Permutations and words

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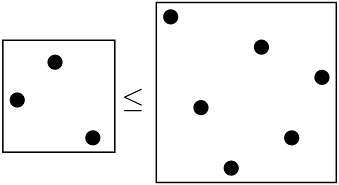
Involvement of permutations

$231 \leq 631524$
$231 \not\leq 54132$



An infinite poset $\mathcal{S}$.

Well founded: no infinite descending chains.

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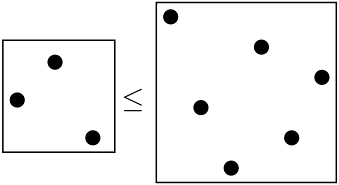

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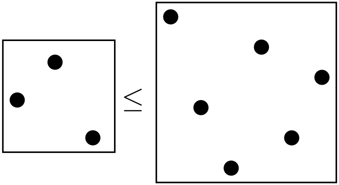

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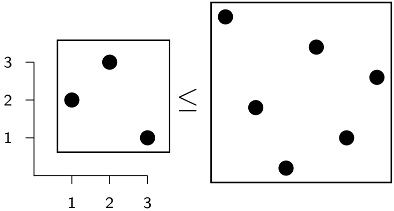

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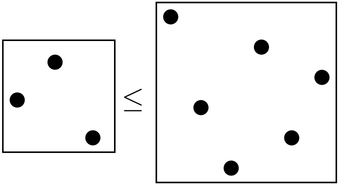

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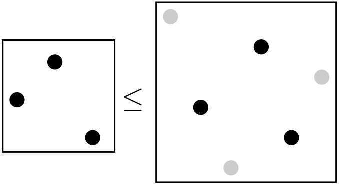

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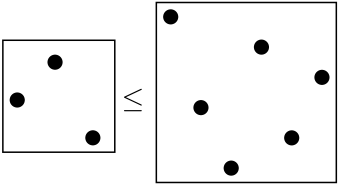

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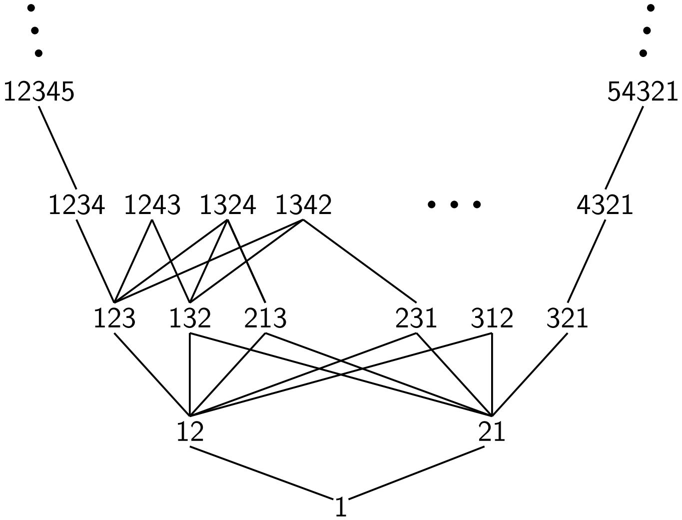
Poset $\mathcal{S}$



Pattern classes

Pattern class of permutations $\mathcal{C}$: a downward closed set (ideal) under $\leq$; i.e. $\sigma \leq \tau \in \mathcal{C} \Rightarrow \sigma \in \mathcal{C}$.

Basis $B = B(\mathcal{C})$: the minimal permutations not in $\mathcal{C}$.

$\mathcal{C} = \{\sigma \,:\, (\forall \beta \in B)(\beta \not\leq \sigma)\} = \mathrm{Av}(B)$.

Bases = antichains in $\mathcal{S}$.

Not partially well ordered = infinite antichains = infinitely based pattern classes.

Context: sorting mechanisms (stacks, queues); mathematical biology; combinatorics of relational structures.

Theory of Pattern Classes

What can be asked of a pattern class $\mathcal{C}$?

- Enumeration: $c_n = |\mathcal{C}_n| =$ the number of permutations of length n in $\mathcal{C}$.
- Generating function $\sum_{n=1}^{\infty} c_n x^n$: rational, algebraic, holonomic?
- Basis: finite, infinite, size?
- Order theoretic properties: antichains, pwo, join property,...
- Computational problems: membership, complexity,...

Sample results

Theorem (folklore)

For any $\mathcal{C} = Av(\pi)$, $|\pi| = 3$, *we have*

$$|\mathcal{C}_n| = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem (Regev 1981; Gessel 1990)

If $\mathcal{C} = Av(12\dots r)$ *then*

$$|\mathcal{C}_n| \sim (r-1)^n.$$

Theorem (Bona 1997)

The generating function for $\mathcal{C} = Av(1342)$ *is*

$$\frac{32x}{-x^2 + 20x + 1 - (1-8x)^{3/2}}.$$

Wilf–Stanley Conjecture (= Markus–Tardos Theorem)

Theorem

For every pattern class $\mathcal{C} \neq \mathcal{S}$ there exists $q \in \mathbb{R}$ such that $|\mathcal{C}_n| \leq q^n$.

