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Math 136 HW 4
Due 02/20/2022

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1. Define the mapping $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{F}(x, y) = (x^2 - y^2, 2xy)$$

for $(x, y) \in \mathbb{R}^2$.

(a) Find the points $(x_0, y_0) \in \mathbb{R}^2$ at which the derivative matrix $\mathbf{DF}(x_0, y_0)$ is invertible.

Proof Starting with the definition of the derivative matrix:

Let $\theta \subseteq \mathbb{R}^n$ be open and suppose $F : \theta \to \mathbb{R}^m$ has first-order partials at $x_0 \in \theta$. The derivative matrix at x_0 is the $m \times n$ matrix

$$\mathbf{DF}(x_0) = [\mathbf{DF}(x_0)_{ij}]_{i=1,j=1}^{m \times n}.$$

Where the matrix has m rows and n columns and $\mathbf{DF}(x)_{ij} = \frac{\partial F_i}{\partial x_j(x_0)}$. Because $\mathbf{F}$ is a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$, this looks like

$$\mathbf{DF}(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1(x)} & \frac{\partial F_1}{\partial x_2(x)} \\ \frac{\partial F_2}{\partial x_1(x)} & \frac{\partial F_2}{\partial x_2(x)} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

We know that the matrix is invertible when the determine is not 0. So,

$$\det \mathbf{F} \neq 0 \implies (2x)(2x) + (2y)(2y) \neq 0 \implies x^2 + y^2 \neq 0,$$

which is only true at $(0, 0)$. so the matrix is invertible at all points expect $(x, y) = (0, 0)$.

□

(b) Find the points $(x_0, y_0) \in \mathbb{R}^2$ at which the differential $\mathbf{dF}(x_0, y_0)$ is an invertible linear mapping.

Proof Starting with the definition of the differential:

Let $\theta \in \mathbb{R}^n$ be open and let $\mathbf{x} \in \theta$. Suppose $F : \theta \to \mathbb{R}^m$ has first-order partial derivatives at $\mathbf{x}$. The linear mapping $\mathbf{dF}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathbf{dF}(\mathbf{x})(\mathbf{h}) = \mathbf{DF}(\mathbf{x})\mathbf{h}$ is called the differential of $F : \theta \to \mathbb{R}^m$ at the point $\mathbf{x}$.
So, the differential $\mathbf{dF}(x) : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear mapping in all $(x_0, y_0) \in \mathbb{R}^2$ expect at $(x, y) = (0, 0)$.

□

2. Suppose the function $h : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Define the function $\eta : \mathbb{R}^3 \to \mathbb{R}$ by

$$\eta(u, v, w) = (3u + 2v)h(u^2 + v^2, uvw), (u, v, w) \in \mathbb{R}^3.$$

Find $D_1\eta(u, v, w)$, $D_2\eta(u, v, w)$, and $D_3\eta(u, v, w)$.

Proof Let $u^2 = x, v^2 = y, uvw = z$ for $h(u^2, v^2, uvw) \in \mathbb{R}^3$. Then, $\eta(u, v, w) = (3u + 2v)h(x, y, z)$. So,

$$D_1\eta(u, v, w) = 3h(u^2, v^2, uvw) + (6u^2 + 2uv)h_w(u^2, v^2, uvw) + (3uv + 2v^2w)h_z(u^2, v^2, uvw)$$

$$D_2\eta(u, v, w) = 3h(u^2, v^2, uvw) + (3uv + 2v^2)hy(u^2, v^2, uvw) + (3u^2w + 2uvw)h_z(u^2, v^2, uvw)$$

$$D_3\eta(u, v, w) = (3u^2v + 2uv^2)h_z(u^2v^2, uvw)$$

□

3. Suppose that the functions $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Express the following two limits in terms of partial derivatives of these functions:

   (a)
   $$\lim_{t \to 0} \frac{f(g(1 + t, 2), h(1 + t, 2)) - f(g(1, 2), h(1, 2))}{t}$$

   Proof Because there is a **t** added within each of the composite functions g, h, this is close to the definition of the derivative with respect to x in both g, h. Rewriting the above in terms of partials,

   $$D_1 f(g(1, 2), h(1, 2) \frac{\partial g(1, 2)}{\partial x} + D_2 f(g(1, 2), h(1, 2) \frac{\partial h(1, 2)}{\partial x}$$

   □

   (b)
   $$\lim_{t \to 0} \frac{f(g(1, 2), h(1, 2)) - f(g(1, 2), h(1, 2))}{t}$$

   Proof Here, t only varies the x component of the function. So,

   $$\frac{\partial}{\partial x} f(g(1, 2), h(1, 2)) = D_1 f(g(1, 2), h(1, 2)).$$

   □

4. Suppose that the continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is some positive number c such that

   $$f'(x) \geq c, \forall x \in R.$$

   Show that the function $f : \mathbb{R} \to \mathbb{R}$ is both one-to-one and onto.

