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Math 136 HW 3
Due 02/13/2022 (extension to 02/16/2022)

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1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives. Let $x_0 \in \mathbb{R}^n$ be such that $\nabla f(x_0) = 0$ and that there exist two points $u, v \in \mathbb{R}^n$ s.t. the following are true:

$$\langle \nabla^2 f(x_0)u, u\rangle > 0,$$
$$\langle \nabla^2 f(x_0)v, v\rangle < 0.$$

Show that the point x_0 is neither a local maximum nor a local minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$.

NOTE: If $f : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O} \subset \mathbb{R}^2$, such a point will be a saddle point of f.

HINT: Consider applying the second-order approx. theorem.

Proof Because $x \in \mathbb{R}$, choose $r > 0$ such that the open ball $B_r(x_0) \subseteq \mathbb{R}^n$, $u, v \in \mathbb{R}^n$. Let $0 < \|u\| < r$ and $0 < \|v\| < r$. Then by the 2nd Order Approximation Theorem,

$$\lim_{u \to 0} \frac{f(x_0 + u) - [f(x_0) + \langle \nabla f(x_0), u\rangle + \frac{1}{2}\langle \nabla^2 f(x_0)u, u\rangle]}{\|u^2\|} = 0.$$

Since $\nabla f(x_0) = 0$,

$$\lim_{u \to 0} \frac{f(x_0 + u) - f(x_0)}{\|u\|^2} = \lim_{u \to 0} \frac{-\frac{1}{2}\langle \nabla^2 f(x_0)u, u\rangle}{\|u\|^2}$$

Because $\langle \nabla^2 f(x_0)u, u\rangle > 0$, $f(x_0 + u) - f(x_0) > 0$ if $f(x_0 + u) - f(x_0) > 0$, then $f(x_0 + u) > f(x_0)$.

$$\implies \lim_{v \to 0} \frac{f(x_0 + v) - [f(x_0) + \langle \nabla f(x_0), v\rangle + \frac{1}{2}\langle \nabla^2 f(x_0)v, v\rangle]}{\|v^2\|} = 0.$$

In turn,

$$\lim_{v \to 0} \frac{f(x_0 + v) - f(x_0)}{\|v\|^2} = \lim_{v \to 0} \frac{-\frac{1}{2}\langle \nabla^2 f(x_0)v, v\rangle}{\|v\|^2}.$$

Since $\langle \nabla^2 f(x_0)v, v\rangle < 0$, $f(x_0+v) - f(x_0) < 0$ and $f(x_0 - v) < f(x_0)$. Since $f(x_0+u) > f(x_0)$ and $f(x_0 - v) < f(x_0)$ for $\|u\|, \|v\| < r$, there must exist a neighborhood around x_0 whose $f(x_0)$ is both the smallest and largest values from two different directions, meaning that x_0 cannot be a local min nor local max.

□

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on an open interval I containing a point a. Suppose that f'' is continuous at $x = a$. Show that

$$\lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2} = f''(a).$$

HINT: Consider Theorem 14.19

Proof Using the difference quotient definition of a derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}$$

Differentiating once again and using the same rearrangement as above,

$$f''(x) = \frac{f'(x) - f'(x-h)}{h} \implies f''(x) = \frac{f(x+h) - f(x) - f(x) + f(x+h)}{h^2}.$$

Let there exist a point $x = a$. Then,

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a+h)}{h^2}.$$

□

3. Let

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that the directional derivative $\frac{\partial f}{\partial p}(0,0) = 0$ for each nonzero point $p \in \mathbb{R}^2$, but f is not differentiable at $(0,0)$.

WTS: $f(x,y)$ has first order partial derivatives at every point in $\mathbb{R}$. Determine if f is continuously differentiable if $(x, y \neq 0$, then $f(x,y)$ is rational with a nonzero denominator, i.e. it is continuously differentiable (CTN first order partials at (x,y)).

Proof Define the following:

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2y)(x^4+y^2) - (x^3y)(4x^3+y^2)}{(x^4+y^2)^2}, (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y} = \frac{(x^3(x4+y^2)) - ((2y+x^4)x^3y)}{(x^4+y^2)^2}, (x,y) \neq (0,0).$$

Note that if $\lim_{f\to 0} \frac{f(x+tp)-f(x)}{t}$ exists, then it is called the directional derivative.

$$\lim_{t\to 0} \frac{f(x+tp) - f(x)}{t} = \lim_{t\to 0} \frac{f(tp_1 + tp_2)}{t} = \lim_{t\to 0} \frac{(tp_1)^3 tp_2}{((tp_1)^4 + (tp_2)^2)t} = \lim_{t\to 0} \frac{t^3p_1^3 + tp_2}{t^5p_1^4 + t^3p_2^2} = 0$$

by asymptotic behavior.

