Finite-Volume, One-Dimensional Hillslope Storage Boussinesq Model

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Finite-Volume, One-Dimensional Hillslope-Storage-Boussinesq (hsB) Model

1 Quickstart & MATLAB Usage

1.1 What this model does

The HSB model solves a 1D hillslope-storage Boussinesq equation for a sloping aquifer with a seepage face at the outlet ($x = 0$) and a no-flow divide at the upslope boundary. It produces time series of subsurface discharge at the outlet, water-table evolution $h(x, t)$, storage $S = f\,w\,h$, and diagnostic mass-balance metrics. Saturation-excess overland flow is tracked when h reaches a finite aquifer thickness D.

1.2 Files and entry points

- `run_hsB.m` — simple, editable *input script*. Define grid, width profile, recharge, parameters, options, and whether to plot. It then calls the solver and (optionally) generates figures.
- `hsB_solver.m` — main solver function. Returns all model states and diagnostics. You can call it directly from your own scripts/functions.

1.3 How to run (script workflow)

1. Open `run_hsB.m`. Edit the input blocks:
   (a) Grid (x) and width profile $w(x)$.
   (b) Time step Δt and number of steps N_t.
   (c) Recharge $N(t)$ in **mm/day** (the script converts to $\mathrm{m\,s^{-1}}$ internally). You can supply time-only ($N_t \times 1$) or space–time ($N_t \times N_x$).
   (d) Physical parameters $\{k, f, \gamma, S_0, D\}$.
   (e) Solver options (θ, Picard settings, etc.) and `do_plot` (true/false).
2. Run the script. It will call:

```
[Qout, Qsurf, Qtotal, h, S, diag] = hsB_solver( ...
    x, w, N_ms, dt, Nt, params, opts, do_plot, sty);
```

3. Inspect returned variables in your workspace. If `do_plot=true`, the figures will be created automatically.

1.4 Direct function call (programmatic usage)

You can bypass the driver and call the solver directly from any script/function once you have assembled inputs (see §1.5):

```
[Qout, Qsurf, Qtotal, h, S, diag] = hsB_solver( ...
    x, w, N_ms, dt, Nt, params, opts, do_plot, sty);
```

where `N_ms` is the recharge in $\mathrm{m\,s^{-1}}$ (convert from mm/day via `N_ms = (N_mmday/1000)/86400;`).

1.5 Inputs at a glance

Grid and geometry

- $\boldsymbol{x} \in \mathbb{R}^{N_x}$: strictly increasing cell-center coordinates [m].
- $\boldsymbol{w} \in \mathbb{R}^{N_x}$: positive planform width at centers [m]. Any shape is allowed (e.g. convergent planform).

Time and recharge

- Δt [s], N_t (number of time steps).
- $\boldsymbol{N}$: recharge. Either $N_t \times 1$ (spatially uniform in time) or $N_t \times N_x$ (space–time varying).
- **Units for input**: specify recharge in $\mathrm{mm\,day^{-1}}$ in `run_hsB.m`; it is converted to $\mathrm{m\,s^{-1}}$ for the solver. For space–time recharge, provide an Nt × Nx matrix (also in mm/day before conversion).

Physical parameters (`params`)

- `k` [$\mathrm{m\,s^{-1}}$]: saturated hydraulic conductivity.
- `f` [−]: drainable porosity.
- `gamma` [rad]: bedrock slope angle.
- `S0` [$\mathrm{m^2}$]: initial storage per unit-x, with $S_0 = f\,w\,h_0$.
- `D` [m]: aquifer thickness cap. If finite, the model limits $h \leq D$ and computes saturation-excess overflow when exceeded.

Solver options (opts)

- `theta`$\in [0,1]$: time weighting (0 explicit, 0.5 Crank–Nicolson, 1 backward Euler).
- `omega`$\in (0,1]$: Picard relaxation.
- `picard_max`, `picard_tol`: fixed-point iteration controls.
- `safeguard` (logical): enforce $S \geq 0$.
- `mass_check` (logical): compute mass-balance residual each step.
- `plot_live` (logical): lightweight live plot of $h(x)$ during the run.
- `no_recharge_outlet` (logical): set $N(:,1) = 0$ to avoid artificial spikes at the seepage face micro-cell.

Plotting controls

- `do_plot` (true/false): produce post-run figures (hydrograph & recharge bars; water-table profiles; $h(x,t)$ image; mass-balance residual).
- `sty` (struct, optional): simple stylistic overrides (axes/line widths, colors, discrete colormap name and levels, number of profiles to overlay, etc.). If omitted, built-in defaults are used.

1.6 Outputs at a glance

- `Qout` $\in \mathbb{R}^{N_t}$ $[\mathrm{m}^3\,\mathrm{s}^{-1}]$: *subsurface* discharge at the outlet face. *Sign convention*: negative = outward. For outward-positive discharge, use $-$`Qout`.
- `Qsurf` $\in \mathbb{R}^{N_t}$ $[\mathrm{m}^3\,\mathrm{s}^{-1}]$: saturation-excess overland flow (outlet-equivalent), nonnegative.
- `Qtotal` $= -$`Qout` $+$ `Qsurf` $[\mathrm{m}^3\,\mathrm{s}^{-1}]$: outward-positive total discharge.
- `h` $\in \mathbb{R}^{N_t \times N_x}$ [m]: head above bedrock.
- `S` $\in \mathbb{R}^{N_t \times N_x}$ $[\mathrm{m}^2]$: storage per unit-x, $S = f\,w\,h$.
- `diag` (struct): diagnostics including `dx`, `xe`, hillslope plan area `Ahs`, mass-balance residual time series, `Qsurf` copy, and overflow flags.

