A Symmetrical Spinning Top

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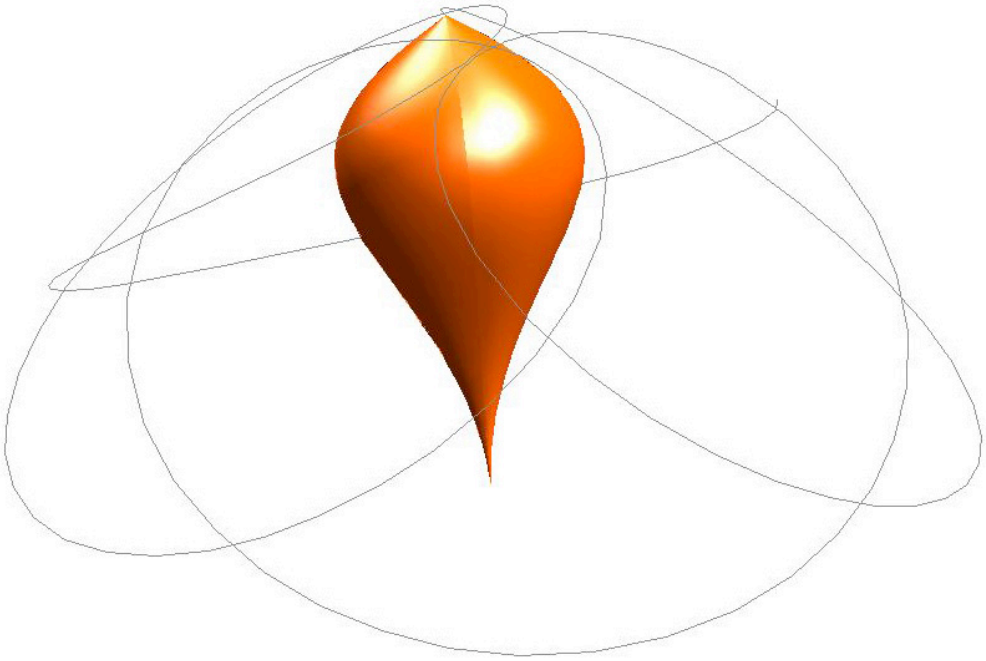


1. Assignments

1.1 Write down explicitly the four first order equations derived from the Hamiltonian of the symmetrical spinning top and determine what are the initial conditions that need to be specified.

1.2 Implement an algorithm that integrates the resulting system of equations using the 4th order Runge – Kutta method. Test the algorithm using $I_1 = I_2 = 3/4, I_3 = 3/10$

1.3 Try to find the initial conditions that lead to different motions, such as nutation, precession and plan spinning.

1.4 Find a couple of instances that leads to time periodic variables.

1.5 Apply a periodic set of kicks to the top by increasing gravity instantaneous by a certain amount. Can you get the top to entrain to the periodic kicks? Are there any chaotic solutions?

2. Theory

In case the motion of the top separates into rotation and translation, which is, by the way, always possible if we choose the center of the mass as a fixed point for the instantaneous axis of rotation, the kinetic energy T is determined by the quadratic form $T = \frac{1}{2}\langle \omega, I\,\omega \rangle$ where **I** is the linear operator for the moment of inertia and ω is the vector of rotation axis. The linear endomorphism **I** can be expressed as follows

$$I = \sum_{body} m_k \left\{ \langle r_k, r_k \rangle - \underbrace{r_k \langle r_k, . \rangle}_{dyadic product} \right\}$$

The problem is to figure out the kinetic energy since an appropriate base system is to be chosen. To keep it easy, we decide for the so-called body system, fixed in the body, in which the matrix elements of **I** are time independent and the matrix is diagonal. The symmetric nature of **I** allows us to choose such a base, consisting of the orthonormalized eigenvectors of **I**. The fixed body system evolves continuously from the space system (has the same origin) described by the rotation matrix R. Time evolution of the body system is described by the simple linear equation: $E' = R(t) \cdot E$, where E is the unit matrix or the space system. Our remaining task is to represent R in very easy manner. Using the eulerian angles is a very useful way of describing the rotation as a product of three easy rotation operators instead of trying to express R by the angular velocity vector. Euler's idea was that general rotation can be considered as consisting of three successive rotations: $R = R_z(\phi) R_x(\theta) R_z(\psi)$. The theory of infinitesimal rotations provides us with a constructive operator differential equation that allows us to express the components of ω in the body system:

