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Math 136 HW 2
Due 02/06/2022

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1. Let $A \subset \mathbb{R}^n$ and let $\mathbf{x}_*$ be a limit point of A that is in A. Let $f : A \to \mathbb{R}^m$. Prove that f is continuous at $\mathbf{x}_*$ if and only if

$$\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x}) = f(\mathbf{x}_*).$$

Proof $(\implies)$ WTS: If f is continuous at $\mathbf{x}_*$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x}) = f(\mathbf{x}).$$

If f is continuous at $\mathbf{x}_*$, then by the $\epsilon - \delta$ condition of continuity

$$(\forall\epsilon > 0)(\exists\delta > 0) : (\|\mathbf{x}_* - \mathbf{x}\| < \delta) \implies (\|f(\mathbf{x}_*) - f(\mathbf{x})\| < \epsilon).$$

Because f is CTN at $\mathbf{x}_*$, let $\{\mathbf{x}_n\}$ be the sequence in A which converges to $\mathbf{x}_*$. Then,

$$\lim_{n\to\infty} \text{dist}(\mathbf{x}_n, \mathbf{x}_*) = 0.$$

By Theorem 11.11 in Fitzpatrick, this is equivalent to saying

$$(\exists\delta > 0) : (\text{dist}(\mathbf{x}, \mathbf{x}_*) < \delta)(\forall x \in A\backslash\{\mathbf{x}_*\}) \implies (\text{dist}(f(\mathbf{x}), f(\mathbf{x}_*)) < \epsilon)$$

which is the logical equivalent of

$$|f(x) - f(x_*)| < \epsilon.$$

Let $f(x_*) = l$ and using Theorem 13.7 in Fitzpatrick, the above statements can be linked and are equivalent to saying

$$\lim_{n\to\infty} f(x_n) = f(x_*) if \lim_{n\to\infty} x_n = x_*.$$

This implies

$$\lim_{x\to\mathbf{x}_*} = l = f(\mathbf{x}_*)$$

$(\impliedby)$ WTS: If

$$\lim_{\mathbf{x}\to\mathbf{x}_*} f(x) = f(\mathbf{x}_*),$$

then f is continuous at $\mathbf{x}_*$.

From Theorem 13.7, we know that if $\mathbf{x}_*$ is a limit point of A, then $(\forall\epsilon > 0)(\exists\delta > 0)$: $(|f(x) - f(x_*)| < \epsilon$ if $x \in A\backslash\{\mathbf{x}_*\}$ and $\text{dist}(x, \mathbf{x}_* < \delta$. This is the definition of a limit point called $\mathbf{x}_*$ and $|f(x) - f(\mathbf{x}_*) < \epsilon| = \text{dist}(f(x), f(\mathbf{x}_*) < \epsilon)$ which is necessary for continuity. Hence, f is continuous.

□

2. Let B be an open ball in $\mathbb{R}^n$ and let $f : B \to \mathbb{R}$ have continuous first order partials ($f \in C^1(B)$). Let $M > 0$ and assume $(\|\nabla f(\mathbf{x})\| \leq M)(\forall \mathbf{x} \in B)$.

   (a) Prove for each $\mathbf{x}, \mathbf{y} \in B$ that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq M\|\mathbf{x} - \mathbf{y}\|$$

Proof First note that the open ball B is convex, so create a line segment joining two points $x, y \in B$. Also note the definition of the MVT in one dimension:

Because M is such that $\|f'(\mathbf{x})\| \leq M$ for all $x \in B$. Then for x and y in B, we have

$$f(y) - f(x) = f'(c)(y - x)$$

for some c between x and y by the MVT.

For the line segment joining x, y there is a number c that lies inside the interval by the properties of connectedness and convexity. Since f is continuously differentiable, apply the MVT for higher dimensions of $\mathbb{R}$:

$$f(x) - f(y) = \langle \nabla f(c), (x - y) \rangle$$
$$f(x) - f(y) = \nabla f(c) \cdot (x - y)$$

Now from the Cauchy-Schwartz inequality (because we proved this in Real Analysis I, I will not prove the following fact here):

$$|\nabla f(c)(x - y)| \leq |\nabla f(c)| \|x - y\|.$$

Furthermore, $f(x) - f(y) \leq |f(x) - f(y)| \leq |\nabla f(c)\|x - y\|$, and because $(\|\nabla f(x)\| \leq M)(\forall x \in B)$, $|f(x) - f(y)| \leq M\|x - y\|$.

□

   (b) Does this proof work if B is replaced by an arbitrary open set? If so, prove the statement in part (a) if f has an arbitrary open set. If not, explain what property of B you used in your proof that does not hold for arbitrary open sets.

