Marcus Rose
Math 136 HW 1
Due 01/30/2022

---

1. Suppose that $f$ is differentiable on $(a, b)$, $f'$ is continuous on $(a, b)$, and that $f'(x) \neq 0$ for all $x \in (a, b)$.

   (a) Prove that $f$ is one-to-one $(a, b)$ and maps $(a, b)$ onto some open interval $(c, d)$.

   **Proof** A function $f$ is said to be one-to-one if $\forall x, y \in (a, b),\ f(a) = f(b) \implies a = b$.

   To begin, note that the derivative of $f$ is never 0. If $f' \neq 0$, then $f(x) > 0$ or $f(x) < 0$. That is, $f$ is strictly monotone. Since $f$ is monotone and continuous over this interval, for every $x \in (a, b)$ there exists a $y \in (c, d)$ such that $f(x) = y$. Because there cannot exist $x_1, x_2 \in (a, b)$ such that $f(x_1) = y = f(x_2)$ (by monotonicity), then $f$ is one-to-one.

   Further, because $f$ is strictly monotone and by the completeness axiom of the real numbers, the following limits both exist:

   $$\lim_{x \to a^+} = c, \lim_{x \to b^-} = d.$$

   Note from (*) mentioned in further discussion below, let $[a', b']$ be a closed interval such that $a' \to a$ and $b' \to b$. Because the function $f$ is said to be differentiable and $(f'(x) \neq 0)(\forall x \in (a, b))$, we need to evoke the IVT **over the closed interval** $[a', b']$ because $f$ is monotone, let $a'$ (resp. $b'$) exist such that $f(a') \in (c, c')$ (resp. $f(b') \in (d, d')$), where $[c', d'] \subset (c, d)$ because the two limits above exist. That is, $a'$ (resp. $b'$) is "close enough" to $a$ (resp. $b$). So, $f(a') \in (c, c')$ (resp. $f(b') \in (d, d')$. Therefore, $f$ maps onto $(c, d)$ because of monotonicity.

   $\square$

   **Further Discussion**

   Rolle's Theorem states that if a function $f$ is continuous on the closed, bounded interval $[a, b]$ and differentiable on the open interval $(a, b)$ such that $f(a) = f(b)$, then $f'(x) = 0$ for some $x$ with $a \leq x \leq b$. This is an immediate contradiction of the initial assumption that $f'(x) \neq 0$ **over the closed interval** $[a, b]$. However, the question asks about the open interval $(a, b)$ (note (*) above). By contradiction, let $x_1, x_2 \in (a, b)$ be two numbers such that $f'(x_1) < 0$ and $f'(x_2) > 0$. By the IVT, there must exist a third number $x_3$ in the given interval such that $f'(x_3) = 0$. This is a contradiction of the initial assumption (i.e. $f'(x) \neq 0$). Furthermore, $f'$ is either strictly positive or strictly negative over the interval. As such, $f$ is strictly monotonic.

   A function $f$ is said to map $(a, b)$ onto some open interval $(c, d)$ if $\forall x \in (a, b)$, there exists $y \in (c, d)$ such that $f(x) = y$. Over the closed interval, this onto condition is true by the IVT. However once again, the question asks about open intervals. Because the limits $c$ and $d$ defined above both exist by monotonicity, and once again establishing the closed, interval $[a', b'] : a' \to a, b' \to b$, then $f$ maps $(a, b)$ onto $(c, d)$ by the IVT and the Monotone Convergence Property.

   (b) Show that $(f^{-1})'$ exists and is continuous on $(c, d)$

**Proof** Given that $f$ is differentiable, we know that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Let

$$y = f(x), f^{-1}(y) = x$$
$$y_0 = f(x_0), f^{-1}(y_0) = x_0$$

be two invertible functions in the given domain. Then,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$$

It follows from the composition property for limits and the quotient property of limits and the definition of differentiability that

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}$$

Therefore, the inverse of the derivative $f$ is given by the above result. Given that the denominator in that fraction is never 0, this inverse's derivative exists.

To prove that $f$ is continuous over $(c, d)$, note the Corollary 4.12 in Fitzpatrick. By composition of functions, note that the $f'$ is continuous and $f$ is continuous, therefore, the composition of two continuous functions is also continuous.

□

2. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = 0$. Also suppose that for each natural number $n$, $f(\frac{1}{n}) = 0$.