   Proof Because $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, there exists a $c > 0$ such that $f'(x) \geq c$ forall $x \in R$. First showing that f is one-to-one, since $f'(x) \geq c, c > 0$, then $f'(x) > 0$. By contradiction, suppose $f(x) = f(y)$ and let $x, y \in \mathbb{R}$ such that $x < y$. Using the MVT,

   $$\frac{f(y) - f(x)}{y - x} = f'(x_0), x_0 \in [x, y].$$

Since $f(y) = f(x), f'(x_0) = 0$. But $f'(x) > 0$, which is a contradiction. Because $f(x) \neq f(y)$ for all $x, y \in \mathbb{R}$, f is injective.

Now showing that f is onto, let $b > 0$ such that $[x - b, x + b] \in \mathbb{R}$. The IVT states that $f(x_0) \in f(x_0) \in [f(x - b), f(y - b)]$. Hence, there exists $x_0 \in (x - b, x + b)$, making f onto.

□

5. Let A be an $n \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^n$. Define $F : \mathbb{R}^n \to \mathbb{R}^n$ by $F(\mathbf{x})\mathbf{b} + A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. F is called an affine function.

   (a) Find the derivative matrix DF and explain why F jas continuous first derivatives (i.e. $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$).

   Proof Note that $\mathbf{DF}$ is the matrix A with $F(\mathbf{x}) = \mathbf{b} + A(\mathbf{x})$ such that $DF(\mathbf{x})_{ij} = \frac{\partial F_i}{\partial x_j}(\mathbf{x})$ and $\mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$.

   Then,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & \ddots & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial dx_1} & \frac{\partial F_1}{\partial dx_2} & \frac{\partial F_1}{\partial dx_3} & \dots & \frac{\partial F_1}{\partial dx_n} \\ \frac{\partial F_2}{\partial dx_1} & \frac{\partial F_2}{\partial dx_2} & \frac{\partial F_2}{\partial dx_3} & \dots & \frac{\partial F_2}{\partial dx_n} \\ \vdots & & \ddots & & \\ \frac{\partial F_n}{\partial dx_1} & \frac{\partial F_n}{\partial dx_2} & \frac{\partial F_n}{\partial dx_n} & \dots & a_{nn} \end{bmatrix} = DF(\mathbf{x}).$$

   Therefore, $DF = A$. Since A exists and by the definition of an affine function, all elements are constants and F has constant first derivatives.

   □

   (b) Find conditions on A for which F is bijective.

   Proof For F to be bijective, the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ must be an invertible linear mapping. Further, Theorem 15.33 of Fitzpatrick states that if $\det A \neq 0$, then the mapping is invertible. Hence, the condition for when F is bijective is when $\det A \neq 0$.

   □

6. Now, let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$. Define $F : \mathbb{R}^n \to \mathbb{R}^n$ by $F(\mathbf{x}) = \mathbf{b} + A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (a) Assume $m > n$. Prove that F is not surjective.

   Proof Assume that $m > n$. Then, for all $\mathbf{y} \in \mathbb{R}^n$, there exists no solution $A\mathbf{x} = \mathbf{y}$ by a fact from elementary linear algebra. Let $\mathbf{z} \in \mathbb{R}^m$ such that $\mathbf{z} = \mathbf{b} + \mathbf{y}$. However, $y \neq A\mathbf{x}$ and there does not exist a solution $A\mathbf{x} = \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n$. Therefore, there does not exist an $\mathbf{x} \in \mathbb{R}^n$ such that $F(\mathbf{x}) = \mathbf{z}$ with $\mathbf{z} = \mathbf{b} + \mathbf{y}$, but $A\mathbf{x} = \mathbf{y}$. Thus, F is not surjective.

   □

   (b) Assume $m < n$. Prove that F is not injective.

Proof Assume $m < n$. Then, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions by a fact of elementary linear algebra. Let $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{y} = \mathbf{z}$ for $\mathbf{z} \in \mathbb{R}^m$. Here, $F(\mathbf{x}+\mathbf{y}) = \mathbf{b}+A(\mathbf{x}+\mathbf{y}) = \mathbf{b}+A\mathbf{x}+Ay = \mathbf{b}+A\mathbf{y} = \mathbf{b}+\mathbf{z}$. Also, $F(\mathbf{y}) = \mathbf{b}+A\mathbf{y} = \mathbf{b}+\mathbf{z}$. Hence, $F(y) = F(x+y)$, but $\mathbf{y} \neq \mathbf{x}+\mathbf{y}$. Therefore, F is not injective.

$\square$