Explanation

From the first part above, we used the fact that open balls are convex sets and hence connected, so we are able to parameterize a line segment and apply the mean value theorem for a function of one variable to then transition to higher dimensions. For any arbitrary **open set**, there is no assurance that said set is convex and we are unable to parameterize a line within the set like the one described in part (a).

3. Let f and g be functions $\mathcal{O}$ to $\mathbb{R}$ and assume g is a kth order approximation to f at $\mathbf{x}_* \in \mathcal{O}$. Prove the following

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\mathbf{x} \in \mathcal{O} \backslash \{\mathbf{x}_0\})(\|\mathbf{x} - \mathbf{x}_0\| < \delta) \implies |f(\mathbf{x}) - g(\mathbf{x}) \leq \epsilon\|\mathbf{x} - \mathbf{x}_0\|^k$$

Proof Two functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{O} \to \mathbb{R}$ are said to be kth-order approximations of one another at the point $\mathbf{x}$ provided that

$$\lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}+\mathbf{h})}{\|\mathbf{h}\|^k} = 0.$$

Define an auxiliary function as follows,

$$f(h) = \begin{cases} \frac{f(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0+\mathbf{h})}{\|\mathbf{h}\|^k} & h \neq 0 \\ 0 & h = 0 \end{cases}.$$

From Theorem 14.2 in Fitzpatrick (First Approximation Thm), if $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and f as defined as the above composition is continuously differentiable with $\mathbf{x}$ being a point in $\mathcal{O}$. Then,

$$\lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle]}{\|\mathbf{h}\|} = 0.$$

As such, there must exist a value containing $h = 0$. Now using the $\epsilon - \delta$ definition of continuity,

$$(|\mathbf{h} - 0| < \delta) \implies (|\frac{f(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}+\mathbf{h})}{\|\mathbf{h}\|^k} - 0| < \epsilon)$$
$$(|\mathbf{h}| < \delta) \implies (|\frac{f(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}+\mathbf{h})}{\|\mathbf{h}\|^k}| < \epsilon)$$

From the difference quotient definition of the derivative, we know that $h = \mathbf{x} - \mathbf{x}_*$. Rewriting the above,

$$(|\mathbf{x} - \mathbf{x}_*| < \delta) \implies (|\frac{f(\mathbf{x} - \mathbf{x}_*) - g(\mathbf{x} - \mathbf{x}_*)}{\|\mathbf{x} - \mathbf{x}_*\|^k}| < \epsilon)$$

Rearranging,

$$(|\mathbf{x} - \mathbf{x}_*| < \delta) \implies (|f(\mathbf{x}) - g(\mathbf{x})| < \epsilon \|\mathbf{x} - \mathbf{x}_0\|^k)$$

With the final implication resulting from Theorem 13.7 in Fitzpatrick.

□

4. Assume $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$ and assume $\mathbf{b} \in \mathbb{R}^n$ satisfies the above definition for f at $\mathbf{x}_0$.

   (a) Explain why $g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0 \rangle$ is an affine first-order approximation to f at $\mathbf{x}_0$.

   Proof Let $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$ s.t. $f(\mathbf{x}_0+\mathbf{h}) = f\mathbf{x})$ and $g(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x}_0 + \mathbf{h} - \mathbf{x}_0 \rangle$. Then, $g(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{h} \rangle$

$$\implies \lim_{h \to 0} \frac{f(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0+\mathbf{h})}{\|\mathbf{h}\|} = \frac{f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{h} \rangle}{\|\mathbf{h}\|} = 0,$$

   as given in the definition provided in the question. As such, $g(\mathbf{x}_0 + \mathbf{h}) = g(\mathbf{x})$ is a first-order approximation to $f(x)$ at point x_0, by Theorem 14.2 in Fitzpatrick.

For g to be affine means g must be equal to a value $c + \sum_{i=1}^{\infty} a_i u_i$. Again, note that $g(\mathbf{x}_0) = f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0 \rangle$.

Recall the above declaration: $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$. Then,

$$g(\mathbf{x}_0) = f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{h} \rangle = f(\mathbf{x}_0) + \sum_{i=1}^{n} \mathbf{b}_i \mathbf{h}_i.$$

This matches the definition of affine and as such, g is an affine first-order approximation to f at $\mathbf{x}_0$.

$\square$

(b) If $\mathbf{b} = (b_1, b_2, ..., b_n)$ show that f has all first partial derivatives at $\mathbf{x}_0$ and $\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = b_i$ for each $i = 1, 2, ..., n$.

