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NOTES : BILINEAR TRANSFORM

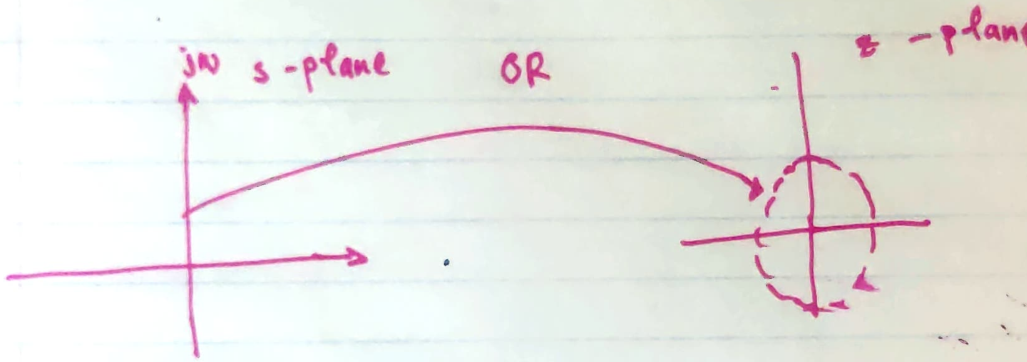
→ Algebraic transformation between s & z

s ↓ Laplace

z ↓ z-plane

→ we want to map discrete frequencies to continuous frequency i.e

$$-\infty < \omega < -\infty \longleftrightarrow -\pi < \Omega < \pi$$



Defn : Bilinear transformation :

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$$

$$H_c(s) \xrightarrow{\text{Bilinear}} H_d(z) = H_c\left(\frac{2}{T_s}\frac{(z-1)}{(z+1)}\right)$$

$H_c(s)$ 2→ continuous-time

Bilinear Transform

Behaviour of Bilinear transformation

→ lets z the subject of the formula;

$$z = \frac{1 + \left(\frac{T_s}{2}\right)s}{1 - \left(\frac{T_s}{2}\right)s}$$

→ substitute $s = \sigma + j\omega$

$$z = \frac{1 + \sigma\left(\frac{T_s}{2}\right) + j\omega T_s/2}{1 - \sigma T_s/2 - j\omega T_s/2}$$

for $\sigma < 0$; $|z| < 1$ { left hand plane folded into interior of unit circle

it follows [for $\sigma > 0$; $|z| > 1$]

this means that stable continuous tf ⟷ stable discrete tf.

next let $s = j\omega$:

$$z = \frac{1 + j\omega \frac{T_s}{2}}{1 - j\omega \frac{T_s}{2}}$$

$|z| = 1$

i.e $j\omega$ gets mapped to the unit circle

BILINEAR TRANSFORMATION

now consider the bilinear transform for $z = e^{j\Omega}$

$$s = \frac{2}{T_d}\left(\frac{e^{j\Omega} - 1}{e^{j\omega} + 1}\right)$$

$$= \frac{2}{T_d} \cdot \frac{e^{j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}{e^{j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)}$$

replace $\omega \leftrightarrow \Omega$

recall $s\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2j}$ $\quad c\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{2}{T_s} \cdot \frac{\cancel{e^{j\Omega/2}} \cdot 2j \sin(\Omega/2)}{\cancel{e^{j\Omega/2}} \cdot 2\cos(\Omega/2)}$$

$s = \sigma + j\omega$ $\quad = \dfrac{2}{T_s}\, j\tan\left(\dfrac{\Omega}{2}\right)$

$\Rightarrow \sigma = 0$ $\quad \omega = \dfrac{2}{T_s}\tan\left(\dfrac{\Omega}{2}\right)$

compare to $\omega = \dfrac{\Omega}{T_s}$ what we expect

$$\Omega = 2\arctan\left(\frac{\omega T_s}{2}\right)$$

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FILTER DESIGN using BILINEAR TRANS

$$\boxed{\begin{array}{c}\text{continuous time specs}\\ \omega_p, \omega_s, \delta_p, \delta_s \ldots\\ \text{etc}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Design continuous filter}\\ H(s)\end{array}}$$

$$\Big\downarrow \; \frac{2}{T_s}\cdot\left(\frac{z-1}{z+1}\right)$$

$$\boxed{\text{IIR filter}}$$

(a) If specs are given in discrete time.

Ω_s , Ω_p → discrete

$\Big\downarrow$ → prewarping $\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right)$

ω_s' , ω_p'

(b) If specs are given in continuous time

ω_s , ω_p → continuous

$\Big\downarrow$ @ sampling period T_s

$\Omega_s = \omega_s T_s ; \; \Omega_p = \omega_p T_s \xrightarrow{\text{prewarp}} \omega_p' ; \omega_s'$

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Bilinear Transformation

Example

Design an IIR filter that meet the following

$0,89125 \leq |H(z)| \leq 1 \qquad 0 \leq \Omega \leq 0,2\pi$

$|H(z)| \leq 0,17783 \qquad 0,3\pi \leq \Omega \leq \pi$

Convert to continuous specifications

$0,89125 \leq |H(s)| \leq 1 \qquad 0 \leq w \leq \frac{2}{T_s}\tan\left(\frac{0,2\pi}{2}\right)$

$|H(s)| \leq 0,17783 \qquad \frac{2}{T_s}\tan\left(\frac{0,3\pi}{2}\right) \leq w \leq \infty$

without prewarping

$0 \leq w \leq \frac{0,2\pi}{T_s}$

$\frac{0,3\pi}{T_s} \leq w \leq \infty$

from we can proceed as normal

→ gives an $N = 6$ Butterworth filter with $w_c = 0,766$

[need an example with continuous time spec]

[compare bilinear to impulse invariance