$\frac{d}{dt}R(t) = \Omega \cdot R(t)$ time evolution of R

$$\Omega \cdot R(t) = \frac{d}{dt}\left\{R_z(\phi)R_x(\theta)R_z(\psi)\right\} = A + B + C$$

$$A = \dot{\phi} \cdot \partial_\phi R_z(\phi)R_x(\theta)R_z(\psi)$$

$$B = \dot{\theta} \cdot R_z(\phi)\partial_\theta R_x(\theta)R_z(\psi)$$

$$C = \dot{\psi} \cdot R_z(\phi)R_x(\theta)\partial_\psi R_z(\psi)$$

$$\Omega = (A + B + C) \cdot R^T$$

The question is what is Ω? This tiny linear operator is strongly associated with ω, the components of ω in the space system can be found as certain matrix elements of the skew symmetric operator Ω, $[\omega]_{space} = (\Omega_{32}, \Omega_{13}, \Omega_{21})^T$

Now we can express ω in body coordinates $[\omega]_{body} = R \cdot [\omega]_{space}$

After performing all this transformations we get

$$[\omega]_{body} = \left(\cos(\phi)\dot{\theta} + \sin(\theta)\sin(\phi)\dot{\psi}, \sin(\phi)\dot{\theta} - \cos(\phi)\sin(\theta)\dot{\psi}, \dot{\phi} + \cos(\theta)\dot{\psi}\right)$$

Thus, the kinetic energy can be written as

$$T = \frac{1}{2}\left(I_1{\psi_1}^2 + I_2{\psi_2}^2 + I_3{\psi_3}^2\right)$$

If we assume that $I_1 = I_2$ and plug in the body coordinates of ω we get

$$T = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos(\theta)\right)^2$$

In a constant gravitational field the potential energy V is given by the formula

$V = -\sum_{body} m_k \langle r_k, g\rangle = -M\langle R, g\rangle$ where R is the center of the mass.

In terms of the eulerian angles we get for the Lagrange function of the symmetric spinning top the following expression

$$L = T - V = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos(\theta)\right)^2 - Mgl\cos(\theta)$$

l is the distance between the center of gravity and the origin.

Since we have powerful tools to get numerical solution of 1st order differential equation systems it is advisable to examine the motion behavior of the top by means of the Hamilton formalism. The transformation from the Langrangian to Hamiltonian formalism corresponds to changing the variables from

$$(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi},t)\xrightarrow[Lendre]{}(\phi,\theta,\psi,p_\phi,p_\theta,p_\psi,t)$$

The procedure for switching variables in such a manner is provided by the Legendre transformation.

$$H(q,p,t)=\sum_i q_i\cdot p_i - L(q,\dot{q}(q,p,t),t)$$
$$p_j=\partial_{\dot{q}_j}L$$

If we apply the described transformation above we get for our system

$$H=\frac{{p_\theta}^2}{2I_1}+\frac{{p_\psi}^2}{2I_3}+\frac{\left(p_\phi-p_\psi\cos(\theta)\right)^2}{2I_1\sin^2(\theta)}+Mgl\cos(\theta)$$

The equations of motion derived from H are

$$\dot{\theta}=\partial_{p_\theta}H=\frac{p_\theta}{I_1}$$
$$\dot{\phi}=\partial_{p_\phi}H=\frac{p_\phi-p_\psi\cos(\theta)}{I_1\sin^2(\theta)}$$
$$\dot{\psi}=\partial_{p_\psi}H=\frac{p_\psi}{I_3}-\frac{p_\phi-p_\psi\cos(\theta)}{I_1\sin^2(\theta)}\cos(\theta)$$
$$\dot{p}_\theta=-\partial_\theta H=glM\sin(\theta)-\frac{p_\phi-p_\psi\cos(\theta)}{I_1\sin(\theta)}\left(p_\psi-\frac{p_\phi-p_\psi\cos(\theta)}{\sin(\theta)}\cot(\theta)\right)$$
$$\underbrace{p_\phi=p_\phi(0)}$$
$$\underbrace{p_\psi=p_\psi(0)}_{1^{st}\text{order system, we need 6 initial conditions for the canonical momenta and the angles}}$$

