APMA 1720: Computational Project 2024

(Due 11:59pm, Saturday, May 4)

Rules and submission: Your answers and MATLAB codes must be submitted before the due time. Late submissions will not be accepted.

1. Please submit your answers, MATLAB codes and the output from MATLAB codes [just like what you have been doing for homework] via Gradescope.

2. Please complete these problems independently. You can discuss them with me or the TAs, but not with other students.

1. Let X and Y be the prices for two underlying stocks. Assume that they are both geometric Brownian motions under the risk neutral probability:

$$\begin{aligned} X_T &= X_0 \exp\left\{\left(r - \frac{1}{2}\sigma_1^2\right)T + \sigma_1\sqrt{T}Z_1\right\} \\ Y_T &= Y_0 \exp\left\{\left(r - \frac{1}{2}\sigma_2^2\right)T + \sigma_2\sqrt{T}Z_2\right\}. \end{aligned}$$

Here (Z_1, Z_2) is assumed to be a jointly normal random vector with distribution

$$N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right).$$

The goal is to simulate the price of a basket call option with maturity T and payoff

$$(c_1 X_T + c_2 Y_T - K)^+.$$

We would like to compare the plain Monte Carlo with the method of conditioning. The latter method uses the tower property

$$\begin{aligned} \text{Price} &= E[e^{-rT}(c_1 X_T + c_2 Y_T - K)^+] \\ &= E[E[e^{-rT}(c_1 X_T + c_2 Y_T - K)^+ | Z_2]]. \end{aligned}$$

(a) Given $Z_2 = z$, what is the conditional distribution of Z_1? No need for any derivation. All you need to do is to state the result.

(b) Given $Z_2 = z$, what is the price of the option? [*Hint:* Be careful that $c_2 Y_T - K$ may be nonnegative.]

(c) Write a MATLAB function to estimate the basket call price and compare plain Monte Carlo with the method of conditioning. The function should have input parameters r, σ_1, σ_2, ρ, X_0, Y_0, T, K, c_1, c_2, and sample size n. Report your estimates and their standard errors for

$$r = 0.1,\ \sigma_1 = 0.2,\ \sigma_2 = 0.3,\ X_0 = 50,\ Y_0 = 50,\ T = 1,\ K = 55,$$

$$\rho = 0.7,\ c_1 = 0.5,\ c_2 = 0.5$$

with sample size $n = 10000$.

2. Consider a stochastic volatility model for stock price [here the volatility is denoted by θ]:

$$\begin{aligned} dS_t &= rS_t\,dt + \theta_t S_t\,dW_t \\ d\theta_t &= a(\Theta - \theta_t)\,dt + \beta\,dB_t. \end{aligned}$$

Here r, a, Θ, β are all positive constants, and (W, B) is a two-dimensional Brownian motion with covariance matrix

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

In this model the volatility θ is itself a (mean-reverting) stochastic process. We are interested in estimating the price of a vertical spread with maturity T. That is,

$$v = e^{-rT}E[(S_T - K_1)^+ - (S_T - K_2)^+], \qquad 0 < K_1 < K_2.$$

By Itô formula, if we define $Y_t = \log S_t$, then

$$dY_t = \left(r - \frac{1}{2}\theta_t^2\right)\,dt + \theta_t\,dW_t.$$

We would like to compare the results from the following two schemes.

(a) Estimate v by straightforward Euler scheme on (Y_t, θ_t);

(b) Estimate v by straightforward Euler scheme on (Y_t, θ_t) combined with control variates.

Please read the following text for some general discussion on how to construct control variables.

> Suppose that the log-stock price process $Y_t = \log S_t$ satisfies the SDE
>
> $$dY_t = \left(r - \frac{1}{2}\theta_t^2\right)\,dt + \theta_t\,dW_t$$
>
> where θ_t is the volatility of the stock price. This volatility θ can be very general: a constant, a deterministic function, a function of S_t, or itself a stochastic process. Suppose we are interested in estimating the price of, say, a call option with strike K and maturity T [that is, the payoff is $(S_T - K)^+ = (\exp(Y_T) - K)^+$]. The Euler scheme on Y_t will be
>
> $$\begin{aligned} \hat{Y}_{t_{i+1}} &= \hat{Y}_{t_i} + \left(r - \frac{1}{2}\hat{\theta}_{t_i}^2\right)(t_{i+1} - t_i) + \hat{\theta}_{t_i}\sqrt{t_{i+1} - t_i}\,Z_{i+1}; \\ \hat{Y}_{t_0} &= Y_0. \end{aligned}$$

Now introduce an *artificial* stochastic process $\bar{Y}$ by

$$\begin{aligned} \bar{Y}_{t_{i+1}} &= \bar{Y}_{t_i} + \left(r - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}; \\ \bar{Y}_{t_0} &= Y_0. \end{aligned}$$

This artificial process uses the *same* sequence $\{Z_i\}$, and a *constant* volatility σ [σ is usually chosen as a typical value of θ]. One can think of $\bar{Y}$ as the logarithm of the price of a virtual stock that follows a classical geometric Brownian motion with drift r and volatility σ. Then we can use the discounted payoff of the call option for this virtual stock, namely,

$$e^{-rT}(\exp(\bar{Y}_T) - K)^+$$

as the control variable. The expected value of this control variable is just the classical Black-Scholes call price with parameter S_0, r, σ, K, T.

In the simulation, divide the time interval $[0, T]$ into m equal-length subintervals. That is,

$$t_i = \frac{i}{m}T, \quad i = 0, 1, 2, \ldots, m.$$

Write a MATLAB function that compares schemes (a) and (b). The function should have input parameters r, a, Θ, β, S_0, θ_0, K_1, K_2, T, m, ρ, and sample size n. Report your estimates and standard errors for

$$r = 0.1,\ a = 3,\ \Theta = 0.2,\ \beta = 0.1,\ S_0 = 20,\ \theta_0 = 0.25,\ T = 1,$$

$$K_1 = 20,\ K_2 = 22,\ \rho = 0.5,\ m = 50,\ n = 10000.$$

Please also indicate which value of σ you have used for the artificial process $\bar{Y}$ and which control variable you have used. Explain your motivation for doing so briefly.

Remark 0.1. For the control variate method, just set $b^* = 1$ for simplicity.