1 Inversions and contractions — practical implementation on the lattice

1.1 Adjoint currents

We the following rule for complex conjugation of a product of Grassmann numbers z_1, z_2

$$(z_1\, z_2)^* = z_2^*\, z_1^*\,. \tag{1}$$

With this convention, the quark-bilinear $\bar{q}q$ (e.g. part of the mass term) is a real quantity

$$(\bar{q}q)^* = \left(q^\dagger\,\gamma_0\,q\right)^* = q^\dagger\,\gamma_0^\dagger\,q = \bar{q}q\,.$$

Adjoint baryon current Thus for a baryon interpolating field Q and its adjoint $\bar{Q}$ we have

$$Q^c_\alpha = \epsilon_{abc}\,\left(q_1^{aT}\,\Gamma\,q_2^b\right)\,q^c_{3\,\alpha} = \epsilon_{abc}\,\left(q^a_{1\,\gamma}\,\Gamma_{\gamma\delta}\,q^b_{2\,\delta}\right)\,q^c_{3\,\alpha} \tag{2}$$

$$\begin{aligned}\bar{Q}^{c'}_\beta &= \left[\epsilon_{a'b'c'}\,\left(q^{a'}_{1\,\gamma}\,\Gamma_{\gamma\delta}\,q^{b'}_{2\,\delta}\right)\,q^{c'}_{3\,\rho}\right]^*\,(\gamma_0)_{\rho\beta} = \epsilon_{a'b'c'}\,\left(q^{b'*}_{2\,\delta}\,\Gamma^\dagger_{\delta\gamma}\,q^{a'*}_{1\,\gamma}\right)\,q^{c'*}_{3\,\rho}\,(\gamma_0)_{\rho\beta}\\ &= -\epsilon_{c'b'a'}\,\bar{q}^{c'}_{3\,\beta}\,\left(\bar{q}^{b'}_{2\,\kappa}\,(\gamma_0)_{\kappa\delta}\,\Gamma^\dagger_{\delta\gamma}\,(\gamma_0)_{\gamma\lambda}\,\bar{q}^{a'}_{1\,\lambda}\right) = -\epsilon_{c'a'b'}\,\bar{q}^{c'}_{3\,\beta}\,\left(\bar{q}^{a'}_2\,\tilde{\Gamma}\,\bar{q}^{b'T}_1\right)\,,\end{aligned} \tag{3}$$

where

$$\tilde{\Gamma} = \gamma_0\,\Gamma^\dagger\,\gamma_0 = \sigma_{\Gamma^{0\dagger}}\,\Gamma$$

and we used the property $\gamma_0^T = \gamma_0$.

Note: We usually multply another Γ' to q_3 in eq. (3). This will lead to an analogous sign $\sigma^{0\dagger}_{\Gamma'}$.

The sign $\sigma^{0\dagger}_\Gamma$ for some relevant structures is

Γ	$C\,\gamma_5$	C	$C\,\gamma_5\,\gamma_0$	$C\,\gamma_0$	$C\,\gamma_{1,3}$	$C\,\gamma_2$	$C\,\gamma_{1,3}\,\gamma_0$	$C\,\gamma_2\,\gamma_0$
$\sigma_{\Gamma^{0\dagger}}$	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	-1

We used $C^\dagger = C$ in the above table.

Adjoint meson current For a meson interpolating field, this entails

$$(\bar{q}_1\,\Gamma\,q_2)^* = q_2^\dagger\,\Gamma^\dagger\,\gamma_0\,q_1 = \bar{q}_2\,\tilde{\Gamma}\,q_1\,.$$

In particular for the charged pions we have $\pi^+ = \bar{d}\,i\gamma_5\,u$ with $\pi^{+*} = \pi^-$.
The values of $\sigma_{\Gamma^{0\dagger}}$ for Γ on of the 16 basis matrices is given by

Γ	γ_0	γ_i	$\mathbb{1}$	γ_5	$\gamma_0\,\gamma_5$	$\gamma_i\,\gamma_5$	$\gamma_0\,\gamma_i$	$\gamma_i\,\gamma_j$
$\sigma_{\Gamma^{0\dagger}}$	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1

Note: I do not include factors of imaginary unit i in the interpolating fields in the contractions. They can be added afterwards as a complex phase to the correlation matrix. This holds for $C = i\gamma_0\,\gamma_2$ as well, which means in the contraction part I only consider the $\gamma_0\,\gamma_2$.

Convention for contraction code For simplicity, I always contract bare quark-field and γ combinations, *without* the minus sign in (3). This means a correlator C from *contract_baryon* must receive a sign

$$C_{Q_1-\bar{Q}_2} \to -\sigma_{\Gamma^{0\dagger}_{Q_2}} C_{Q_1\bar{Q}_2} \tag{4}$$

$$C_{Q_1-M_2^\dagger\bar{Q}_2} \to -\sigma_{\Gamma^{0\dagger}_{Q_2}} \sigma_{\Gamma^{0\dagger}_{M_2}} C_{Q_1-M_2^\dagger\bar{Q}_2} \tag{5}$$

$$C_{M_1Q_1-M_2^\dagger\bar{Q}_2} \to -\sigma_{\Gamma^{0\dagger}_{Q_2}} \sigma_{\Gamma^{0\dagger}_{M_2}} C_{M_1Q_1-M_2^\dagger\bar{Q}_2} \tag{6}$$

In the *old code version, which produces the full diagrams in one step*, the overall sign from the adjoint operator at source can be added e.g. via the *comp_list_sign*.
The following table gives the final phase, which according to the above convention must be multiplied to the result of *contract_baryon*.

interpolator	source phase	sink phase
$N^+\,(C\gamma_5,\,\mathbb{1})$	$-i$	$+i$
$N^+\,(C,\,\gamma_5)$	$-i$	$+i$
$N^+\,(C\gamma_5\gamma_0,\,\mathbb{1})$	$-i$	$+i$
$N^+\,(C\gamma_0,\,\gamma_5)$	$+i$	$+i$
$\Delta^{++}\,(C\gamma_{1,3},\,\mathbb{1})$	$-i$	$+i$
$\Delta^{++}\,(C\gamma_2,\,\mathbb{1})$	$+i$	$+i$
$\Delta^{++}\,(C\gamma_{1,3}\gamma_0,\,\mathbb{1})$	$-i$	$+i$
$\Delta^{++}\,(C\gamma_2\gamma_0,\,\mathbb{1})$	$+i$	$+i$
$N^+\,(C\gamma_5,\,\mathbb{1})\,\pi^+$	$+1$	-1
$N^+\,(C,\,\gamma_5)\,\pi^+$	$+1$	-1
$N^+\,(C\gamma_5\gamma_0,\,\mathbb{1})\,\pi^+$	$+1$	-1
$N^+\,(C\gamma_0,\,\gamma_5)\,\pi^+$	-1	-1