Let $y = x$ be a sequence of points in $\mathbb{R}^2$ along the line $y = x$ which converges to $(0,0)$. Then,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4y^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^2} = 1.$$

Similarly, let $y = -x$ be a sequence of points in $\mathbb{R}^2$ along the line $y = -x$ which converges to $(0,0)$. As such,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4y^2} = \lim_{x\to 0} \frac{-x^4}{x^4 - x^2} = -1.$$

Because the limit must approach the same value from the left and the right, f is not differentiable at $(0,0)$.

$\square$

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives and that $x_0 \in \mathbb{R}^n$ is a point such that $\nabla f(x_0) = 0$. Suppose that there exists $\lambda > 0$ s.t. $\langle \nabla^2 f(x_0)y, y\rangle \geq \lambda \|y\|^2\rangle$ for all $y \in \mathbb{R}^2$. Prove that there exist two positive numbers $\delta, c > 0$ s.t. if $\|h\| < \delta$, then

$$f(x_0 + h) - f(x_0) \geq c\|h\|^2.$$

Proof Let there exist $\lambda > 0$ such that $\langle \nabla^2 f(x_0)y, y\rangle \geq \lambda\|y\|^2 \in \mathbb{R}^n$. Using the Second Order Approximation Theorem, for $x_0 \in \mathbb{R}^n$,

$$\lim_{h\to 0} \frac{f(x_0+h) - [f(x_0) + \langle \nabla f(x_0), h\rangle + \frac{1}{2}\langle \nabla^2 f(x_0)h, h\rangle]}{\|h^2\|} > 0.$$

Rearrange the above by letting $h \geq y$ such that

$$\lim_{y\to 0} \frac{f(x_0+y) - [f(x_0) + \langle \nabla f(x_0), y\rangle + \frac{1}{2}\langle \nabla^2 f(x_0)y, y\rangle]}{\|y^2\|} = 0$$

This implies,

$$\lim_{y\to 0} \frac{f(x_0+y) - f(x_0)}{\|y^2\|} = \lim_{y\to 0} \frac{\frac{1}{2}\langle \nabla^2 f(x_0)y, y\rangle}{\|y\|^2}.$$

Since $\frac{1}{2}\langle \nabla^2 f(x_0)y, y\rangle \geq \lambda\|y\|^2$ for some $\lambda > 0$, then $\frac{1}{2}\langle \nabla^2 f(x_0)y, y\rangle \geq \frac{1}{2}\lambda\|y\|^2$.
Hence, $\frac{\frac{1}{2}\langle \nabla^2 f(x_0)y,y\rangle}{\|y\|^2} \geq \frac{1}{2}\lambda$, which implies $\frac{f(x_0+y)-f(x_0)}{\|y\|^2} = \frac{\frac{1}{2}\langle \nabla^2 f(x_0)y,y\rangle}{\|y\|^2} \geq \frac{1}{2}\lambda$.
Thus, $f(x_0+y) - f(x_0) \geq \frac{1}{2}\lambda\|y\|^2 < \lambda\delta$. Because $\frac{1}{2}\|y\|^2 < \delta$ and $\frac{1}{2}\lambda\|y\|^2 < \lambda\delta$,

$$f(x_0+y) - f(x_0) < \lambda\delta$$

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If we let $c > 0$ such that $c < \frac{1}{2}\lambda$, then $f(x_0+y)-f(x_0) \geq \frac{1}{2}\lambda\|y\|^2 c\lambda\delta$, $f(x_0+y)-f(x_0) \geq c\|y\|^2$.

With $h > y$, write $f(x_0+h) - f(x_0) \geq c\|h\|^2$ if $\|h\| < \delta$ and $c < \frac{1}{2}\lambda$ given $c, d > 0$.

$\square$

5. Let f, g be twice continuously differentiable on $\mathbb{R}$ and define

$$F(x,y) = f(x+ay) + g(x-ay)$$

for $(x,y) \in \mathbb{R}^2$ and $a \in \mathbb{R}$. Show that F satisfies the "wave equation":

$$a^2 \frac{\partial^2 F(x,y)}{\partial x^2} = \frac{\partial^2 F(x,y)}{\partial y^2}$$

Proof First, note the following partial derivatives,

$$\frac{\partial F}{\partial x} = f'(x+ay) - g'(x-ay),$$

$$\frac{\partial^2 F}{\partial x^2} = f'(x+ay) - g'(x-ay),$$

$$\frac{\partial F}{\partial y} = af'(x+ay) - ag'(x-ay),$$

$$\frac{\partial^2 F}{\partial y^2} = a^2 f''(x+ay) + a^2 g(x-ay).$$

If $\frac{\partial^2 F}{\partial x^2} = f''(x+ay) + g''(x-ay)$ and $\frac{\partial^2 F}{\partial y^2} = a^2[f''(x+ay) + g''(x-ay)]$,

then, $a^2 \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2}$.

Therefore, $\frac{\partial^2 F}{\partial y^2} - a^2 \frac{\partial^2 F}{\partial x^2} = 0$ and $F(x,y) = f(x+ay) + g(x-ay)$ is a solution of the "wave equation."

$\square$