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Z-transform

Motivation

- consider a sys with impulse response $h[n]$
→ let $x[n] = z^n$ i.e. complex exponential

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$\Rightarrow \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}\right)}_{H(z)} \cdot z^n$$

Definition

The z-transform of a discrete time function $f[n]$ is

$$F(z) = \mathcal{Z}\{f[n]\} = \sum_{n=-\infty}^{\infty} f[n]\, z^{-n}$$

sequence ⟷ complex valued function

one sided (unilateral) z-transform

$$F(z) = Z\{f[n]\} = \sum_{n=0}^{\infty} f[n] z^{-n}$$

useful for causal signals

The inverse z-transform is given by

$$f[n] = \frac{1}{2\pi j} \oint F[z]\, z^{n-1}\, dz$$

contour integration around a closed path in the complex z-plane.

Region of convergence

→ Region in the z-plane where $F(z)$ "z-transform" is defined.

Example

① $x[n] = \alpha^n u[n]$:

$$X(z) = \sum_{n=0}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

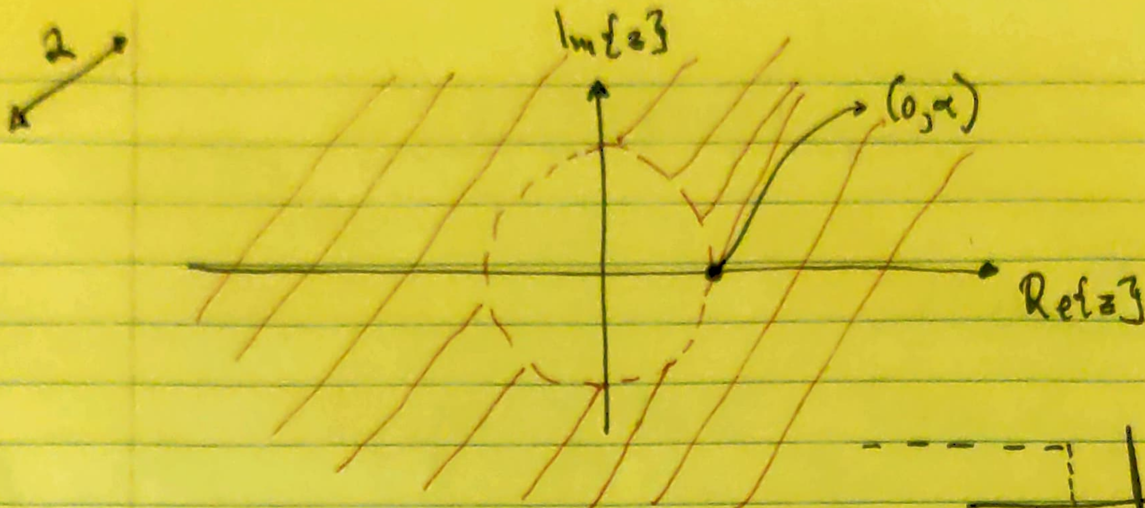
Geometric series

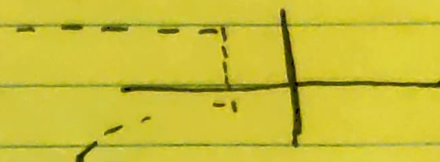
$r = \alpha z^{-1}$

$\Rightarrow X(z) = \dfrac{1}{1 - \alpha z^{-1}}$ defined over $|\alpha z^{-1}| < 1$

$|z$

R.O.C → $|z| > |\alpha|$





② $\cancel{x[n] = -\alpha^k u[-k}$ $x[n] = -\alpha^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$