1.2 Δ^{++} to $\pi^+ N^+$ 3-point function

Motivated by the considerations in the previous section **??** we continue with the practical implementation of the estimation of the matrix elements in lattice QCD. We start from

$$\langle J_{\Delta^{++}}(x_f)\, J_M^\dagger(x_{i_2})\, \bar J_N(x_{i_1})\rangle = -\sigma_{\Gamma_N^{0\dagger}}\,\sigma_{\Gamma_M^{0\dagger}}\,\langle \underbrace{J^\alpha_{\Delta^{++}}}_{(u^T\,C\Gamma_\Delta\,u)\,u}(t_f,\,\vec{x})\;\underbrace{J^\dagger_{\pi^+}}_{\bar u\,\gamma_5\,d}(t_i,\,\vec{y})\;\underbrace{\bar J^\beta_N}_{(\bar d\,C\Gamma_N\,\bar u^T)\,\bar u}(t_i,\,\vec{z})\rangle \tag{7}$$

$$\langle\Big[u^a_\gamma(x_f)\,(C\Gamma_\Delta)_{\gamma\delta}\,u^b_\delta(x_f)\,u^c_\alpha(x_f)\Big]\;\Big[\bar u^d_\sigma(x_{i_2})\,(\Gamma_M)_{\sigma\tau}\,d^d_\tau(x_{i_2})\Big]\;\Big[\bar d^l_\kappa(x_{i_1})\,(C\Gamma_N)_{\kappa\lambda}\,\bar u^m_\lambda(x_{i_1})\,\bar u^n_\beta(x_{i_1})\Big]\rangle_f = \tag{8}$$

$$\begin{aligned}
&-T(x_f,x_{i_1})\,C\Gamma_N\;\left(C\Gamma_\Delta\,U(x_f,x_{i_1})\right)^t\,U(x_f,x_{i_1})\\
&-T(x_f,x_{i_1})\,C\Gamma_N\,U(x_f,x_{i_1})^t\,C\Gamma_\Delta\,U(x_f,x_{i_1})\\
&-U(x_f,x_{i_1})\;\left(C\Gamma_\Delta\,T(x_f,x_{i_1})\,C\Gamma_N\right)^t\,U(x_f,x_{i_1})\\
&-U(x_f,x_{i_1})\;\left(T(x_f,x_{i_1})\,C\Gamma_N\right)^t\,C\Gamma_\Delta\,U(x_f,x_{i_1})\\
&-U(x_f,x_{i_1})\,\mathrm{Tr}\left(T(x_f,x_{i_1})\,C\Gamma_N\,U(x_f,x_{i_1})^t\,C\Gamma_\Delta\right)\\
&-U(x_f,x_{i_1})\,\mathrm{Tr}\left(T(x_f,x_{i_1})\,C\Gamma_N\;\left(C\Gamma_\Delta\,U(x_f,x_{i_1})\right)^t\right)\\
&= T_1+T_2+T_3+T_4+T_5+T_6
\end{aligned} \tag{9}$$

Eq. (8) defines the triangle diagrams $T_1,\ldots,T_6$.
We use the notation

$$T(x_f,x_i) = T^{f_1 f_2\,ab}_{\alpha\beta}(x_f;t,\vec{q};x_i) = \sum_{\vec{y}}\left(S_{f_1}(x_f;\,t,\vec{y})\,\Gamma_M\,\mathrm{e}^{i\vec{q}\vec{y}}\,S_{f_2}(t,\vec{y};\,x_i)\right)^{ab}_{\alpha\beta} \tag{10}$$

for the sequential propagator with

- flavors "f_1 after f_2";
- sequential source timeslice t;
- sequential source momentum $\vec{q}$;
- sequential source Dirac structure Γ_M.

In particular we shall use the notation

$$T_{fii} = T(x_f;t_i,\vec{q};x_i) \tag{11}$$

$$T_{ffi} = T(x_f;t_f,\vec{q};x_i) \tag{12}$$

for 1-step sequential propagators.

Quantum numbers for Delta, pion and nucleon:

	Δ^{++}	π^+	$N^+ =$ Proton
J	$\frac{3}{2}$	0	$\frac{1}{2}$
I	$\frac{3}{2}$	1	$\frac{1}{2}$
I_3	$+\frac{3}{2}$	$+1$	$+\frac{1}{2}$
P	$+1$	-1	$+1$

1.3 Δ^{++} to Δ^{++}

$$\langle J_\Delta(x_f)\,\bar{J}_\Delta(x_i)\rangle_f = -\sigma_{\Gamma_i^{0\dagger}}\,\langle \left(u^T\,C\Gamma_f\,u\right)\,u(x_f)\,\left(\bar{u}^T\,C\Gamma_i\,\bar{u}\right)\,\bar{u}(x_i)\rangle \tag{13}$$

