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Math 136 HW 6
Due 03/06/2022

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1. Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int}_a^b f \leq 0 \leq \overline{\int}_a^b f.$$

Proof Note that $f : [a, b] \to \mathbb{R}$ is a bounded function on the the closed interval $[a, b]$ with

$$M = \sup_I f(x)dx, m = \inf_I f(x)dx.$$

If $p = \{I_1, I_2, ..., I_n\}$ is a partition of I, let $M_k = \sup_{I_k} f(x)dx$. Because $f(x)dx$ is bounded, we define the upper Riemann sum of $f(x)dx$ w.r.t. the partition p by

$$U(f, p) = \sum_{k=1}^{n} M_k|I_k| = \sum_{k=1}^{n} M_k(x_k, x_{k-1})$$

and the lower Riemann sum of f w.r.t. the partition by

$$L(f, p) = \sum_{k=1}^{n} m_k|I_k| = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

This first sum U is the sum of the area of the rectangle over the given interval above the graph of $f(x)dx$. The second sum L is the sum of the area of the rectangle that lies below the graph $f(x)dx$. Because

$$m(b - a) \leq L(f, p) \leq U(f, p) \leq M(b - a)$$

by the Lemma proved in class, define the upper Riemann integral of $f(x)dx$ over the closed interval $[a, b]$ by

$$U(f) = \overline{\int}_a^b f(x)dx = \inf_p U(f, p).$$

Symmetrically, the lower Riemann integral is defined by

$$L(f) = \underline{\int}_a^b f(x)dx = \sup_p L(f, p).$$

As an extension of the above Lemma,

$$L(f) \leq U(f) \implies \underline{\int}_a^b f(x)dx \leq \overline{\int}_a^b f(x)dx.$$

Then with $f(x) = 0$ for each $x \in [a, b]$,

$$\underline{\int_a^b} f(x)dx \leq 0 \leq \overline{\int_a^b} f(x)dx$$

□

2. Suppose that the two bounded functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ have the property that

$$g(x) \leq f(x)$$

for all $x \in [a, b]$.

(a) For P a partition of $[a, b]$, show that $L(g, P) \leq L(f, P)$.

Proof Consider the function $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be two functions s.t.

$$g(x) \leq f(x) \forall x \in [a, b].$$

Let P be a partition in $[a, b]$. Specifically,

$$P = \{a = x_0, x_1, x_2, ..., x_n = b\}.$$

Let m_i and m_i^* be an intersection of $f(x)$ and $g(x)$ in the sub-interval $[x_{i-1}, x_i] \in P$. As $g(x) \leq f(x) \forall x \in [a, b]$, $m_i \leq m_i^*, \forall i \in \mathbb{N} \implies m_i \Delta x_i \leq m_i^* \Delta x_i \forall i \in \mathbb{N}$.

Now using the definition of the Darboux sums, this implies that

$$\sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} m_i^* \Delta x_i \implies L(g, p) \leq L(f, p).$$

□

(b) Use (a) to show that $\underline{\int_a^b} g \leq \overline{\int_a^b} f$.

Proof From part (a), $L(g, p) \leq L(f, p), \forall P \in [a, b]$. Then noting the definition of an integral,

$$\int_a^b g(x)dx = \sup_p L(g, p)$$

$$\int_a^b f(x)dx = \sup_p L(f, p).$$

Therefore,

$$\sup_p L(g, p) \leq \sup_p L(f, p) \implies \int_a^b g(x)dx \leq \int_a^b f(x)dx.$$

□

3. Let $a < b$ and $c \in (a, b)$. Assume $f : [a, b] \to \mathbb{R}$ is bounded and integrable on both $[a, c]$ and on $[c, b]$. Prove that f is integrable on $[a, b]$ and that $\int_a^b f = \int_a^c f + \int_c^b f$. Note that this is the converse of the addition of endpoints theorem at the end of the Fitzpatrick.

Proof We assume that f is integrable on $[a, b]$. Then, given $\epsilon > 0$, there is a partition $P \in [a, b]$ s.t. $U(f; P) - L(f; P) < \epsilon$. Let $P^* = P \cup \{c\}$ be the refinement of P, i.e. the new partition adding c t the endpoints of P. Then $P^* = P_1 \cup P_2$, with $P_1 = P^* \cap [a, c]$ and $P_2 = P^* \cup [c, b]$ are partitions of $[a, c]$ and $[c, b]$ respectively. As such,

$$U(f, P^*) = U(f, P_1) + U(f, P_2), L(f, P^*) = L(f, P_1) + L(f, P_2).$$

Moving things around,

$$U(f, P_1) - L(f, P_1) = U(f, P^*) - L(f, P^*) - [U(f, P_2) - L(f, P_2)] \leq U(f, P) - L(f, P) < \epsilon.$$

This shows that f is integrable on $[a, c]$. Just change P_1 and P_2 to show the case where f is integrable over $[c, b]$.