   (a) Prove that $f(0) = 0$

   **Proof** Note that because for each natural number $n$, $f(\frac{1}{n}) = 0$,

$$f(0) = 0 \implies f(0) = f(\lim_{n \to \infty} \frac{1}{n}) = \lim_{n \to \infty} f(\frac{1}{n}) = 0$$

□

   (b) Prove that $f'(0) = 0$

   **Proof** Let $x_n$ be the given sequence $\frac{1}{n}, n = 1, 2, ....$Because in part (a) we showed that $f(0) = 0$ and we know that $f$ is differentiable at $x = 0$, i.e. $f'(0)$ exists,

$$\lim_{n \to \infty} \frac{f(x_n) - f(0)}{x_n - 0} = f'(0)$$

Following the given path to 0 for $x_n$ along the sequence $\frac{1}{n}, n = 1, 2, 3, ...$ (the way this sequence is defined in the question, meaning that it is zero at all $n \in \mathbb{N}$),

$$\lim_{n \to \infty} \frac{f(\frac{1}{n}) - 0}{\frac{1}{n}} = \lim_{n \to \infty} \frac{0}{\frac{1}{n}} = 0 = f'(0)$$

Since each $f(\frac{1}{n})$ is given to be 0, and $\lim 0 = 0$, $f'(0) = 0$.

$\square$

3. Decide whether

$$\lim_{(x,y)\to(0,0)} \frac{\tan(x^4+y^2)}{x^4+y^2}$$

exists and prove your result. If the limit exists, also determine the value of the limit.

**Proof** Clearly, plugging in 0 for each x and y component of the two-variable function produces an indeterminate result. In order to determine the value of the limit,

First, note that

$$\lim_{(x,y)\to(0,0)} \frac{\tan(x^4+y^2)}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^2)}{x^4+y^2} \frac{1}{\cos(x^4+y^2)}$$

.

Note that the first term in the expanded limit can be written as the composition $g(f(x,y))$, where

$$g(z) = \begin{cases} \frac{\sin(z)}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is continuous and $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Hence, the limit of the first term approaches $g(0) = 1$ as $z \to 0$ (this is from Calc I... using L'Hopitals you can see that in one variable, the limit of $\frac{sin(z)}{z}$ can be evaluated at $\frac{cos(z)}{1}$).

Note the second term in the expanded limit is simply $\frac{1}{\cos(x^4+y^2)}$, or $\sec(x^4+y^2)$. This term is continuous and approaches 1 as $(x,y) \to (0,0)$ using a similar composition. Hence, the limit of the overall term tends to $1 \times 1 = 1$.

$\square$

4. Prove the following

Let $A$ be a subset of $\mathbb{R}^n$. and let $x_*$ be a limit point of $A$. For a function, $f : A \to \mathbb{R}$ and a real number L, the following two assertions are equivalent:

(a)

$$\lim_{\mathbf{x}\to x_*} f(\mathbf{x}) = L$$

(b) For each positive number $\epsilon$, there exists a positive number $\delta$ such that

$$|f(\mathbf{x}) - L| < \epsilon$$

if $\mathbf{x}$ is in $A\backslash\{x_*\}$ and $\text{dist}(\mathbf{x}, x_*) < \delta$.

**Proof** The following proof requires equality, so I will begin my proving the forward direction (a) $\rightarrow$ (b), i.e. given that $\lim_{x \rightarrow x_*} f(\mathbf{x}) = L$, show that $\forall \epsilon > 0$, there exists a positive number $\delta$ such that $|f(\mathbf{x}) - L| > \epsilon$ if $\mathbf{x}$ is in $A \backslash \{x_*\}$ and $(\mathbf{x}, x_*) > \delta$

Suppose by contradiction that $\exists \epsilon_*$ such that for all $\delta > 0$, there exists $\mathbf{x} \in A \backslash \{x_*\}$ such that

$$\|\mathbf{x} - x_*\| \leq \delta \implies |f(\mathbf{x}) - L| \geq \epsilon_*$$

Let $\delta$ be the sequence $\frac{1}{n}$ for all natural numbers $n$. Then,

$$\|\mathbf{x} - x_*\| \leq \frac{1}{n} \implies \|\mathbf{x}_2 - x_*\| \leq \frac{1}{2} \implies \|f(\mathbf{x}) - L\| \geq \epsilon_*$$

for all natural numbers $n$. Now, construct a sequence $\{\mathbf{x}_k\}$ such that