$$\begin{aligned}
&\langle \Big[\,\epsilon_{abc}\,u_\gamma^a(x_f)\,\left(C\Gamma_f\right)_{\gamma\delta}\,u_\delta^b(x_f)\,u_\alpha^c(x_f)\Big]\;\Big[\epsilon_{lmn}\,\bar{u}_\kappa^l(x_i)\,\left(C\Gamma_i\right)_{\kappa\lambda}\,\bar{u}_\lambda^m(x_i)\,\bar{u}_\beta^n(x_i)\Big]\rangle_f = \\
&\quad \epsilon_{abc}\,\epsilon_{lmn}\,\left(C\Gamma_f\right)_{\gamma\delta}\,\left(C\Gamma_i\right)_{\kappa\lambda}\,\Big\{ \\
&\qquad + U_{\alpha\beta}^{cn}\,\left(U_{\delta\kappa}^{bl}\,U_{\gamma\lambda}^{am} - U_{\delta\lambda}^{bm}\,U_{\gamma\kappa}^{al}\right) \\
&\qquad - U_{\alpha\lambda}^{cm}\,\left(U_{\delta\kappa}^{bl}\,U_{\gamma\beta}^{an} - U_{\delta\beta}^{bn}\,U_{\gamma\kappa}^{al}\right) \\
&\qquad + U_{\alpha\kappa}^{cl}\,\left(U_{\delta\lambda}^{bm}{}_{\gamma\beta}^{an} - U_{\delta\beta}^{bn}\,U_{\gamma\lambda}^{am}\right) \\
&\Big\} = \\
&\quad - U(x_f,x_i)\,C\Gamma_i\,\left(C\Gamma_f\,U(x_f,x_i)\right)^t\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,C\Gamma_i\,U(x_f,x_i)^t\,C\Gamma_f\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,\left(C\Gamma_f\,U(x_f,x_i)\,C\Gamma_i\right)^t\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,\left(U(x_f,x_i)\,C\Gamma_i\right)^t\,C\Gamma_f\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,\mathrm{Tr}\left(C\Gamma_f\,U(x_f,x_i)\,C\Gamma_i\,U(x_f,x_i)^t\right) \\
&\quad - U(x_f,x_i)\,\mathrm{Tr}\left(C\Gamma_f\,U(x_f,x_i)\,\left(U(x_f,x_i)\,C\Gamma_i\right)^t\right) \\
&= D_1 + D_2 + D_3 + D_4 + D_5 + D_6\,.
\end{aligned} \tag{14}$$

This define the $I = 3/2$ diagrams $D_1, \ldots, D_6$.

Adjoint correlator Using γ_5-Hermiticity, parity and time reversal we expect, that

$$\begin{aligned}
C_{\mu\nu}^{\alpha\beta}(x,y) &= \langle J_{\Delta\mu}^\alpha(x)\,\bar{J}_{\Delta\nu}^\beta(y)\rangle \\
C_{\mu\nu}(x,y) &= \sigma_\mu^{02}\,\sigma_\mu^{02}\,C_{\mu\nu}^{\tilde{\dagger}} \\
\sigma_\mu^{02} &= \begin{cases} +1 & \mu = 0,\,2 \\ -1 & \mu = 1,\,3 \end{cases},
\end{aligned}$$

where $\tilde{\dagger}$ denotes the conjugate with respect to the spinor indices. This relation should hold exactly in the free case (gauge field $U = \mathbb{1}$) and at the level of the gauge average in the non-free case.

$$\begin{aligned}
(t_x,\,t_y) \sim t_x - t_y &\xrightarrow{\gamma_5-\text{Hermiticity}} (t_y,\,t_x) \sim t_y - t_x \\
&\xrightarrow{\mathcal{T}} (T - t_y,\,T - t_x) \sim (T - t_y) - (T - t_x) = t_x - t_y\,.
\end{aligned}$$

1.4 N^+ to N^+

$$\langle J_N(x_f)\, \bar{J}_N(x_i)\rangle_f = -\sigma_{\Gamma_i^{0\dagger}}\, \langle \big(u^T\, C\Gamma_f\, d\big)\ u(x_f)\ \big(\bar{d}^T\, C\Gamma_i\, \bar{u}\big)\ \bar{u}(x_i)\rangle \tag{15}$$

$$\begin{aligned}
&\langle \Big[\, \epsilon_{abc}\, u^a_\gamma(x_f)\ (C\Gamma_f)_{\gamma\delta}\ d^b_\delta(x_f)\, u^c_\alpha(x_f)\Big]\ \Big[\epsilon_{lmn}\, \bar{d}^l_\kappa(x_i)\ (C\Gamma_i)_{\kappa\lambda}\ \bar{u}^m_\lambda(x_i)\, \bar{u}^n_\beta(x_i)\Big]\rangle_f = \\
&\quad - U(x_f,x_i)\ (C\Gamma_f\, D(x_f,x_i)\, C\Gamma_i)^t\ U(x_f,x_i) \\
&\quad - U(x_f,x_i)\, \mathrm{Tr}\left((C\Gamma_f\, D(x_f,x_i)\, C\Gamma_i)^t\ U(x_f,x_i)\right) \\
&= N_1 + N_2\,.
\end{aligned} \tag{16}$$

This defines the diagrams N_1, N_2.

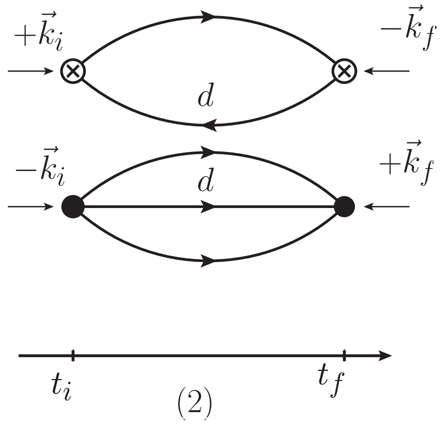

Figure 1: Graphical representation of the quark-disconnected contribution to the 4-pt. function $\pi\, N \to \pi\, N$

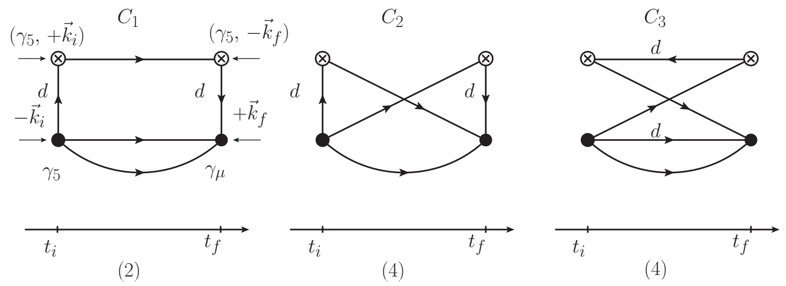

Figure 2: Graphical representation of the quark-connected contribution to the 4-pt. function $\pi\, N \to \pi\, N$ at zero total 3-momentum, $\vec{Q} = 0$