Proof Let $i \in \{1, 2, ..., n\}$ and let $\{x_k\} \in \mathbb{R}$ be a sequence s.t. $x_k \to 0$ $\forall k \in \mathbb{N}$ $x_k \neq 0$ and $\forall k \in \mathbb{N}$, $x_0 + t_k e_i \in \mathcal{O}$. Let $\{h_k\} = \{t_k e_i\}$. We know there exists a vector $\mathbf{b} \in \mathbb{R}^n$ s.t.

$$\lim_{\mathbf{h} \to \mathbf{0}} f(\mathbf{x}_0 + \mathbf{h}) - \frac{[f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{h} \rangle]}{\|\mathbf{h}\|}.$$

$$\implies {}^{(*)} \lim_{k \to \infty} \left| \frac{f(\mathbf{x}_0 + t_k e_i) - [f(\mathbf{x}_0) + \langle \mathbf{b}, t_k \mathbf{e}_i \rangle]}{t_k \mathbf{e}_i} \right| = 0$$

The partial derivative defined in the direction e_i is defined as

$$\frac{\partial f}{\partial \mathbf{e}_i}(\mathbf{x}_0) = \lim_{k \to \infty} \frac{f(\mathbf{x}_0) + t_k \mathbf{e}_i) - f(\mathbf{x}_0)}{t_k}$$

Let $\mathbf{x} = \mathbf{x}_0 + t_k e_i$. Note that x_0 is in $\mathcal{O}$ and as such, there exists an ball B of radius r about the point in $\mathcal{O}$. Call this ball $B(\mathbf{x}_0, r) \subseteq \mathcal{O}$. Hence,

$$\|t_k\| < r, \|\mathbf{x} - \mathbf{x}_0\| = \|\mathbf{x}_0 + t_k \mathbf{e}_i - \mathbf{x}_0\| = \|t_k \mathbf{e}_i\| = \|t_k\| \|\mathbf{e}_i\| = \|t_k\| < r$$

Then,

$$\mathbf{x}_0 + t_k \mathbf{e}_i \in B(\mathbf{x}_0, r) \subseteq \mathcal{O}.$$

Furthermore, as $t_k \to 0$, $\mathbf{x} \to \mathbf{x}_0$. Because f is differentiable at $x_0 \in \mathcal{O}$, by (*) above,

$$\lim_{k \to \infty} \left| \frac{f(\mathbf{x}_0 + t_k \mathbf{e}_i) - [f(\mathbf{x}_0) + t_k \sum_{i=1}^{n} b_i \mathbf{e}_i]}{t_k} \right| = \lim_{k \to \infty} \frac{f(\mathbf{x}_0 + t_k \mathbf{e}_i) - f(\mathbf{x}_0)}{t_k} - \sum_{i=1}^{n} b_i \mathbf{e}_i = 0$$

$$\implies \lim_{k \to \infty} \frac{f(\mathbf{x}_0 + t_k \mathbf{e}_i) - f(\mathbf{x}_0)}{t_k} = \sum_{i=1}^{n} b_i \mathbf{e}_i$$

Now, there must exist a $\frac{\partial f}{\partial \mathbf{e}_i} \mathbf{x}_0$ where $\frac{\partial f}{\partial \mathbf{e}_i} \mathbf{x}_0 = \sum_{i=1}^{n} b_i \mathbf{e}_i$.

From here, WTS: $\frac{\partial f}{\partial x_i} \mathbf{x}_0 = b_i$ for each $i = 1, 2, ..., n$. Because

$$\frac{\partial f}{\partial \mathbf{e}_i}\mathbf{x}_0 = \sum_{i=1}^{n} b_i \mathbf{e}_i,$$

by Theorem 13.16 in Fitzpatrick,

$$\frac{\partial f}{\partial e_i}\mathbf{x}_0 = \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial f}{\partial x_i}\mathbf{x}_0.$$

Therefore, $b_i = \frac{\partial f}{\partial x_i}\mathbf{x}_0, \ \ i = 1, 2, ..., n.$

□

(c) Prove that f is continuous at x_0.