3. Implementation

It is obvious that it would be very hard to find a analytical solution for the motion problem therefore we will apply a 4th order Runge – Kutta method to find at least a numerical answer to the problem. The basic idea of all single step methods to which the Runge – Kutta algorithm belongs is to estimate the solution iteratively obeying the following pattern:

$$y_{n+1} = y_n + \phi\left(h, t_n, y_n; f\right)$$

solves $\dot{y} = f(t, y)$ and $y(0) = y_0$

The 4th order Runge – Kutta uses the following function for ϕ

$$\phi = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$k_1 = h \cdot f(x_n, y_n)$$
$$k_2 = h \cdot f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$
$$k_3 = h \cdot f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$$
$$k_4 = h \cdot f(x_n + h, y_n + k_3)$$

As the name of this method suggests the global error is $O\left(h^4\right)$, we can therefore expect that our numerical solution will be very close to the analytical solution of the problem, and if the solution smooth enough we can choose larger step sizes.

To make the method more efficient we will change the step size during the calculation. The following algorithm can be implemented to keep the error within a given accuracy and to accelerate the evaluation in case we can increase the step size.

Given: a, b, y_0

start step size $0 < h \le b - a$

tolerance ε

method function ϕ of order p (in our case $p = 4$)

securityfactor $\rho = 0.9$

increasing factor q

maximum step size $h_{\max}$

while $t_j < b$

$h = \min\{h, b - t_j\}$

$H = h$

$u = u_j + \frac{H}{2}\phi\left(t_j, u_j, H/2\right)$

$\underbrace{u = u + \frac{H}{2}\phi\left(t_j + H/2, u, H/2\right)}_{\text{estimate with step size H/2}}$

$\underbrace{v = u_j + H\phi\left(t_j, u_j, H\right)}_{\text{estimate with H}}$

$error_j = \frac{1}{2^p - 1}\|u - v\|$

if $error_j < \varepsilon$

$t_{j+1} = t_j + H$

$u_{j+1} = u$

$j = j + 1$

endif

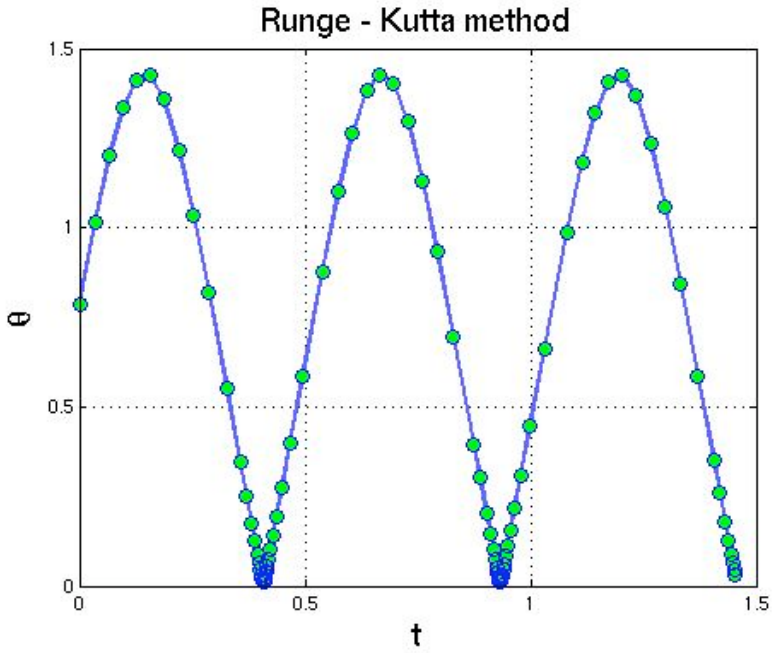
$h = \min\left\{qH, \rho H \sqrt[p+1]{\varepsilon / error_j}, h_{\max}\right\}$

end while

A very important question remains: How good is my numerical solution? Fortunately it can be proved that the energy of the system is conserved, since the Hamiltonian is time independent. Hence, this value can be regarded as a global error. One could use this fact to develop a step size controlling routine but this is apparently very sophisticated since one is supposed to figure out a new step size. Only doubling and halving slow down the algorithm in many simple cases.