1.5 $\pi^+ N^+$ to $\pi^+ N^+$

In total these sum up to 12 contributions; we can check that is the right number 2 combinations of down quarks $\times$ 3! combinations of up quarks
We introduce some notation to write out the necessary contractions, the 2-step sequential propagators P

$$P^{f_1 f_2 f_1\, ab}_{\alpha\beta}(x_f;\, t_1, \vec{q}_1;\, t_2, \vec{q}_2;\, x_i) = \tag{17}$$
$$\sum_{\vec{z}_1, \vec{z}_2} \Big(S_{f_1}(x_f;\, t_1, \vec{z}_1)\, \Gamma_1\, S_{f_2}(t_1,\, \vec{z}_1;\, t_2, \vec{z}_2)\, \Gamma_2\, S_{f_1}(t_2, \vec{z}_2;\, x_i) \Big)^{ab}_{\alpha\beta} \mathrm{e}^{i(\vec{q}_1 \vec{z}_1 + \vec{q}_2 \vec{z}_2)}$$

where $f_{1/2} \in \{u,\, d\}$ and $f_1 \neq f_2$. In particular we shall use

$$P_{fifi} = P^{udu}(x_{f_1}; t_i, \vec{q}_{i_2}; t_f, \vec{q}_{f_2}; x_{i_1}) \tag{18}$$
$$P_{ffii} = P^{dud}(x_{f_1}; t_f, \vec{q}_{f_2}; t_i, \vec{q}_{i_2}; x_{i_1}) \tag{19}$$

with $t_{f_1} = t_{f_2} = t_f$ and $t_{i_1} = t_{i_2} = t_i$.

With these generalized propagators we can write the contractions in a short way.

$$\begin{aligned} &\langle J_{\pi^+ N^+}(x_{f_1}; x_{f_2})\, \bar{J}_{\pi^+ N^+}(x_{i_1}; x_{i_2}) \rangle_f \\ &= -\sigma_{\Gamma^{0\dagger}_{N_i}}\, \sigma_{\Gamma^{0\dagger}_{M_i}}\, \langle \left(u^t\, C\Gamma_{N_f}\, d \right) u(x_{f_1})\, \bar{d}\, \Gamma_{M_f}\, u(x_{f_2})\, \bar{u}\, \Gamma_{M_i}\, d(x_{i_2})\, \left(\bar{d}\, C\Gamma_{N_i}\, u \right) u(x_{i_1}) \rangle \end{aligned} \tag{20}$$

$$\begin{aligned} &\langle \Big[\epsilon_{abc}\, u^a_\gamma(x_{f_1})\, \left(C\Gamma_{N_f} \right)_{\gamma\delta}\, d^b_\delta(x_{f_1})\, u^c_\alpha(x_{f_1}) \Big]\, \Big[\bar{d}^d_\sigma(x_{f_2})\, \left(\Gamma_{M_f} \right)_{\sigma\tau}\, u^d_\tau(x_{f_2}) \Big] \times \\ &\quad \Big[\bar{u}^e_\mu(x_{i_2})\, (\Gamma_{M_i})_{\mu\nu}\, d^e_\nu(x_{i_2}) \Big]\, \Big[\epsilon_{lmn}\, \bar{d}^l_\kappa(x_{i_1})\, (C\Gamma_{N_i})_{\kappa\lambda}\, \bar{u}^m_\lambda(x_{i_1})\, \bar{u}^n_\beta(x_{i_1}) \Big] \rangle \\ &= C_B + C_W + C_Z + C_{\text{disconnected}} \end{aligned} \tag{21}$$

Quark-disconnected contribution — direct diagram

$$C_{\text{disconnected}} = -\mathrm{Tr}\left(U(x_{f_2}, x_{i_2})\, \Gamma_{M_i}\, D(x_{i_2}, x_{f_2})\, \Gamma_{M_f} \right) \times (N_1 + N_2)\ . \tag{22}$$

Quark-connected contributions — B, W and Z diagrams The connected contractions $C_{B,W,Z}$ are

$$
\begin{aligned}
C_B = & \\
& - U(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, P_{ffii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(C\Gamma_{N_f}\, P_{ffii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, U(x_{f_1}, x_{i_1})^t\right) \\
= & \, B_1 + B_2
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
C_W = & \\
& - T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i} \left(C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1})\right)^t\, U(x_{f_1}, x_{i_1}) \\
& - T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, U(x_{f_1}, x_{i_1})^t\, C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1}) \left(T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i} \left(C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1})\right)^t\right) \\
= & \, C_{W_1} + C_{W_2} + C_{W_3} + C_{W_4}
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
C_Z = & \\
& - P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1}) \\
& - P^{udu}_{fifi}(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(\left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1})\right) \\
& - U(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\right) \\
= & \, C_{Z_1} + C_{Z_2} + C_{Z_3} + C_{Z_4}
\end{aligned} \tag{25}
$$

Comments

- note that with degenerate up and down quarks we can use $T^{ud} = T^{du}$ and $P^{udu} = P^{dud}$. Moreover in the case at hand we will always have $\vec{k}_i = \vec{k}_f$
- the straightforward contractions using the *Seq²Propagators* from point sources would be demanding; for each momentum vector $\vec{k} = \vec{k}_f = \vec{k}_i$ we would need $N_s \times N_c \times (3N_{t_f} + 2)$ inversions to produce all required S, T, P fields. N_s, N_c, N_{t_f} are the numbers of spinor, color components and the number of sink timeslices
- but (1): from the investigations of nucleon/delta matrix elements with current insertions it is know that $N_{t_f} = \mathcal{O}\left(T/4/a\right)$, a quarter of the temporal extent of the lattice
- but (2): using momenta from the class represented by $\vec{k} = (0, 0, 1)$, could one save computing time by combining $\vec{k}$ and $-\vec{k}$ using coherent sources ?
- What about stochastic timeslice sources? Issue: we need spin and color dilution (do we?); would thus still require $\mathcal{O}\left(N_s\, N_c\, T/a\right)$ inversions, same order of magnitude; needs inversions for both sources with and without momentum
- timeslice sources may require $\mathbb{Z}_5$ sources
- would be nice if chosen method would be applicable to Δ^{++}-, N^+-2-point functions and $\Delta^{++} \to \pi^+ N^+$ 3-point function as well

2 Post-poned spin-color reduction and factorization for $I = 3/2,\ I^3 = +3/2$

2.1 B-diagrams

$$P^{dud}_{ffii} = \phi^d(f_1)\,\xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,T^{ud}(f_2,i_2,i_1) \tag{26}$$