Then, let $P = P_1 \cup P_2$. It follows that

$$U(f, P) - L(f, P) = U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \epsilon,$$

which proves that f is integrable $[a, b]$. If f is integrable, then with the partitions P, P_1, P_2,

$$\int_a^b f \leq U(f, P) = U(f, P_1) + U(f, P_2) < L(f, P_1) + L(f, P_2) + \epsilon < \int_a^c f + \int_c^b f + \epsilon.$$

Symmetrically,

$$\int_a^b f \geq L(f, P) = L(f, P_1) + L(f, P_2) > U(f, P_1) + U(f, P_2) - \epsilon > \int_a^c f + \int_c^b f - \epsilon.$$

Note that this is true for any positive ϵ and therefore, $\int_a^b f = \int_a^c f + \int_c^b f$.

□

4. Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (a) Assume f is CTN on $[a, b] f(x) \geq 0$ for all $x \in [a, b]$ and assume $f(x_0) > 0$ for some $x_0 \in [a, b]$. Prove that $\int_a^b f > 0$.

   Proof Let f be continuous at x_0. There exists a $\delta > 0$ such that

$$f(x) \geq \frac{f(x_0)}{2} \forall x \in [a, b] : |x - x_0| < \delta.$$

   Let δ be a small number such that $[x_0 - \delta, x_0 + \delta] \subseteq [a, b]$. Then,

$$\int_{x_0-\delta}^{x_0+\delta} f(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2} dx = 2\delta \frac{f(x_0)}{2} = \delta f(x_0) > 0$$

   as $\delta, f(x_0) > 0$. Therefore,

$$\int_{x_0-\delta}^{x_0+\delta} f(x)dx > 0$$

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Since $\int_a^{x_0-\delta} f(x)dx \geq 0$ and $\int_{x_0+\delta}^{b} f(x)dx \geq 0$. Furthermore because δ is small,

$$\int_a^{x_0} f(x)dx \geq 0 + \int_{x_0}^{b} f(x)dx \geq 0$$

$$\implies \int_a^{x_0} f(x)dx + \int_{x_0}^{b} f(x)dx = \int_a^b f(x)dx > 0.$$

□

(b) Is the conclusion in part (a) true if one assumes f is integrable on $[a,b]$, $f(x) \geq 0, \forall x \in [a,b]$ and assume $f(x_0) > 0$ for some $x_0 \in [a,b]$. Either prove it or give a counterexample.

Proof No. Counterexample:

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & 1 \end{cases}.$$

The point discontinuity is countable and integrable, but the function is not continuous.

□

5. Note *Riemann's Condition for Integrability*: Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is integrable iff for each $\epsilon > 0$, there is a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \epsilon$.

   Given the Riemann Condition, prove f is integrable.

   Proof Using the Riemann Condition, for any partition P,

$$L(f,P) \leq \underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx \leq U(f,P).$$

   Then, the Riemann Condition implies that

$$\overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \epsilon$$

   for all $\epsilon > 0$. In other words, the LHS of the above inequality is getting closer and closer to each other for a positive tolerance ϵ. Therefore,

$$\underline{\int_a^b} f dx = \overline{\int_a^b} f dx,$$

   which by the definition of integrability makes f integrable.

□

6. Let $f : [a,b] \to \mathbb{R}$ be Lipschitz. Prove, using the definition of integrability or Riemann's Condition (problem 5 above) that f is integrable. Please do not use the theorem that continuous functions are integrable.

Proof Let $\epsilon > 0$ and let $\delta > 0$ be the norm of the partition P. Then by the EVT (which we can use because Lipschitz functions are CTN and this is a closed, bounded interval), for each partition, choose $u_i, v_i \in [x_{i-1}, x_i]$ such that $m_i = f(u_i) \leq f(x) \leq f(v_i) = M_i, \forall x \in [x_{i-1}, x_i]$. Specifically,

$$\sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(f(v_i) - f(u_i))(x_i - x_{i-1})$$

Then, incorporating a constant M by the definition of a Lipschitz function,

$$\leq \sum_{i=1}^{n} M|v_i - u_i|(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M\delta(x_i - x_{i-1}) = M\delta \sum_{i=1}^{n}(x_i - x_{i-1}) = M\delta.$$

With $0 < \delta = \frac{\epsilon}{M}$, we see that $U(f,P) - L(f,P) < \epsilon$.

$\square$