1.7 Units and conventions

- Recharge: $\mathrm{mm\,day^{-1}}$ in the input script; internally converted to $\mathrm{m\,s^{-1}}$.
- All other quantities: SI units (m, s, $\mathrm{m^3\,s^{-1}}$).
- Outlet discharge sign: model-internal `Qout` < 0 means flow exiting the domain; use $-$`Qout` when plotting an outward-positive hydrograph, and add `Qsurf` if you want total discharge.

1.8 Common adaptations

- **Grid refinement near outlet:** adjust (M, r, d_0) in `run_hsB.m` for micro-cells by the seepage face.
- **Planform convergence/divergence:** edit $\boldsymbol{w}(x)$ to test geometric effects on storage and discharge.
- **Recharge scenarios:** switch between time-only $(N_t \times 1)$ and space–time $(N_t \times N_x)$ forcing; build multi-pulse storms, seasonal cycles, or spatial gradients.
- **Boundary condition nuance:** set `opts.no_recharge_outlet=true` to remove recharge in the first (seepage) micro-cell.
- **Finite thickness and overflow:** pick a site-realistic D. The solver caps $h \leq D$ and reports saturation-excess overflow via `Qsurf`.

2 Model Inputs and Parameters

2.1 Spatial Grid and Geometry

$\boldsymbol{x} \in \mathbb{R}^{N_x}$ Strictly increasing cell-center coordinates [m].

$\boldsymbol{w} \in \mathbb{R}^{N_x}$ Planform width at centers [m], strictly positive. Any shape allowed (constant, convergent/divergent, piecewise).

A helper pattern in `run_hsB.m` clusters cells near the outlet via a short geometric progression (parameters M, r, d_0); this is optional and fully user-editable.

2.2 Time and Recharge

Δt Time step [s].

N_t Number of time steps.

N Recharge forcing. Supply either

1. $N_t \times 1$ (spatially uniform in time), or
2. $N_t \times N_x$ (space–time varying).

Units in the input script: $\mathrm{mm\,day^{-1}}$. The driver converts to $\mathrm{m\,s^{-1}}$ before calling the solver: `N_ms` = (`N_mmday`/1000)/86400.

2.3 Physical Parameters (`params`)

Field	Units	Meaning
`k`	$\mathrm{m\,s^{-1}}$	Saturated hydraulic conductivity
`f`	$(-)$	Drainable porosity
`gamma`	rad	Bedrock slope angle
`S0`	$\mathrm{m^2}$	Initial storage per unit-x ($S_0 = f\,w\,h_0$)
`D`	m	Aquifer thickness cap; if finite, triggers saturation-excess handling

Table 1: Physical parameter fields in `params`.

2.4 Solver Options (`opts`)

Field	Range/Type	Role
`theta`	$[0, 1]$	Time-weighting (0 explicit, 0.5 CN, 1 backward Euler)
`omega`	$(0, 1]$	Picard relaxation weight (with mild adaptivity)
`picard_max`	integer	Max Picard iterations per step
`picard_tol`	real > 0	L^∞ tolerance on Picard increment (storage)
`safeguard`	logical	Enforce $S \geq 0$ at each update
`mass_check`	logical	Track mass-balance residual per step
`plot_live`	logical	Lightweight live plot of $h(x)$ during runtime
`no_recharge_outlet`	logical	Set $N(:, 1) = 0$ in the seepage micro-cell

Table 2: Algorithmic and modeling options in `opts`.

2.5 Styling (Optional, for Plots)

The solver accepts an optional struct `sty` passed through `run_hsB.m` to control line/axis widths, fonts, colors, discrete colormaps, and the number of water-table profiles shown. If omitted, the driver provides sensible defaults. Key fields include: `axlw`, `plotlw`, `tickdir`, `ticklength`, `fs`, color fields under `sty.c`, and discrete colormap controls `cmap_name`, `cmap_levels`, `cmap_clim`, plus `n_profiles`.

3 Outputs, States, and Diagnostics

3.1 Primary Time Series

`Qout` [$\mathrm{m^3\,s^{-1}}$] Subsurface discharge at the outlet face. *Sign convention*: negative means outward. For outward-positive hydrographs, use −`Qout`.

`Qsurf` [$\mathrm{m^3\,s^{-1}}$] Saturation-excess overland flow (outlet-equivalent), nonnegative. Nonzero if the finite-thickness cap $h \le D$ is active and would otherwise be exceeded.

`Qtotal` = −`Qout` + `Qsurf` Outward-positive total discharge.

3.2 State Histories

`h` [m] Matrix $N_t \times N_x$: head above bedrock at all times and cells.

`S` [$\mathrm{m^2}$] Matrix $N_t \times N_x$: storage per unit-x, $S = f\,w\,h$.

3.3 Diagnostics (`diag` struct)

`dx, xe` Grid metrics (cell widths and edges).

`Ahs` Hillslope plan area $\sum_i w_i\,\Delta x_i$ [$\mathrm{m^2}$].

`mass_residual` Time series of global mass-balance residual [$\mathrm{m^3}$] per step (near zero when options are consistent).

`Qsurf` Copy of saturation-excess discharge time series.

`overflow_mask`, `overflow_steps`, `max_Qsurf` Convenience fields indicating whether/when overflow occurred and its maximum magnitude.

4 Boundary Conditions, Conventions, and Assumptions

4.1 Boundary Conditions

- **Outlet (left face, $x = 0$):** Dirichlet seepage boundary $h(0) = 0$.
- **Divide (right face):** No-flow boundary $Q = 0$.

4.2 Conventions

- **Discharge sign:** `Qout` < 0 means water leaving the domain. For outward-positive plots, flip sign and add `Qsurf` if total discharge is desired.
- **Recharge units:** Provide $\mathrm{mm\,day^{-1}}$ in the input script; the driver converts to $\mathrm{m\,s^{-1}}$ internally for the solver.
- **Thickness cap and overflow:** If $D < \infty$, the solver caps $h \leq D$ and accounts for the excess as `Qsurf` to preserve mass balance. A warning summarizes any overflow events.