B_1

$$\begin{aligned}
(B_1)_{\alpha\beta} &= -b_{1\phi}\,(\vec{p}_{f_1},\Gamma_{f_1})^m_{\beta\alpha\delta}\ (\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\delta\gamma}\ b_{1\xi}\,(\vec{p}_{f_2},\Gamma_{f_2})^m_{\gamma} \\
b^m_{1\xi;\,\gamma} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,T_{fii}(f_2,i_1)^{\cdot;m}_{\cdot;\gamma} &(27)\\
b^m_{1\phi;\,\beta\alpha\delta} &= \epsilon_{mnl}\,\left[\epsilon_{bca}\,\phi(f_1)^b_\kappa\ (C\,\Gamma_{f_1})^t_{\kappa\lambda}\ U(f_1,i_1)^{cn}_{\lambda,\beta}\right]\ U(f_1,i_1)^{al}_{\alpha,\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} &(28)
\end{aligned}$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

B_2

$$\begin{aligned}
(B_2)_{\alpha\beta} &= -b_{2\phi}\,(\vec{p}_{f_1},\Gamma_{f_1})^m_{\delta\alpha\beta}\ (\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\delta\gamma}\ b_{2\xi}\,(\vec{p}_{f_2},\Gamma_{f_2})^m_{\gamma} \\
b^m_{2\xi;\,\gamma} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,T_{fii}(f_2,i_1)^{\cdot;m}_{\cdot;\gamma} = b_{1\xi;\,\gamma} &(29)\\
b^m_{2\phi;\,\delta\alpha\beta} &= \epsilon_{mnl}\,\left[\epsilon_{bca}\,\phi(f_1)^b_\kappa\ (C\,\Gamma_{f_1})^t_{\kappa\lambda}\ U(f_1,i_1)^{cn}_{\lambda\delta}\right]\ U(f_1,i_1)^{al}_{\alpha,\beta}\,\mathrm{e}^{i\vec{p}_{f_1}} &(30)\\
&= b^m_{1\phi;\,\delta\alpha\beta}
\end{aligned}$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

2.2 W-diagrams

$$T^{du}_{ffi} = \phi^d(f_1)\,\xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)\,. \tag{31}$$

Note, that we can use γ_5-hermiticity and write

$$T^{du}_{ffi} = \gamma_5\,\xi(f_1)\,\phi^u(f_2)^\dagger\,\gamma_5\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)\,, \tag{32}$$

where the Hermitean conjugation is with respect to spin-color indices. This means we can make corresponding replacements in $w_{n\xi}$ and $w_{n\phi}$ to separate contractions involving sequential propagators from those involving stochastic propagators (using $\phi^u = \phi^d$ in the case of Wilson-clover fermions). We can use $\gamma_5^t = \gamma_5$.

W_1

$$\begin{aligned}
(W_1)_{\alpha\beta} &= -w^m_{1\phi;\,\beta\alpha\delta}\ (C\,\Gamma_{i_1})_{\delta\gamma}\ w^m_{1\xi;\,\gamma}\,\mathrm{e}^{i\vec{p}_{i_1}} \\
w^m_{1\xi;\,\gamma} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;m}_{\cdot;\gamma} &(33)\\
w^m_{1\phi;\,\beta\alpha\delta} &= \epsilon_{mnl}\,\left[\epsilon_{bca}\,\phi(f_1)^b_\kappa\ (C\,\Gamma_{f_1})^t_{\kappa\lambda}\ U(f_1,i_1)^{cn}_{\lambda\beta}\right]\ T_{fii}(f_1,i_1)^{al}_{\alpha\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} &(34)
\end{aligned}$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

W_2

$$
\begin{aligned}
(W_2)_{\alpha\beta} &= -w^n_{2\phi;\,\gamma\alpha\delta}\ (C\,\Gamma_{i_1}(\vec{p}_{i_1}))_{\delta\gamma}\ w^n_{2\xi;\beta} \\
w^n_{2\xi;\beta} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;n}_{\cdot,\beta} = w^n_{1\xi;\beta} && (35) \\
w^n_{2\phi;\,\gamma\alpha\delta} &= \epsilon_{nml}\ \left[\epsilon_{cba}\,\phi(f_1)^c_\kappa\ (C\,\Gamma_{f_1})^t_{\kappa\lambda}\ U(f_1,i_1)^{bm}_{\lambda\gamma}\right]\ T_{fii}(f_1,i_1)^{al}_{\alpha\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (36) \\
&= w^n_{1\phi;\,\beta\alpha\delta}
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

W_3

$$
\begin{aligned}
(W_3)_{\alpha\beta} &= -w^n_{3\phi;\,\gamma\alpha\delta}\ (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\delta\gamma}\ w^n_{3\xi;\beta} \\
w^n_{3\xi;\beta} &= \xi^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;n}_{\cdot,\beta} = w^n_{1\xi;\beta} && (37) \\
w^n_{3\phi;\,\gamma\alpha\delta} &= \epsilon_{nml}\ \left[\epsilon_{cba}\,\phi(f_1)^c_\lambda\ (C\,\Gamma_{f_1})^t_{\lambda\kappa}\,T_{fii}(f_1,i_1)^{bm}_{\kappa\gamma}\right]\ U(f_1,i_1)^{al}_{\alpha\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (38)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

W_4

$$
\begin{aligned}
(W_4)_{\alpha\beta} &= -w^n_{4\phi;\,\gamma\alpha\beta}\ (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\delta\gamma}\ w^n_{4\xi;\delta} \\
w^n_{4\xi;\delta} &= \xi^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;n}_{\cdot,\delta} = w^n_{1\xi;\delta} && (39) \\
w^n_{4\phi;\,\gamma\alpha\beta} &= \epsilon_{nml}\ \left[\epsilon_{cba}\,\phi(f_1)^c_\lambda\ (C\,\Gamma_{f_1})^t_{\lambda\kappa}\,T_{fii}(f_1,i_1)^{bm}_{\kappa,\gamma}\right]\ U(f_1,i_1)^{al}_{\alpha\beta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (40) \\
&= w^n_{3\phi;\,\gamma\alpha\beta}
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Exchanging $\phi^d \to \phi^u$ In all the W diagrams we exchange $\phi^d\,\xi^\dagger \to (\gamma_5\,\xi)\ (\gamma_5\,\phi^u)^\dagger$ and thereby achieve the separation of sequential propagators T_{fii} and stochastic propagators $\phi^{u/d}$ into separate diagrams. Thus the contractions can be split accordingly into

- one part using only forward and sequential propagators and stochastic sources;
- one part using only forward and stochastic propagators.