Conjecture

$\sqrt[n]{|\mathcal{C}_n|}$ has a limit as $n \to \infty$.

Permutations and words

Do permutations, subject to pattern avoidance restrictions, behave like words in some sense?

Words

A – an alphabet; A^* – words over A.

Subword ordering: $abac \leq cacbabca$, $abac \not\leq bacbca$.

[Word = sequence; subword = subsequence.]

Downward closed set: $u \leq v \in W \Rightarrow u \in W$.

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Higman's Theorem

Theorem (Higman)

A^* is PWO under $\leq$ (no infinite antichains).

Corollary

Every downward closed set has a finite basis.

Corollary

Every downward closed set W can be expressed as

$$W = A^* \setminus \bigcup_{i=1}^{n} A^* a_{i,1} A^* a_{i,2} A^* \dots A^* a_{i,l_i} A^*.$$

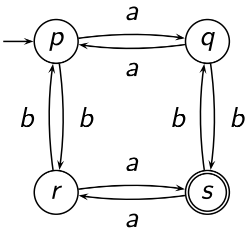
Corollary

Every downward closed set is regular.

Regular languages

Theorem (Kleene)

A language is regular iff it is accepted by a finite state automaton.



$$
\begin{aligned}
p_n &= q_{n-1} + r_{n-1} \\
q_n &= p_{n-1} + s_{n-1} \\
r_n &= p_{n-1} + s_{n-1} \\
s_n &= r_{n-1} + q_{n-1}
\end{aligned}
$$

Corollary

Every regular set has a rational generating function, which can be effectively computed.

Back to permutations

Can we encode permutations by words, while preserving the nice properties of the subword ordering?

[Murphy] took the biscuits carefully out of the packet and laid them face upward on the grass, in order as he felt of edibility. They were the same as always, a Ginger, an Osborne, a Digestive, a Petit Beurre and one anonymous. He always ate the first-named last, because he liked it the best, and the anonymous first, because he thought it very likely the least palatable. The order in which he ate the remaining three was indifferent to him and varied irregularly from day to day. On his knees now before the 5 it struck him for the first time that this reduced to a paltry six the number of ways in which he could make his meal. (...) Even if he conquered his prejudice against the anonymous, still there would be only twenty-four ways in which the biscuits could be eaten. But were he to take the final step and overcome his infatuation with the ginger, then the assortment would spring to life before him, dancing the radiant measure of its total permutability, edible in a hundred and twenty ways!

(*S. Beckett*)

Rank encoding

Albert, Atkinson, NR 2003.

Replace every entry by the number of smaller entries after it $+1$.

Example: $\pi = 2451637$, $\rho(\pi) = 2331211$.

In general: need an infinite alphabet. But for some classes finite alphabet suffices.

$\Omega_k = \{\sigma \in \mathcal{S} \,:\, \rho(\sigma) \in [k]^*\}$ – a pattern class!

Example

$\mathcal{C} = \mathrm{Av}(\{321, 312\})$. For every entry there is at most one smaller entry to the right of it; $\mathcal{C} = \Omega_2$.

Example

Permutations generated by a system with finite memory (a graph).

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Involvement and rank encoding

Difficulty: Subpermutations don't correspond to subwords.

$\rho(2451637) = 2331211$, $\rho(234156) = 222111$.

Hence: Pattern classes not encoded by downward closed sets of words.

Fortunately, limited damage:

Proposition

The set

$$\{(\rho(\sigma), \rho(\tau)) \,:\, \sigma, \tau \in \Omega_k, \sigma \leq \tau\}$$

is recognised by a transducer (finite state translator).

Rank encoding : results

Theorem

Let $\mathcal{C} \subseteq \Omega_k$, and let B be its basis. Then $\rho(\mathcal{C})$ is regular iff $\rho(B)$ is regular. In particular all finitely based subclasses of Ω_k are regular.

Corollary

If $\mathcal{C}$ is a finitely based (or indeed regular) subclass of Ω_k then:

(i) *the generating function of $\mathcal{C}$ is rational;*

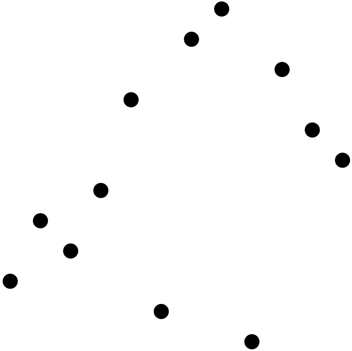
(ii) *it can be effectively computed from the basis of $\mathcal{C}$;*

(iii) *the membership problem in $\mathcal{C}$ is decidable in linear time.*

Follow-on development: Insertion encoding; Albert, Linton, NR, 2005.