$$\begin{aligned}
\exists \mathbf{x}_1 \in A \backslash \{x_*\} : \|\mathbf{x}_1 - x_*\| < 1 &\implies |f(\mathbf{x}_1) - L| \geq \epsilon_*, \\
\exists \mathbf{x}_2 \in A \backslash \{x_*\} : \|\mathbf{x}_2 - x_*\| < \frac{1}{2} &\implies |f(\mathbf{x}_2) - L| \geq \epsilon_*, \\
\exists \mathbf{x}_3 \in A \backslash \{x_*\} : \|\mathbf{x}_3 - x_*\| < \frac{1}{3} &\implies |f(\mathbf{x}_3) - L| \geq \epsilon_*,
\end{aligned}$$

so on and so forth. Furthermore,

$$0 \leq \|\mathbf{x}_n - x_*\| < \frac{1}{n}.$$

For all natural numbers $n$, $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, so using the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} \|\mathbf{x}_n - x_*\| = 0 \implies \lim_{n \rightarrow \infty} \mathbf{x}_n = x_* \implies (\exists \epsilon_* > 0) : (|f(\mathbf{x}_n) - L| \geq \epsilon_0, \forall n \in \mathbb{N}) \implies \lim_{n \rightarrow \infty} f(\mathbf{x}_n) \neq L.$$

This contradicts the original statement (the converse of the below) and therefore,

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\|\mathbf{x} - x_*\| \leq \delta) \implies (\|f(\mathbf{x}) - L\| \leq \epsilon)$$

Now proving (b) $\rightarrow$ (a), let $\{x_k\} A \backslash \{x_*\}$ such that $\lim_{k \rightarrow \infty} x_k = x_*$. Let $\epsilon > 0$. From forward direction above, there must exist $\delta > 0$ such that $(\forall x \in A \backslash \{x_0\})(\|x - x_k\| < \delta) \implies (\|f(x) - L\| < \epsilon)$. For $\delta$, there exists $K \geq 1$ such that $\forall k \geq K$, $\|x_k - x_*\| \leq \delta$ by the definition of convergence, i.e. $x_k \rightarrow x_*$. By this definition,

$$\lim_{x \rightarrow x_*} f(x) = L$$

$\square$

5. Let $I$ be a neighborhood of $x_0$ and let $f^{-1} : f(I) \rightarrow \mathbb{R}$ be continuous, strictly monotone, and differentiable at $x_0$. Assume that $f'(x_0) = 0$. Use the characteristic property of the inverses,

$$f^{-1}(f(x)) = x, \forall x \in I,$$

and the Chain Rule to prove that the inverse function $f^{-1} : f(I) \rightarrow \mathbb{R}$ is not differentiable at $f(x_0)$.

**Proof** Given $f^{-1}(f(x)) = x$, we can differentiate both sides to get

$$f^{-1'}(f(x))f'(x) = 1 \implies f^{-1'}(f(x_0))f'(x_0) = 1$$

when evaluated at $x_0$ (chain rule). If $f'(x_0)$, then $0 \neq 1$ and $f^{-1'}(f(x_0))$ does not exist. Therefore, $f^{-1}$ is not differentiable at $f(x_0)$.

□

6. Let the function $f : \mathbb{R} \to \mathbb{R}$ have the property that there is a positive number $c$ such that

$$|f(u) - f(v)| \leq c(u - v)^2$$

for all $u, v \in \mathbb{R}$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is constant.

**Proof** Let $c \in \mathbb{R}$. Take $\Delta c \in \mathbb{R}$ and $u = c + \Delta c, v = c$ from the provided inequality. Then,

$$|f(c + \Delta c) - f(c)| \leq c(\Delta c)^2$$

$$-c(\Delta c) < \frac{f(c + (\Delta c)) - f(c)}{h} \leq c(\Delta c)$$

Because

$$\lim_{\Delta c \to 0^+} (-c(\Delta c)) = \lim_{\Delta c \to 0^+} (c(\Delta c)) = 0,$$

we conclude that

$$\lim_{\Delta c \to 0^+} \frac{f(c + \Delta c) - f(c)}{\Delta c} = 0.$$

Similarly,

$$\lim_{\Delta c \to 0^-} \frac{f(c + \Delta c) - f(c)}{\Delta c} = 0.$$

.

Therefore,

$$f'(c) = \lim_{\Delta c \to 0} \frac{f(c + \Delta c) - f(c)}{\Delta c} = 0.$$

This is the logical equivalent to the definition of the derivative in Fitzpatrick (i.e. this is the difference quotient definition), and since $c \in \mathbb{R}$ is arbitrary, $f'(x) = 0$ for every $x \in \mathbb{R}$. Therefore, $f$ is constant on $\mathbb{R}$.

□