



## 9. A Baseline HANK Model

Adv. Macro: Heterogenous Agent Models

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Jeppe Druedahl & Patrick Moran

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# Introduction

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  1. Linearized IRF
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- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
  1. Based on the [GEModelTools](#) package
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- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools

## HANK model

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- **Central bank:** Set nominal interest rate

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- **Static** problem for representative final good firm:

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- **Note:** Zero profits (can be used to derive price index)

# Derivation of demand curve

- FOC wrt.  $y_{jt}$

$$0 = P_t \mu \left( \int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu}-1} - p_{jt} \Leftrightarrow$$

$$\frac{p_{jt}}{P_t} = \left( \int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow$$

$$\left( \frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu-1}} = \left( \int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow$$

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- **Dynamic problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$
$$\text{s.t. } y_{jt} = Z_t n_{jt}, \quad y_{jt} = \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$
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- **NKPC** derived from FOC wrt.  $p_{jt}$  and envelope condition:

$$\log(1 + \pi_t) = \kappa \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied dividends:**  $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

# Derivation of NKPC

- FOC wrt.  $p_{jt}$ :

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{Z_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}} \\ - \frac{\mu}{\mu - 1} \frac{1}{\kappa} \frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$



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- FOC + Envelope + Symmetry +  $\pi_t = P_t/P_{t-1} - 1$

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- **Household problem:** Distribution,  $\mathbf{D}_t$ , over  $z_t$  and  $a_{t-1}$

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}[v_{t+1}(z_{t+1}, a_t) | z_t, a_t]$$

$$\text{s.t. } a_t = (1 + r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

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- **Effective labor-supply:**  $n_t = z_t \ell_t$

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t [v_{a,t}(z_t, a_{t-1})] = \mathbb{E} [(1 + r_t)c_t^{-\sigma}]$$



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- **Endogenous grid method:** Vary  $z_t$  and  $a_t$  to find

$$c_t = (\beta \underline{v}_{a,t+1}(z_t, a_t))^{-\frac{1}{\sigma}}$$

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- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_t, a_{t-1}) \text{ and } \ell^*(z_t, a_{t-1}) \text{ with } m_t = (1+r_t)a_{t-1} + (w_t \ell_t^* - \tau_t + d_t)z_t$$

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- **Savings:**  $a^*(z_t, a_{t-1}) = (1 + r_t)a_{t-1} - c_t^* + (w_t \ell_t^* - \tau_t + d_t)z_t$

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3. Return to step 1



- **Monetary policy:** Follow Taylor-rule:

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where  $i_t^*$  is a shock

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where  $i_t^*$  is a shock

- **Fisher relationship:**

$$r_t = (1 + i_{t-1}) / (1 + \pi_t) - 1$$

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- **Government:** Choose  $\tau_t$  to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

# Market clearing

1. Labor:  $N_t = \int n_t^*(z_t, a_{t-1}) d\mathbf{D}_t$  (in effective units)
2. Assets:  $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t$
3. Goods:  $Y_t = \int c_t^*(z_t, a_{t-1}) d\mathbf{D}_t + G_t + \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

## As an equation system

$$\begin{aligned} H(\pi, w, Y, i^*, Z, \underline{D}_0) &= 0 \\ \left[ \begin{array}{c} \log(1 + \pi_t) - \left[ \kappa \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}} \right] \\ N_t - \int n_t^*(z_t, a_{t-1}) d\mathbf{D}_t \\ B_{ss} - \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t \end{array} \right] &= 0 \end{aligned}$$

The rest of the model is given by

$$\mathbf{X} = M(\pi, w, Y, i^*, Z)$$

# As a DAG (from Auclert et al., 2021)

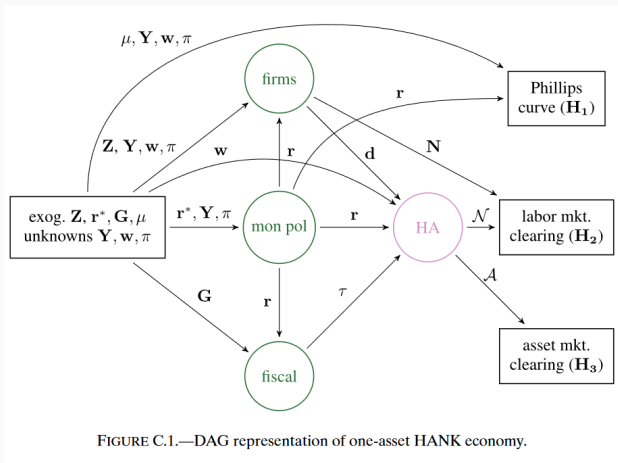


FIGURE C.1.—DAG representation of one-asset HANK economy.

