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Math 126 HW 1
Due 2/02/23

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1. Linear Algebra Review: Matrix Multiplication

Proof WTS the product $A \in \mathbb{R}^{m\times m}$ of an invertible matrix $C \in \mathbb{R}^{m\times m}$ and a singular matrix $B \in \mathbb{R}^{m\times m}$ is also singular.

A matrix is singular if and only if its determinant is zero. Because our *square* matrix exists in a vector space where commutativity is defined, rewrite the determinant as

$$\det(A) = \det(BC) = \det(B)\det(C)$$

Because B was defined to be singular, that means that its determinant is necessarily 0. Therefore, A is necessarily singular by the linearity of the determinant in square matrices.

□

2. Given a $n \times n$ invertible, upper-triangular matrix U, the goal is to generalize the backward substitution to solve $U\mathbf{x} = \mathbf{c}$. Foreward substitution was already performed, as we are starting with an upper-triangular matrix. Backward substitution begins with the n^{th} entry of $\mathbf{x}$ and $\mathbf{c}$. That is,

$$x_n = \frac{c_n}{u_{nn}}$$

the last equation is the first to solve. Next,

$$x_i = \frac{c_i \sum_{j=i+1}^{n} u_{ij} x_i}{u_{ii}}$$

for $i = n, n-1, ..., 1$. The pseudocode is as follows

Algorithm 1: Backward substitution

Data: Upper-triangular matrix U
for $i = 1, ..., n-1$ **do**
  sum $= 0$
  for $j = i+1, j+2, ..., n$ **do**
    sum $=$ sum $+ u_{ij}x_j$
  end
  $x_i = \frac{c_i - \text{sum}}{u_{ii}}$
end

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3. Note that LU decomp functions by reducing matrix A (dim. 3) such that

$$A = LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Start first by defining U as follows

$$U = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{15}{4} & 1 \\ 0 & 1 & 4 \end{bmatrix} \quad R_2 \leftarrow R_2 - \tfrac{1}{4}R_1 \quad \to \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{15}{4} & 1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix} \quad R_3 \leftarrow R_3 - \tfrac{4}{15}R_2$$

Next, define L by filling in the pivots of the identity matrix using the appropriate multipliers used for the row reduction for matrix U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{15} & 1 \end{bmatrix} \quad \begin{array}{l} \text{Replace } l_{21} \text{ with the respective multiplier } \frac{1}{4} \\ \text{Replace } l_{32} \text{ with the respective multiplier } \frac{4}{15} \end{array}$$

Hence, the full decomposition is

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{15} & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{15}{4} & 1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix}.$$

4. An example of *catastrophic collision*:

The relative error between two numbers $x, y \in \mathbb{F}$ and their measured, respective counterparts $\hat{x}, \hat{y}$ is defined as

$$\frac{|x - \hat{x}|}{|x|}.$$

Then, for x

$$x_{err} = \frac{|3.1415927 \times 10^0 - 3.1415929 \times 10^0|}{|3.1415927 \times 10^0|} = 6.3661976296 \times 10^{-8}.$$

Now, for y

$$y_{err} = \frac{|3.1415916 \times 10^0 - 3.1415914 \times 10^0|}{|3.1415916 \times 10^0|} = 6.3661978323 \times 10^{-8}.$$

Now computing the differences between relative errors and actual errors, i.e.

$$\hat{x} - \hat{y} = 3.1415929 \times 10^0 - 3.1415914 \times 10^0 = 0.0000015 = 1.5 \times 10^{-6},$$
$$x - y = 3.1415927 \times 10^0 - 3.1415916 \times 10^0 = 0.0000011 = 1.1 \times 10^{-6}.$$

So overall percentage of error between the differences between these variables is

$$\frac{||x - y| - |\hat{x} - \hat{y}||}{|x - y|} = 0.\overline{36}.$$

The key to what's happening is in the 8th decimal position. This is the number where the subtraction occurs, and it involves a number that is rounded. The only significant figures that remain after the subtraction are the figures that are most reduced. As it happens, numbers are most prone to error because of their location near the end of these indexed digits.

5. Performing Gaussian Elimination

$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \to \begin{bmatrix} 1000 & 999 \\ 0 & 998 - \frac{999^2}{1000} \end{bmatrix} \quad R_1 \leftarrow R_1 \frac{999}{1000}$

Now, doing the same thing with 5 digits of precision

$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \to \begin{bmatrix} 1000 & 999 \\ 0 & 998.001 \end{bmatrix} \quad R_1 \leftarrow R_1 \frac{999}{1000} \quad \to \begin{bmatrix} 1000 & 999 \\ 0 & 0 \end{bmatrix} \quad R_2 \leftarrow R_2 - 999R_1$

6. Exercise 6 asks us to find the cause of the 100% error in x_1 of the linear system

$$\begin{bmatrix} 10^{-9} & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Performing the Gaussian Elimination *without* pivoting

$$\begin{bmatrix} 10^{-9} & 1 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 10^{-9} & 1 \\ 0 & 2 - 10^9 \end{bmatrix} \quad R_2 \leftarrow R_2 - 10^9 R_1$$

or $(2 - 10^9)x^2 = 1 - 10^9 \implies x_2 = \frac{1-10^9}{2-10^9} = 1 \times 10^9$.

Using 8 digits of precision, $x_2 = 1$ (this is where the error lies). Then solving for x_1,

$$10^{-9} x_1 + 1 = 1 \implies x_1 = 0.$$

This is precisely where the error lies. The 8-digits of precision removes that vital significant figure at the end of x_2, hence causing the subtraction in the backward substitution to produce a 0 value because the value in the matrix with the rounding error happened to round to the exact same value as its respective value in the column vector.

7. The matrix from question 5 is clearly linearly dependent. That is, there exists a constant (namely $\frac{999}{1000}$) whose product with the first column produces exactly the second. However, the matrix above is linearly independent. To quote Professor Börgers, the two vectors in the transformation "are not even close to being dependent." Hence, the key difference between the two matrices is linear dependence. Specifically, linear independence allows for the solution to be more stable (through partial pivoting) than linear dependent systems.

   The second matrix provides the column space for the matrix: it has two fully independent columns, so it is full rank in dimension 2. Because the columns of the linear dependent span a lower-dimensional subspace, there may not exist a unique solution. This means that there exists a unique solution that is relatively unaffected by small changes in the data.

8. $$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & 6 & 6 \end{bmatrix}, L_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, L_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

   Now

$$PL_{21}A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 6 & 6 \\ 0 & -3 & 1 \end{bmatrix},$$

$$L_{32}PL_{21}A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 6 & 6 \\ 0 & 0 & 4 \end{bmatrix} = U,$$

as defined in the question. Then because $P = P^T$,

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

This is still upper-triangular. Next, solving $L_{32}PL_{12}P^TPA = L_{32}\tilde{L}IA \implies PA = \tilde{L}^{-1}L_{32}^{-1}U$ and $L_{32}\tilde{L}PA = U$. So

$$\tilde{L}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}, \tilde{L}^{-1}L_{32}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 6 & 6 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 6 & 6 \\ 2 & -2 & 1 \end{bmatrix}.$$

Then,

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 6 & 6 \\ 2 & -2 & 1 \end{bmatrix}.$$

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