4.3 Modeling Assumptions (high level)

- 1D hillslope representation with planform width $w(x)$ and bedrock slope γ.
- Boussinesq-type storage–discharge relation with storage $S = f\,w\,h$.
- Homogeneous hydraulic conductivity k and drainable porosity f (spatial variability can be introduced by modifying the code where needed).
- Semi-implicit time integration with Picard linearization; stabilized flux splitting (diffusion + drift).

5 Interfaces and Minimal Examples

5.1 Function Signature (Solver)

```
[Qout, Qsurf, Qtotal, h, S, diag] = hsB_solver( ...
    x, w, N_ms, dt, Nt, params, opts, do_plot, sty);
```

Arguments

- `x`, `w`: $N_x \times 1$ vectors (SI units).
- `N_ms`: recharge in $\mathrm{m\,s^{-1}}$: either $N_t \times 1$ or $N_t \times N_x$.
- `dt` [s], `Nt` (integer).
- `params`, `opts`: as described above.
- `do_plot`: logical toggle for post-run figures.
- `sty`: optional style struct (may be empty or omitted; defaults applied in the driver).

5.2 Minimal Call (from any script)

```
% Suppose you already built x, w, N_mmday, dt, Nt, params, opts
N_ms = (N_mmday/1000)/86400;       % mm/day -> m/s
do_plot = true;                     % or false
[Qout, Qsurf, Qtotal, h, S, diag] = hsB_solver( ...
    x, w, N_ms, dt, Nt, params, opts, do_plot);
```

5.3 Space–Time Recharge Example

```
% Build Nt x Nx recharge in mm/day
N_mmday = zeros(Nt, Nx);
N_mmday(1:72, :) = 5;                 % 3-day storm, 5 mm/day everywhere
N_mmday(:, 1:5) = N_mmday(:, 1:5) + 2;   % +2 mm/day near the outlet
N_ms = (N_mmday/1000)/86400;          % convert to m/s
```

6 Optional Plotting

If `do_plot=true`, the driver produces:

1. Outlet hydrograph (outward-positive) with inverted recharge bars in $\mathrm{mm\,day^{-1}}$.
2. n equally-spaced water-table profiles $h(x,t)$.
3. $h(x,t)$ image with a discrete colormap of configurable levels.
4. Mass-balance residual time series.

All styling is controlled by `sty`; if omitted, defaults are applied.

7 Governing Equations (Integral Form)

The primary state is the storage per unit-x,

$$S(x,t) = f\,w(x)\,h(x,t), \qquad h(x,t) \ge 0, \tag{1}$$

where f is the drainable porosity, $w(x) > 0$ the planform width, and h the water-table height above bedrock. Conservation over a control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S\,\mathrm{d}x \;=\; Q\Big(x_{i-\frac{1}{2}},t\Big) - Q\Big(x_{i+\frac{1}{2}},t\Big) \;+\; \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} N(x,t)\,w(x)\,\mathrm{d}x, \tag{2}$$

with Boussinesq/Dupuit discharge

$$Q(x,t) = -\frac{k}{f}\left[\,S(x,t)\,\cos\gamma\,\frac{\partial h}{\partial x}(x,t) \;+\; S(x,t)\,\sin\gamma\,\right]. \tag{3}$$

A finite aquifer thickness D imposes the cap $0 \le h \le D$.

8 Spatial Discretization (Finite Volumes)

Let x_i denote cell centers, edges $x_{i\pm\frac{1}{2}}$, and $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. We store $S_i \approx S(x_i, t)$ and construct face fluxes $Q_{i+\frac{1}{2}}$ by flux splitting:

$$Q_{i+\frac{1}{2}} = Q^{\text{diff}}_{i+\frac{1}{2}} + Q^{\text{adv}}_{i+\frac{1}{2}} + Q^{\text{num}}_{i+\frac{1}{2}}, \tag{4}$$

$$Q^{\text{diff}}_{i+\frac{1}{2}} = -\frac{k\cos\gamma}{f}\, S^H_{i+\frac{1}{2}}\, \frac{h_{i+1} - h_i}{x_{i+1} - x_i}, \tag{5}$$

$$Q^{\text{adv}}_{i+\frac{1}{2}} = -\frac{k\sin\gamma}{f}\, S^{\text{up}}_{i+\frac{1}{2}}, \qquad S^{\text{up}}_{i+\frac{1}{2}} = S_i \quad \text{(downslope)}, \tag{6}$$

$$Q^{\text{num}}_{i+\frac{1}{2}} = \tfrac{1}{2}\, a\, (S_i - S_{i+1})\,, \qquad a = \left|\frac{k\sin\gamma}{f}\right|. \tag{7}$$

The diffusive coefficient uses a harmonic mean with a small "film" to avoid degeneration near the seepage face:

$$S^H_{i+\frac{1}{2}} \;=\; \frac{2\,\widehat{S}_i\,\widehat{S}_{i+1}}{\widehat{S}_i + \widehat{S}_{i+1}}, \qquad \widehat{S}_j \;=\; \max\Big(S_j,\; f\,\bar{w}_{i+\frac{1}{2}}\,h_{\text{floor}}\Big)\,, \quad \bar{w}_{i+\frac{1}{2}} = \tfrac{1}{2}\,(w_i + w_{i+1}), \tag{8}$$

with $h_{\text{floor}} \sim 10^{-4}\,\text{m}$. At the outlet ($x = 0$) we impose $h(0, t) = 0$ and use a one-sided gradient in (5); at the divide $Q = 0$.

