10 + 11
use rectangles

4.2 Area

Ex1: a) $\sum_{i=1}^{6} i = 1+2+3+4+5+6$

b) $\sum_{i=0}^{5} (i+1) = (0+1)+(1+1)+(2+1)+(3+1)+(4+1)+(5+1)$

$= 1 + 2 + 3 + 4 + 5 + 6$

c) $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d) $\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$

e) $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \frac{1}{n}(3^2+1) + \dots + \frac{1}{n}(n^2+1)$

f) $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$

Ex2:

$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(i+1) = \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$

$$= \frac{1}{n^2} \cdot \left(\frac{n(n+1)}{2} + n\right)$$

$$= \frac{1}{n^2}\left(\frac{n^2+n}{2} + \frac{2n}{2}\right) = \frac{1}{n^2}\left(\frac{n^2+3n}{2}\right)$$

$$= \frac{n^2+3n}{2n^2} = \frac{n+3}{2n}$$

a) $n=10$: $\frac{10+3}{2(10)} = \frac{13}{20}$

b) $n=100$: $\frac{100+3}{2(100)} = \frac{103}{200}$

c) $n=1000$: $\frac{1000+3}{2(1000)} = \frac{1003}{2000}$

Ex3: $f(x) = -x^2+5$, $n=5$

a) lower sum

$f(\frac{2}{5}) = 4.84$ $\quad$ $f(\frac{6}{5}) = 3.56$ $\quad$ $f(\frac{10}{5}) = 1$

$f(\frac{4}{5}) = 4.36$ $\quad$ $f(\frac{8}{5}) = 2.44$

$$\frac{2}{5}(4.84 + 4.36 + 3.56 + 2.44 + 1) = 6.48$$

b) upper sum

$f(0) = 5$ $\quad$ $f(\frac{2}{5}), f(\frac{4}{5}), f(\frac{6}{5}), f(\frac{8}{5})$

$$\frac{2}{5}(5 + 4.84 + 4.36 + 3.56 + 2.44) = 8.08$$

- lower sum:

$$\sum_{i=1}^{5} \overbrace{f(\tfrac{2}{5}i)}^{\text{height}} \cdot \overbrace{(\tfrac{2}{5})}^{\text{width}} = \sum_{i=1}^{5}\left(-(\tfrac{2}{5}i)^2+5\right)(\tfrac{2}{5})$$

$$= \sum_{i=1}^{5}\left(\frac{-4}{25}i^2+5\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left(\frac{-8}{125}i^2+2\right)$$

$$= \frac{-8}{125}\sum_{i=1}^{5} i^2 + \sum_{i=1}^{5} 2 = \frac{-8}{125} \cdot \frac{5(5+1)(2\cdot5+1)}{6} + 2\cdot5$$

$$= \frac{-8}{125} \cdot \frac{330}{6} + 10 = \frac{-2640}{750} + 10 = 6.48$$

- upper sum:

$$\sum_{i=1}^{5} f(\tfrac{2}{5}(i-1))(\tfrac{2}{5}) = \frac{2}{5} \cdot \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2+5\right] = \frac{2}{5}\sum_{i=1}^{5}\left[-\left(\frac{4i^2-8i+4}{25}\right)+5\right]$$

$$= \frac{2}{5}\sum_{i=1}^{5}\left(\frac{-4i^2+8i-4}{25}\right) + \sum_{i=1}^{5} \frac{2}{\cancel{5}}\cancel{5}$$

$$= \frac{2}{125}\left[-4\sum_{i=1}^{5} i^2 + 8\sum_{i=1}^{5} i - \sum_{i=1}^{5} 4\right] + 10$$

$$= \frac{2}{125}\left[-4 \cdot \frac{5(6)(11)}{6} + 8 \cdot \frac{5(6)}{2} - 4\cdot5\right] + 10$$

$$= \frac{2}{125}\left[-220 + 120 - 20\right] + 10$$

$$= -1.92 + 10 = 8.08$$

Ex: $f(x) = x^3 \quad [0,1]$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \frac{1}{n}$$

if the interval has width 1 and is divided into n subintervals, each subinterval has width $\frac{1}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right)$$

right endpoint: $c_i = \frac{i}{n} = \frac{1}{n}\cdot i$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\cdot\frac{n^2(n^2+2n+1)}{4} = \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{4n^4}$$

$$= \frac{1}{4}\cdot \lim_{n\to\infty}\left(\frac{n^4}{n^4}+\frac{2n^3}{n^4}+\frac{n^2}{n^4}\right) = \frac{1}{4}\cdot(1+0+0) = \frac{1}{4}$$

Ex: $f(x) = 4 - x^2 \ ; [1,2] \qquad \Delta x = \frac{2-1}{n} = \frac{1}{n} \qquad c_i = a + i\Delta x = 1 + \frac{i}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n}\left(4-\left(1+\frac{i}{n}\right)^2\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(4-1-\frac{2i}{n}-\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(3-\frac{2i}{n}-\frac{i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}3 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\cdot 3n - \frac{2}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[3 - \frac{2n^2+2n}{2n^2} - \cancel{n(2n^2+3n+1)}\ \frac{2n^3+3n^2+n}{6n^3}\right]$$

$$= 3 - \frac{2}{2} - \frac{2}{6}$$

$$= 3 - 1 - \frac{1}{3} = 2 - \frac{1}{3} = \frac{6}{3} - \frac{1}{3} = \frac{5}{3}$$