In the $SU(2)$ symmetric case, this comes at no additional cost or complexity, since $\phi^u = \phi^d$. We just change $\Gamma_{f_2} \to \gamma_5\,\Gamma_{f_2}$ and $C\,\Gamma_{f_1} \to \gamma_5\,C\,\Gamma_{f_1} = C\,\gamma_5\,\Gamma_{f_1}$.

2.3 Z-diagrams

$$
P^{udu}_{fifi} = \phi^{u(\gamma)}(f_1)\ (\Gamma_{i_2}(\vec{p}_{i_2})\,\gamma_5)_{\gamma\delta}\ \phi^{u(\delta)}(f_2)^\dagger\,\gamma_5\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)\,. \tag{41}
$$

We omit the $(\Gamma_{i_2}(\vec{p}_{i_2})\,\gamma_5)$ in the following.

Z_1

$$
\begin{aligned}
(Z_1)_{\alpha\beta} &= -z^l_{1\xi;\,\gamma}\;(C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\gamma\delta}\;z^l_{1\phi;\,\alpha\delta\beta} \\
z^l_{1\xi;\,\gamma} &= \phi(f_2)^\dagger\,\gamma_5\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{;\,l}_{;\,\gamma} && (42)\\
z^l_{1\phi;\,\alpha\delta\beta} &= \phi(f_1)^a_\alpha\,\left[\epsilon_{abc}\,\epsilon_{lmn}\,D(f_1,i_1)^{bm\,t}_{\delta,\kappa}\;(C\,\Gamma_{f_1})^t_{\kappa\lambda}\;U(f_1,i_1)^{cn}_{\lambda\beta}\right]\,\mathrm{e}^{i\vec{p}_{f_1}} && (43)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Z_2

$$
\begin{aligned}
(Z_2)_{\alpha\beta} &= -z^l_{2\phi;\,\alpha\delta\gamma}\;(C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\gamma\delta}\;z^l_{2\xi;\,\beta} \\
z^l_{2\xi;\,\beta} &= z^l_{1\xi;\,\beta} && (44)\\
z^l_{2\phi;\,\alpha\delta\gamma} &= z^l_{1\phi;\,\alpha\delta\gamma} && (45)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Z_3

$$
\begin{aligned}
(Z_3)_{\alpha\beta} &= -z^n_{3\phi;\,\delta\alpha\gamma}\;(C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\gamma\delta}\;z^n_{3\xi;\,\beta} \\
z^n_{3\xi;\,\beta} &= z^n_{1\xi;\,\beta} && (46)\\
z^n_{3\phi;\,\delta\alpha\gamma} &= \epsilon_{nml}\;\left[\epsilon_{cba}\,\phi(f_1)^c_\kappa\;(C\,\Gamma_{f_1})_{\kappa\lambda}\;D(f_1,i_1)^{bm}_{\lambda\delta}\right]\;U(f_1,i_1)^{al}_{\alpha\gamma}\,\mathrm{e}^{i\vec{p}_{f_1}} && (47)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Z_4

$$
\begin{aligned}
(Z_4)_{\alpha\beta} &= -z^n_{4\xi;\,\gamma}\;(C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\gamma\delta}\;z^n_{4\phi;\,\delta\alpha\beta} \\
z^n_{4\xi;\,\gamma} &= z^n_{1\xi;\,\gamma} && (48)\\
z^n_{4\phi;\,\delta\alpha\beta} &= \epsilon_{nml}\;\left[\epsilon_{cba}\,\phi(f_1)^c_\kappa\;(C\,\Gamma_{f_1})_{\kappa\lambda}\;D(f_1,i_1)^{bm}_{\lambda\delta}\right]\;U(f_1,i_1)^{al}_{\alpha\beta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (49)\\
&= z^n_{3\phi;\,\delta\alpha\beta}
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

2.4 Conclusion

For the B, W, Z-type diagrams we need

B: $b_{1\xi}$, $b_{1\phi}$;

W: $w_{1\xi}$, $w_{1\phi}$, $w_{3\phi}$;

Z: $z_{1\xi}$, $z_{1\phi}$, $z_{3\phi}$.

diagram type	object	function
B	$b_{1\xi}$	$\mathcal{V}_3$
	$b_{1\phi}$	$\mathcal{V}_2$
W	$w_{1\xi}$	$\mathcal{V}_3$
	$w_{1\phi}$	$\mathcal{V}_2$
	$w_{3\phi}$	$\mathcal{V}_2$
Z	$z_{1\xi}$	$\mathcal{V}_3$
	$z_{1\phi}$	$\mathcal{V}_4$
	$z_{3\phi}$	$\mathcal{V}_2$

Note: Γ_{f_2} covers the pion vertex at sink (2-point function diagrams) as well as the vector current vertex at the current insertion (3-point function diagrams).

2.5 Reduction operations

2.5.1 Fermion - Propagator scalar products

These are of the form

$$V_3(x,y)^b_\beta = \mathcal{V}_3(\xi,\,S) = \sum_{a,\alpha}\sum_{\vec{x}} \xi(t_x,\vec{x})^{a*}_\alpha\,\Gamma\,\mathrm{e}^{i\vec{p}\vec{x}}\,S(t_x,\vec{x};t_y,\vec{y})^{ab}_{\alpha\beta} \tag{50}$$

This is the same we use for the meson-meson contractions.

2.5.2 Propagator - Propagator ϵ products

$$A(x,y)^{al}_{\alpha_{i_1},\alpha_{i_2}} = \sum_{b,c}\sum_{m,n}\sum_{\alpha_{i_3},\alpha_{i_4}} \epsilon_{abc}\,\epsilon_{lmn}\,S(x,y)^{bm}_{\alpha_1\alpha_2}\,S(x,y)^{cn}_{\alpha_3\alpha_4}\,\delta_{\alpha_{i_3},\alpha_{i_4}} \tag{51}$$

with $i_1,i_2,i_3,i_4\,\in\,\{1,2,3,4\}$ and $i_k\neq i_l\quad\forall k\neq l$ and $i_1<i_2,\,i_3<i_4$.