The semi-discrete balance is

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} \;=\; -\frac{Q_{i+\frac{1}{2}} - Q_{i-\frac{1}{2}}}{\Delta x_i} \;+\; N_i\,w_i. \tag{9}$$

9 Time Integration (Theta Method) and Nonlinear Iteration

With time step Δt and $\theta \in [0, 1]$ ($\theta = 1$ by default), define

$$S^{n+\theta} \;=\; (1-\theta)\,S^n + \theta\,S^{n+1}. \tag{10}$$

The fully discrete update is

$$\frac{S_i^{n+1} - S_i^n}{\Delta t} \;=\; -\frac{Q_{i+\frac{1}{2}}(S^{n+\theta}) - Q_{i-\frac{1}{2}}(S^{n+\theta})}{\Delta x_i} \;+\; N_i^n\,w_i. \tag{11}$$

Because Q depends nonlinearly on S via $h = S/(f\,w)$, we solve (11) by Picard iteration with relaxation:

$$\begin{aligned}
&\text{Initialize } S^{(0)} = S^n, \quad \text{for } m = 0, 1, \ldots \\
&S^{n+\theta,(m)} = (1-\theta)S^n + \theta S^{(m)}, && (12)\\
&\text{Build } Q^{(m)} \text{ from } S^{n+\theta,(m)} \text{ using (4)–(8)}, && (13)\\
&\text{Form div}\, Q_i^{(m)} = -\frac{Q^{(m)}_{i+\frac{1}{2}} - Q^{(m)}_{i-\frac{1}{2}}}{\Delta x_i}, && (14)\\
&S^{\text{new}} = S^n + \Delta t \left(\text{div}\, Q^{(m)} + N^n \odot w\right), && (15)\\
&S^{\text{new}} \leftarrow \max(S^{\text{new}}, 0), \qquad S^{\text{new}} \leftarrow \min(S^{\text{new}},\, f \odot w \odot D) \quad \text{(if finite } D\text{)}, && (16)\\
&S^{(m+1)} = (1-\omega)\, S^{(m)} + \omega\, S^{\text{new}}, \qquad \omega \in (0, 1], && (17)\\
&\text{Stop if } \|S^{(m+1)} - S^{(m)}\|_\infty < \text{tol}. && (18)
\end{aligned}$$

A simple adaptive damping reduces ω (not below 0.2) when the Picard increment grows between iterations.

10 Head Cap and Saturation-Excess Overflow

If $D < \infty$, the physical cap $h \le D$ translates to

$$S \;\le\; S_{\max} \;\equiv\; f\,w\,D. \tag{19}$$

At each step, the *raw* (unclipped) update

$$S^{\text{raw}} \;=\; S^n + \Delta t\big(\text{div}\, Q + N\,w\big) \tag{20}$$

is clipped to $S^{\text{clip}} = \min\big(\max(S^{\text{raw}}, 0), S_{\max}\big)$. The per-cell *excess volume* that cannot enter storage is

$$E \;=\; \max\big(S^{\text{raw}} - S^{\text{clip}},\, 0\big), \tag{21}$$

interpreted as saturation-excess overland flow. The domain-integrated, outlet-equivalent discharge is

$$Q^{n+1}_{\text{surf}} \;=\; \frac{1}{\Delta t} \sum_i E_i\, \Delta x_i. \tag{22}$$

11 Mass-Balance Diagnostic

Using the *accepted* state and fluxes,

$$\text{div}\, Q_i^\star = -\frac{Q^\star_{i+\frac{1}{2}} - Q^\star_{i-\frac{1}{2}}}{\Delta x_i}, \qquad R_i^{\text{excess}} = \frac{E_i}{\Delta t}, \tag{23}$$

the per-cell residual is

$$\mathcal{R}_i \;=\; \frac{S_i^{n+1} - S_i^n}{\Delta t} - \left(\text{div}\, Q_i^\star + N_i^n w_i - R_i^{\text{excess}}\right), \tag{24}$$

and the reported diagnostic is the volume imbalance

$$\mathrm{MB}^{n+1} = \sum_i \mathcal{R}_i \, \Delta x_i \quad [\mathrm{m}^3], \tag{25}$$

which should remain close to zero (up to solver tolerances).

12 Boundary Conditions and Sign Convention

At the outlet ($x = 0$) a seepage face enforces

$$h(0,t) = 0 \quad \Rightarrow \quad Q_{1/2} = Q_{1/2}^{\mathrm{diff}} + Q_{1/2}^{\mathrm{adv}} + Q_{1/2}^{\mathrm{num}}, \tag{26}$$

computed with a one-sided gradient in (5). At the divide ($x = L$),

$$Q_{N+\frac{1}{2}} = 0. \tag{27}$$

Internally, the model stores $Q_{1/2} < 0$ for outward discharge. For hydrographs, report outward-positive $-Q_{1/2}$, and total outward discharge as $-Q_{1/2} + Q_{\mathrm{surf}}$.

13 Algorithm Summary

1. Build nonuniform grid $\{x_i\}$ and edges $\{x_{i\pm\frac{1}{2}}\}$, widths w_i, spacings Δx_i.
2. Initialize S^0 (dry or user-specified) and precompute $S_{\max} = f\,w\,D$ if $D < \infty$.
3. For $n = 0, \ldots, N_t - 1$:
   (a) Set $S^{(0)} = S^n$; iterate Picard with θ-time averaging to obtain S^{n+1}: build Q from (4)–(8), update S, clip, relax, test convergence.
   (b) Recompute Q at S^{n+1}; store $Q_{1/2}$, $h^{n+1} = S^{n+1}/(f\,w)$.
   (c) Compute overflow Q_{surf}^{n+1} via (22).
   (d) Optionally evaluate mass-balance residual and quick live plot.