Gridding a permutation

$$3, 5, 4, 6, 9, 2, 11, 12, 1, 10, 8, 7$$



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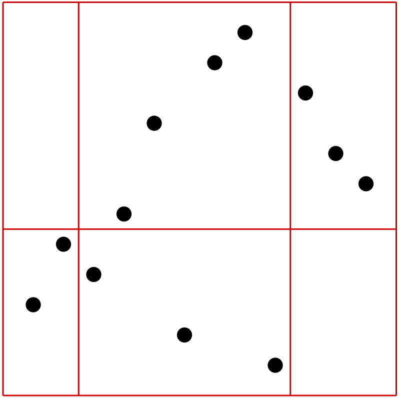

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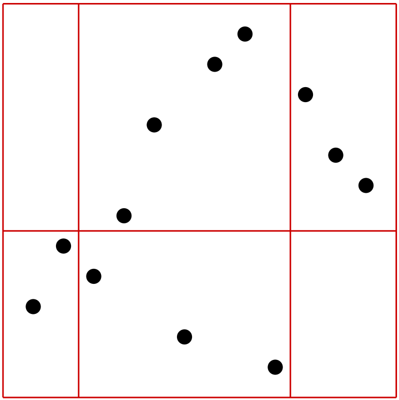

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Geometric grid classes

Albert, Atkinson, Bouvel, NR, Vatter, to appear.

	1	−1
1	−1	

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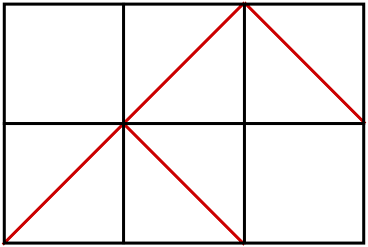

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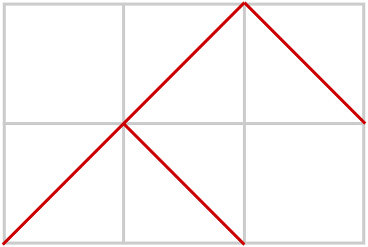

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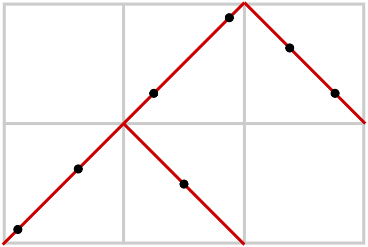

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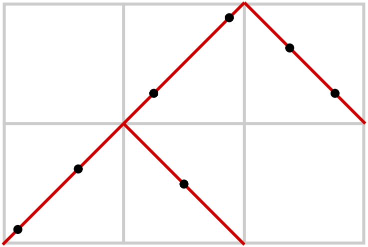

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Geometric grid classes: examples

Example

(1 1) defines the class of juxtapositions of two increasing permutations.

Example (Atkinson 1999)

$\mathrm{Av}(321, 2143)$ is the union of ggc classes of (1 1) and $(1\ 1)^T$.

Example (Murphy 2003)

$\mathrm{Av}(132, 4312)$ is the ggc class of

$$\mathrm{Av}(132, 4312) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Natural encoding

Natural encoding: properties

- maps words to permutations;
- order preserving (subwords $\mapsto$ subpermutations);
- onto (every word has a code);
- finite to one;
- obstacles to $1-1$: non-uniqueness of griddings; independent cells;
- can be resolved without leaving regular languages.

Geometric grid classes: results

Theorem

The following hold for every geometric grid class $\mathcal{C}$:

(i) *$\mathcal{C}$ can be encoded by a regular language in a bijective and order-preserving manner;*

(ii) *$\mathcal{C}$ is finitely based;*

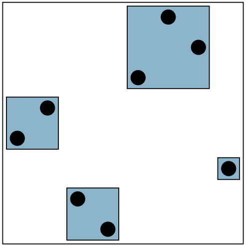
(iii) *$\mathcal{C}$ is partially well ordered;*

(iv) *$\mathcal{C}$ has a rational generating function.*

Theorem

Every subclass of a geometric grid class is a finite union of geometric grid classes and has all the above properties.

Extension: inflations

$3142[12,21,132,1] =$  $= 45216873$

$$\mathcal{C}[\mathcal{D}] = \{\sigma[\delta_1, \dots, \delta_m] \,:\, \sigma \in \mathcal{C},\ |\sigma| = m,\ \delta_i \in \mathcal{D}\}.$$

$\mathcal{C}$ strongly rational: all subclasses have rational general functions (Albert, Atkinson, Vatter, 2012).

Theorem (Albert, NR, Vatter)

If $\mathcal{C}$ is a geometric grid class, and $\mathcal{D}$ is strongly rational, then $\mathcal{C}[\mathcal{D}]$ is strongly rational.

Application: small classes

Theorem (Vatter 2011)

The unique real root $\kappa \approx 2.20557$ of $x^3 - 2x^2 - 1$ is the largest real number such that there are only countably many pattern classes with growth rate $< \kappa$ (small classes).

For each small class $\mathcal{C}$ there exists a geometric grid class $\mathcal{G}$ and $k > 0$ such that

$$\mathcal{C} \subseteq \underbrace{\mathcal{G}[\mathcal{G}[\ldots\mathcal{G}[\mathcal{G}]\ldots]]}_{k}$$

Theorem (Albert, NR, Vatter)

Every small pattern class has a rational generating function.

Thank you