**Notation:**  $i^* = r^*$ ,  $\mu$  is a shock,  $A^{hh} = \mathcal{A}$ ,  $N^{hh} = \mathcal{N}$

# Steady state

- **Chosen:**  $B_{ss}$ ,  $G_{ss}$ ,  $r_{ss}$
- **Analytically:**
  1. **Normalization:**  $Z_{ss} = N_{ss} = 1$
  2. **Zero-inflation:**  $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^* = (1 + r_{ss})(1 + \pi_{ss}) - 1$
  3. **Firms:**  $Y_{ss} = Z_{ss}N_{ss}$ ,  $w_{ss} = \frac{Z_{ss}}{\mu}$  and  $d_{ss} = Y_{ss} - w_{ss}N_{ss}$
  4. **Government:**  $\tau_{ss} = r_{ss}B_{ss} + G_{ss}$
  5. **Assets:**  $A_{ss} = B_{ss}$
- **Numerically:** Choose  $\beta$  and  $\varphi$  to get market clearing

# The HANK example from GEModelToolsNotebooks I

- **Presentation:** I go through the code for finding the transition path
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
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## IRFs and simulation

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- **Previously:** Full non-linear transition path to an MIT-shock

# Linearized IRFs

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- **Today:** Just consider the first order solution

# Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution
  1. Solve for IRFs for unknowns

$$H(\mathbf{U}, \mathbf{Z}) = 0 \Rightarrow \mathbf{H}_U d\mathbf{U} + \mathbf{H}_Z d\mathbf{Z} = 0 \Leftrightarrow d\mathbf{U} = \underbrace{-\mathbf{H}_U^{-1} \mathbf{H}_Z}_{\equiv \mathbf{G}_U} d\mathbf{Z}$$

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2. Derive all other IRFs for

$$\begin{aligned} \mathbf{X} = \mathbf{M}(\mathbf{U}, \mathbf{Z}) &\Rightarrow d\mathbf{X} = \mathbf{M}_U d\mathbf{U} + \mathbf{M}_Z d\mathbf{Z} \\ &= \underbrace{(-\mathbf{M}_U \mathbf{H}_U^{-1} \mathbf{H}_Z + \mathbf{M}_Z)}_{\equiv \mathbf{G}} d\mathbf{Z} \end{aligned}$$

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  1. Imprecise for *large* shocks

# Linearized IRFs

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- **Limitations:**
  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities*  
(direct in aggregate equations or through micro-behavior)



## Basic linearized simulation

- **Shocks:** Write the shocks as an  $MA(\infty)$  with coefficients  $d\mathbf{Z}_s$  for  $s \in \{0, 1, \dots\}$  driven by the innovation  $\epsilon_t$ .

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  3. Calculate the time series of other aggregate variables as

$$d\tilde{\mathbf{X}}_t = \sum_{s=0}^T d\mathbf{X}_s \tilde{\epsilon}_{t-s}$$

where  $d\mathbf{X}_s$  is the IRF to a unit-shock after  $s$  periods

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- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:**
  1. Same result if we linearize all aggregated equations and write the model in  $MA(\infty)$  form
  2. The state space form can also be recovered (not needed)

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations



# Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution** (advanced):

# Advanced linearized simulation

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- **Full distribution** (advanced):
  1. The IRF for grid point  $i_g$  in a policy function can be calculated as

$$da_{i_g, s}^* = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \mathbf{X}^{hh}} \frac{\partial a_{i_g}^*}{\partial X_{s'-s}^{hh}} dX_{s'-s}^{hh}.$$

where  $\partial a_{i_g}^* / \partial X_k^{hh}$  is the derivative to a  $k$ -period ahead shock to input  $X^{hh}$  (calculated in fake news algorithm)

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where  $\partial a_{i_g}^* / \partial X_k^{hh}$  is the derivative to a  $k$ -period ahead shock to input  $X^{hh}$  (calculated in fake news algorithm)

2. The policy function can there be simulated as

$$a_{i_g}^* = \sum_{s=0}^T da_{i_g,s}^* \tilde{\epsilon}_{t-s}$$

# Advanced linearized simulation

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$$a_{i_g}^* = \sum_{s=0}^T da_{i_g,s}^* \tilde{\epsilon}_{t-s}$$

3. Distribution can then be simulated forwards

# The HANK example from GEModelToolsNotebooks II

- **Presentation:** I go through the code for *finding the linearized IRFS and simulating the model*
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
  2. Write at least one question on [https://padlet.com/jeppe\\_drue Dahl/advmacrohet](https://padlet.com/jeppe_drue Dahl/advmacrohet)

## Exercise

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# Exercise = Assignment II

*You can start working on Assignment II: The HANK model*

## Summary

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# Summary and next week

- **Today:**

1. A baseline HANK model
2. Linearized IRFs and simulation

- **Next week:** Analytical Properties of HANK models

- **Homework:**

1. Work on Assignment II
2. Read: Auclert et al. (2018), »The Intertemporal Keynesian Cross«