric series, yields the final results given as:

$$H[z] = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.2455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}}$$

$$+ \frac{1.8577 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}$$