Proof Because f is differentiable at $\mathbf{x}_0$ and letting $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - [f(\mathbf{x}_0) + \sum_{i=1}^{n} b_i(x_i - x_0)]}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

$$\implies f(\mathbf{x}) - f(\mathbf{x}_0) = f(\mathbf{x}) - [f(\mathbf{x}_0) + \sum_{i=1}^{n} b_i(x_i - x_0)] + \sum_{i=1}^{n} b_i(x_i - x_0)$$

$$= \sum_{i=1}^{n} b_i(x_i - x_0) + f(\mathbf{x}) - f(\mathbf{x}_0) - \sum_{i=1}^{n} b_i(x_i - x_0).$$

The first step of this proof is using the Triangle Inequality as follows,

$$0 \leq \|f(\mathbf{x}) - f(\mathbf{x}_0)\| = |\langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle + f(\mathbf{x}, f(\mathbf{x}_0) - \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle|$$

$$\leq |\langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle| + |f(\mathbf{x}) - f(\mathbf{x}_0) - \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle|$$

$$\leq \|\mathbf{b}(\mathbf{x} - \mathbf{x}_0)\| + \|f(\mathbf{x}_0) - \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle\|$$

$$= \|\mathbf{b}\|\|\mathbf{x} - \mathbf{x}_0\| + |f(\mathbf{x}) - f(\mathbf{x}_0) - \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle|$$

with that last inequality resulting from the Cauchy-Schwarz Inequality. Rearranging,

$$\|\mathbf{x} - \mathbf{x}_0\|(\|\mathbf{b}\| + \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle|}{\|\mathbf{x} - \mathbf{x}_0\|})$$

From here, as $\mathbf{x} \to \mathbf{x}_0$, the above approaches 0 because

$$\frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle|}{\|\mathbf{x} - \mathbf{x}_0\|} \to 0$$

as $x \to x_0$ and $0(\|\mathbf{b}\| + 0) \to 0$.

Finally, using the Squeeze Theorem, $\lim_{\mathbf{x}\to\mathbf{x}_0} \|f(\mathbf{x}) - f(\mathbf{x}_0)\| = 0$ and $f(\mathbf{x}) \to f(\mathbf{x}_0)$ as $\mathbf{x} \to \mathbf{x}_0$. Therefore, f is continuous.

□

5. Let a, b, c be real numbers. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{ax^2 + bxy + y^2}{\sqrt{x^2 + y^2}} = 0$$

Proof By Corollary 14.3 in Fitzpatrick, suppose $\mathcal{O}$ an an open subset of the plane $\mathbb{R}^2$ that contains point $(x_0, y_0) \in \mathcal{O}$ and that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then, there is a tangent plane to the graph of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$. This tangent plane is the graph of the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined for $(x, y) \in \mathbb{R}^2$ by

$$\psi(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

By Theorem 14.2 in Fitzpatrick (First Order Approximation),

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \psi(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

Now, consider the continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for all $(x, y) \in \mathbb{R}^2$

$$f(x, y) = ax^2 + bxy + y^2$$

and replacing $(x, y) = (0, 0)$ in $f(x, y) = ax^2 + bxy + y^2$,

$$f(0, 0) = 0$$

$\square$

6. Define

$$g(x, y) = \begin{cases} \frac{x^2y^4}{(x^2+y^2)} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives. Is the function $g : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable?

Proof For $(x, y) \neq (0, 0)$, let D be the partial derivative. Calculate the partials as follows,

$$\frac{\partial g}{\partial x} = \frac{2xy^6}{(x^2 + y^2)^2}$$
$$\frac{\partial g}{\partial y} = \frac{4x^4y^3 + 2x^2y^5}{(x^2 + y^2)^2}$$

Now, using the definition of partial derivatives to evaluate the derivatives at $(0, 0)0$,

$$\frac{\partial g(0, 0)}{\partial x} = \lim_{t\to 0} \frac{0 - 0}{t} = 0$$

$$\frac{\partial g(0,0)}{\partial y} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Because both partials at $(0,0)$ are rational, they are continuous (excluding asymptotes, i.e. $(0,0)$, which is a condition that we will check further down). To check continuity of both partials, check

$$\lim_{(x,y)\to(0,0)} \frac{\partial g_i}{\partial x} = 0$$

for $i = 1, 2$. For $(x, y) \neq (0, 0)$,

$$|\frac{\partial g}{\partial x_1} - 0| = |\frac{2xy^6}{(x^2+y^2)^2}| = |\frac{2xy^6}{(\frac{x^4}{y^4} + \frac{2x^2}{y^2} + 1})| \leq 2|x|y^2 \to 0$$

$$|\frac{\partial g}{\partial x_2} - 0| = |\frac{4x^4y^3 + 2x^2y^5}{(x^4 + 2x^2y^2 + y^4}| = |\frac{4x^2y + 2y^3}{(\frac{x^2}{y^2} + \frac{y^2}{x^2} + 2})| \leq 2x^2|y| + |y|^3 \to 0$$

as $(x, y) \to (0, 0)$ for both of the above.

Therefore, $\frac{\partial g}{\partial x_1}$ and $\frac{\partial g}{\partial x_2}$ are both continuous, and g is therefore continuously differentiable.

□