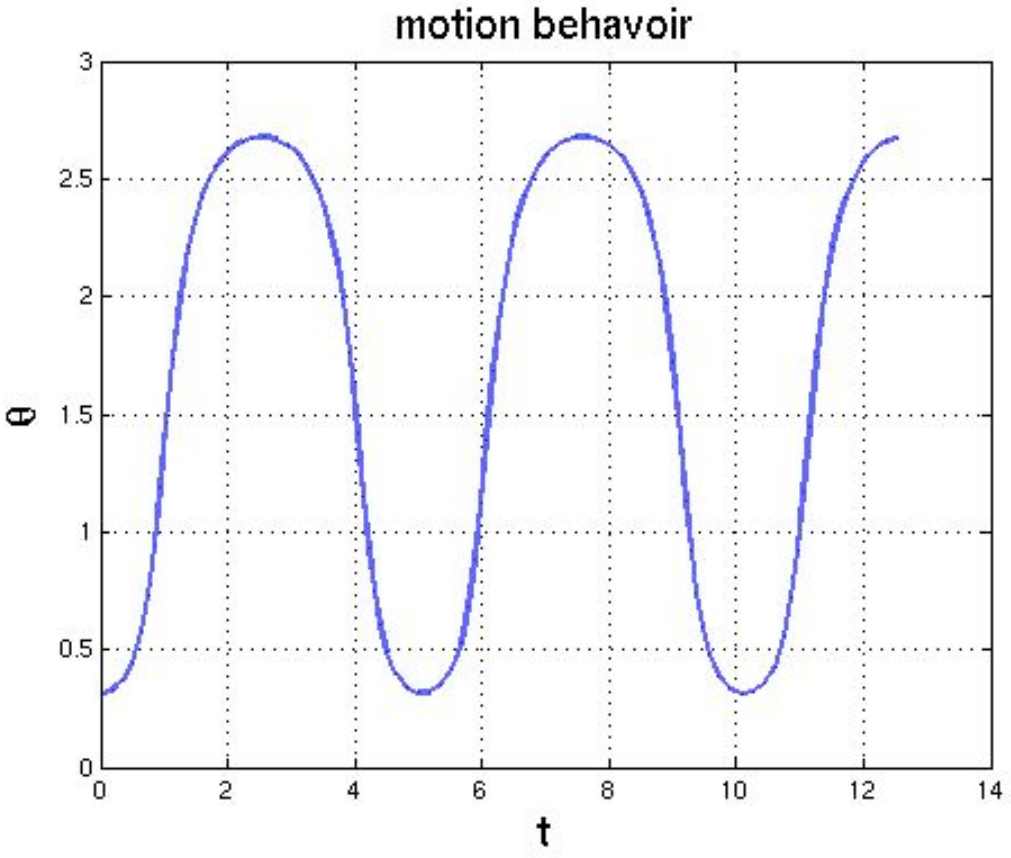
4. Results

4.1 Step size changing



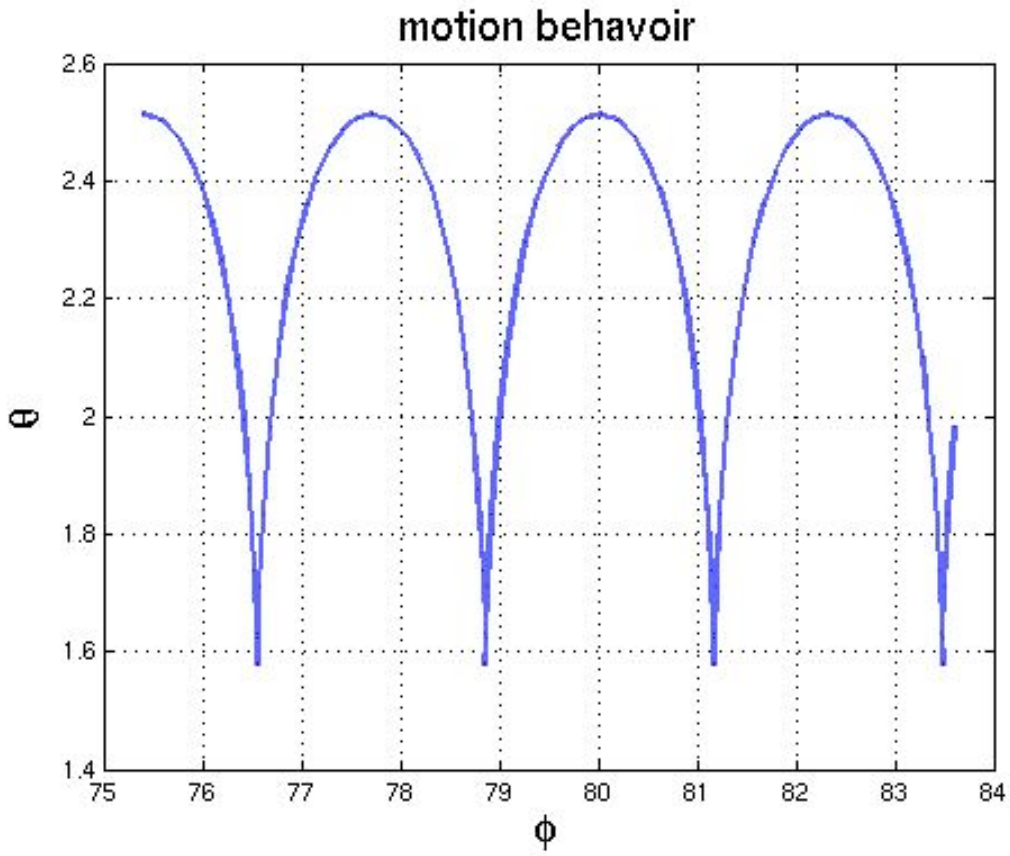
In the picture above I have figure out the solution with adapting the step size to the error, if I would try to use a Runge-Kutta method without step size controlling but with the same amount of points using equal step size the procedure would blow up. One can see the more rapidly the solution changes the smaller the step size (lower edges of the graph above).

