



# 10. A Baseline HANK Model

## Adv. Macro: Heterogenous Agent Models

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Jeppe Druedahl & Patrick Moran  
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# Introduction

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  1. Linearized IRF
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- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
  1. Based on the [GEModelTools](#) package
  2. Example from [GEModelToolsNotebooks/HANK](#)
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools

## HANK model

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1. Differ by stochastic idiosyncratic productivity and savings
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- **Central bank:** Set nominal interest rate

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- **Note:** Zero profits (can be used to derive price index)

# Derivation of demand curve

- FOC wrt.  $y_{jt}$

$$0 = P_t \mu \left( \int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu}-1} - p_{jt} \Leftrightarrow$$

$$\frac{p_{jt}}{P_t} = \left( \int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow$$

$$\left( \frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu-1}} = \left( \int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow$$

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- **Dynamic problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

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- **NKPC** derived from FOC wrt.  $p_{jt}$  and envelope condition:

$$\log(1 + \pi_t) = \kappa \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied dividends:**  $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2$

# Derivation of NKPC

- FOC wrt.  $p_{jt}$ :

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{Z_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}} \\ - \frac{\mu}{\mu - 1} \frac{1}{\kappa} \frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$



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- FOC + Envelope + Symmetry +  $\pi_t = P_t/P_{t-1} - 1$

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- **Household problem:** Distribution,  $\mathbf{D}_t$ , over  $z_t$  and  $a_{t-1}$

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}[v_{t+1}(z_{t+1}, a_t) \mid z_t, a_t]$$

$$\text{s.t. } a_t + c_t = (1 + r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t \geq 0$$

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- **Effective labor-supply:**  $n_t = z_t \ell_t$

- **Beginning-of-period value function:**

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- **Endogenous grid method:** Vary  $z_t$  and  $a_t$  to find

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$$\ell_t = \left( \frac{w_t z_t}{\varphi} c_t^{-\sigma} \right)^{\frac{1}{\nu}}$$

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- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_t, a_{t-1}) \text{ and } \ell^*(z_t, a_{t-1}) \text{ with } m_t = (1+r_t)a_{t-1} + (w_t \ell_t^* - \tau_t + d_t)z_t$$

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- **Savings:**  $a^*(z_t, a_{t-1}) = (1+r_t)a_{t-1} - c_t^* + (w_t \ell_t^* - \tau_t + d_t)z_t$

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3. Return to step 1



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- **Fisher relationship:**

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where  $i_t^*$  is a shock

- **Fisher relationship:**

$$r_t = (1 + i_{t-1}) / (1 + \pi_t) - 1$$

- **Government:** Choose  $\tau_t$  to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

# Market clearing

1. Goods:  $Y_t = \int c_t^*(z_t, a_{t-1}) d\mathbf{D}_t$
2. Labor:  $N_t = \int n_t^*(z_t, a_{t-1}) d\mathbf{D}_t$  (in effective units)
3. Assets:  $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t$

## As an equation system

$$\begin{aligned} H(\pi, w, Y, i^*, Z, \underline{D}_0) &= 0 \\ \left[ \begin{array}{c} \log(1 + \pi_t) - \left[ \kappa \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}} \right] \\ N_t - \int n_t^*(z_t, a_{t-1}) d\mathbf{D}_t \\ B_{ss} - \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t \end{array} \right] &= 0 \end{aligned}$$

The rest of the model is given by

$$\mathbf{X} = M(\pi, w, Y, i^*, Z)$$

# Steady state

- **Analytically:**

1. **Normalization:**  $Z_{ss} = N_t = 1 \Rightarrow Y_{ss} = 1$
2. **Zero-inflation:**  $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^* = r_{ss}$
3. **Phillips-curve:**  $w_{ss} = \frac{1}{\mu}$  and  $d_t = 1 - w_{ss}$
4. **Government:**  $\tau_t = G_{ss}$

- **Numerically:** Choose  $\beta$  and  $\varphi$  to get market clearing

# The HANK example from GEModelToolsNotebooks I

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## **IRFs and simulation**

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# Linearized IRFs

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  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities*  
(direct in aggregate equations or through micro-behavior)

## Basic linearized simulation

- **Shocks:** Write the shocks as an  $MA(\infty)$  with coefficients  $d\mathbf{Z}_s$  for  $s \in \{0, 1, \dots\}$  driven by the innovation  $\epsilon_t$ .



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where  $d\mathbf{X}_s$  is the IRF to a unit-shock after  $s$  periods

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- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:**
  1. Same result if we linearized all aggregated equations and write the model in  $MA(\infty)$  form
  2. The state space form can also be recovered (not needed)

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations

# Advanced linearized simulation

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  1. The IRF for grid point  $i_g$  in a policy function can be calculated as

$$da_{i_g,s}^* = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \mathbf{X}^{hh}} \frac{\partial a_{i_g}^*}{\partial X_{s'-s}^{hh}} dX_{s'-s}^{hh}.$$

where  $\partial a_{i_g}^* / \partial X_k^{hh}$  to a  $k$ -period ahead shock to input  $X^{hh}$  (calculated in fake news algorithm)

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$$a_{i_g}^* = \sum_{s=0}^T da_{i_g,s}^* \tilde{\epsilon}_{t-s}$$

3. Distribution can then be simulated forwards

# The HANK example from GEModelToolsNotebooks II

- **Presentation:** I go through the code for *finding the linearized IRFS and simulating the model*
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
  2. Write at least one question on [https://padlet.com/jeppe\\_drue Dahl/advmacrohet](https://padlet.com/jeppe_drue Dahl/advmacrohet)

# Exercises

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- **Understand the model dynamics**

1. Illustrate through which channels tighter monetary policy affect aggregate consumption.
2. Illustrate the effect of tighter monetary policy on inequality.

# Exercises: Model mechanism and extension

- **Understand the model dynamics**

1. Illustrate through which channels tighter monetary policy affect aggregate consumption.
2. Illustrate the effect of tighter monetary policy on inequality.

- **Sticky prices vs. sticky wages:**

1. Assume that labor supply and wages are chosen by union
2. Assume wage inflation,  $\pi_t^w = \frac{w_t P_t}{w_{t-1} P_{t-1}}$ , solves

$$\log(1 + \pi_t^w) = \kappa_w \left( \varphi N_t^{1+\nu} - \frac{w_t N_t}{\mu_w} \int z_{it} c_{it}^{-\sigma} d\mathbf{D}_t \right) + \beta \log(1 + \pi_{t+1}^w)$$

**Micro-foundation:** Based on unions maximizing average utility of household.

3. Investigate how model dynamics change with changes in wage and price stickiness through  $\kappa$  and  $\kappa_w$

## Summary

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# Summary and next week

- **Today:**

1. A baseline HANK model
2. Linearized IRFs and simulation

- **Next week:** Adding unemployment to the HANK model

- **Homework:**

1. Work on completing the model extension exercise
2. Read: Broer et. al. (2022), »The Unemployment-Risk Channel in Business-Cycle Fluctuations«