14 Convergence and Stability Notes

- Backward Euler ($\theta = 1$) is unconditionally stable for the linearized diffusion part and robust with Picard iteration for the nonlinear flux.
- Relaxation $\omega \in [0.4, 0.7]$ is effective; reduce adaptively if the Picard increment grows.
- The harmonic mean with a small head floor maintains ellipticity near the seepage face and prevents $S^H \to 0$.
- When D is small and recharge large, frequent clipping is expected; the explicit accounting through Q_{surf} preserves mass and flags saturation-excess events.

15 Troubleshooting (Common Issues)

- **Dimension mismatch for recharge:** Ensure `N_ms` is either $N_t \times 1$ or $N_t \times N_x$.
- **Non-increasing grid:** $\boldsymbol{x}$ must be strictly increasing; check clustering calculations when changing M, r, d_0.
- **Overflow warnings:** If D is small or recharge is intense/persistent, saturation-excess (`Qsurf`¿0) may occur; this is expected. Inspect `diag.overflow_steps` and `diag.max_Qsurf`.
- **Convergence issues:** Increase `omega` damping (e.g., 0.4), increase `theta` toward 1, or reduce Δt.

16 Discrete Conservation and Complexity

The finite-volume form *exactly conserves* storage at the semi-discrete level because the interior-face fluxes cancel in the domain sum. With time stepping,

$$\sum_i \frac{S_i^{n+1} - S_i^n}{\Delta t} \Delta x_i \;=\; -Q_{\frac{1}{2}}^{n+\theta} \;+\; Q_{N+\frac{1}{2}}^{n+\theta} \;+\; \sum_i N_i^n w_i \Delta x_i \;-\; \sum_i R_i^{\text{excess}} \Delta x_i, \tag{28}$$

where $Q_{N+\frac{1}{2}} = 0$ and R^{excess} is the saturation-excess sink. This is the identity monitored by the mass-balance residual (reported in m^3).

Per Picard iteration the algorithm is $O(N_x)$: one pass to build face fluxes and one for the update. For N_{it} iterations per step and N_t steps, the total cost is $O(N_x N_{\text{it}} N_t)$.

17 Verification Tests

17.1 Dry-to-dry (null test)

Set $S^0 = \mathbf{0}$, $N = \mathbf{0}$. Then for every step

$$S^{n+1} = \mathbf{0}, \qquad Q_{\frac{1}{2}}^{n+1} = 0, \qquad \text{MB}^{n+1} = 0. \tag{29}$$

This checks sign conventions, boundary setup, and conservation.

17.2 Steady recharge over a very long time

For uniform width $w = w_0$ and small γ (nearly horizontal), use a small constant recharge $N = N_0 > 0$ and integrate to quasi-steady. The hydrograph should approach a plateau where

$$-Q_{\frac{1}{2}} \;\approx\; N_0 \, A_{\text{hs}}, \tag{30}$$

until the head cap $h \leq D$ activates if D is finite, in which case overflow Q_{surf} appears and the total outward discharge approaches $N_0 A_{\text{hs}}$ with part routed through Q_{surf}.

17.3 Grid refinement

Halve Δx (increase N_x) while keeping Δt and options fixed. Time series of outward discharge and volume-integrated storage should converge. The mass-balance residual norm

$$\max_n |\text{MB}^n| \tag{31}$$

should decrease with refinement (subject to Picard tolerance).

17.4 Time-step sensitivity

Reduce Δt by factors of two. For backward Euler ($\theta = 1$), solutions are first-order in time for smooth transients; hydrographs and head fields should approach a limit. If the Picard tolerance is tight, the changes should become negligible.

18 Choosing Discretization and Solver Options

Grid clustering near the outlet Face gradients are steep near the seepage face ($h \to 0$). Use a few geometrically stretched cells adjacent to $x = 0$. A practical recipe is $M \in [2, 5]$, $r \in [1.10, 1.20]$, d_0 of a few meters.

Time step For fully implicit integration set $\theta = 1$. Start with Δt comparable to the signal in $N(t)$ (e.g., hourly if recharge is hourly). If the Picard iterations saturate at the cap $h \leq D$ frequently, reducing Δt improves temporal resolution of overflow events.

Picard damping and tolerance A fixed $\omega \in [0.4, 0.7]$ works well. The adaptive reduction (if the increment grows) promotes robustness. Use

$$\text{tol} \in [10^{-10}, 10^{-8}]\ \text{m}^2 \tag{32}$$

for the S-increment (S has units of m^2 per unit-x). Tighten if you need mass-residuals $\ll 1\,\text{m}^3$.

Seepage micro-cell recharge Optionally set $N(:, 1) = 0$ to avoid an artificial spike exactly at the Dirichlet head boundary. This is governed by the flag in the driver: `opts.no_recharge_outlet = true`.

19 Overflow Interpretation

If $D < \infty$, exceeding $S_{\max} = f\, w\, D$ triggers a storage clip. The excess

$$E_i = \max\Big(S_i^{\mathrm{raw}} - S_i^{\mathrm{clip}},\, 0\Big), \qquad Q_{\mathrm{surf}} = \frac{1}{\Delta t}\sum_i E_i\, \Delta x_i, \tag{33}$$

is reported as saturation-excess overland flow (outlet-equivalent). When overflow occurs, the solver prints a warning summarizing the number of affected steps and the maximum Q_{surf}.