This is *qcd.quarkContractAB* or *_fp_eq_fp_eps_contractAB_fp*. With the special choice of $A=1,\,B=3$ we get $\mathcal{V}_4$.

$$V_4(x,y)^l_{\alpha\beta\gamma} = \mathcal{V}_4\left(\phi,\,S,\,S\right) = \sum_a \phi(x)^a_\alpha\;\;\mathrm{contract13}\left(S(x,y),\,S(x,y)\right)^{al}_{\beta\gamma}\,. \tag{52}$$

2.5.3 Fermion - Propagator ϵ product

This reduction will be useful for a fermion ϕ and a point-to-all propagator S.

$$V_1(x,y)^{am}_{\alpha_2} = \mathcal{V}_1\left(\phi,\,S\right) = \sum_{\alpha_1}\sum_{c,b} \epsilon_{cba}\,\phi(x)^c_{\alpha_1}\,S(x,y)^{bm}_{\alpha_1\alpha_2}\,. \tag{53}$$

The V_1s is a mixed type with 2 color indices and 1 Dirac index. (Can be done per spin-color component (α_2, m) if need be).

The final reduction step will then be of the form

$$V_2(x,y)^n_{\alpha_1\alpha_2\alpha_3} = \mathcal{V}_2\,(V_1,\,S) = \sum_a \sum_{l,m} \epsilon_{nml}\, V_1(x,y)^{am}_{\alpha_1}\, S(x,y)^{al}_{\alpha_2\,\alpha_3}\,, \tag{54}$$

which gives again a mixed type with 1 color index and 3 Dirac indices.
V_2 will be Fourier transformed and stored to disk.
We will thus need the functios $\mathcal{V}_1$, $\mathcal{V}_2$ in eqs. (53), (54) above.

In *QLUA* The function $\mathcal{V}_1$ and $\mathcal{V}_2$ are implemented as functions called by

- $\mathcal{V}_1$: ColorVector[36] $v1 = qcd.contractV1$ (DiracFermion F, DiracPropagator P)
  indexing $v1(a,b,\alpha) = v1[4\cdot(3\cdot a+b)+\alpha]$
- $\mathcal{V}_2$: ColorVector[192] $v2 = qcd.contractV2$ (ColorVector[36] $v1$, DiracPropagator P)
  indexing $v2(\alpha,\beta,\gamma,a) = v2[3\cdot(4\cdot(4\cdot\alpha+\beta)+\gamma)+a]$
- $\mathcal{V}_3$ $qcd.contractV3$
- $\mathcal{V}_4$ use existing $qcd.quarkContractAB$

To do If need be, generalized contraction functions for B_k, W_k, Z_k

2.6 Adding the ρ-channel, $I=1$, $I_3=0$

We add the diagrams for $\pi\pi$-scattering in the ρ-channel with $I=1$ and $I_3=0$. The interpolating fields of interest are given by

$$O_{\pi\pi}(t,\vec{p}_1,\vec{p}_2) = \pi^+(t,\vec{p}_1)\,\pi^-(t,\vec{p}_2) - \pi^-(t,\vec{p}_1)\,\pi^+(t,\vec{p}_2) \tag{55}$$

$$\pi^\pm(t,\vec{p}) = \sum_{\vec{x}} \bar{\psi}(t,\vec{x})\,\gamma_5\,\tau^\pm\,\psi(t,\vec{x})\,\mathrm{e}^{i\vec{p}\vec{x}}$$

$$O_\rho(t,\vec{P}) = \bar{\psi}(t,\vec{x})\,\Gamma\,\tau^3\,\psi(t,\vec{x})\,\mathrm{e}^{i\vec{P}\vec{x}} \tag{56}$$

2.6.1 Contractions in position space

$\pi\pi - \pi\pi$ We have two types contributions for $I=1$, $I_3=0$, the box diagram and the direct diagram.[1]

[1] Wick contractions also give a diagram with a charged pion propagating within the source and sink timeslice, respective. This diagram vanishes for $I=1$, $I_3=0$ for any momentum combination at source and sink.

From the *box diagram* we get

$$
\begin{aligned}
&C^{\text{box}}_{\pi\pi-\pi\pi} && (57)\\
&\quad = -\mathrm{tr}\left(U(x_{i_1},x_{i_2})\,\gamma_5\;D(x_{i_2},x_{f_2})\,\gamma_5\,U(x_{f_2},x_{f_1})\,\gamma_5\,D(x_{f_1},x_{i_1})\,\gamma_5\right)\\
&\qquad - \mathrm{tr}\left(U(x_{i_1},x_{f_1})\,\gamma_5\;D(x_{f_1},x_{f_2})\,\gamma_5\,U(x_{f_2},x_{i_2})\,\gamma_5\,D(x_{i_2},x_{i_1})\,\gamma_5\right)\\
&\quad = -2\,\mathrm{Re}\left(\mathrm{tr}\left(U(x_{i_1},x_{f_1})\,\gamma_5\;D(x_{f_1},x_{f_2})\,\gamma_5\,U(x_{f_2},x_{i_2})\,\gamma_5\,D(x_{i_2},x_{i_1})\,\gamma_5\right)\right)\\
&\quad = C^{\text{box}}_1 + C^{\text{box}}_2 && (58)
\end{aligned}
$$