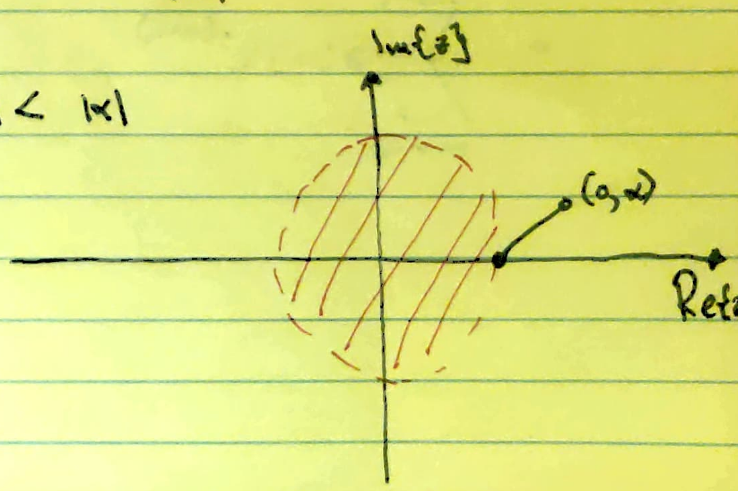
$$= -\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n = \cancel{-\sum_{n=1}^{\infty} (\alpha z^{-1})^n}$$

$$= \cancel{\frac{\alpha z^{-1}}{1 - \alpha z^{-1}}} \qquad m = -n \Rightarrow -\sum_{m=1}^{\infty} (\alpha z^{-1})^{-m}$$

R.O.C $\Rightarrow$ $\cancel{|\alpha z^{-1}|}$ $\qquad \Rightarrow \sum_{m=1}^{\infty} (\alpha^{-1} z)^m$

$$|\alpha^{-1} z| < 1 \Rightarrow \frac{|z|}{|\alpha|} < 1$$

R.O.C $\Rightarrow$ $|z| < |\alpha|$



PROPERTIES

Linearity

if $x_1[n] \xleftrightarrow{Z} X_1(z)$ ROC = R_1

$x_2[n] \xrightarrow{Z} X_2(z)$ ROC = R_2

then $a x_1[n] + b x_2[n] \xrightarrow{Z} a X_1(z) + b X_2(z)$

R.O.C $R_1 \cap R_2$

Time Shifting

$x[n] \xleftrightarrow{Z} X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\cancel{Z\{x[n]\}} \; Z\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n}$$

let $m = n - n_0 \Rightarrow n = m + n_0$

$$\Rightarrow \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)} = \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) z^{-n_0}$$

$$z^{-n_0} X(z)$$

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Frequency Shifting

$x[n] \longleftrightarrow X(z)$

$e^{j\Omega_0 n} x[n] \longleftrightarrow X(e^{-j\Omega_0} z)$

Time Reversal

$x[n] \longleftrightarrow X(z) \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$x[-n] \xleftrightarrow{Z} Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$

let $m = -n$: $\sum_{m=-\infty}^{\infty} x[m] \cdot z^{m} \cdot \left(\cancel{\frac{z^{-m}}{z^{-m}}}\right)$

$= \sum_{m=-\infty}^{\infty} x[m] \cdot (z^{-1})^{m} = X(z^{-1})$

Convolution Property

$x_1[n] \xleftrightarrow{Z} X_1(z)$

$x_1[n] * x_2[n] \longleftrightarrow X_1(z) X_2(z)$

$x_2[n] \xleftrightarrow{Z} X_2(z)$

Differentiation Property

$x[n] \xleftrightarrow{Z} X(z)$

$n\, x[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z \to \infty} F(z)$$

Final Value Theorem

$$\lim_{n \to \infty} f[n] = \lim_{z \to 1} (z-1) F(z)$$

INVERSE Z - TRANSFORM

* Contour integration will not use

① Using tables

→ requires familiarity with z-transform pairs & properties of the z-transform.

Example ①

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$

from tables we know :

$$x[n] = \alpha^n u[n] \xleftrightarrow{z} \frac{z}{z-\alpha}$$

using linearity

$$X(z) = \mathcal{Z}\{(\tfrac{1}{2})^n u[n]\} + \mathcal{Z}\{(\tfrac{1}{3})^n u[n]\}$$

$$= \frac{z}{z - 1/2} + \frac{z}{z - \frac{1}{3}}$$

Example 2

$x[n] = n a^n u[n]$

from Tables we have $x[n] = a^n u[n] \longleftrightarrow \dfrac{z}{z-a}$

from differentiation property

$$n x[n] \longleftrightarrow \frac{-z\, d X(z)}{dz}$$

$$\therefore X(z) = -z \frac{d}{dz}\left(\frac{z}{z-a}\right) = \frac{az}{(z-a)^2}$$