20 Common Pitfalls and Remedies

- **Non-convergence of Picard.** Reduce Δt, reduce ω, or increase the head floor. Check width $w(x) > 0$ and monotone x.
- **Large mass residuals.** Tighten Picard tolerance; ensure mass residual is computed with the accepted $Q^{\star}$ and the explicit overflow sink; confirm unit consistency for N (mm/day $\to$ m/s).
- **No discharge with nonzero recharge.** Verify boundary at the outlet is $h = 0$ (not $Q = 0$). Confirm that the area-averaging of N is correct if spatially varying recharge is provided.
- **Oscillatory hydrograph under large slopes.** Increase geometric clustering near the outlet and decrease Δt; the upwind drift plus LF viscosity stabilizes advection but very steep slopes benefit from finer resolution.
- **Immediate overflow.** If D is too small or N too large, the system saturates quickly. Increase D (if physically justified), reduce N, or interpret the result as saturation-excess dominated response.

21 Recharge–Release Test and Linear-Reservoir Comparison

This driver configures a simple *recharge–release* scenario to verify hsB baseflow recession behavior against a canonical linear reservoir. It (i) builds a hillslope grid from preprocessed distance-to-stream bins, (ii) applies constant recharge for N_{on} days followed by N_{off} dry days, (iii) runs the solver on a native forcing grid, and (iv) compares the hsB recession (depth-equivalent) to

$$Q_{\mathrm{LR}}(t) \;=\; a\, \mathrm{e}^{-b\,t}, \tag{34}$$

with a [$\mathrm{m}^3\,\mathrm{s}^{-1}$] and b [day^{-1}].

21.1 Inputs and Geometry

The script expects `input_data.mat` containing uniform distance-to-stream bins x_H [m], per-bin widths H [m], and the maximum hillslope length $L = $ `metrics.max_hillslope_length_m`. One ghost bin is added on each side to define a smooth width function; $w(x)$ is then interpolated to the model centers and area-closed so that

$$A_{\text{hs}} = \sum_{i=1}^{N_x} w_i\,\Delta x_i \approx \sum_{j=1}^{N_H} H_j\,\Delta x_H. \tag{35}$$

21.2 Forcing and Units

Let $\{t_k\}_{k=0}^{n_F-1}$ (days) be the native forcing stamps. The recharge is piecewise-constant:

$$N(t_k) = \begin{cases} N_{\text{on}} & \text{if } t_k < N_{\text{recharge\_days}}, \\ 0 & \text{otherwise,} \end{cases} \qquad N_{\text{on}} = \texttt{recharge\_rate\_mmday}. \tag{36}$$

Conversion to SI uses

$$N_{\text{m/s}} = \frac{N_{\text{mm/day}}}{1000 \cdot 86400}. \tag{37}$$

21.3 Solver Call (Native Forcing)

The driver calls the native-forcing path of the wrapper (see §22):

```
Forcing.t_days = t_days;      % [nF x 1], days
Forcing.N_mps  = N_mps;       % [nF x 1] or [nF x Nx], m/s
[Qout,Qsurf,Qtotal,h,S,diag] = hsB_solver(x, w, Forcing, params, opts, fals
```

By convention, the core returns `Qout` < 0 for outward subsurface discharge; outward-positive total discharge is $-$`Qout` + `Qsurf`.

21.4 Linear-Reservoir Fit and Metrics

The recession start index k_0 is the first stamp with $N(t_{k_0}) \le 0$. Define the recession clock (days)

$$t_{\text{rec}} = t - t_{k_0}. \tag{38}$$

Observed hsB discharge (positive outward) and the linear reservoir are

$$Q_{\text{obs}}^{(\text{m}^3/\text{s})}(t) = -Q_{\text{out}}(t), \qquad Q_{\text{LR}}(t) = a\,\mathrm{e}^{-b\,t_{\text{rec}}}. \tag{39}$$

Depth-equivalent conversion uses plan area A_{hs}:

$$\left[\text{m}^3\,\text{s}^{-1}\right] \xrightarrow{\times\,\frac{1000\cdot 86400}{A_{\text{hs}}}} \left[\text{mm}\,\text{day}^{-1}\right]. \tag{40}$$

The performance metrics reported are

$$\text{RMSE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(Q_{\text{LR},i} - Q_{\text{obs},i}\right)^2}, \tag{41}$$

$$\text{NSE} = 1 - \frac{\sum_{i=1}^{M}\left(Q_{\text{LR},i} - Q_{\text{obs},i}\right)^2}{\sum_{i=1}^{M}\left(Q_{\text{obs},i} - \overline{Q}_{\text{obs}}\right)^2}, \tag{42}$$

$$\text{PBIAS [\%]} = 100\,\frac{\sum_{i=1}^{M}\left(Q_{\text{LR},i} - Q_{\text{obs},i}\right)}{\sum_{i=1}^{M} Q_{\text{obs},i}}. \tag{43}$$

Helpers. `add_ghosts_to_width_function` appends one ghost bin at each end so that w is well-defined and flat near the boundaries; `compare_recession_LR_days` performs the sign flip, unit conversion (40), computes (41)–(43), and renders a compact comparison plot with a LaTeX metrics box.

22 Extended Solver Wrapper: Legacy vs. Native Forcing

22.1 Two Call Styles

The wrapper `hsB_solver` dispatches to:

1. **Legacy (fixed step):**

$$[Q_{\text{out}}, Q_{\text{surf}}, Q_{\text{total}}, h, S, \text{diag}] = \texttt{hsB\_solver}(x, w, N, dt, N_t, \texttt{params}, \texttt{opts}, \texttt{do\_plot}, \texttt{sty}). \tag{44}$$

2. **Native (preferred; internal substeps between forcing stamps):**

$$[Q_{\text{out}}, Q_{\text{surf}}, Q_{\text{total}}, h, S, \text{diag}] = \texttt{hsB\_solver}(x, w, \texttt{Forcing}, \texttt{params}, \texttt{opts}, \texttt{do\_plot}, \texttt{sty}), \tag{45}$$

where `Forcing` has fields `t_days` [nF×1] and `N_mps` [nF×1 or nF×N_x].