Adding the momenta and using the sequential propagator and the stochastic source and propagator, we get

$$
\begin{aligned}
&C^{\text{box}}_2\,(t_f,t_i;\vec{p}_{f1}\,\vec{p}_{f2},\vec{p}_{i_2}) =\\
&\mathrm{tr}\left(U(x_{i_1},x_{f_1})\,\Gamma_{f_1}(\vec{p}_{f_1})\,D(x_{f_1},x_{f_2})\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(x_{f_2},x_{i_2})\,\Gamma_{i_2}(\vec{p}_{i_2})\,D(x_{i_2},x_{i_1})\,\Gamma_{i_1}(\vec{p}_{i_1})\right)\\
&\;= \mathrm{tr}\left(D(x_{f_1},x_{i_1})^{\tilde{\dagger}}\,\gamma_5\,\Gamma_{f_1}\,\mathrm{e}^{i\vec{p}_{f_1}\vec{x}_{f_1}}\;\bar{\phi}^r_{t_{f_2}}(x_{f_1})\,\xi^{r\tilde{\dagger}}_{t_{f_2}}(x_{f_2})\,\Gamma_{f_2}\,\mathrm{e}^{i\vec{p}_{f_2}\vec{x}_{f_2}}\,T^{ud}(x_{f_2};t_{i_2},\vec{p}_{i_2};x_{i_1})\,\Gamma_{i_1}\,\gamma_5\right)
\end{aligned}
\quad (59)
$$

$$
C^{\text{box}}_{\pi\pi-\pi\pi} = \left[C^{\text{box}}_2\,(t_f,t_i;\vec{p}_{f1}\,\vec{p}_{f2},\vec{p}_{i_2}) + C^{\text{box}}_2\,(t_f,t_i;-\vec{p}_{f1}\;-\vec{p}_{f2},-\vec{p}_{i_2})^*\,\prod_l \sigma^{5\tilde{\dagger}}_{\Gamma_l}\right]\,\mathrm{e}^{i\vec{p}_{i_1}\vec{x}_{i_1}}\;, \quad (60)
$$

with $\sigma^{5\tilde{\dagger}}_{\Gamma_l} = 1 \quad \forall\, l\, \in\, \{i_1,i_2,f_1,f_2\}$ in the present case.

Thus we can construct this contraction from

$$
\begin{aligned}
V_3 &: \xi^{r\tilde{\dagger}}_{t_{f_2}}(x_{f_2})\,\Gamma_{f_2}\,\mathrm{e}^{i\vec{p}_{f_2}\vec{x}_{f_2}}\,T^{ud}(x_{f_2};t_{i_2},\vec{p}_{i_2};x_{i_1}) && (61)\\
V_3^* &: D(x_{f_1},x_{i_1})^{\tilde{\dagger}}\,\gamma_5\,\Gamma_{f_1}\,\mathrm{e}^{i\vec{p}_{f_1}\vec{x}_{f_1}}\;\bar{\phi}^r_{t_{f_2}}(x_{f_1})\\
&\quad = \left[\bar{\phi}^r_{t_{f_2}}(x_{f_1})^{\tilde{\dagger}}\;(\gamma_5\,\Gamma_{f_1})^{\tilde{\dagger}}\;\mathrm{e}^{-i\vec{p}_{f_1}\vec{x}_{f_1}}\;D(x_{f_1},x_{i_1})\right]^* && (62)
\end{aligned}
$$

Note that $\Gamma_{i_1} = \gamma_5 = \Gamma_{f_1}$, thus $\Gamma_{i_1}\,\gamma_5 = \mathbb{1}$ and $(\gamma_5\,\Gamma_{f_1})^{\tilde{\dagger}} = \mathbb{1}$.

The *direct diagram* gives

$$
\begin{aligned}
&C^{\text{direct}}_{\pi\pi-\pi\pi} && (63)\\
&\quad = \mathrm{tr}\left(U(x_{i_1},x_{f_1})\,\Gamma_{f_1}(\vec{p}_{f_1})\,D(x_{f_1},x_{i_1})\,\Gamma_{i_1}(\vec{p}_{i_1})\right)\;\mathrm{tr}\left(U(x_{f_2},x_{i_2})\,\Gamma_{i_2}(\vec{p}_{i_2})\,D(x_{i_2},x_{f_2})\,\Gamma_{f_2}(\vec{p}_{f2})\right)
\end{aligned}
$$

The first factor on the right-hand side of (63) is obtained from point-source-propagator contractions, the second factor from oet-stochastic timeslice propagator contractions.

$\pi\pi - \rho$ We get

$$
\begin{aligned}
& C^{\text{triangle}}_{\pi\pi-\rho} \\
&\quad = \mathrm{tr}\left(U(x_{i_1}, x_{i_2})\,\Gamma_{i_2}(\vec{p}_{i_2})\,D(x_{i_2}, x_f)\,\Gamma_f(\vec{p}_f)\,D(x_f, x_{i_1})\,\Gamma_{i_1}(\vec{p}_{i_1})\right) \\
&\qquad - \mathrm{tr}\left(U(x_{i_1}, x_f)\,\Gamma_f(\vec{p}_f)\,U(x_f, x_{i_2})\,\Gamma_{i_2}(\vec{p}_{i_2})\,D(x_{i_2}, x_{i_1})\,\Gamma_{i_1}(\vec{p}_{i_1})\right) \\
&\quad = C^{\text{triangle}}_1 + C^{\text{triangle}}_2
\end{aligned} \tag{64}
$$

$$
C^{\text{triangle}}_2(t_f, t_i; \vec{p}_f, \vec{p}_{i_2}) = -\mathrm{tr}\left(D(x_f, x_{i_1})^{\tilde{\dagger}}\,\gamma_5\,\Gamma_f(\vec{p}_f)\,T^{ud}(x_f; t_i, \vec{p}_{i_2}; x_{i_1})\,\Gamma_{i_1}(\vec{p}_{i_1})\,\gamma_5\right) \tag{65}
$$

$$
\begin{aligned}
& C^{\text{triangle}}_{\pi\pi-\rho} \\
&\quad = \left[C^{\text{triangle}}_2(t_f, t_i; \vec{p}_f, \vec{p}_{i_2}) - \sigma^{5\tilde{\dagger}}_{\Gamma_{i_1}}\,\sigma^{5\tilde{\dagger}}_{\Gamma_{i_2}}\,\sigma^{5\tilde{\dagger}}_{\Gamma_f}\,C^{\text{triangle}}_2(t_f, t_i; -\vec{p}_f, -\vec{p}_{i_2})^*\right]\,\mathrm{e}^{i\vec{p}_{i_1}\vec{x}_{i_1}}
\end{aligned} \tag{66}
$$

Note, that with $\Gamma_{i_1} = \gamma_5 = \Gamma_{i_2}$ we have $\sigma^{5\tilde{\dagger}}_{\Gamma_{i_{1,2}}} = 1$ and with $\Gamma_f = \gamma_j$, $\gamma_j\,\gamma_0$ we have $\sigma^{5\tilde{\dagger}}_{\Gamma_f} = -1$.