POWER SERIES EXPANSION METHOD

① Example :

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

$$\begin{array}{r l}
 & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} \\
z - a & \overline{)\, z} \\
 & \underline{z - a} \\
 & -a \\
 & \underline{a - a^2z^{-1}} \\
 & a^2z^{-1} \\
 & \underline{a^2z^{-1} - a^3z^{-2}} \\
 & a^3z^{-2}
\end{array}$$

$$X(z) = \underset{x[0]}{\underset{\uparrow}{1}} + \underset{x[1]}{\underset{\uparrow}{az^{-1}}} + \underset{x[2]}{\underset{\uparrow}{a^2z^{-2}}} + \underset{x[3]}{\underset{\uparrow}{a^3z^{-3}}}$$

we can also see a pattern here

$x[n] = a^n u[n]$

(b) $X(z) = \log(1 + az^{-1})$

recall Taylor series expansion

$$\log(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{w^n}{n}$$

$$X(z) = \cancel{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x[n] = \begin{cases} \frac{(-1)^{n+1} a^n}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

PARTIAL FRACTION METHOD

→ use this for rational $X[z]$

i.e. $X(z) = \frac{N(z)}{D(z)}$

$$= k \cdot \frac{(z-z_1) \cdots (z-z_m)}{(z-p_1) \cdots (z-p_n)}$$

zeros

poles

(a) $n \geq m$ all poles p_k are simple (no repeated poles)

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \dots + \frac{c_n}{z-p_n}$$

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$$X(z) = c_0 + \frac{c_1 z}{z - p_1} + \cdots + \frac{c_n z}{z - p_n}$$

use tables

$$\alpha^n u[n] = \frac{z}{z-\alpha}$$

~~$c_0 \delta_0[n] + c_1 \ldots$~~

$$x[z] = c_0 \delta_0[n] + c_1 \cdot p_1^n \cdot u[n] + \cdots + c_n \cdot p_n^n \cdot u[n]$$

(b) · If $m > n$

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum c_k \frac{z}{z - p_k}$$

Example

$$X(z) = \frac{2z^3 - 5z^2 + z + 3}{(z-1)(z-2)}$$

→ long division

$$\begin{array}{r} 2z + 1 \\ z^2 - 3z + 2 \,\big)\, 2z^3 - 5z^2 + z + 3 \\ \underline{2z^3 - 6z^2 + 4z} \\ z^2 - 3z + 3 \\ \underline{z^2 - 3z + 2} \\ 1 \end{array}$$

$$X(z) = 2z + 1 + \frac{1}{(z-1)(z-2)} \rightarrow X^*(z)$$

$$= \cancel{2z+1} +$$

$$\frac{X^*(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2} \cdot \frac{1}{z-2}$$

$$X^*(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{1}{2}\frac{z}{z-2}$$

$$X(z) = 2z + \frac{3}{2} - \frac{z}{z-1} + \frac{1}{2}\frac{z}{z-2}$$

$$= 2\delta[n+1] + \frac{3}{2}\delta[n] - u[n] + \frac{1}{2}(2)^n u[n]$$

C Repeated poles

If p_i has multiplicity r; we have the following associated term.

$$\frac{\lambda_r}{z-p_i} + \frac{\lambda_{r-1}}{(z-p_i)^2} + \dots + \frac{\lambda_1}{(z-p_i)^r}$$

where
$$\lambda_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[((z-p_i))^r \frac{X(z)}{z}\right]\Bigg|_{z=p_i}$$

Example:

$$X(z) = \frac{z}{z(z-1)(z-2)^2}$$

$$\frac{X(z)}{z} = \frac{1}{z(z-1)(z-2)^2} = \cancel{\frac{A}{z} + \dots}$$

$$= \frac{A}{c} + \frac{B}{z-1} + \frac{\lambda_2}{z-2} + \frac{\lambda_1}{(z-2)^2}$$