4.2 Initial conditions



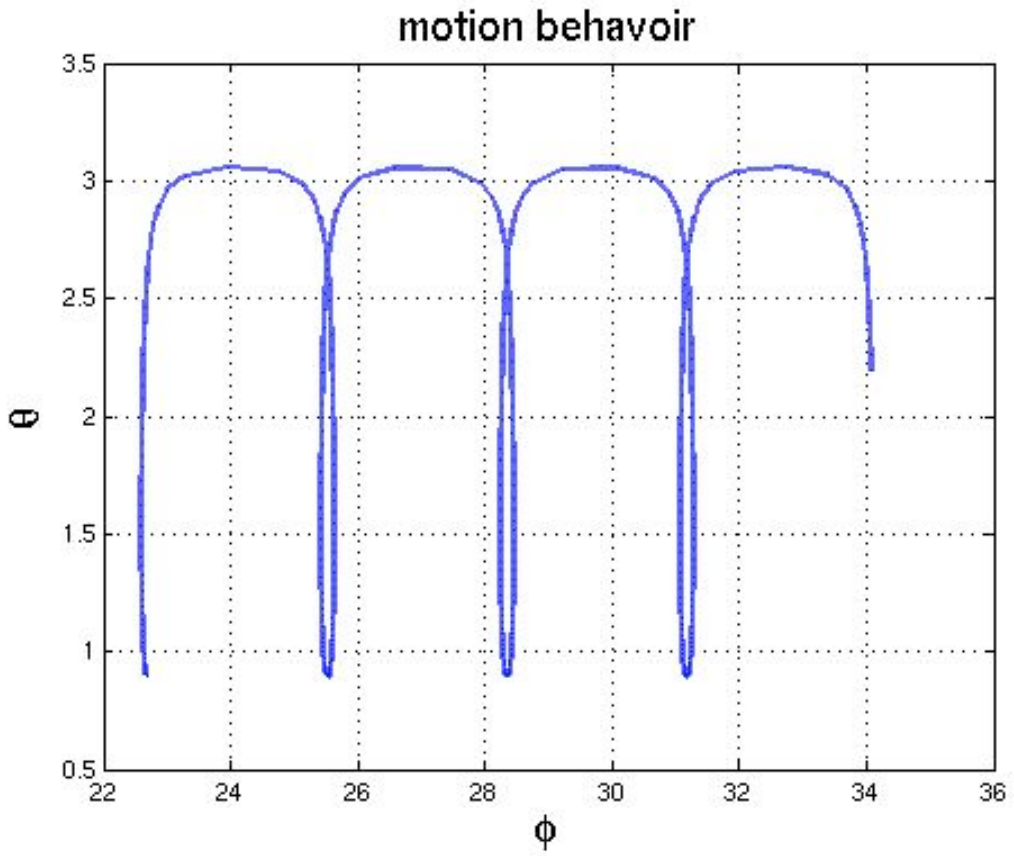
$M = 1.05, g = 9.81, I_1 = 0.15, I_3 = 5.10,$

$\theta_0 = \pi/10, \phi_0 = 0, \psi_0 = 0, p_{\theta 0} = 0, p_{\phi 0} = 5.30, p_{\psi 0} = 4.80$



$M = 1.15, g = 9.81, I_1 = 0.23, I_3 = 5.10,$

$\theta_0 = 0.8 \cdot \pi, \phi_0 = 24 \cdot \pi, \psi_0 = 30 \cdot \pi, p_{\theta 0} = 0, p_{\phi 0} = 0.0, p_{\psi 0} = 13.3$



$M = 1.65, g = 9.81, I_1 = 0.75, I_3 = 5.10,$

$\theta_0 = 0.5 \cdot \pi, \phi_0 = 0, \psi_0 = 0, p_{\theta 0} = 35, p_{\phi 0} = 0.2, p_{\psi 0} = 4.4$

4.3 Closed Orbits

Before we start to find closed orbits we have to define what does it mean that an orbit is closed. When we trace the cusp of the top, which can easily described by polar coordinates, we can visualize the motion in a $\phi-\theta-$ diagram like the graphs above. Every 2π-step on the ϕ- Axes represents a full rotation of the cusp in the spatial coordinate system. Suppose the Hamilton system provides us with a periodic function θ in ϕ it is persuasive that if the period say T is a rational multiple of π, the cusp comes to same point after a special number of rotations. It is essential that T can be written as $T = q \cdot \pi$ where $q \in \mathbb{Q}$, so we can figure out after how many full revolutions the motion will repeat: $\#\,revolutions = \min\left\{ n \in \mathbb{N} \,\middle|\, \frac{2\pi n}{T} \in \mathbb{N} \right\}$, otherwise there is no minimum since in case q is irrational the set is empty.
Whenever we get a numeric solution that seems to be periodic (as fare we can determine it), the period T can be written in the required form since computer numbers are never irrational. Apparently, following this pure mathematical approach it doesn't make sense to ask for closed orbits. In practice, it is sensible to require that in case of a periodic solution the number of revolutions that are necessary to pass the same point is adequate small.

The following parameters give almost a closed orbit, but it depends how accurate the solution is. There are also same trivial cases: plan spinning and when there is no motion.

$$M = 1.65, g = 9.81, I_1 = 0.75, I_3 = 5.10,$$
$$\theta_0 = 0.5 \cdot \pi, \phi_0 = 0, \psi_0 = 0, p_{\theta 0} = 0, p_{\phi 0} = 0, p_{\psi 0} = 258.6$$