22.2 Sign Convention and Totals

The core integrator retains the seepage-face sign convention:

$$Q_{\frac{1}{2}} \equiv Q_{\text{out}} < 0 \quad \text{(outward subsurface)}. \tag{46}$$

The outward-positive total discharge reported by the wrapper is

$$Q_{\text{total}} = -Q_{\text{out}} + Q_{\text{surf}}. \tag{47}$$

22.3 Internal Substepping (Native Mode)

Between two forcing stamps $[t_k, t_{k+1}]$, the wrapper advances hsB using internal steps Δt_{sub} controlled by conservative stability/accuracy estimates. With $c_{\text{adv}} = (k/f)\sin\iota$ and effective diffusive scale $\nu_{\text{eff}} \sim (k/f)\cos\iota\, S^\star$,

$$\Delta t_{\text{CFL,adv}} = \frac{C_{\text{adv}}\ \min_i \Delta x_i}{\max(c_{\text{adv}}, \varepsilon)}, \qquad \Delta t_{\text{CFL,diff}} = \frac{C_{\text{diff}}\ \min_i \Delta x_i^2}{\max(\nu_{\text{eff}}, \varepsilon)}. \tag{48}$$

A fractional-change limiter further restricts the step when

$$\eta \equiv \max_i \frac{\left|S_i^{n+1} - S_i^n\right|}{\max\big(|S_i^n|,\, \epsilon\big)} > \eta_{\max}, \tag{49}$$

forcing $\Delta t_{\text{sub}} \downarrow$. The accepted substeps exactly tile $[t_k, t_{k+1}]$.

22.4 Mass-Balance Diagnostics (Native Mode)

For each forcing interval, the wrapper accumulates strict volumes from the core:

$$\text{diag.forcing\_Rvol}(k) = \sum_{\text{substeps}} \left(\int N\, w\, \mathrm{d}x\right) \Delta t, \tag{50}$$

$$\text{diag.forcing\_Outvol}(k) = \sum_{\text{substeps}} \left(-Q_{\text{out}} + Q_{\text{surf}}\right) \Delta t, \tag{51}$$

$$\text{diag.forcing\_dV}(k) = \sum_{\text{substeps}} \left(\int \Delta S\, \mathrm{d}x\right). \tag{52}$$

These track recharge input, outward-positive discharge, and storage change over $[t_k, t_{k+1}]$ and enable coarse-grained mass-balance checks.

22.5 Plotting (Both Modes)

If `do_plot=true`, the wrapper generates (i) a semi-log hydrograph of subsurface and surface components with inverted recharge bars (mm day^{-1}), (ii) a set of water-table profiles $h(x,t)$, and (iii) an $h(x,t)$ image. In legacy mode, an additional per-step mass-balance residual figure is provided.

Notes.

- Setting `opts.no_recharge_outlet=true` enforces $N(:,1) = 0$ in the seepage micro-cell to avoid artificial spikes at $x = 0$.
- The native interface reports Q_{out}, Q_{surf}, Q_{total}, h, S at the *forcing stamps*; internal substeps are only for stability/accuracy and are summarized via (50)–(52).

23 Recharge–Release Calibration of k and f (minimizing RMSE)

This driver extends the recharge–release experiment (§21) to *calibrate* two hsB parameters—the saturated hydraulic conductivity k [m s^{-1}] and the drainable porosity f [–]—by minimizing the depth-equivalent RMSE between the hsB recession hydrograph and a target linear reservoir

$$Q_{\mathrm{LR}}(t) \;=\; a\,\mathrm{e}^{-b\,t}, \tag{53}$$

with a [m^3 s^{-1}] and b [day^{-1}] provided by the user.

23.1 Experiment Setup

The forcing follows the recharge–release pattern:

$$N(t_k) \;=\; \begin{cases} N_{\mathrm{on}} & \text{if } t_k < N_{\mathrm{recharge\_days}}, \\ 0 & \text{otherwise}, \end{cases} \qquad N_{\mathrm{on}} \text{ in mm day}^{-1}, \tag{54}$$

converted to SI via

$$N_{\mathrm{m/s}} \;=\; \frac{N_{\mathrm{mm/day}}}{1000 \cdot 86400}. \tag{55}$$

Let k_0 be the first index with $N(t_{k_0}) \le 0$; the recession clock is

$$t_{\mathrm{rec}} \;=\; t - t_{k_0}. \tag{56}$$

Observed hsB discharge (positive outward) during recession is

$$Q_{\mathrm{obs}}^{(\mathrm{m}^3/\mathrm{s})}(t) \;=\; -\,Q_{\mathrm{out}}(t), \tag{57}$$

and both series are converted to depth-equivalent units using the plan area A_{hs}:

$$Q_{\mathrm{mm/day}} \;=\; Q_{\mathrm{m^3/s}} \cdot \frac{1000 \cdot 86400}{A_{\mathrm{hs}}}. \tag{58}$$

23.2 Calibration Problem Statement

We estimate parameters $p = [k,\ f]^{\top}$ by solving a bound-constrained, smooth least-squares proxy:

$$\min_{p} \;\; \mathrm{RMSE}_{\mathrm{mm/day}}(p) \quad \text{s.t.} \quad \ell \;\le\; p \;\le\; u, \tag{59}$$

where ℓ and u are sensible physical bounds (e.g., $k \in [10^{-7}, 10^{-2}]$ m s^{-1}, $f \in [10^{-3}, 0.5]$). The objective is

$$\mathrm{RMSE}_{\mathrm{mm/day}}(p) \;=\; \sqrt{\frac{1}{M}\sum_{i=1}^{M}\Big(Q_{\mathrm{LR},i} - Q_{\mathrm{obs},i}(p)\Big)^2}, \tag{60}$$

with Q_{LR} from (53) and Q_{obs} obtained by running the hsB model under parameters p on $[t_{k_0},\ t_{\mathrm{end}}]$.