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Heaviside formula for residues

$$\frac{N(z)}{D(z)} = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

$$= \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \cdots + \frac{k_n}{z-p_n}$$

First order poles (i.e no repeating poles)

$$k_r = \left[(z-p_r)\cdot\frac{N(z)}{D(z)}\right]_{z=p_r}$$

Higher order poles.

$$\frac{N(z)}{D(z)} = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_{r-1})(z-p_r)^q}$$

p_r is a repeated root of multiplicity q

$$\frac{N(z)}{D(z)} = \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \cdots + \frac{k_{r-1}}{z-p_{r-1}} + \frac{k_{r,1}}{z-p_r} + \frac{k_{r,2}}{(z-p_r)^2}$$

$$+ \cdots + \frac{k_{r,q}}{(z-p_r)^q}$$

$k_1 \longleftrightarrow k_{r-1}$ & $k_{r,q}$ are calculated as above for $k_{r,m}$ where $1 \leq m \leq (q-1)$

we have

$$\cancel{k_{r,m} = \left[(z - p_r)^m \cdot N(z) \right]}$$

$$k_{r,m} = \frac{1}{(q-m)!} \left[\frac{d^{q-m}}{dz^{q-m}} (z - p_r)^q \cdot \frac{N(z)}{D(z)} \right]_{z = p_r}$$

Example repeated root

$$X(z) = \frac{z^2}{(z-1)(z-0{,}5)^2}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0{,}5)^2}$$

$$= \frac{k_1}{z-1} + \frac{k_{2,1}}{z-0{,}5} + \frac{k_{2,2}}{(z-0{,}5)^2}$$

$k_1 = 4$

$k_{2,2} = -1$

$$k_{2,1} = \frac{1}{(2-1)!} \left[\frac{d}{dz} \frac{z}{z-1} \right]_{z = 1/2}$$

$$= \left[(z-1)^{-1} + (-1) z (z-1)^{-2} \right]_{z = 1/2} = -4$$

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$$\frac{X(z)}{z} = \frac{4}{z-1} - \frac{4}{z-0.5} - \frac{1}{(z-0{,}5)^2}$$

$$X(z) = \frac{4z}{z-1} - \frac{4z}{z-0{,}5} - \frac{z}{(z-0{,}5)^2}$$

$$x[n] = 4u[n] - 4\left(\tfrac{1}{2}\right)^n u[n] - 2n\left(\tfrac{1}{2}\right)^n u[n]$$
$$= 4 - 4(0.5)^n -$$

$$\left\{ \begin{array}{l} \alpha^n u[n] \longleftrightarrow \dfrac{1}{1-\alpha z^{-1}} \\ n\alpha^n u[n] \longleftrightarrow \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \quad \dfrac{\alpha z}{(z-\alpha)^2} \end{array} \right.$$

Example: Complex Poles

$$X(z) = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$$

$$\frac{X(z)}{z} = \frac{2(3z+17)}{(z-1)\underbrace{(z^2-6z+25)}_{\text{complex roots}}} = \frac{c_0}{z-1} + \frac{c_1 z + c_2}{z^2-6z+25}$$

c_0 by cover-up method: $c_0 = 2$

$$6z + 34 = 2(z^2-6z+25) + (z-1)(c_1 z + c_2)$$
$$= 2z^2 - 12z + 50 + c_1 z^2 + c_2 z - c_1 z - c_2$$
$$= z^2(2+c_1) + z(c_2 - c_1 - 12) + (50 - c_2)$$

comparing co-efficients $2 + c_1 = 0 \Rightarrow c_1 = -2$

$50 - c_2 = 34 \Rightarrow c_2 = 16$

$\left\{ \begin{array}{l} 16+2-12 \\ = 6 \end{array} \right.$

$$\frac{X(z)}{z} = \frac{2}{z-1} + \frac{16-2z}{z^2-6z+25}$$

$$X(z) = \frac{2z}{z-1} + \frac{z(16-2z)}{z^2-6z+25}$$

$$\frac{2z}{z-1} \rightarrow 2u[n] \qquad \frac{z(16-2z)}{z^2-6z+25} \rightarrow \text{from tables}$$

{ from tables