Optional fit diagnostics. In addition to (60), the script reports

$$\mathrm{NSE} \;=\; 1 - \frac{\sum_{i=1}^{M}\big(Q_{\mathrm{LR},i} - Q_{\mathrm{obs},i}\big)^2}{\sum_{i=1}^{M}\big(Q_{\mathrm{obs},i} - \overline{Q}_{\mathrm{obs}}\big)^2}, \qquad \mathrm{PBIAS}\,[\%] \;=\; 100\,\frac{\sum_{i=1}^{M}\big(Q_{\mathrm{LR},i} - Q_{\mathrm{obs},i}\big)}{\sum_{i=1}^{M} Q_{\mathrm{obs},i}}. \tag{61}$$

23.3 Optimization Variables and Scaling

For numerical robustness, the calibration is performed in base-10 log space:

$$q \;=\; \begin{bmatrix}\log_{10} k\\ \log_{10} f\end{bmatrix}, \qquad \text{with bounds} \quad q_{\min} \;=\; \log_{10}\ell, \;\; q_{\max} \;=\; \log_{10} u. \tag{62}$$

The mapping back to linear space is

$$k \;=\; 10^{q_1}, \qquad f \;=\; 10^{q_2}. \tag{63}$$

Working in q improves scale awareness and line-search behavior. In `fmincon`, `TypicalX`$=$ $[\log_{10} k_{\mathrm{typ}},\ \log_{10} f_{\mathrm{typ}}]$ provides additional scaling cues.

23.4 Solver and Policy

We use `fmincon` with a bound-constrained algorithm (e.g., `sqp` or `interior-point`). Recommended settings include:

$$\texttt{OptimalityTolerance} \in [10^{-8}, 10^{-6}], \quad \texttt{MaxFunctionEvaluations} \gtrsim 100\text{–}300. \tag{64}$$

Finite-difference gradients are taken by the optimizer; `UseParallel` can be enabled if the Parallel Computing Toolbox is available.[1]

23.5 Objective Evaluation (Per Iteration)

For a candidate q:

1. Map to p via (63) and update `params.k`, `params.f`.
2. Run `hsB_solver` (native forcing) to obtain $Q_{\mathrm{out}}(t)$.
3. Form Q_{obs} using (57), convert with (58).
4. Compute $\mathrm{RMSE}_{\mathrm{mm/day}}$ as in (60).

Numerical safeguards penalize failures or non-finite outputs with a large surrogate objective, keeping the search in feasible regions.

[1] Parallel gradients can reduce wall time because each function evaluation runs the hsB model.

23.6 Outputs

The script prints the best-fit parameters and objective:

$$k^\star = \arg\min_k \mathrm{RMSE}_{\mathrm{mm/day}}(k, f), \qquad f^\star = \arg\min_f \mathrm{RMSE}_{\mathrm{mm/day}}(k, f), \qquad \mathrm{RMSE}_{\mathrm{mm/day}}(k^\star, f^\star). \tag{65}$$

It then re-runs hsB at $(k^\star, f^\star)$ and generates a recession plot ($\mathrm{mm\,day}^{-1}$) annotated with N_{on}, the on/off-day counts, $(k^\star, f^\star)$, and the metrics (61).

Practical guidance.

- **Bounds.** Start with physically broad bounds; if the solution hits a bound, consider widening or revisiting priors.
- **Initialization.** Choose (k, f) close to site knowledge to accelerate convergence.
- **Tolerance vs. noise.** If the objective surface is noisy (due to tight solver tolerances or frequent overflow clipping), loosen optimality tolerances or reduce time-step stringency in hsB substepping.
- **Identifiability.** k and f can co-vary in recession fits. Consider fixing one, or complementing with additional diagnostics (e.g., fill-phase dynamics) when identifiability is weak.

24 Extending the Model

Spatially variable parameters Allow $k(x)$, $f(x)$, and $\gamma(x)$ by storing cell-wise vectors and replacing scalars in the flux assembly. The formulas are unchanged; only k/f and $\cos\gamma$, $\sin\gamma$ become local.

Alternative outlet boundary A Cauchy-type boundary (linear head–discharge relation) can replace $h = 0$ if a measurement weir rating or baseflow constraint is preferred. Implement by modifying the left-face flux closure.

Lateral exchange or channel losses Add a sink/source term $L(x, t)$ to (9). It enters the update as $+ L_i$ and the mass residual with the proper sign.

Rainfall–runoff coupling To include infiltration-excess (Hortonian) processes, split N into an infiltration component and a surface runoff component based on infiltration capacity. The present framework directly accepts a space–time $N(x, t)$ field from an external runoff module.

25 Units and Conversions

Inputs/outputs follow SI except recharge, which the driver accepts and plots in mm/day for user convenience. Convert with

$$1 \text{ mm/day} = \frac{10^{-3}}{86400} \text{ m s}^{-1} \approx 1.1574 \times 10^{-8} \text{ m s}^{-1}. \tag{66}$$

The hydrograph is presented in mm/day as an areal flux over the hillslope plan area, computed from the model discharge divided by A_{hs}.

26 Reproducibility and Versioning

Record in your scripts: the commit/version of `hsB_solver.m`, the exact `opts` and `params` structs, grid construction parameters (L, N_x, M, r, d_0), the width function used, and the recharge series. Store figure seeds only if you randomize any inputs (the distributed driver does not).

How to Cite This Model

If you use this model in your research or applications, please cite:

> Marcus N. Gomes Jr. (2025).
> *Finite-Volume, One-Dimensional Hillslope Storage Boussinesq Model.*
> GitHub Repository: `https://github.com/marcusnobrega-eng/1D_hsB`
> DOI (if available): `10.5281/zenodo.xxxxxxx`

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